

Light-cone gauge for black-hole perturbation theory

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The geometrical meaning of the Eddington-Finkelstein coordinates of Schwarzschild spacetime is well understood: (i) the advanced-time coordinate v is constant on incoming light cones that converge toward $r = 0$, (ii) the angles θ and ϕ are constant on the null generators of each light cone, (iii) the radial coordinate r is an affine-parameter distance along each generator, and (iv) r is an areal radius, in the sense that $4\pi r^2$ is the area of each two-surface $(v, r) = \text{constant}$. The light-cone gauge of black-hole perturbation theory, which is formulated in this paper, places conditions on a perturbation of the Schwarzschild metric that ensure that properties (i)–(iii) of the coordinates are preserved in the perturbed spacetime. Property (iv) is lost, in general, but it is retained in exceptional situations that are identified in this paper. Unlike other popular choices of gauge, the light-cone gauge produces a perturbed metric that is expressed in a meaningful coordinate system; this is a considerable asset that greatly facilitates the task of extracting physical consequences. We illustrate the use of the light-cone gauge by calculating the metric of a black hole immersed in a uniform magnetic field. We construct a three-parameter family of solutions to the perturbative Einstein-Maxwell equations and argue that it is applicable to a broader range of physical situations than the exact, two-parameter Schwarzschild-Melvin family.

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I. INTRODUCTION

The theory of perturbations of the Schwarzschild spacetime is a well-developed one [1], and it may seem surprising that authors are still writing on this venerable topic almost 50 years after its inception in the work of Regge and Wheeler [2] (see also Refs. [3,4]). This is (at least partially) explained by the fact that the field has witnessed a resurgence of sorts in the last several years, motivated by new applications that include the gravitational self-force problem [5–8], the “close-limit” collision of two black holes [9], and the study of the dynamics of black holes placed in tidal environments [10–14]. The theory has been presented in various sophisticated packages [15–21], and it has reached what is likely to be its definitive form.

We wish to make an additional contribution to this body of literature by formulating a useful, attractive, and simple gauge condition for black-hole perturbation theory. We believe that this gauge, which we call *the light-cone gauge*, is preferable in many ways to most popular gauges, including the oft-used Regge-Wheeler gauge. We believe that the use of the light-cone gauge will be a great benefit to any researcher faced with the task of computing and interpreting a perturbation of the Schwarzschild spacetime.

The idea is simple. The difficulties of the Schwarzschild coordinates (t, r, θ, ϕ) across the black-hole horizon are well documented, and it is well known that the transformation $v = t + r + 2M \ln|r/2M - 1|$ brings the metric to a form that is well behaved on the event horizon. The Eddington-Finkelstein coordinates (v, r, θ, ϕ) have a clear geometrical meaning. The null coordinate v (called advanced time) is constant on incoming light cones that converge toward $r = 0$, the angles θ and ϕ are constant on the null generators of each light cone $v = \text{constant}$, and

r is an affine-parameter distance along each generator. In addition, r doubles as an areal radius, in the sense that $4\pi r^2$ is the area of each two-sphere $(v, r) = \text{constant}$.

The light-cone gauge places conditions on the metric perturbation that ensure that the geometrical meaning of the coordinates is preserved in the perturbed spacetime. The advanced-time coordinate v therefore continues to label incoming light cones that converge toward $r = 0$, the angles θ and ϕ continue to label the generators of each light cone, and r continues to be an affine-parameter distance along each generator. One geometrical aspect of the coordinates that must generally be given up is the role of r as an areal radius; we shall show, however, that this property also can be preserved in special circumstances. The light-cone gauge therefore produces a perturbed metric that is expressed in a meaningful coordinate system. This is a considerable asset that greatly facilitates the task of extracting the physical properties of the spacetime.

The light-cone gauge is developed in Sec. II of this paper. The gauge conditions are introduced in Sec. IIB, after we present in Sec. IIA a brief review of the Schwarzschild metric in Eddington-Finkelstein coordinates. In Secs. IIC and IID we decompose the metric perturbation in spherical harmonics and explore the space of gauge transformations that keep the perturbation within the light-cone gauge. This remaining gauge freedom is convenient, as it can be exploited to simplify the form of the perturbed metric to the fullest extent possible. In Sec. IIE we determine the conditions under which r retains its interpretation as an areal radius. The answer turns out to be simple: This is possible whenever the component $T^{vv} = T_{rr}$ of the perturbing energy-momentum tensor vanishes. In Sec. IIF we summarize our construction and discuss its merits; in particular, we compare our light-cone gauge to

the very widely used, but far less compelling, Regge-Wheeler gauge.

In Sec. III we present an illustrating application of the light-cone gauge for black-hole perturbation theory: We examine a black hole immersed in a uniform magnetic field, and calculate its metric accurately through second order in the strength of the magnetic field. The physical situation is described in Sec. III A. The magnetic field and its energy-momentum tensor are computed in Sec. III B. In Sec. III C and III D we integrate the equations of black-hole perturbation theory for this situation. The solution is presented in Sec. III E, and in Sec. III F we examine the structure of the perturbed horizon. Finally, in Sec. III G we compare our perturbative solution to the exact Schwarzschild-Melvin solution [22–24], which also describes a magnetized black hole. We conclude that the perturbative solution is applicable to a broader range of physical situations.

In the Appendix we provide a complete listing of the linearized field equations in the light-cone gauge.

Our developments in this paper rely heavily on the recent work of Martel and Poisson [21], which presents a covariant and gauge-invariant formalism for black-hole perturbation theory. They also incorporate key ideas from a companion paper [25] devoted to the construction of light-cone coordinates centered on a geodesic world line of an arbitrary curved spacetime.

We point out that the light-cone gauge constructed here is adapted specifically to incoming light cones $v = \text{constant}$ that converge toward $r = 0$. It would be exceedingly straightforward to adapt the construction to outgoing light cones $u = \text{constant}$ that expand toward $r = \infty$. (In Schwarzschild spacetime, the retarded time coordinate u is defined by $u = t - r - 2M \ln|r/2M - 1|$.) While the incoming light-cone gauge is well suited to study the properties of the perturbed event horizon, the outgoing light-cone gauge would be well suited to study the gravitational radiation escaping toward future null infinity. We suggest this adaptation as an exercise for the reader.

Throughout the paper we work in geometrized units ($c = G = 1$) and adhere to the conventions of Misner, Thorne, and Wheeler [26].

II. LIGHT-CONE GAUGE: DEFINITION AND PROPERTIES

A. Schwarzschild metric in light-cone coordinates

The transformation $v = t + r + 2M \ln|r/2M - 1|$ brings the Schwarzschild metric from its usual form to the Eddington-Finkelstein form

$$ds^2 = -fdv^2 + 2dvdr + r^2d\Omega^2, \quad (2.1)$$

where

$$f := 1 - \frac{2M}{r} \quad (2.2)$$

and

$$d\Omega^2 := \Omega_{AB}d\theta^A d\theta^B := d\theta^2 + \sin^2\theta d\phi. \quad (2.3)$$

The parameter M is the black-hole mass and $\theta^A = (\theta^2, \theta^3) = (\theta, \phi)$ are angles that span the two-spheres $(v, r) = \text{constant}$.

The coordinates (v, r, θ, ϕ) are well behaved across the event horizon, and they possess a clear geometrical meaning. We note first that the vector

$$l_\alpha := -\nabla_\alpha v = (-1, 0, 0, 0) \quad (2.4)$$

is null, which implies that each surface $v = \text{constant}$ is a null hypersurface of the Schwarzschild spacetime; these are, in fact, incoming light cones that converge toward the black-hole singularity. The fact that l_α is a gradient implies that l^α is everywhere tangent to a congruence of null geodesics; these are affinely parameterized, and they are the generators of each light cone $v = \text{constant}$. Using the metric of Eq. (2.1) to raise indices, we find that

$$l^\alpha = (0, -1, 0, 0). \quad (2.5)$$

This relation implies that $\theta^A = \text{constant}$ on the generators, so that the angles θ^A can be used as generator labels. Furthermore, Eq. (2.5) reveals that the affine parameter on each generator is $-r$. The geometrical meaning of the coordinates is therefore the following: The null coordinate v (called advanced time) is constant on incoming light cones that converge toward $r = 0$, θ^A labels the generators of each light cone, and r is an affine-parameter distance along each generator. The radial coordinate r also doubles as an areal radius, meaning that $4\pi r^2$ is the area of each two-sphere $(v, r) = \text{constant}$.

B. Perturbed metric in light-cone coordinates

We introduce a perturbation $p_{\alpha\beta}$ of the Schwarzschild metric, defined by the statement

$$g_{\alpha\beta}^{\text{perturbed}} = g_{\alpha\beta} + p_{\alpha\beta}, \quad (2.6)$$

where $g_{\alpha\beta}^{\text{perturbed}}$ is the metric of the perturbed spacetime, while $g_{\alpha\beta}$ represents the Schwarzschild solution, which we express in the light-cone coordinates of Eq. (2.1). *We wish to place conditions on the metric perturbation that ensure that the meaning of the light-cone coordinates will be preserved in the perturbed spacetime.* Specifically, we demand that, in the perturbed spacetime, v continues to be constant on incoming light cones that converge toward $r = 0$, θ^A continue to be constant on the null generators of each light cone, and r continues to be an affine-parameter distance along each generator. In exceptional circumstances that will be identified in Sec. II E below, r also reprises its role as an areal radius, but, in general, this property will not be preserved in the perturbed spacetime.

The geometrical meaning of the coordinates will be preserved if Eqs. (2.4) and (2.5) continue to hold in the

perturbed spacetime. We now have $l_\alpha = (g_{\alpha\beta} + p_{\alpha\beta})l^\beta$, and we infer that the perturbation must satisfy the *gauge conditions*

$$p_{\alpha\beta}l^\beta = 0 \quad \Rightarrow \quad p_{vr} = p_{rr} = p_{rA} = 0. \quad (2.7)$$

There are four conditions, which we refer to as the *light-cone gauge conditions*. A metric perturbation $p_{\alpha\beta}$ that satisfies Eqs. (2.7) will be said to be in a light-cone gauge. As we shall see below, Eqs. (2.7) do not completely fix the gauge, and the remaining gauge freedom can be exploited to simplify the form of the perturbed metric.

The gauge conditions leave p_{vv} , p_{vA} , and p_{AB} as nonvanishing components of the metric perturbation. The trace of the perturbation is

$$p := g^{\alpha\beta} p_{\alpha\beta} = r^{-2} \Omega^{AB} p_{AB}, \quad (2.8)$$

where Ω^{AB} is the matrix inverse of Ω_{AB} , defined by Eq. (2.3). The determinant of the perturbed metric is given by

$$\sqrt{-g^{\text{perturbed}}} = r^2 \sin\theta \left(1 + \frac{1}{2}p\right), \quad (2.9)$$

and r will retain its role as an areal radius whenever the metric perturbation has a vanishing trace. In Sec. II E we will determine under what conditions this happens.

C. Even-parity sector

The even-parity sector refers to those components of the metric perturbation that can be expanded in terms of even-parity spherical harmonics Y^{lm} , Y_A^{lm} , $\Omega_{AB} Y^{lm}$, and Y_{AB}^{lm} . (Throughout the paper we use the notation of Martel and Poisson [21].) The scalar harmonics Y^{lm} are the usual spherical-harmonic functions, the vectorial harmonics are defined by $Y_A^{lm} := D_A Y^{lm}$ (where D_A is the covariant derivative operator compatible with Ω_{AB}), and the tensorial harmonics are defined by $Y_{AB}^{lm} := [D_A D_B + \frac{1}{2}l(l+1)\Omega_{AB}]Y^{lm}$; these are trace-free by virtue of the eigenvalue equation for the spherical harmonics: $\Omega^{AB} Y_{AB}^{lm} = [\Omega^{AB} D_A D_B + l(l+1)]Y^{lm} = 0$.

The even-parity sector is

$$p_{ab} = \sum_{lm} h_{ab}^{lm}(x^a) Y^{lm}(\theta^A), \quad (2.10)$$

$$p_{aB} = \sum_{lm} j_a^{lm}(x^a) Y_B^{lm}(\theta^A), \quad (2.11)$$

$$p_{AB} = r^2 \sum_{lm} [K^{lm}(x^a) \Omega_{AB} Y^{lm}(\theta^A) + G^{lm}(x^a) Y_{AB}^{lm}(\theta^A)], \quad (2.12)$$

where $x^a = (x^0, x^1) = (v, r)$. The sums over the integer l begin at $l = 0$ in the case of Eq. (2.10) and the K^{lm} term in Eq. (2.12), at $l = 1$ in the case of Eq. (2.11), and at $l = 2$ in the case of the G^{lm} term in Eq. (2.12). The sums over the

integer m go from $-l$ to $+l$. The light-cone gauge conditions are

$$h_{vr}^{lm} = h_{rr}^{lm} = j_r^{lm} = 0. \quad (2.13)$$

The components h_{vv}^{lm} , j_v^{lm} , K^{lm} , and G^{lm} are nonzero in the light-cone gauge.

Even-parity gauge transformations are generated by a dual vector field $\Xi_\alpha = (\Xi_a, \Xi_A)$ that can be expanded as

$$\Xi_a = \sum_{lm} \xi_a^{lm}(x^a) Y^{lm}(\theta^A), \quad (2.14)$$

$$\Xi_A = \sum_{lm} \xi_A^{lm}(x^a) Y_A^{lm}(\theta^A). \quad (2.15)$$

According to Eqs. (4.6)–(4.9) of Martel and Poisson [21], such a transformation will preserve the conditions of Eq. (2.13) provided that ξ_v , ξ_r , and ξ satisfy the equations

$$0 = \frac{\partial \xi_v}{\partial r} + \frac{\partial \xi_r}{\partial v} + \frac{2M}{r^2} \xi_r, \quad 0 = \frac{\partial \xi_r}{\partial r},$$

$$0 = \frac{\partial \xi}{\partial r} + \xi_r - \frac{2}{r} \xi.$$

(We henceforth omit the spherical-harmonic indices for brevity. Our considerations momentarily exclude the special cases $l = 0$ and $l = 1$, which will be handled separately below.) This means that a gauge transformation generated by

$$\xi_v = -\dot{a}(v)r - fa(v) + b(v), \quad (2.16)$$

$$\xi_r = a(v), \quad (2.17)$$

$$\xi = a(v)r + c(v)r^2 \quad (2.18)$$

will keep a perturbation within the light-cone gauge. The remaining gauge freedom is therefore characterized by three arbitrary functions $a(v)$, $b(v)$, $c(v)$, and the overdot in Eq. (2.16) indicates differentiation with respect to v . The gauge transformation changes the nonvanishing components of the perturbation field according to

$$h_{vv} \rightarrow h'_{vv} = h_{vv} + 2\ddot{a}r + 2\left(1 - \frac{3M}{r}\right)\dot{a} - 2\dot{b} + \frac{2M}{r^2}b, \quad (2.19)$$

$$j_v \rightarrow j'_v = j_v + fa - b - \dot{c}r^2, \quad (2.20)$$

$$K \rightarrow K' = K + 2\dot{a} + \frac{l(l+1)}{r}a - \frac{2}{r}b + l(l+1)c, \quad (2.21)$$

$$G \rightarrow G' = G - \frac{2}{r}a - 2c. \quad (2.22)$$

The lower multipoles $l = 0$ and $l = 1$ must be considered separately. For $l = 0$ the spherical harmonics Y_A and Y_{AB} are identically zero, and the only relevant perturbation

fields are h_{ab} and K ; j_a and G are not defined. A gauge transformation is then generated by $\Xi_a = \xi_a Y^{00}$, $\Xi_A = 0$, and ξ is not defined. It is easy to check that the light-cone gauge will be preserved with a ξ_a that is still given by Eqs. (2.16) and (2.17). In this case the remaining gauge freedom is characterized by two arbitrary functions, $a(v)$ and $b(v)$. The corresponding change in h_{vv} is still given by Eq. (2.19), while K' is obtained by setting $l=0$ in Eq. (2.21). For $l=1$ the tensorial harmonics Y_{AB} are identically zero, and only G is not defined. The gauge transformation of Eqs. (2.16), (2.17), and (2.18) is still seen to preserve the light-cone gauge, and the changes in the perturbation fields are still described by Eqs. (2.19), (2.20), and (2.21), in which we must set $l(l+1)=2$; Eq. (2.22) is irrelevant when $l=1$.

It is easy to verify that the dual vector field of Eqs. (2.14), (2.15), (2.16), (2.17), and (2.18) generates the (small) coordinate transformation

$$v \rightarrow v' = v + a(v, \theta^A), \quad (2.23)$$

$$r \rightarrow r' = \left(1 - \frac{\partial a}{\partial v}\right)r + b(v, \theta^A), \quad (2.24)$$

$$\theta^A \rightarrow \theta'^A = \theta^A + \Omega^{AB} \frac{\partial}{\partial \theta^B} \left[\frac{a}{r} + c(v, \theta^A) \right], \quad (2.25)$$

where $a(v, \theta^A) := \sum_{lm} a^{lm}(v) Y^{lm}(\theta^A)$, with similar equations defining $b(v, \theta^A)$ and $c(v, \theta^A)$. This transformation leaves the conditions of Eq. (2.7) intact.

D. Odd-parity sector

The odd-parity sector refers to those components of the metric perturbation that can be expanded in terms of odd-parity spherical harmonics X_A^{lm} and X_{AB}^{lm} . The vectorial harmonics are defined by $X_A^{lm} := -\varepsilon_A{}^B D_B Y^{lm}$, where ε_{AB} is the Levi-Civita tensor on the unit two-sphere (with the independent component $\varepsilon_{\theta\phi} = \sin\theta$), and where $\varepsilon_A{}^B := \Omega^{BC} \varepsilon_{AC}$. The tensorial harmonics are $X_{AB}^{lm} := -\frac{1}{2}(\varepsilon_A{}^C D_B + \varepsilon_B{}^C D_A) D_C Y^{lm}$; these are trace-free by virtue of the antisymmetry of the Levi-Civita tensor: $\Omega^{AB} X_{AB}^{lm} = -\varepsilon^{AB} D_A D_B Y^{lm} = 0$.

The odd-parity sector is

$$p_{ab} = 0, \quad (2.26)$$

$$p_{aB} = \sum_{lm} h_a^{lm}(x^a) X_B^{lm}(\theta^A), \quad (2.27)$$

$$p_{AB} = \sum_{lm} h_2^{lm}(x^a) X_{AB}^{lm}(\theta^A). \quad (2.28)$$

The sums over the integer l begin at $l=1$ in the case of Eq. (2.27), and at $l=2$ in the case of Eq. (2.28). The light-cone gauge conditions are

$$h_r^{lm} = 0. \quad (2.29)$$

The components h_v^{lm} and h_2^{lm} are nonzero in the light-cone gauge.

Odd-parity gauge transformations are generated by a dual vector field $\Xi_\alpha = (\Xi_a, \Xi_A)$ that can be expanded as

$$\Xi_a = 0, \quad (2.30)$$

$$\Xi_A = \sum_{lm} \xi^{lm}(x^a) X_A^{lm}(\theta^A). \quad (2.31)$$

According to Eqs. (5.5) of Martel and Poisson [21], such a transformation will preserve the conditions of Eq. (2.29) provided that ξ satisfies $\partial \xi / \partial r - 2\xi/r = 0$. (We henceforth omit the spherical-harmonic indices for brevity. Our considerations momentarily exclude the special case $l=1$, which will be handled separately below.) This means that a gauge transformation generated by

$$\xi = \alpha(v) r^2 \quad (2.32)$$

will keep a perturbation within the light-cone gauge. The remaining gauge freedom is therefore characterized by a single arbitrary function $\alpha(v)$. The gauge transformation changes the nonvanishing components of the perturbation field according to

$$h_v \rightarrow h'_v = h_v - \dot{\alpha} r^2, \quad (2.33)$$

$$h_2 \rightarrow h'_2 = h_2 - 2\alpha r^2, \quad (2.34)$$

where an overdot indicates differentiation with respect to v .

The situation is the same for the special case $l=1$, except that X_{AB} is then identically zero and h_2 is not defined. The gauge transformation of Eq. (2.32) is still seen to preserve the light-cone gauge, and it still changes h_v according to Eq. (2.33); Eq. (2.34) is then irrelevant.

It is easy to verify that the dual vector field of Eqs. (2.30), (2.31), and (2.32) generates the (small) coordinate transformation

$$\theta^A \rightarrow \theta'^A = \theta^A - \varepsilon^{AB} \frac{\partial}{\partial \theta^B} \alpha(v, \theta^A), \quad (2.35)$$

where $\alpha(v, \theta^A) := \sum_{lm} \alpha^{lm}(v) Y^{lm}(\theta^A)$ and $\varepsilon^{AB} := \Omega^{AC} \Omega^{BD} \varepsilon_{CD}$. This transformation leaves the conditions of Eq. (2.7) intact.

E. When is r an areal radius?

According to Eqs. (2.8) and (2.9), r keeps its interpretation as an areal radius whenever $\Omega^{AB} p_{AB} = 0$. And, according to Eq. (2.12), this happens when $K^{lm}(v, r) = 0$ for all values of l and m . In this subsection we determine under what circumstances it is possible to impose this condition.

The light-cone gauge produces a very convenient decoupling of the equation that governs the behavior of K^{lm} from the equations that determine the remaining perturbation fields. According to the field equations listed in the Appendix, we have

$$Q_{lm}^{vv} = Q_{rr}^{lm} = -\frac{\partial^2}{\partial r^2} K^{lm} - \frac{2}{r} \frac{\partial}{\partial r} K^{lm}, \quad (2.36)$$

where, for example, $Q_{rr}^{lm} := 8\pi \int T_{rr} \bar{Y}^{lm} d\Omega$, with $d\Omega = \sin\theta d\theta d\phi$, are the spherical-harmonic projections of the rr component of the energy-momentum tensor. When $T^{vv} = T_{rr} = 0$, Eq. (2.36) reveals that K^{lm} must be of the form $p^{lm}(v) + q^{lm}(v)/r$, where p^{lm} and q^{lm} are arbitrary functions of v . But it is possible to exploit the remaining gauge freedom contained in Eqs. (2.16), (2.17), and (2.18) to set $K^{lm} = 0$. As can be seen from Eq. (2.23), this condition constrains the functions $b^{lm}(v)$ and $c^{lm}(v)$, which must now be related to $a^{lm}(v)$. The remaining gauge freedom is therefore restricted to transformations characterized by a single arbitrary function, $a^{lm}(v)$. Our conclusion is that K^{lm} can be set equal to zero whenever $T^{vv} = T_{rr} = 0$, and that this operation still does not fully exhaust the gauge freedom.

We have established the following theorem: When the energy-momentum tensor responsible for the metric perturbation is such that

$$T_{\alpha\beta} l^\alpha l^\beta = 0, \quad (2.37)$$

the light-cone gauge can be refined to include the trace-free condition

$$p := g^{\alpha\beta} p_{\alpha\beta} = 0 \quad (2.38)$$

in addition to the four conditions of Eq. (2.7). In such circumstances, $\sqrt{-g^{\text{perturbed}}} = r^2 \sin\theta$ and r retains its interpretation as an areal radius. In these circumstances the light-cone gauge becomes the ‘‘incoming radiation gauge’’ of Chrzanowski [27–30].

The theorem relating the trace-free condition of Eq. (2.38) to the vanishing of $T_{\alpha\beta} l^\alpha l^\beta$ is a new result. The theorem was established independently by Price, Shankar, and Whiting [31,32] in work that has not yet been published, except for a statement of the result made in Sec. 4.1 of Ref. [33]. Remarkably, these authors were able to extend the theorem from Schwarzschild spacetime to all Petrov type-II spacetimes.

F. Discussion; comparison with the Regge-Wheeler gauge

The light-cone gauge possesses two main virtues. The first is that it involves simple algebraic conditions on the metric perturbation; these were stated in covariant form in Eq. (2.7), $p_{\alpha\beta} l^\beta = 0$, and they were stated in expanded form in Eqs. (2.13) and (2.29), $h_{vr}^{lm} = h_{rr}^{lm} = j_r^{lm} = h_r^{lm} = 0$. The second is that the gauge conditions preserve the geometrical meaning of the original coordinate system (v, r, θ^A) ; as was shown in Sec. II B, the advanced-time coordinate v continues to label incoming light cones that converge toward $r = 0$, the angles θ^A continue to label the generators of each light cone, and the radial coordinate r continues to be an affine-parameter distance along each

generator. The task of extracting the physical properties of a perturbed spacetime will be greatly facilitated by the use of such meaningful coordinates.

Most of the literature on black-hole perturbation theory employs an alternative gauge known as the ‘‘Regge-Wheeler gauge’’ [2]. The gauge conditions in this case are $j_v^{lm} = j_r^{lm} = G^{lm} = h_2^{lm} = 0$. The Regge-Wheeler gauge also has the advantage of involving simple algebraic conditions on the metric perturbation. But unlike the light-cone gauge, the Regge-Wheeler gauge produces a coordinate system that does not possess a clear geometrical meaning; this is a disadvantage. And, indeed, the coordinates can sometimes be pathological. For example, the Regge-Wheeler gauge produces metric components that do not display asymptotically flat behavior near future null infinity, even when the source of the perturbation is spatially bounded [9]. This problem is associated with the fact that, by imposing $G^{lm} = h_2^{lm} = 0$, the Regge-Wheeler gauge is actually setting to zero the transverse–trace-free part of the metric perturbation, thereby effectively ‘‘gauging away’’ its gravitational-wave content. (The gravitational-wave modes are still present in the metric perturbation, but in the Regge-Wheeler gauge they are encoded in unnatural places.) The end result is a meaningless coordinate system, a metric perturbation that fails to be asymptotically flat, and a spacetime that does not easily reveal its radiation content. These problems are not present in the light-cone gauge.

Another approach that has been followed in the literature on black-hole perturbation theory is to avoid fixing the gauge, and to work instead with a gauge-invariant formalism [15–21]. Such an approach can be very useful, especially when an application calls for a switch from one gauge to another. We would argue, however, that a *good choice of gauge* can be even more useful in concrete applications. After all, most relativists would begin an investigation of the Schwarzschild spacetime by making a specific choice of coordinate system; few relativists would insist on staying coordinate-free. And most relativists would agree that the Eddington-Finkelstein system (v, r, θ^A) is more convenient to work with than the Schwarzschild coordinates (t, r, θ^A) when one is concerned with the event horizon; these relativists would say that the Eddington-Finkelstein coordinates are *good coordinates*. These attitudes need not change when one goes slightly away from the Schwarzschild spacetime, and the light-cone gauge provides a *good coordinate system* to investigate the perturbed spacetimes.

III. BLACK HOLES IN A MAGNETIC FIELD

A. Physical situation

To illustrate the use of the light-cone gauge in black-hole perturbation theory, we work through a model problem involving a black hole immersed in a uniform magnetic field. We have in mind a situation in which a large me-

chanical structure, such as a giant solenoid, is set up in outer space and made to produce a uniform magnetic field of strength B . The structure has a mass M' and its linear extension is of the order of the length scale¹ a ; the magnetic field is imagined to be uniform over a region of this size. A black hole of mass M , initially isolated, is then brought to the structure and inserted within the magnetic field. This process is quasistatic, and, as we shall justify in Sec. III F below, the black hole's surface area stays constant during the immersion. We wish to study how the black hole distorts the magnetic field within the structure, and how the magnetic field distorts the geometry of the black hole.

We suppose that the perturbation created by the magnetic field is small and that its effects can be adequately calculated with the equations of black-hole perturbation theory. We shall see below that the criterion for this is $r^2 B^2 \ll 1$, where r is the distance from the black hole. If we restrict our attention to the interior of the mechanical structure and impose the inequality $r < a$, then the perturbative criterion is

$$a^2 B^2 \ll 1. \quad (3.1)$$

In addition to Eq. (3.1) we assume that the structure is situated in the black hole's weak-field region, so that

$$\frac{M}{a} \ll 1. \quad (3.2)$$

While $a^2 B^2$ and M/a must both be small, their relative sizes are not constrained. We may imagine that M/a is either much smaller than, comparable to, or much larger than $a^2 B^2$; black-hole perturbation theory can handle all these situations. Below we will be particularly (but not exclusively) interested in the first possibility, $M/a \ll a^2 B^2$ or $M/a^3 \ll B^2$. In this situation there exists an asymptotic region (described by $M \ll r < a$) in which the gravitational effects of the magnetic field, though small, are larger than those of the black hole.

Another aspect of our model problem is the tidal gravity exerted by the mechanical structure. Because the structure has a mass M' and is situated at a distance a from the black hole, the tidal field (or Weyl-curvature) it produces near the black hole is $\mathcal{E} \sim M'/a^3$. This quantity \mathcal{E} , which will be formally introduced below, is an additional parameter that

characterizes the physical situation. Below we will imagine that \mathcal{E} is of the same order of magnitude as B^2 , so that $M'/a^3 \sim B^2$. Our results, however, will not be tied to this assumption; they will be just as valid when \mathcal{E} is much smaller than (or indeed much larger than) B^2 .

The perturbed black-hole solution that we construct below is, in fact, a three-parameter family of solutions; each solution is characterized by the black-hole mass M , the magnetic field strength B , and the tidal gravity \mathcal{E} . The solution is obtained perturbatively through order (B^2, \mathcal{E}) .

B. Magnetic field

We first calculate the electromagnetic field $F_{\alpha\beta}$ that surrounds the black hole. Because we seek to determine the perturbed metric accurately through order B^2 , it is sufficient to calculate $F_{\alpha\beta}$ to order B . And because the metric corrections of order B^2 do not enter this calculation, we may let the spacetime have an unperturbed Schwarzschild metric.

To find the electromagnetic field, we rely on Wald's observation [34] that, in a vacuum spacetime, any Killing vector can be identified with the vector potential of a test electromagnetic field. The fact that the vector satisfies Killing's equation ensures that the resulting $F_{\alpha\beta}$ satisfies the source-free Maxwell equations. To produce a magnetic field that is asymptotically uniform when $r \gg M$, we set

$$A^\alpha = \frac{1}{2} B \phi^\alpha, \quad (3.3)$$

where $\phi^\alpha := (0, 0, 0, 1)$ is the rotational Killing vector of the unperturbed Schwarzschild spacetime; we use the ordering (v, r, θ, ϕ) of the unperturbed light-cone coordinates.

The vector potential gives rise to an electromagnetic field tensor $F_{\alpha\beta} = \nabla_\alpha A_\beta - \nabla_\beta A_\alpha$. To display its components it is useful to decompose it in an orthonormal tetrad e_μ^α that is oriented along the ‘‘Cartesian directions’’ associated with the ‘‘spherical coordinates’’ (r, θ, ϕ) ; here the superscript α is the usual vectorial index, and the subscript μ is a frame index that identifies each member of the tetrad. We thus introduce the tetrad

$$e_0^\alpha = (f^{-1/2}, 0, 0, 0), \quad (3.4)$$

$$e_1^\alpha = (f^{-1/2} \sin\theta \cos\phi, f^{1/2} \sin\theta \cos\phi, r^{-1} \cos\theta \cos\phi, -r^{-1} \sin\phi / \sin\theta), \quad (3.5)$$

$$e_2^\alpha = (f^{-1/2} \sin\theta \sin\phi, f^{1/2} \sin\theta \sin\phi, r^{-1} \cos\theta \sin\phi, r^{-1} \cos\phi / \sin\theta), \quad (3.6)$$

¹The constant a is not to be confused with the functions $a^{lm}(v)$ introduced in Sec. II C.

$$e_3^\alpha = (f^{-1/2} \cos\theta, f^{1/2} \cos\theta, -r^{-1} \sin\theta, 0). \quad (3.7)$$

We may think of e_1^α as pointing in the “ x direction,” e_2^α as pointing in the “ y direction,” and e_3^α as pointing in the “ z direction.” In this tetrad, the nonvanishing frame components of the electromagnetic field tensor are

$$B_1 := F_{23} := F_{\alpha\beta} e_2^\alpha e_3^\beta = B(1 - \sqrt{f}) \sin\theta \cos\theta \cos\phi, \quad (3.8)$$

$$B_2 := F_{31} := F_{\alpha\beta} e_3^\alpha e_1^\beta = B(1 - \sqrt{f}) \sin\theta \cos\theta \sin\phi, \quad (3.9)$$

$$B_3 := F_{12} := F_{\alpha\beta} e_1^\alpha e_2^\beta = B[\sqrt{f} + (1 - \sqrt{f})\cos^2\theta], \quad (3.10)$$

where $f := 1 - 2M/r$. The field is purely magnetic. Asymptotically, when $r \gg M$, $B_1 \sim 0$, $B_2 \sim 0$, $B_3 \sim B$; the magnetic field is uniform and aligned with the z direction. Closer to the black hole the magnetic field is distorted; as $r \rightarrow 2M$ we have $|\mathbf{B}|^2 := B_1^2 + B_2^2 + B_3^2 \rightarrow B^2 \cos^2\theta$, which indicates that the field is strongest at the poles. We note that the tetrad is defined outside the event horizon only; as a consequence, Eqs. (3.8), (3.9), and (3.10) are valid for $r > 2M$ only. The electromagnetic field, however, extends below the event horizon, and its components in the coordinates (v, r, θ, ϕ) are well behaved across $r = 2M$.

The electromagnetic field produces an energy-momentum tensor given by

$$T^{\alpha\beta} = \frac{1}{4\pi} \left(F^{\alpha\gamma} F_\gamma^\beta - \frac{1}{4} g^{\alpha\beta} F^{\gamma\delta} F_{\gamma\delta} \right). \quad (3.11)$$

Its nonvanishing components are

$$T^{vv} = \frac{B^2}{4\pi} \sin^2\theta, \quad (3.12)$$

$$T^{vr} = \frac{B^2}{8\pi} \frac{1}{r} [r - 2M - 2(r - M)\cos^2\theta], \quad (3.13)$$

$$T^{rr} = \frac{B^2}{8\pi} \frac{r - 2M}{r^2} [r - 2M - 2(r - M)\cos^2\theta], \quad (3.14)$$

$$T^{v\theta} = \frac{B^2}{4\pi} \frac{1}{r} \sin\theta \cos\theta, \quad (3.15)$$

$$T^{r\theta} = \frac{B^2}{4\pi} \frac{r - 2M}{r^2} \sin\theta \cos\theta, \quad (3.16)$$

$$T^{\theta\theta} = -\frac{B^2}{8\pi} \frac{1}{r^3} [r - 2M - 2(r - M)\cos^2\theta], \quad (3.17)$$

$$T^{\phi\phi} = \frac{B^2}{8\pi} \frac{1}{r^3 \sin^2\theta} [r - 2M + 2M\cos^2\theta]. \quad (3.18)$$

These equations are valid both above and below the horizon. This energy-momentum tensor is the source of the metric perturbation that will be calculated in the following subsections.

It is easy to see from Eqs. (3.12), (3.13), (3.14), (3.15), (3.16), (3.17), and (3.18) that the angular dependence of the energy-momentum tensor is contained entirely in spherical-harmonic functions of degrees $l = 0$ and $l = 2$; and, because there is no dependence on ϕ , only functions with an azimuthal index $m = 0$ are involved. It can also be seen that the angular dependence of the energy-momentum tensor has an even parity. Our solution to the equations of black-hole perturbation theory will therefore have the following properties: (i) it will be axially symmetric; (ii) it will contain even-parity spherical-harmonic modes with $(l, m) = \{(0, 0), (2, 0)\}$ only; and (iii) it will be stationary. The metric perturbation will contain a term of magnetic origin, and it will also contain a homogeneous term associated with the ambient Weyl curvature.

C. Integrating the perturbation equation: $l = 0$

As discussed in Sec. IV D of Martel and Poisson [21] (see also the Appendix of this paper), the relevant projections of the energy-momentum tensor when $l = 0$ are Q^{ab} and Q^b , which are defined in the Appendix. Using $Y^{00} = 1/\sqrt{4\pi}$ and the energy-momentum tensor of Eqs. (3.12), (3.13), (3.14), (3.15), (3.16), (3.17), and (3.18), we obtain

$$Q^{vv} = 4b^2, \quad (3.19)$$

$$Q^{vr} = b^2 \frac{r - 4M}{r}, \quad (3.20)$$

$$Q^{rr} = b^2 \frac{(r - 2M)(r - 4M)}{r^2}, \quad (3.21)$$

$$Q^b = 2b^2, \quad (3.22)$$

where² $b^2 := 2\sqrt{\pi}B^2/3$.

We now integrate the perturbation equations for the two relevant functions $K(r)$ and $h_{vv}(r)$ —please refer to the listing of field equations in the Appendix. We first substitute Eq. (3.19) into Eq. (2.36) and solve for K . The general solution is $K(r) = -\frac{2}{3}b^2 r^2 + p + q/r$, where p and q are arbitrary constants. As discussed in Secs. II C and II E, we may exploit the remaining gauge freedom to set them equal to zero. We have, therefore,

$$K = -\frac{2}{3}b^2 r^2. \quad (3.23)$$

The remaining field equations provide a number of equivalent differential equations for h_{vv} . The general solution is $h_{vv}(r) = -\frac{1}{3}b^2 r(3r - 8M) + 2\delta M/r$. It involves an arbi-

²The constant b is not to be confused with the functions $b^{lm}(v)$ introduced in Sec. II C.

trary constant δM that can be interpreted as a shift in M , the black-hole mass parameter. To reflect the fact that we wish our perturbed black hole to have the same surface area as our original, unperturbed black hole (this was motivated back in Sec. III A), we set $\delta M = 0$. We will verify in Sec. III F that this condition does indeed lead to a preservation of the horizon area. We have, therefore,

$$h_{vv} = -\frac{1}{3}b^2r(3r - 8M). \quad (3.24)$$

Substituting Eqs. (3.23) and (3.24) into Eqs. (2.10), (2.11), and (2.12) yields

$$p_{vv} = -\frac{1}{9}B^2r(3r - 8M), \quad (3.25)$$

$$p_{\theta\theta} = -\frac{2}{9}B^2r^4, \quad (3.26)$$

$$p_{\phi\phi} = -\frac{2}{9}B^2r^4\sin^2\theta \quad (3.27)$$

for the $l = 0$ sector of the metric perturbation.

D. Integrating the perturbation equation: $l = 2$

The relevant spherical-harmonic functions are $Y^{20} = \frac{1}{4}\sqrt{5/\pi}(3\cos^2\theta - 1)$, $Y_{\theta}^{20} = -\frac{3}{2}\sqrt{5/\pi}\sin\theta\cos\theta$, $Y_{\phi}^{20} = 0$, $Y_{\theta\theta}^{20} = \frac{3}{4}\sqrt{5/\pi}\sin^2\theta$, $Y_{\phi\phi}^{20} = 0$, and $Y_{\phi\phi}^{20} = -\frac{3}{4}\sqrt{5/\pi}\sin^4\theta$. The required projections of the energy-momentum tensor are defined in Eqs. (4.17)–(4.20) of Martel and Poisson [21] (see also the Appendix of this paper); they are

$$Q^{vv} = -b^2, \quad (3.28)$$

$$Q^{vr} = -b^2\frac{r - M}{r}, \quad (3.29)$$

$$Q^{rr} = -b^2\frac{(r - 2M)(r - M)}{r^2}, \quad (3.30)$$

$$Q^v = -b^2r, \quad (3.31)$$

$$Q^r = -b^2(r - 2M), \quad (3.32)$$

$$Q^b = b^2, \quad (3.33)$$

$$Q^\# = -b^2r(r - 2M), \quad (3.34)$$

where the constant³ b^2 has been reassigned to $b^2 := \frac{8}{3} \times \sqrt{\pi/5}B^2$.

We now integrate the field equations for the four functions $h_{vv}(r)$, $j_v(r)$, $K(r)$, and $G(r)$. The equation for K decouples, as was shown in Eq. (2.36), and it involves the source term Q^{vv} . Exploiting the remaining gauge freedom to set all integration constants to zero, we take the solution

to be

$$K = \frac{1}{6}b^2r^2. \quad (3.35)$$

The remaining field equations form a set of coupled ordinary differential equations for the remaining quantities h_{vv} , j_v , and G . These equations are easily decoupled by taking additional derivatives, and we easily obtain general solutions to the higher-order equations. These would-be solutions involve a number of integration constants that are not part of the true solution space; these are determined by substituting the would-be solutions into the original system of second-order equations, and making sure that the solutions stay valid. At the end of this process we obtain $h_{vv} = -c_1M/r^2 - 3c_2(r - 2M)^2 + \frac{1}{3}b^2M(r - 3M)$, $j_v = \frac{1}{3}c_1(r + M)/r - c_2r^2(r - 2M) + \frac{1}{6}b^2Mr^2$, and $G = \frac{1}{3}c_1/r - c_2(r^2 - 2M^2) + \frac{1}{6}b^2(r^2 + M^2)$, where c_1 and c_2 are the remaining constants of integration. As we shall show below, the gauge freedom that is still at our disposal can be exploited to set $c_1 = 0$. Setting also $c_2 = \frac{1}{6}b^2 - \frac{1}{3}\varepsilon$ for later convenience (thus discarding c_2 in favor of the new constant ε), our solutions are

$$h_{vv} = -\frac{1}{6}b^2(3r^2 - 14Mr + 18M^2) + \varepsilon(r - 2M)^2, \quad (3.36)$$

$$j_v = -\frac{1}{6}b^2r^2(r - 3M) + \frac{1}{3}\varepsilon r^2(r - 2M), \quad (3.37)$$

$$G = \frac{1}{2}b^2M^2 + \frac{1}{3}\varepsilon(r^2 - 2M^2). \quad (3.38)$$

Substituting Eqs. (3.35), (3.36), (3.37), and (3.38) into Eqs. (2.10), (2.11), and (2.12) yields

$$p_{vv} = -\frac{1}{9}B^2(3r^2 - 14Mr + 18M^2)(3\cos^2\theta - 1) + \varepsilon(r - 2M)^2(3\cos^2\theta - 1), \quad (3.39)$$

$$p_{v\theta} = \frac{2}{3}B^2r^2(r - 3M)\sin\theta\cos\theta - 2\varepsilon r^2(r - 2M) \times \sin\theta\cos\theta, \quad (3.40)$$

$$p_{\theta\theta} = \frac{1}{9}B^2r^4(3\cos^2\theta - 1) + B^2M^2r^2\sin^2\theta + \varepsilon r^2(r^2 - 2M^2)\sin^2\theta, \quad (3.41)$$

$$p_{\phi\phi} = \frac{1}{9}B^2r^4\sin^2\theta(3\cos^2\theta - 1) - B^2M^2r^2\sin^4\theta - \varepsilon r^2(r^2 - 2M^2)\sin^4\theta \quad (3.42)$$

for the $l = 2$ sector of the metric perturbation. We have introduced the constant $\varepsilon := \frac{1}{4}\sqrt{5/\pi}\varepsilon$; its interpretation as a tidal gravitational field will be examined below.

We must now explain why it was admissible to set $c_1 = 0$ in our solutions. We go back to Eqs. (2.19), (2.20), (2.21), and (2.22) and consider the subclass of gauge transformations that leave K unchanged (in addition to h_{vr} , h_{rr} , and j_r , which are all zero in the light-cone gauge). We see that, when $l = 2$, the subclass is characterized by a single

³The constant b is not to be confused with the functions $b^{lm}(v)$ introduced in Sec. II C.

function $a(v)$, with the other functions related to it by $b(v) = 3a$ and $c(v) = -\frac{1}{3}\dot{a}$. Taking a to be a constant produces $c = 0$, and we observe that, under such a gauge transformation, h_{vv} changes by a term $6aM/r^2$, j_v changes by a term $-2a(r + M)/r$, and G changes by a term $-2a/r$. We then see that selecting $a = \frac{1}{6}c_1$ produces a gauge transformation that effectively sets c_1 to zero. There is therefore no loss of generality in making this assignment.

E. Perturbed metric

Combining Eqs. (3.25), (3.26), and (3.27) from Sec. III C and Eqs. (3.39), (3.40), (3.41), and (3.42) from Sec. III D gives us the metric of our perturbed black hole. Its non-vanishing components are

$$g_{vv} = -f - \frac{1}{9}B^2r(3r - 8M) - \frac{1}{9}B^2(3r^2 - 14Mr + 18M^2) \times (3\cos^2\theta - 1) + \mathcal{E}(r - 2M)^2(3\cos^2\theta - 1), \quad (3.43)$$

$$g_{vr} = 1, \quad (3.44)$$

$$g_{v\theta} = \frac{2}{3}B^2r^2(r - 3M)\sin\theta\cos\theta - 2\mathcal{E}r^2(r - 2M) \times \sin\theta\cos\theta, \quad (3.45)$$

$$g_{\theta\theta} = r^2 - \frac{2}{9}B^2r^4 + \frac{1}{9}B^2r^4(3\cos^2\theta - 1) + B^2M^2r^2\sin^2\theta + \mathcal{E}r^2(r^2 - 2M^2)\sin^2\theta, \quad (3.46)$$

$$g_{\phi\phi} = r^2\sin^2\theta - \frac{2}{9}B^2r^4\sin^2\theta + \frac{1}{9}B^2r^4\sin^2\theta(3\cos^2\theta - 1) - B^2M^2r^2\sin^4\theta - \mathcal{E}r^2(r^2 - 2M^2)\sin^4\theta, \quad (3.47)$$

and we observe that the perturbation is small whenever $r^2B^2 \ll 1$ and $r^2\mathcal{E} \ll 1$, as was anticipated in Sec. III A. This is a three-parameter family of solutions to the Einstein-Maxwell equations, accurate through order (B^2, \mathcal{E}) . The electromagnetic field is generated by the vector potential of Eq. (3.3); it is accurate through order B . The parameters of the family are the black-hole mass M , the magnetic field strength B , and the tidal gravitational field \mathcal{E} .

The interpretation of \mathcal{E} as a tidal-gravity (Weyl-curvature) parameter comes from an examination of the asymptotic behavior of the metric when $r \gg M$ (keeping $r \ll 1/B$, as was discussed in Sec. III A). In this regime Eqs. (3.43), (3.44), (3.45), (3.46), and (3.47) reduce to

$$g_{vv} \sim -1 - \frac{1}{3}B^2r^2 - \frac{1}{3}B^2r^2(3\cos^2\theta - 1) + \mathcal{E}r^2(3\cos^2\theta - 1), \quad (3.48)$$

$$g_{vr} = 1, \quad (3.49)$$

$$g_{v\theta} \sim \frac{2}{3}B^2r^3\sin\theta\cos\theta - 2\mathcal{E}r^3\sin\theta\cos\theta, \quad (3.50)$$

$$g_{\theta\theta} \sim r^2 - \frac{2}{9}B^2r^4 + \frac{1}{9}B^2r^4(3\cos^2\theta - 1) + \mathcal{E}r^4\sin^2\theta, \quad (3.51)$$

$$g_{\phi\phi} \sim r^2\sin^2\theta - \frac{2}{9}B^2r^4\sin^2\theta + \frac{1}{9}B^2r^4\sin^2\theta(3\cos^2\theta - 1) - \mathcal{E}r^4\sin^4\theta. \quad (3.52)$$

The asymptotic metric no longer refers to the central black hole. It is the metric of a spacetime that contains only a uniform magnetic field, expressed in an advanced coordinate system that is adapted to the incoming light cones of an observer situated at $r = 0$; the metric is limited to a domain $r < a$, where a is a length scale such that $a^2B^2 \ll 1$. The observer, of course, is fictitious, as $r = 0$ is actually occupied by the black-hole singularity; nevertheless, the observer may be thought to exist in an unphysical extension of the asymptotic spacetime beyond its domain of validity, $r \gg M$.

The metric of an arbitrary spacetime in light-cone coordinates was thoroughly investigated in our companion paper [25]. By comparing our Eqs. (3.48), (3.49), (3.50), (3.51), and (3.52) to Eqs. (4.9)–(4.12) of the companion paper, we infer that the asymptotic spacetime is characterized by the following irreducible quantities: $\rho := B^2/(8\pi)$ is the mass-energy density of the magnetic field as measured by the observer at $r = 0$, $S_{11} = S_{22} = -\frac{1}{2}S_{33} := B^2/(12\pi)$ are the nonvanishing components of the trace-free part of the field's stress tensor, and $T := B^2/(8\pi)$ is the trace of the stress tensor; these assignments are precisely what should be expected for a uniform magnetic field. The comparison reveals also that $\mathcal{E}_{11} = \mathcal{E}_{22} = -\frac{1}{2}\mathcal{E}_{33} := \mathcal{E}$ are the nonvanishing components of the spacetime's Weyl curvature tensor. (The irreducible quantities are all defined in our companion paper.) The comparison therefore gives us an operational meaning for the parameter \mathcal{E} : As was already anticipated, it is the Weyl curvature (the tidal gravitational field) of the asymptotic spacetime as measured by an observer comoving with the black hole in the region $M \ll r \ll 1/B$.

F. Perturbed event horizon

The perturbed black-hole spacetime retains ϕ^α as a rotational Killing vector, and it retains $t^\alpha = (1, 0, 0, 0)$ as a time-translation Killing vector. This vector is timelike outside the black hole, but it becomes null on the event horizon (which is therefore a Killing horizon). Setting $g_{\alpha\beta}t^\alpha t^\beta = g_{vv} = 0$ and involving Eq. (3.43) informs us that the event horizon is now described by

$$r = r_{\text{horizon}}(\theta) := 2M(1 + \frac{2}{3}M^2B^2\sin^2\theta). \quad (3.53)$$

It is interesting to note that $r_{\text{horizon}}(\theta)$ involves B^2 but not \mathcal{E} .

The horizon's intrinsic geometry is obtained by inserting Eq. (3.53) into the perturbed metric. It is described by the two-dimensional line element

$$ds_{\text{horizon}}^2 = 4M^2[1 + M^2(B^2 + 2\mathcal{E})\sin^2\theta]d\theta^2 + 4M^2\sin^2\theta[1 - M^2(B^2 + 2\mathcal{E})\sin^2\theta]d\phi^2. \quad (3.54)$$

The element of surface area on the horizon is $4M^2 \sin\theta d\theta d\phi$, and the integrated area is

$$A_{\text{horizon}} = 16\pi M^2. \quad (3.55)$$

As was anticipated in Sec. III A, the perturbed black hole has the same surface area as the original Schwarzschild black hole.

To see that the horizon area cannot change during the quasistatic immersion of the black hole within the magnetic field, we recall the Hawking-Hartle formula [35], which states that, during a quasistatic process in which a black hole is perturbed by an energy-momentum tensor $T_{\alpha\beta}$, its area changes according to

$$\frac{dA_{\text{horizon}}}{dv} = \frac{8\pi}{\kappa} \oint T_{\alpha\beta} t^\alpha t^\beta dS,$$

where κ is the horizon's surface gravity (see below), and where the integration is over a two-dimensional cross-section of the event horizon. The component $T_{\alpha\beta} t^\alpha t^\beta = T_{vv} = T^{rr}$ of the energy-momentum tensor measures the flux of energy that crosses the horizon, and, in general, this flux will produce an increase in area. But according to Eq. (3.14), in the present case of a magnetic perturbation, T^{rr} vanishes on the event horizon (apart from corrections of order B^4 which are not under our control). It follows that dA_{horizon}/dv vanishes (to order B^2) during the quasistatic immersion of our black hole within the magnetic field. This conclusion was anticipated back in Sec. III C, when we set $\delta M = 0$; it is this choice that produced the result of Eq. (3.55).

The distortion of the event horizon can be measured by the Ricci scalar associated with the two-dimensional metric of Eq. (3.54). This is

$$R = \frac{1}{2M^2}[1 + 2M^2(B^2 + 2\mathcal{E})(3\cos^2\theta - 1)]. \quad (3.56)$$

The distortion has a quadrupolar structure. The larger concentration of curvature at the poles reflects the greater strength of the magnetic field there; as was observed back in Sec. III B, the square of the magnetic field is given by $|\mathbf{B}|^2 = B^2 \cos^2\theta$.

It is interesting to note that, in accordance with the zeroth law of black-hole mechanics, the horizon's surface gravity displays no trace of this distortion. The surface gravity κ is defined by the statement that, on the horizon, t^α satisfies the generalized form of the geodesic equation: $t^\beta \nabla_\beta t^\alpha = \kappa t^\alpha$. A short calculation based on this equation reveals that $\kappa = 1/(4M)$ plus terms of order B^4 , $B^2\mathcal{E}$, and \mathcal{E}^2 . The surface gravity is uniform on the horizon, and it keeps its unperturbed, Schwarzschild value.

G. Comparison with the Schwarzschild-Melvin solution

There exists an exact solution to the Einstein-Maxwell equations that describes a nonrotating black hole immersed in Melvin's magnetic universe [36–38]. Known as the Schwarzschild-Melvin solution [22–24], it has a metric given by

$$ds^2 = \Lambda^2(-fdt^2 + f^{-1}d\bar{r}^2 + \bar{r}^2d\theta^2) + \Lambda^{-2}\bar{r}^2\sin^2\theta d\phi^2 \quad (3.57)$$

and a vector potential given by

$$A^\alpha = \frac{1}{2}B\Lambda\phi^\alpha, \quad (3.58)$$

where $\phi^\alpha := (0, 0, 0, 1)$ is the spacetime's rotational Killing vector. We have $f := 1 - 2M/\bar{r}$ as before, and we introduced the function

$$\Lambda := 1 + \frac{1}{4}B^2\bar{r}^2\sin^2\theta. \quad (3.59)$$

This is a two-parameter family of black-hole solutions; the first parameter is the black-hole mass M , and the second is the magnetic field strength B .

The solution of Eqs. (3.57), (3.58), and (3.59) is exact, and we wish to compare it with the perturbative solution of Eqs. (3.43), (3.44), (3.45), (3.46), and (3.47). We must first linearize the exact solution with respect to B^2 and transform the coordinates from the original system $(t, \bar{r}, \theta, \phi)$ to the light-cone system (v, r, θ, ϕ) . The transformation from t to v is the same as for the Schwarzschild spacetime:

$$v = t + \bar{r} + 2M \ln|\bar{r}/2M - 1|. \quad (3.60)$$

The transformation from \bar{r} to r is designed to change the g_{vr} component of the metric tensor from its current value $\Lambda^2 \simeq 1 + \frac{1}{2}B^2\bar{r}^2\sin^2\theta$ to the new value of 1. It is given by

$$r = \bar{r}[1 + \frac{1}{6}B^2\bar{r}^2\sin^2\theta + O(B^4)]. \quad (3.61)$$

The angular coordinates (θ, ϕ) are not affected by the transformation.

These manipulations bring the Schwarzschild-Melvin metric to the new form

$$g_{vv} = -f - \frac{1}{6}B^2r(3r - 8M)\sin^2\theta + O(B^4), \quad (3.62)$$

$$g_{vr} = 1 + O(B^4), \quad (3.63)$$

$$g_{v\theta} = -\frac{1}{3}B^2r^3 \sin\theta \cos\theta + O(B^4), \quad (3.64)$$

$$g_{\theta\theta} = r^2 + \frac{1}{6}B^2r^4\sin^2\theta + O(B^4), \quad (3.65)$$

$$g_{\phi\phi} = r^2\sin^2\theta - \frac{5}{6}B^2r^4\sin^4\theta + O(B^4). \quad (3.66)$$

Comparison with Eqs. (3.43), (3.44), (3.45), (3.46), and (3.47) reveals that the solutions are identical provided that we restrict the parameter freedom of the perturbative solution. Indeed, to get a match we must set

$$\mathcal{E} = \frac{1}{2}B^2. \quad (3.67)$$

The Weyl curvature of the Schwarzschild-Melvin solution is intimately related to its magnetic field. This feature is, in fact, inherited from Melvin's pure magnetic universe, as can be inferred from reading Sec. V B of our companion paper [25].

We conclude with the following statement: While the Schwarzschild-Melvin solution has the advantage of being an exact solution to the Einstein-Maxwell equations, the perturbative solution of Eqs. (3.43), (3.44), (3.45), (3.46), and (3.47) has the advantage of possessing a larger number of parameters. The perturbative solution can therefore represent a wider class of physical situations. In particular, it provides the description of a magnetized black-hole spacetime in which the tidal gravity is not directly tied to the magnetic field.

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APPENDIX: PERTURBATION EQUATIONS IN THE LIGHT-CONE GAUGE

In the even-parity sector, the nonvanishing perturbation fields are $h_{vv}(v, r)$, $j_v(v, r)$, $K(v, r)$, and $G(v, r)$. According to Eqs. (4.13)–(4.16) of Martel and Poisson [21], they satisfy the field equations

$$\begin{aligned} Q^{vv} &= -\frac{\partial^2}{\partial r^2}K - \frac{2}{r}\frac{\partial}{\partial r}K, \\ Q^{vr} &= \frac{\partial^2}{\partial v \partial r}K + \frac{2}{r}\frac{\partial}{\partial v}K - \frac{1}{r}\frac{\partial}{\partial r}h_{vv} + \frac{\lambda}{2r^2}\frac{\partial}{\partial r}j_v + \frac{r-M}{r^2}\frac{\partial}{\partial r}K - \frac{1}{r^2}h_{vv} + \frac{\lambda}{r^3}j_v - \frac{\mu}{2r^2}K - \frac{\mu\lambda}{4r^2}G, \\ Q^{rr} &= -\frac{\partial^2}{\partial v^2}K + \frac{r-M}{r^2}\frac{\partial}{\partial v}K + \frac{1}{r}\frac{\partial}{\partial v}h_{vv} - \frac{\lambda}{r^2}\frac{\partial}{\partial v}j_v - \frac{f}{r}\frac{\partial}{\partial r}h_{vv} + \frac{(r-M)f}{r^2}\frac{\partial}{\partial r}K + \frac{\mu r + 4M}{2r^3}h_{vv} + \frac{\lambda(r-M)}{r^4}j_v \\ &\quad - \frac{\mu f}{2r^2}K - \frac{\mu\lambda f}{4r^2}G, \\ Q^v &= \frac{\partial^2}{\partial r^2}j_v - \frac{\partial}{\partial r}K - \frac{\mu}{2}\frac{\partial}{\partial r}G - \frac{2}{r^2}j_v, \\ Q^r &= -\frac{\partial^2}{\partial v \partial r}j_v + \frac{2}{r}\frac{\partial}{\partial v}j_v - \frac{\partial}{\partial v}K - \frac{\mu}{2}\frac{\partial}{\partial v}G - \frac{\mu f}{2}\frac{\partial}{\partial r}G - f\frac{\partial}{\partial r}K + \frac{\partial}{\partial r}h_{vv} - \frac{2}{r^2}j_v, \\ Q^b &= 2\frac{\partial^2}{\partial v \partial r}K + \frac{2}{r}\frac{\partial}{\partial v}K + f\frac{\partial^2}{\partial r^2}K - \frac{\partial^2}{\partial r^2}h_{vv} - \frac{2}{r}\frac{\partial}{\partial r}h_{vv} + \frac{\lambda}{r^2}\frac{\partial}{\partial r}j_v + \frac{2(r-M)}{r^2}\frac{\partial}{\partial r}K, \\ Q^\# &= -2r^2\frac{\partial^2}{\partial v \partial r}G - 2r\frac{\partial}{\partial v}G - r^2f\frac{\partial^2}{\partial r^2}G - 2(r-M)\frac{\partial}{\partial r}G + 2\frac{\partial}{\partial r}j_v, \end{aligned}$$

where $\lambda := l(l+1) = \mu + 2$ and $\mu := (l-1)(l+2) = \lambda - 2$. According to Eqs. (4.17)–(4.20) of Martel and Poisson [21], the source terms are

$$\begin{aligned} Q^{ab} &= 8\pi \int T^{ab} \bar{Y}^{lm} d\Omega, \\ Q^a &= \frac{16\pi r^2}{l(l+1)} \int T^{aB} \bar{Y}_B^{lm} d\Omega, \\ Q^b &= 8\pi r^2 \int T^{AB} \Omega_{AB} \bar{Y}^{lm} d\Omega, \\ Q^\# &= \frac{32\pi r^4}{(l-1)l(l+1)(l+2)} \int T^{AB} \bar{Y}_{AB}^{lm} d\Omega, \end{aligned}$$

where $x^a = (v, r)$. The perturbation equations are not all independent; they are linked by the Bianchi identities

$$\begin{aligned} 0 &= \frac{\partial}{\partial v}Q^{vv} + \frac{\partial}{\partial r}Q^{vr} + \frac{M}{r^2}Q^{vv} + \frac{2}{r}Q^{vr} - \frac{\lambda}{2r^2}Q^v - \frac{1}{r}Q^b, \\ 0 &= \frac{\partial}{\partial v}Q^{vr} + \frac{\partial}{\partial r}Q^{rr} + \frac{Mf}{r^2}Q^{vv} - \frac{2M}{r^2}Q^{vr} + \frac{2}{r}Q^{rr} \\ &\quad - \frac{\lambda}{2r^2}Q^r - \frac{f}{r}Q^b, \\ 0 &= \frac{\partial}{\partial v}Q^v + \frac{\partial}{\partial r}Q^r + \frac{2}{r}Q^r + Q^b - \frac{\mu}{2r^2}Q^\#. \end{aligned}$$

When $l = 0$ the only nonvanishing perturbation fields are h_{vv} and K , and the only relevant equations are those involving Q^{ab} and Q^b . When $l = 1$ the only nonvanishing perturbation fields are h_{vv} , j_v , and K , and the only relevant equations are those involving Q^{ab} , Q^a , and Q^b .

In the odd-parity sector the nonvanishing perturbation fields are $h_v(v, r)$ and $h_2(v, r)$. According to Eqs. (5.8) and (5.9) of Martel and Poisson [21], they satisfy the field

equations

$$\begin{aligned}
 P^v &= \frac{\partial^2}{\partial r^2} h_v - \frac{\mu}{2r^2} \frac{\partial}{\partial r} h_2 - \frac{2}{r^2} h_v + \frac{\mu}{r^3} h_2, \\
 P^r &= -\frac{\partial^2}{\partial v \partial r} h_v + \frac{2}{r} \frac{\partial}{\partial v} h_v - \frac{\mu}{2r^2} \frac{\partial}{\partial v} h_2 - \frac{\mu f}{2r^2} \frac{\partial}{\partial r} h_2 \\
 &\quad + \frac{\mu}{r^2} h_v + \frac{\mu f}{r^3} h_2, \\
 P &= -\frac{\partial^2}{\partial v \partial r} h_2 + \frac{1}{r} \frac{\partial}{\partial v} h_2 - \frac{f}{2} \frac{\partial^2}{\partial r^2} h_2 + \frac{r-3M}{r^2} \frac{\partial}{\partial r} h_2 \\
 &\quad + \frac{\partial}{\partial r} h_v - \frac{r-4M}{r^3} h_2,
 \end{aligned}$$

where $\lambda := l(l+1) = \mu + 2$ and $\mu := (l-1)(l+2) = \lambda - 2$. According to Eqs. (5.10) and (5.11) of Martel and

Poisson [21], the source terms are

$$\begin{aligned}
 P^a &= \frac{16\pi r^2}{l(l+1)} \int T^{aB} \bar{X}_B^{lm} d\Omega, \\
 P &= \frac{16\pi r^4}{(l-1)l(l+1)(l+2)} \int T^{AB} \bar{X}_{AB}^{lm} d\Omega.
 \end{aligned}$$

The perturbation equations are not all independent; they are linked by the Bianchi identity

$$0 = \frac{\partial}{\partial v} P^v + \frac{\partial}{\partial r} P^r + \frac{2}{r} P^r - \frac{\mu}{r^2} P.$$

When $l = 1$ the only nonvanishing perturbation field is h_v , and the only relevant equations are those involving P^a .

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- [1] S. Chandrasekhar, *The Mathematical Theory of Black Holes* (Clarendon Press, Oxford, England, 1983).
- [2] T. Regge and J. A. Wheeler, *Phys. Rev.* **108**, 1063 (1957).
- [3] C. V. Vishveshwara, *Phys. Rev. D* **1**, 2870 (1970).
- [4] F. J. Zerilli, *Phys. Rev. D* **2**, 2141 (1970).
- [5] E. Poisson, *Living Rev. Relativity* **7**, 6 (2004), <http://www.livingreviews.org/lrr-2004-6>.
- [6] S. Detweiler, *Classical Quantum Gravity* **22**, S681 (2005).
- [7] W. Hikida, H. Nakano, and M. Sasaki, *Classical Quantum Gravity* **22**, S753 (2005).
- [8] S. Drasco, E. E. Flanagan, and S. A. Hughes, *Classical Quantum Gravity* **22**, S801 (2005).
- [9] R. Gleiser, C. Nicasio, R. Price, and J. Pullin, *Phys. Rep.* **325**, 41 (2000).
- [10] K. Alvi, *Phys. Rev. D* **61**, 124013 (2000).
- [11] K. Alvi, *Phys. Rev. D* **64**, 104020 (2001).
- [12] K. Alvi, *Phys. Rev. D* **67**, 104006 (2003).
- [13] N. Yunes, W. Tichy, B. J. Owen, and B. Bruegmann, gr-qc/0503011.
- [14] N. Yunes and J. Gonzalez, *Phys. Rev. D* **73**, 024010 (2006).
- [15] V. Moncrief, *Ann. Phys. (N.Y.)* **88**, 323 (1974).
- [16] U. H. Gerlach and U. K. Sengupta, *Phys. Rev. D* **19**, 2268 (1979).
- [17] O. Sarbach and M. Tiglio, *Phys. Rev. D* **64**, 084016 (2001).
- [18] C. Gundlach and J. M. Martin-Garcia, *Phys. Rev. D* **61**, 084024 (2000).
- [19] C. Clarkson and R. Barrett, *Classical Quantum Gravity* **20**, 3855 (2003).
- [20] A. Nagar and L. Rezzolla, *Classical Quantum Gravity* **22**, R167 (2005); **23**, 4297 (2006).
- [21] K. Martel and E. Poisson, *Phys. Rev. D* **71**, 104003 (2005).
- [22] F. J. Ernst, *J. Math. Phys. (N.Y.)* **17**, 54 (1976).
- [23] W. A. Hiscock, *J. Math. Phys. (N.Y.)* **22**, 1828 (1981).
- [24] F. J. Ernst and W. J. Wild, *J. Math. Phys. (N.Y.)* **17**, 182 (1976).
- [25] B. Preston and E. Poisson, preceding Article, *Phys. Rev. D* **74**, 064009 (2006).
- [26] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973).
- [27] P. L. Chrzanowski, *Phys. Rev. D* **11**, 2042 (1975).
- [28] R. M. Wald, *Phys. Rev. Lett.* **41**, 203 (1978).
- [29] C. O. Lousto and B. F. Whiting, *Phys. Rev. D* **66**, 024026 (2002).
- [30] A. Ori, *Phys. Rev. D* **67**, 124010 (2003).
- [31] L. Price, K. Shankar, and B. F. Whiting, "On the existence of radiation gauges in Petrov type II spacetimes (to be published).
- [32] L. Price, "Towards Metric Perturbations of Kerr," in *Proceedings of the 8th Capra Meeting on Radiation Reaction*, 2005, <http://www.sstsd.rl.ac.uk/capra/Price.pdf>.
- [33] B. F. Whiting and L. R. Price, *Classical Quantum Gravity* **22**, S589 (2005).
- [34] R. M. Wald, *Phys. Rev. D* **10**, 1680 (1974).
- [35] S. W. Hawking and J. B. Hartle, *Commun. Math. Phys.* **27**, 283 (1972).
- [36] M. A. Melvin, *Phys. Lett.* **8**, 65 (1964).
- [37] M. A. Melvin, *Phys. Rev.* **139**, B225 (1965).
- [38] K. S. Thorne, *Phys. Rev.* **139**, B244 (1965).