Light-cone coordinates based at a geodesic world line

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Continuing work initiated in an earlier publication [Phys. Rev. D **69**, 084007 (2004)], we construct a system of light-cone coordinates based at a geodesic world line of an arbitrary curved spacetime. The construction involves (i) an advanced-time or a retarded-time coordinate that labels past or future light cones centered on the world line, (ii) a radial coordinate that is an affine parameter on the null generators of these light cones, and (iii) angular coordinates that are constant on each generator. The spacetime metric is calculated in the light-cone coordinates, and it is expressed as an expansion in powers of the radial coordinate in terms of the irreducible components of the Riemann tensor evaluated on the world line. The formalism is illustrated in two simple applications, the first involving a comoving world line of a spatially flat cosmology, the other featuring an observer placed on the axis of symmetry of Melvin's magnetic universe.

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I. INTRODUCTION

We continue a research program initiated in Ref. [1], which aims to construct and exploit light-cone coordinates based at an arbitrary world line γ of an arbitrary curved spacetime. The *retarded coordinates* introduced in Ref. [1] are denoted (u, r, θ, ϕ) and are adapted to the *future light cone* of each point z on the selected world line. In this paper we construct *advanced coordinates* (v, r, θ, ϕ) which are instead adapted to the *past light cone* of each point z on the world line; for simplicity we take the world line to be a geodesic of the spacetime. (The advanced coordinates were introduced briefly in Ref. [2]; this paper provides details that were not given in the earlier work.) We collectively denote the light-cone coordinates by (w, r, θ, ϕ), with w standing for either u or v depending on the context. In both cases the null coordinate w is constant on each light cone, and it agrees with proper time τ at the cone's apex. The radial coordinate r is an affine parameter on the cone's null generators, and it measures the distance away from the world line. The angular coordinates $\theta^A = (\theta, \phi)$ are constant on each one of these generators. The geometrical meaning of the light-cone coordinates is clear, and this is one of their main virtues.

The formalism developed in Ref. [1] and in this paper incorporates ideas formulated many years ago by Bondi and his collaborators [3,4], and it complements a line of research that was initiated by Synge [5] and pursued by Ellis and his collaborators [6–11] in their work on observational cosmology. While the central ideas exploited here are the same as with Synge and Ellis, our implementation is substantially different: While Synge and Ellis sought definitions for their optical or observational coordinates that apply to large regions of the spacetime, our considerations are limited to a small neighborhood of the world line.

The introduction of retarded coordinates was motivated by the desire to construct solutions to wave equations for massless fields that are produced by a pointlike source moving on the world line. The retarded coordinates naturally incorporate the causal relation that exists between the source and the field, and for this reason the solution takes a simple explicit form (in the neighborhood in which the coordinates are defined). The introduction of advanced coordinates is motivated instead by the desire to construct solutions to the Einstein field equations that describe black holes placed in a distribution of matter or in a tidal environment. Such an application was described in Ref. [2], in which the metric of a tidally distorted black hole was presented in advanced coordinates. In a companion paper [12] we use the guidance offered by the advanced coordinates to formulate a light-cone gauge for black-hole perturbation theory, and to calculate the metric of a black hole immersed in a uniform magnetic field.

A quasi-Cartesian version of the advanced coordinates is introduced first in Sec. II B, after reviewing some necessary geometrical elements in Sec. II A. The metric tensor in advanced coordinates is constructed gradually in Secs. II C, II D, II E, and II F, and its quasi-Cartesian form is displayed in Eqs. (2.27)–(2.29).

In Sec. III A we combine the results obtained in Sec. II with the earlier results of Ref. [1] and present the metric in a general form suitable for both advanced and retarded coordinates. At this stage the metric is expressed in terms of the Riemann tensor evaluated on the world line γ. In Sec. III B we begin to refine the form of the metric by decomposing the Riemann tensor into its Weyl and Ricci parts, and by further decompositing the Weyl and energy-momentum tensors into their irreducible components. This leads us, in Sec. III C, to introduce *tidal and matter potentials* that make the basic building blocks of the metric tensor. The potentials are displayed in Table I, and the refined form of the metric is displayed in Eqs. (3.14)–(3.16).

In Sec. IV we carry out a transformation of the metric from the quasi-Cartesian coordinates $\hat{x}^a$ to the quasispher-

TABLE I. Tidal and matter potentials. Each potential is identified with a sans-serif superscript that specifies its multipole content. A potential labeled with a "q" is a quadrupole field, and one labeled with a "d" is a dipole field. The vector and tensor potentials are all orthogonal to Ω^a, and all tensors are symmetric and trace-free.

${}^{\mathsf{q}}\mathcal{E} = \mathcal{E}_{cd}\Omega^c\Omega^d$
${}^{\mathsf{q}}\mathcal{E}_a = (\delta_a{}^c - \Omega_a\Omega^c)\mathcal{E}_{cd}\Omega^d$
${}^{\mathsf{q}}\mathcal{E}_{ab} = 2(\delta_a{}^c - \Omega_a\Omega^c)(\delta_b{}^d - \Omega_b\Omega^d)\mathcal{E}_{cd} + (\delta_{ab} - \Omega_a\Omega_b){}^{\mathsf{q}}\mathcal{E}$
${}^{\mathsf{q}}\mathcal{B}_a = \varepsilon_{apq}\Omega^p\mathcal{B}^q{}_c\Omega^c$
${}^{\mathsf{q}}\mathcal{B}_{ab} = \varepsilon_{apq}\Omega^p\mathcal{B}^q{}_c(\delta^c{}_b - \Omega^c\Omega_b) + \varepsilon_{bpq}\Omega^p\mathcal{B}^q{}_c(\delta^c{}_a - \Omega^c\Omega_a)$
${}^{\mathsf{d}}j = j_c\Omega^c$
${}^{\mathsf{d}}j_a = (\delta_a{}^c - \Omega_a\Omega^c)j_c$
${}^{\mathsf{q}}S = S_{cd}\Omega^c\Omega^d$
${}^{\mathsf{q}}S_a = (\delta_a{}^c - \Omega_a\Omega^c)S_{cd}\Omega^d$

TABLE II. Scalar and vectorial harmonics of degree $l = 1$, labeled by the abstract index $\mathsf{m} = \{0, 1c, 1s\}$. The odd-parity vectorial harmonics $X_A^{1\mathsf{m}}$ are not required, and the tensorial harmonics $Y_{AB}^{1\mathsf{m}}$ and $X_{AB}^{1\mathsf{m}}$ vanish identically.

m	0	$1c$	$1s$
$Y^{1\mathsf{m}}$	$\cos\theta$	$\sin\theta\cos\phi$	$\sin\theta\sin\phi$
$Y_\theta^{1\mathsf{m}}$	$-\sin\theta$	$\cos\theta\cos\phi$	$\cos\theta\sin\phi$
$Y_\phi^{1\mathsf{m}}$	0	$-\sin\theta\sin\phi$	$\sin\theta\cos\phi$

ical coordinates (r, θ^A). The transformation, introduced in Sec. IV A, is the familiar one from flat spacetime: $\hat{x}^a = r\Omega^a(\theta^A)$ or, more explicitly, $\hat{x} = r\sin\theta\cos\phi$, $\hat{y} = r\sin\theta\sin\phi$, and $\hat{z} = r\cos\theta$. This final expression for the metric tensor, in the coordinates (w, r, θ^A), is displayed in Eqs. (4.9)–(4.12). It involves the angular components of the tidal and matter potentials introduced in Sec. III. As shown in Table IV (with the results derived in Secs. IV B and IV C), these are naturally expressed as expansions in scalar, vector, and tensor harmonics. The required spherical-harmonic functions are listed in Tables II and III.

In Sec. V we present two simple applications of the light-cone coordinates. In Sec. V A we apply the formalism to a comoving world line of a spatially flat cosmology. In Sec. V B we examine the metric near the axis of symmetry of Melvin's magnetic universe [13–15].

Throughout the paper we work in geometrized units ($G = c = 1$) and adhere to the conventions of Misner, Thorne, and Wheeler [16].

II. ADVANCED COORDINATES

The presentation in this section follows very closely Sec. II of Ref. [1]. The material is very similar but, because of important differences of sign that occur in various places, we present here the details that are specific to the advanced coordinates. Other details are omitted and can be obtained from Ref. [1].

A. Geometrical elements

We first introduce some geometrical elements on the world line γ at which the advanced coordinates are based. The world line is described by parametric relations $z^\mu(\tau)$ in which τ denotes proper time. Its normalized tangent vector is $u^\mu = dz^\mu/d\tau$, and we assume that this satisfies the geodesic equation $Du^\mu/d\tau = 0$. The world line is therefore a geodesic of the curved spacetime, and this assumption represents a loss of generality relative to the construction of retarded coordinates presented in Ref. [1]. While it would be a simple matter to restore this level of generality, we refrain from doing so in this work. Throughout we use Greek indices μ, ν, λ, ρ, etc. to refer to tensor fields defined, or evaluated, on the world line.

We install on γ an orthonormal tetrad that consists of the tangent vector u^μ and three spatial vectors e_a^μ. These are parallel transported on the world line, so that $De_a^\mu/d\tau = 0$. It is easy to check that this is compatible with the requirement that the tetrad (u^μ, e_a^μ) be orthonormal everywhere on γ.

TABLE III. Scalar, vectorial, and tensorial harmonics of degree $l = 2$, labeled by the abstract index $\mathsf{m} = \{0, 1c, 1s, 2c, 2s\}$.

m	0	$1c$	$1s$	$2c$	$2s$
$Y^{2\mathsf{m}}$	$-(3\cos^2\theta - 1)$	$2\sin\theta\cos\theta\cos\phi$	$2\sin\theta\cos\theta\sin\phi$	$\sin^2\theta\cos2\phi$	$\sin^2\theta\sin2\phi$
$Y_\theta^{2\mathsf{m}}$	$6\sin\theta\cos\theta$	$2(2\cos^2\theta - 1)\cos\phi$	$2(2\cos^2\theta - 1)\sin\phi$	$2\sin\theta\cos\theta\cos2\phi$	$2\sin\theta\cos\theta\sin2\phi$
$Y_\phi^{2\mathsf{m}}$	0	$-2\sin\theta\cos\theta\sin\phi$	$2\sin\theta\cos\theta\cos\phi$	$-2\sin^2\theta\sin2\phi$	$2\sin^2\theta\cos2\phi$
$Y_{\theta\theta}^{2\mathsf{m}}$	$-3\sin^2\theta$	$-2\sin\theta\cos\theta\cos\phi$	$-2\sin\theta\cos\theta\sin\phi$	$(\cos^2\theta + 1)\cos2\phi$	$(\cos^2\theta + 1)\sin2\phi$
$Y_{\theta\phi}^{2\mathsf{m}}$	0	$2\sin^2\theta\sin\phi$	$-2\sin^2\theta\cos\phi$	$-2\sin\theta\cos\theta\sin2\phi$	$2\sin\theta\cos\theta\cos2\phi$
$Y_{\phi\phi}^{2\mathsf{m}}$	$3\sin^4\theta$	$2\sin^3\theta\cos\theta\cos\phi$	$2\sin^3\theta\cos\theta\sin\phi$	$-\sin^2\theta(\cos^2\theta + 1)\cos2\phi$	$-\sin^2\theta(\cos^2\theta + 1)\sin2\phi$
$X_\theta^{2\mathsf{m}}$	0	$2\cos\theta\sin\phi$	$-2\cos\theta\cos\phi$	$2\sin\theta\sin2\phi$	$-2\sin\theta\cos2\phi$
$X_\phi^{2\mathsf{m}}$	$6\sin^2\theta\cos\theta$	$2\sin\theta(2\cos^2\theta - 1)\cos\phi$	$2\sin\theta(2\cos^2\theta - 1)\sin\phi$	$2\sin^2\theta\cos\theta\cos2\phi$	$2\sin^2\theta\cos\theta\sin2\phi$
$X_{\theta\theta}^{2\mathsf{m}}$	0	$-2\sin\theta\sin\phi$	$2\sin\theta\cos\phi$	$2\cos\theta\sin2\phi$	$-2\cos\theta\cos2\phi$
$X_{\theta\phi}^{2\mathsf{m}}$	$-3\sin^3\theta$	$-2\sin^2\theta\cos\theta\cos\phi$	$-2\sin^2\theta\cos\theta\sin\phi$	$\sin\theta(\cos^2\theta + 1)\cos2\phi$	$\sin\theta(\cos^2\theta + 1)\sin2\phi$
$X_{\phi\phi}^{2\mathsf{m}}$	0	$2\sin^3\theta\sin\phi$	$-2\sin^3\theta\cos\phi$	$-2\sin^2\theta\cos\theta\sin2\phi$	$2\sin^2\theta\cos\theta\cos2\phi$

TABLE IV. Spherical-harmonic decomposition of the tidal and matter potentials. In the first part of the table we list the definitions of ${}^{\mathsf{d}}j_{\mathsf{m}}(w)$, ${}^{\mathsf{q}}\mathcal{E}_{\mathsf{m}}(w)$, ${}^{\mathsf{q}}\mathcal{B}_{\mathsf{m}}(w)$, and ${}^{\mathsf{q}}S_{\mathsf{m}}(w)$ in terms of $j_a(w)$, $\mathcal{E}_{ab}(w)$, $\mathcal{B}_{ab}(w)$, and $S_{ab}(w)$. In the second part of the table we display the spherical-harmonic decompositions of the tidal and matter potentials.

${}^{\mathsf{d}}j_0 = j_3$	${}^{\mathsf{q}}\mathcal{E}_0 = \frac{1}{2}(\mathcal{E}_{11} + \mathcal{E}_{22})$	${}^{\mathsf{q}}\mathcal{B}_0 = \frac{1}{2}(\mathcal{B}_{11} + \mathcal{B}_{22})$	${}^{\mathsf{q}}S_0 = \frac{1}{2}(S_{11} + S_{22})$
${}^{\mathsf{d}}j_{1c} = j_1$	${}^{\mathsf{q}}\mathcal{E}_{1c} = \mathcal{E}_{13}$	${}^{\mathsf{q}}\mathcal{B}_{1c} = \mathcal{B}_{13}$	${}^{\mathsf{q}}S_{1c} = S_{13}$
${}^{\mathsf{d}}j_{1s} = j_2$	${}^{\mathsf{q}}\mathcal{E}_{1s} = \mathcal{E}_{23}$	${}^{\mathsf{q}}\mathcal{B}_{1s} = \mathcal{B}_{23}$	${}^{\mathsf{q}}S_{1s} = S_{23}$
	${}^{\mathsf{q}}\mathcal{E}_{2c} = \frac{1}{2}(\mathcal{E}_{11} - \mathcal{E}_{22})$	${}^{\mathsf{q}}\mathcal{B}_{2c} = \frac{1}{2}(\mathcal{B}_{11} - \mathcal{B}_{22})$	${}^{\mathsf{q}}S_{2c} = \frac{1}{2}(S_{11} - S_{22})$
	${}^{\mathsf{q}}\mathcal{E}_{2s} = \mathcal{E}_{12}$	${}^{\mathsf{q}}\mathcal{B}_{2s} = \mathcal{B}_{12}$	${}^{\mathsf{q}}S_{2s} = S_{12}$
${}^{\mathsf{d}}j = \sum_{\mathsf{m}} {}^{\mathsf{d}}j_{\mathsf{m}} Y^{1\mathsf{m}}$	${}^{\mathsf{q}}\mathcal{E} = \sum_{\mathsf{m}} {}^{\mathsf{q}}\mathcal{E}_{\mathsf{m}} Y^{2\mathsf{m}}$		${}^{\mathsf{q}}S = \sum_{\mathsf{m}} {}^{\mathsf{q}}S_{\mathsf{m}} Y^{2\mathsf{m}}$
${}^{\mathsf{d}}j_A = \sum_{\mathsf{m}} {}^{\mathsf{d}}j_{\mathsf{m}} Y^{1\mathsf{m}}_A$	${}^{\mathsf{q}}\mathcal{E}_A = \frac{1}{2}\sum_{\mathsf{m}} {}^{\mathsf{q}}\mathcal{E}_{\mathsf{m}} Y^{2\mathsf{m}}_A$	${}^{\mathsf{q}}\mathcal{B}_A = \frac{1}{2}\sum_{\mathsf{m}} {}^{\mathsf{q}}\mathcal{B}_{\mathsf{m}} X^{2\mathsf{m}}_A$	${}^{\mathsf{q}}S_A = \frac{1}{2}\sum_{\mathsf{m}} {}^{\mathsf{q}}S_{\mathsf{m}} Y^{2\mathsf{m}}_A$
	${}^{\mathsf{q}}\mathcal{E}_{AB} = \sum_{\mathsf{m}} {}^{\mathsf{q}}\mathcal{E}_{\mathsf{m}} Y^{2\mathsf{m}}_{AB}$	${}^{\mathsf{q}}\mathcal{B}_{AB} = \sum_{\mathsf{m}} {}^{\mathsf{q}}\mathcal{B}_{\mathsf{m}} X^{2\mathsf{m}}_{AB}$	

From the tetrad on γ we define a dual tetrad (e^0_μ, e^a_μ) with the relations $e^0_\mu = -u_\mu$ and $e^a_\mu = \delta^{ab} g_{\mu\nu} e^\nu_b$. The dual vectors e^a_μ also are parallel transported on the world line. The tetrad and its dual give rise to the completeness relations

$$g^{\mu\nu} = -u^\mu u^\nu + \delta^{ab} e^\mu_a e^\nu_b, \qquad g_{\mu\nu} = -e^0_\mu e^0_\nu + \delta_{ab} e^a_\mu e^b_\nu \tag{2.1}$$

for the metric and its inverse evaluated on the world line.

The advanced coordinates are constructed with the help of a null geodesic segment that links a given point x to the world line. This geodesic segment must be unique, and we thus restrict x to be within the normal convex neighborhood of γ. We denote by β the unique, future-directed null geodesic segment that goes from x to the world line, and $x' \equiv z(v)$ is β's point of arrival on the world line; v is the value of the proper-time parameter at this point. To tensors at x we assign the Greek indices α, β, γ, δ, etc.; to tensors at x' we assign the indices α', β', γ', δ', and so on.

From the tetrad $(u^{\alpha'}, e^{\alpha'}_a)$ at x' we obtain another tetrad (e^α_0, e^α_a) at x by parallel transport on β. By raising the frame index and lowering the vectorial index, we obtain also a dual tetrad at x: $e^0_\alpha = -g_{\alpha\beta} e^\beta_0$ and $e^a_\alpha = \delta^{ab} g_{\alpha\beta} e^\beta_b$. The metric at x can then be expressed as

$$g_{\alpha\beta} = -e^0_\alpha e^0_\beta + \delta_{ab} e^a_\alpha e^b_\beta, \tag{2.2}$$

and the parallel propagator [5] (also known as the bivector of geodetic parallel displacement [17]) from x' to x is given by

$$g^\alpha{}_{\alpha'}(x, x') = -e^\alpha_0 u_{\alpha'} + e^\alpha_a e^a_{\alpha'}, \qquad g^{\alpha'}{}_\alpha(x', x) = u^{\alpha'} e^0_\alpha + e^{\alpha'}_a e^a_\alpha. \tag{2.3}$$

This is defined such that, if A^α is a vector that is parallel transported on β, then $A^\alpha(x) = g^\alpha{}_{\alpha'}(x, x') A^{\alpha'}(x')$ and $A^{\alpha'}(x') = g^{\alpha'}{}_\alpha(x', x) A^\alpha(x)$. Similarly, if p_α is a dual vector that is parallel transported on β, then $p_\alpha(x) = g^{\alpha'}{}_\alpha(x', x) p_{\alpha'}(x')$ and $p_{\alpha'}(x') = g^\alpha{}_{\alpha'}(x, x') p_\alpha(x)$.

The last ingredient we shall need is Synge's world function $\sigma(z, x)$ [5] (also known as the biscalar of geodetic interval [17]). This is defined as half the squared geodesic distance between the world-line point $z(\tau)$ and a neighboring point x. The derivative of the world function with respect to z^μ is denoted $\sigma_\mu(z, x)$; this is a vector at z (and a scalar at x) that is known to be tangent to the geodesic linking z and x. The derivative of $\sigma(z, x)$ with respect to x^α is denoted $\sigma_\alpha(z, x)$; this vector at x (and scalar at z) is also tangent to the geodesic. We use a similar notation for multiple derivatives; for example, $\sigma_{\mu\alpha} \equiv \nabla_\alpha \nabla_\mu \sigma$ and $\sigma_{\alpha\beta} \equiv \nabla_\beta \nabla_\alpha \sigma$, where ∇_α denotes a covariant derivative at x while ∇_μ indicates covariant differentiation at z.

The vector $-\sigma^\mu(z, x)$ can be thought of as a separation vector between x and z, pointing from the world line to x. When x is close to γ, $-\sigma^\mu(z, x)$ is small and can be used to express bitensors in terms of ordinary tensors at z [5,17]. For example,

$$\sigma_{\mu\nu} = g_{\mu\nu} - \tfrac{1}{3} R_{\mu\lambda\nu\rho} \sigma^\lambda \sigma^\rho + \cdots, \tag{2.4}$$

$$\sigma_{\mu\alpha} = -g^\nu{}_\alpha (g_{\mu\nu} + \tfrac{1}{6} R_{\mu\lambda\nu\rho} \sigma^\lambda \sigma^\rho + \cdots), \tag{2.5}$$

where $g^\mu{}_\alpha \equiv g^\mu{}_\alpha(z, x)$ is the parallel propagator and $R_{\mu\lambda\nu\rho}$ is the Riemann tensor evaluated on the world line.

B. Definition of the advanced coordinates

In their quasi-Cartesian version, the retarded coordinates are defined by

$$\hat{x}^0 := v, \qquad \hat{x}^a := -e^a_{\alpha'}(x') \sigma^{\alpha'}(x, x'), \qquad \sigma(x, x') = 0. \tag{2.6}$$

The last statement indicates that $x' \equiv z(v)$ and x are linked by the null geodesic segment β, and we demand that this be future-directed from x to x'.

From the fact that $\sigma^{\alpha'}$ is a null vector, we obtain

$$r := (\delta_{ab} \hat{x}^a \hat{x}^b)^{1/2} = -u_{\alpha'} \sigma^{\alpha'}, \tag{2.7}$$

and r is a positive quantity because $\sigma^{\alpha'}$ is a future-directed vector. We will see in Sec. II C that $-r$ is an affine parameter on β; this property adds credibility to the idea that r is a meaningful measure of the distance from x to $x' \equiv z(v)$.

Another consequence of Eq. (2.6) is that

$$\sigma^{\alpha'} = r(u^{\alpha'} - \Omega^a e_a^{\alpha'}), \tag{2.8}$$

where $\Omega^a := \hat{x}^a/r$ is a frame vector that satisfies $\delta_{ab}\Omega^a\Omega^b = 1$.

A straightforward calculation reveals that, under a displacement of the point x (which induces a displacement of x'), the advanced coordinates change according to

$$dv = -l_\alpha dx^\alpha, \tag{2.9}$$

$$d\hat{x}^a = -e^a_{\alpha'}\sigma^{\alpha'}{}_{\beta'}u^{\beta'}dv - e^a_{\alpha'}\sigma^{\alpha'}{}_{\beta}dx^\beta, \tag{2.10}$$

where $l_\alpha := -\sigma_\alpha/r$ is a future-directed null vector at x that is tangent to β.

C. Advanced distance; null vector field

If we keep x' linked to x by the relation $\sigma(x, x') = 0$, then

$$r(x) = -\sigma_{\alpha'}(x, x')u^{\alpha'}(x') \tag{2.11}$$

can be viewed as an ordinary scalar field defined in a neighborhood of γ. We can compute the gradient of r by finding how r changes under a displacement of x (which induces a displacement of x'). The result is

$$\nabla_\beta r = (\sigma_{\alpha'\beta'}u^{\alpha'}u^{\beta'})l_\beta - \sigma_{\alpha'\beta}u^{\alpha'}. \tag{2.12}$$

Similarly, we can view

$$l^\alpha(x) = -\frac{\sigma^\alpha(x, x')}{r(x)} \tag{2.13}$$

as an ordinary vector field, which is tangent to the congruence of null geodesics that converge to x'. It is easy to check that Eqs. (2.12) and (2.13) imply

$$l^\alpha \nabla_\alpha r = -1. \tag{2.14}$$

In addition, combining the general statement $\sigma^\alpha = -g^\alpha{}_{\alpha'}\sigma^{\alpha'}$ with Eq. (2.8) gives

$$l^\alpha = g^\alpha{}_{\alpha'}(u^{\alpha'} - \Omega^a e_a^{\alpha'}); \tag{2.15}$$

the vector at x is therefore obtained by parallel transport of $u^{\alpha'} - \Omega^a e_a^{\alpha'}$ on β. From this and Eq. (2.3) we get the alternative expression

$$l^\alpha = e_0^\alpha - \Omega^a e_a^\alpha, \tag{2.16}$$

which confirms that l^α is a future-directed null vector field (recall that $\Omega^a = \hat{x}^a/r$ is a unit frame vector).

The covariant derivative of l_α can be computed by finding how the vector changes under a displacement of x. This calculation reveals that l^α satisfies the geodesic equation in affine-parameter form, $l^\beta\nabla_\beta l^\alpha = 0$, and Eq. (2.14) informs us that the affine parameter is in fact $-r$. A displacement along a member of the congruence is therefore described by $dx^\alpha = -l^\alpha dr$. Specializing to the advanced coordinates, and using Eqs. (2.9), (2.10), and (2.13), we find that this statement becomes $dv = 0$ and $d\hat{x}^a = (\hat{x}^a/r)dr$, which integrate to $v =$ constant and $\hat{x}^a = r\Omega^a$, respectively, with Ω^a representing a constant unit vector. We have found that the congruence of null geodesics that converge to x' is described by

$$v = \text{constant}, \qquad \hat{x}^a = r\Omega^a(\theta^A) \tag{2.17}$$

in the advanced coordinates. Here, the two angles θ^A ($A = 1, 2$) serve to parametrize the unit vector Ω^a, which is independent of r.

Finally, we state without proof that l^α is hypersurface orthogonal (the proof is contained in Ref. [1]). This, together with the property that l^α satisfies the geodesic equation in affine-parameter form, implies that there exists a scalar field $v(x)$ such that

$$l_\alpha = -\nabla_\alpha v. \tag{2.18}$$

This scalar field was already identified in Eq. (2.9): it is numerically equal to the proper-time parameter of the world line at x'. We conclude that the geodesics to which l^α is tangent are the generators of the light cone $v =$ constant. As Eq. (2.17) indicates, a specific generator is selected by choosing a direction Ω^a (which can be parame-

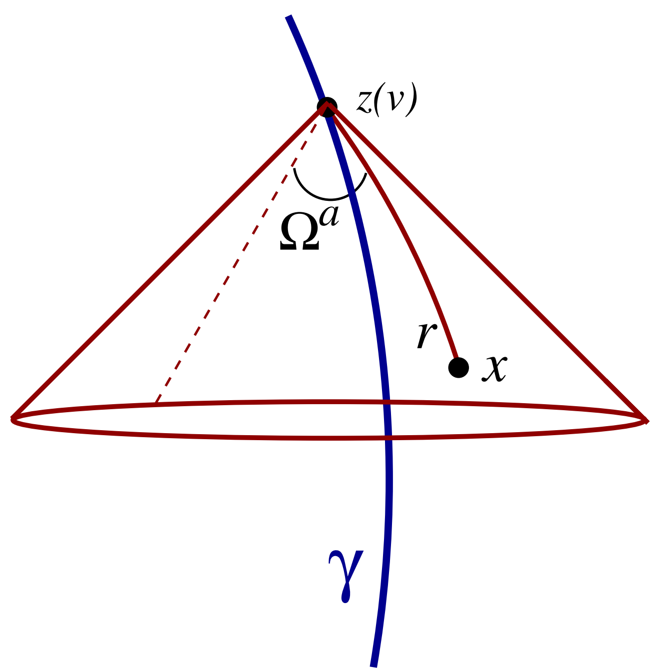

FIG. 1 (color online). Advanced coordinates of a point x relative to a world line γ. The advanced time v selects a particular light cone, the unit vector $\Omega^a := \hat{x}^a/r$ selects a particular generator of this light cone, and the advanced distance r selects a particular point on this generator.

trized by two angles θ^A), and $-r$ is an affine parameter on each generator. The geometrical meaning of the advanced coordinates is now completely clear; the construction is illustrated in Fig. 1.

D. Frame components of tensors on the world line

The metric at x in the advanced coordinates will be expressed in terms of frame components of tensors evaluated on γ. We shall need, in particular,

$$\begin{aligned} R_{a0b0}(v) &:= R_{\alpha'\gamma'\beta'\delta'} e^{\alpha'}_a u^{\gamma'} e^{\beta'}_b u^{\delta'}, \\ R_{abc0}(v) &:= R_{\alpha'\gamma'\beta'\delta'} e^{\alpha'}_a e^{\gamma'}_b e^{\beta'}_c u^{\delta'}, \\ R_{acbd}(v) &:= R_{\alpha'\gamma'\beta'\delta'} e^{\alpha'}_a e^{\gamma'}_c e^{\beta'}_b e^{\delta'}_d. \end{aligned} \tag{2.19}$$

These are the frame components of the Riemann tensor evaluated on γ; these quantities depend on v only (recall that v is numerically equal to the proper-time parameter on the world line). We next form the useful combinations

$$P_{ab} := R_{a0b0} - R_{acb0}\Omega^c - R_{bca0}\Omega^c + R_{acbd}\Omega^c\Omega^d = P_{ba}, \tag{2.20}$$

$$P_a := P_{ab}\Omega^b = R_{a0b0}\Omega^b - R_{abc0}\Omega^b\Omega^c, \tag{2.21}$$

$$P := P_a\Omega^a = R_{a0b0}\Omega^a\Omega^b, \tag{2.22}$$

in which the quantities $\Omega^a := \hat{x}^a/r$ depend on the angles θ^A only—they are independent of v and r.

E. Coordinate displacements near γ

The changes in the quasi-Cartesian advanced coordinates under a displacement of x are given by Eqs. (2.9) and (2.10). In these, we substitute the expansions for $\sigma_{\alpha'\beta'}$ and $\sigma_{\alpha'\beta}$ that appear in Eqs. (2.4) and (2.5), as well as Eqs. (2.8) and (2.16). After a straightforward calculation, we obtain the following expressions for the coordinate displacements:

$$dv = (e^0_\alpha dx^\alpha) + \Omega_a(e^b_\alpha dx^\alpha), \tag{2.23}$$

$$\begin{aligned} d\hat{x}^a &= [\tfrac{1}{2}r^2P^a + O(r^3)](e^0_\alpha dx^\alpha) \\ &\quad + [\delta^a{}_b + \tfrac{1}{6}r^2(P^a{}_b + 2P^a\Omega_b) + O(r^3)](e^b_\alpha dx^\alpha). \end{aligned} \tag{2.24}$$

Notice that the result for dv is exact, but that $d\hat{x}^a$ is expressed as an expansion in powers of r.

F. Metric near γ

It is straightforward to invert the relations of Eqs. (2.23) and (2.24) and solve for $e^0_\alpha dx^\alpha$ and $e^a_\alpha dx^\alpha$. The results are

$$\begin{aligned} e^0_\alpha dx^\alpha &= [1 + \tfrac{1}{2}r^2P + O(r^3)]dv \\ &\quad - [\Omega_a - \tfrac{1}{6}r^2(P_a - P\Omega_a) + O(r^3)]d\hat{x}^a, \end{aligned} \tag{2.25}$$

$$\begin{aligned} e^a_\alpha dx^\alpha &= [-\tfrac{1}{2}r^2P^a + O(r^3)]dv \\ &\quad + [\delta^a_b - \tfrac{1}{6}r^2(P^a_b - P^a\Omega_b) + O(r^3)]d\hat{x}^b. \end{aligned} \tag{2.26}$$

These relations, when specialized to the advanced coordinates, give us the components of the dual tetrad (e^0_α, e^a_α) at x. The metric is then computed by involving the completeness relations of Eq. (2.1). We find

$$g_{vv} = -1 - r^2P + O(r^3), \tag{2.27}$$

$$g_{va} = \Omega_a - \tfrac{2}{3}r^2(P_a - P\Omega_a) + O(r^3), \tag{2.28}$$

$$\begin{aligned} g_{ab} &= \delta_{ab} - \Omega_a\Omega_b - \tfrac{1}{3}r^2(P_{ab} - P_a\Omega_b - \Omega_aP_b \\ &\quad + P\Omega_a\Omega_b) + O(r^3). \end{aligned} \tag{2.29}$$

We see that the metric possesses a directional ambiguity on the world line: The metric at $r = 0$ still depends on the vector $\Omega^a := \hat{x}^a/r$ that specifies the direction to the point x. The advanced coordinates are therefore singular on the world line, and tensor components cannot be defined on γ. This poses no particular difficulty because we can always work, as we have been doing, with *frame components* of tensors instead of tensorial components.

III. LIGHT-CONE COORDINATES; DECOMPOSITION OF THE RIEMANN TENSOR

A. Retarded and advanced coordinates

The developments of Sec. II parallel very closely the construction of retarded coordinates described in Ref. [1]. The combined set of results is a coordinate system $(w, \hat{x}^a)$ that refers either to past light cones (advanced coordinates, $w \equiv v$, $\eta \equiv +1$) or to future light cones (retarded coordinates, $w \equiv u$, $\eta \equiv -1$) centered on a geodesic world line. In either case the metric is expressed as

$$g_{ww} = -1 - r^2P + O(r^3), \tag{3.1}$$

$$\eta g_{wa} = \Omega_a - \tfrac{2}{3}r^2(P_a - P\Omega_a) + O(r^3), \tag{3.2}$$

$$\begin{aligned} g_{ab} &= \delta_{ab} - \Omega_a\Omega_b - \tfrac{1}{3}r^2(P_{ab} - P_a\Omega_b - \Omega_aP_b \\ &\quad + P\Omega_a\Omega_b) + O(r^3), \end{aligned} \tag{3.3}$$

with

$$P_{ab} := R_{a0b0} - \eta R_{acb0}\Omega^c - \eta R_{bca0}\Omega^c + R_{acbd}\Omega^c\Omega^d \tag{3.4}$$

$$P_a := R_{a0b0}\Omega^b - \eta R_{abc0}\Omega^b\Omega^c, \tag{3.5}$$

$$P := R_{a0b0}\Omega^a\Omega^b. \tag{3.6}$$

The Riemann tensor is evaluated at the advanced/retarded point $x' \equiv z(w)$, and its frame components are defined as in Eqs. (2.19). Our subsequent developments will apply to the general metric of Eqs. (3.1)–(3.3). They will not distinguish between the advanced and retarded coordinates, and we will collectively refer to them as *light-cone coordinates*. As was stated previously, w stands for either v or u, and η is an indicator that takes the value $+1$ for the advanced coordinates and the value -1 for the retarded coordinates.

B. Decomposition of the Riemann tensor

To bring the metric to a more explicit form, we decompose the Riemann tensor into its Weyl and Ricci parts, and we involve the Einstein field equations to relate the Ricci tensor to the energy-momentum tensor of the matter distribution. This gives

$$R_{\alpha'\beta'\gamma'\delta'} = C_{\alpha'\beta'\gamma'\delta'} + 8\pi(g_{\alpha'[\gamma'}T_{\delta']\beta'} - g_{\beta'[\gamma'}T_{\delta']\alpha'}) - \frac{16\pi}{3} g_{\alpha'[\gamma'}g_{\delta']\beta'}{}^4T, \tag{3.7}$$

where $C_{\alpha'\beta'\gamma'\delta'}$ is the Weyl tensor, $T_{\alpha'\beta'}$ is the energy-momentum tensor, ${}^4T := T^{\alpha'}{}_{\alpha'}$ is its four-dimensional trace, and the square brackets indicate antisymmetrization of the enclosed indices.

We next project the Weyl tensor onto the tetrad $(u^{\alpha'}, e^{\alpha'}_a)$ and decompose the projections into irreducible components, according to [18]

$$C_{a0b0} = \mathcal{E}_{ab}, \qquad C_{abc0} = \varepsilon_{abp}\mathcal{B}^p{}_c,$$
$$C_{abcd} = \delta_{ac}\mathcal{E}_{bd} - \delta_{bc}\mathcal{E}_{ad} - \delta_{ad}\mathcal{E}_{bc} + \delta_{bd}\mathcal{E}_{ac}, \tag{3.8}$$

where ε_{abc} is the flat-space permutation symbol. The electric components of the Weyl tensor are denoted $\mathcal{E}_{ab}$, while the magnetic components are denoted $\mathcal{B}_{ab}$. These tensors are symmetric and trace-free, so that, for example, $\mathcal{E}_{ba} = \mathcal{E}_{ab}$ and $\mathcal{B}^a{}_a = 0$. Because the Weyl tensor is evaluated on the world line, $\mathcal{E}_{ab}$ and $\mathcal{B}_{ab}$ are functions of the null coordinate w only.

We perform similar operations on the energy-momentum tensor, and introduce the notation

$$T_{00} = \rho, \qquad T_{0a} = -j_a, \qquad T_{ab} = S_{ab} + \tfrac{1}{3}\delta_{ab}T, \tag{3.9}$$

where S_{ab} is symmetric and trace-free, and $T := \delta^{ab}T_{ab} = T_{\alpha'\beta'}(\delta^{ab}e^{\alpha'}_a e^{\beta'}_b) = T_{\alpha'\beta'}(g^{\alpha'\beta'} + u^{\alpha'}u^{\beta'}) = {}^4T + \rho$ is the spatial trace of the energy-momentum tensor. The quantity ρ represents the mass-energy density measured by an observer moving on the world line γ, j_a is the flux of mass-energy traveling in the direction of the base vector $e^{\alpha'}_a$, S_{ab} is the trace-free part of the stress tensor, and $\frac{1}{3}T$ is an isotropic pressure. These quantities also are functions of w only.

Substituting Eqs. (3.8) and (3.9) into Eq. (3.7), and then this into Eq. (3.4), produces a decomposition of P_{ab} into its irreducible pieces. We obtain

$$\begin{aligned} P_{ab} &= 2\mathcal{E}_{ab} - \Omega_a\mathcal{E}_{bc}\Omega^c - \Omega_b\mathcal{E}_{ac}\Omega^c + \delta_{ab}\mathcal{E}_{cd}\Omega^c\Omega^d \\ &\quad - \eta(\varepsilon_{apq}\Omega^p\mathcal{B}^p{}_b + \varepsilon_{bpq}\Omega^p\mathcal{B}^p{}_a) \\ &\quad + \frac{4\pi}{3}(3\delta_{ab} - 2\Omega_a\Omega_b)\rho - 4\pi\eta(j_a\Omega_b + j_b\Omega_a \\ &\quad - 2\delta_{ab}j_c\Omega^c) - 4\pi(\Omega_a S_{bc}\Omega^c + \Omega_b S_{ac}\Omega^c \\ &\quad - \delta_{ab}S_{cd}\Omega^c\Omega^d) + \frac{4\pi}{3}\delta_{ab}T. \end{aligned} \tag{3.10}$$

C. Tidal and matter potentials

At this stage it is useful to involve the irreducible quantities $\mathcal{E}_{ab}(w)$ and $\mathcal{B}_{ab}(w)$ in the definition of a number of *tidal potentials*. We also involve $j_a(w)$ and $S_{ab}(w)$ in the definition of *matter potentials*. These potentials, which are displayed in Table I, form the elementary building blocks of the metric tensor. Each potential is identified by its multipole content. For example, ${}^{\mathrm{q}}\mathcal{E} := \mathcal{E}_{ab}\Omega^a\Omega^b$ is a quadrupolar potential by virtue of the fact that $\mathcal{E}_{ab}$ is symmetric and trace-free. As another example, ${}^{\mathrm{d}}j := j_a\Omega^a$ is a dipolar potential. Table I also introduces vectorial and tensorial potentials that possess the property of being transverse, meaning that each vector or tensor is orthogonal to the unit frame vector Ω^a. Finally, the tensor potentials ${}^{\mathrm{q}}\mathcal{E}_{ab}$ and ${}^{\mathrm{q}}\mathcal{B}_{ab}$ have the additional property of being symmetric and trace-free.

It is easy to check that P_{ab} is expressed in terms of the tidal and matter potentials as

$$\begin{aligned} P_{ab} &= {}^{\mathrm{q}}\mathcal{E}_{ab} + 2\Omega_{(a}{}^{\mathrm{q}}\mathcal{E}_{b)} + \Omega_a\Omega_b{}^{\mathrm{q}}\mathcal{E} - \eta({}^{\mathrm{q}}\mathcal{B}_{ab} + 2\Omega_{(a}{}^{\mathrm{q}}\mathcal{B}_{b)}) \\ &\quad + 4\pi(\delta_{ab} - \Omega_a\Omega_b)\rho + \frac{4\pi}{3}\Omega_a\Omega_b\rho \\ &\quad + 8\pi\eta[(\delta_{ab} - \Omega_{ab})^{\mathrm{d}}j - \Omega_{(a}{}^{\mathrm{d}}j_{b)}] \\ &\quad + 4\pi(\delta_{ab} - \Omega_a\Omega_b)^{\mathrm{q}}S - 8\pi\Omega_{(a}{}^{\mathrm{q}}S_{b)} - 4\pi\Omega_a\Omega_b{}^{\mathrm{q}}S \\ &\quad + \frac{4\pi}{3}(\delta_{ab} - \Omega_a\Omega_b)T + \frac{4\pi}{3}\Omega_a\Omega_b T. \end{aligned} \tag{3.11}$$

We observe that P_{ab} is now decomposed into transverse-transverse components that are fully orthogonal to Ω_a, transverse-longitudinal components that are partly orthogonal to and partly aligned with Ω_a, and longitudinal-longitudinal components that are proportional to $\Omega_a\Omega_b$. Contracting Eq. (3.11) with Ω^b produces

$$\begin{aligned} P_a &= {}^{\mathrm{q}}\mathcal{E}_a + \Omega_a{}^{\mathrm{q}}\mathcal{E} - \eta{}^{\mathrm{q}}\mathcal{B}_a + \frac{4\pi}{3}\Omega_a\rho - 4\pi\eta{}^{\mathrm{d}}j_a \\ &\quad - 4\pi{}^{\mathrm{q}}S_a - 4\pi\Omega_a{}^{\mathrm{q}}S + \frac{4\pi}{3}\Omega_a T, \end{aligned} \tag{3.12}$$

and contracting this with Ω^a gives

$$P = {}^{\mathrm{q}}\mathcal{E} + \frac{4\pi}{3}\rho - 4\pi{}^{\mathrm{q}}S + \frac{4\pi}{3}T. \tag{3.13}$$

Substituting Eqs. (3.11)–(3.13) into Eqs. (3.1)–(3.3) produces our final expression for the metric tensor in the quasi-Cartesian version of the light-cone coordinates. We obtain, after simplification,

$$g_{ww} = -1 - r^2 {}^{\mathsf{q}}\mathcal{E} - \frac{4\pi}{3} r^2(\rho - 3{}^{\mathsf{q}}S + T) + O(r^3), \tag{3.14}$$

$$\eta g_{wa} = \Omega_a - \frac{2}{3} r^2({}^{\mathsf{q}}\mathcal{E}_a - \eta {}^{\mathsf{q}}\mathcal{B}_a) + \frac{8\pi}{3}\eta r^2({}^{\mathsf{q}}S_a + \eta {}^{\mathsf{d}}j_a) + O(r^3), \tag{3.15}$$

$$g_{ab} = \delta_{ab} - \Omega_a\Omega_b - \frac{1}{3} r^2({}^{\mathsf{q}}\mathcal{E}_{ab} - \eta {}^{\mathsf{q}}\mathcal{B}_{ab}) - \frac{4\pi}{3} r^2(\delta_{ab} - \Omega_a\Omega_b)\Big(\rho + 2\eta {}^{\mathsf{d}}j + {}^{\mathsf{q}}S + \frac{1}{3}T\Big) + O(r^3). \tag{3.16}$$

We observe that the metric is neatly expressed in terms of the monopolar "potentials" ρ and T, the dipolar potentials ${}^{\mathsf{d}}j$ and ${}^{\mathsf{d}}j_a$, the quadrupolar tidal potentials ${}^{\mathsf{q}}\mathcal{E}$, ${}^{\mathsf{q}}\mathcal{E}_a$, ${}^{\mathsf{q}}\mathcal{E}_{ab}$, ${}^{\mathsf{q}}\mathcal{B}_a$, ${}^{\mathsf{q}}\mathcal{B}_{ab}$, and the quadrupolar matter potentials ${}^{\mathsf{q}}S$ and ${}^{\mathsf{q}}S_a$. We also observe that g_{wa} contains both longitudinal and transverse pieces, while g_{ab} is fully transverse. We recall that the coordinates are advanced when $\eta = +1$ (then $w \equiv v$) and that they are retarded when $\eta = -1$ (then $w \equiv u$).

IV. ANGULAR COORDINATES

A. Transformation to angular coordinates

Because the frame vector $\Omega^a := \hat{x}^a/r$ satisfies $\delta_{ab}\Omega^a\Omega^b = 1$, it can be parametrized by two angles θ^A. A canonical choice for the parametrization is

$$\Omega^a = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta). \tag{4.1}$$

It is then convenient to perform a coordinate transformation from $\hat{x}^a$ to (r, θ^A) using the relations $\hat{x}^a = r\Omega^a(\theta^A)$. (Recall from Sec. III C that the angles θ^A are constant on the generators of the light cones $w =$ constant, and that $\pm r$ is an affine parameter on these generators. The relations $\hat{x}^a = r\Omega^a$ therefore describe the behavior of the generators.) The differential form of the coordinate transformation is

$$d\hat{x}^a = \Omega^a dr + r\Omega^a_A d\theta^A, \tag{4.2}$$

where the transformation matrix

$$\Omega^a_A := \frac{\partial \Omega^a}{\partial \theta^A} \tag{4.3}$$

satisfies the identity $\Omega_a\Omega^a_A = 0$.

We introduce the quantities

$$\Omega_{AB} := \delta_{ab}\Omega^a_A\Omega^b_B, \tag{4.4}$$

which act as a (nonphysical) metric on the submanifold spanned by the angular coordinates. In the canonical parametrization of Eq. (4.1), $\Omega_{AB} = \mathrm{diag}(1, \sin^2\theta)$, and the metric is that of a round two-sphere of unit radius. We use the inverse of Ω_{AB}, denoted Ω^{AB}, to raise uppercase Latin indices. We then define the new object

$$\Omega^A_a := \delta_{ab}\Omega^{AB}\Omega^b_B \tag{4.5}$$

which satisfies the identities

$$\Omega^A_a\Omega^a_B = \delta^A_B, \qquad \Omega^a_A\Omega^A_b = \delta^a{}_b - \Omega^a\Omega_b. \tag{4.6}$$

The first result is a direct consequence of the definition, and the second result follows from the fact that both sides are symmetric in a and b, orthogonal to Ω_a and Ω^b, and have the same trace.

The Levi-Civita tensor on S^2 is constructed as

$$\varepsilon_{AB} := \varepsilon_{abc}\Omega^a_A\Omega^b_B\Omega^c, \tag{4.7}$$

where ε_{abc} is the Cartesian permutation symbol; in the canonical coordinates we have $\varepsilon_{\theta\phi} = \sin\theta$.

We let D_A denote the covariant derivative operator compatible with Ω_{AB}, so that $D_A\Omega_{BC} = 0$. It is easy to show that $D_A\varepsilon_{BC} = 0$ and

$$\Omega^a_{AB} := D_B\Omega^a_A = -\Omega^a\Omega_{AB}. \tag{4.8}$$

When we apply the coordinate transformation of Eq. (4.2) to the metric of Eqs. (3.14)–(3.16), we find that the only nonvanishing components of the metric tensor are now given by

$$g_{ww} = -1 - r^2 {}^{\mathsf{q}}\mathcal{E} - \frac{4\pi}{3} r^2(\rho - 3{}^{\mathsf{q}}S + T) + O(r^3), \tag{4.9}$$

$$g_{wr} = \eta, \tag{4.10}$$

$$g_{wA} = -\frac{2}{3}\eta r^3({}^{\mathsf{q}}\mathcal{E}_A - \eta {}^{\mathsf{q}}\mathcal{B}_A) + \frac{8\pi}{3}\eta r^3({}^{\mathsf{q}}S_A + \eta {}^{\mathsf{d}}j_A) + O(r^4), \tag{4.11}$$

$$g_{AB} = r^2\Omega_{AB} - \frac{1}{3} r^4({}^{\mathsf{q}}\mathcal{E}_{AB} - \eta {}^{\mathsf{q}}\mathcal{B}_{AB}) - \frac{4\pi}{3} r^4\Omega_{AB}\Big(\rho + 2\eta {}^{\mathsf{d}}j + {}^{\mathsf{q}}S + \frac{1}{3}T\Big) + O(r^5), \tag{4.12}$$

where ${}^{\mathsf{q}}\mathcal{E}_A := {}^{\mathsf{q}}\mathcal{E}_a\Omega^a_A$, ${}^{\mathsf{q}}\mathcal{E}_{AB} := {}^{\mathsf{q}}\mathcal{E}_{ab}\Omega^a_A\Omega^b_B$, and so on. The results $g_{wr} = \eta$, $g_{rr} = 0$, and $g_{rA} = 0$ are exact, and they follow from the light-cone nature of the coordinates. [For example, for the advanced coordinates we have $l_\alpha = (-1, 0, 0, 0)$ and $l^\alpha = (0, -1, 0, 0)$, where we use the ordering (v, r, θ, ϕ); these relations imply that $g_{vr} = 1$ and $g_{rr} = g_{r\theta} = g_{r\phi} = 0$.] Once more we recall that the coordinates are advanced when $\eta = +1$ (then $w \equiv v$), and that they are retarded when $\eta = -1$ (then $w \equiv u$).

B. Tidal and matter potentials in spherical coordinates

According to Table I, the tidal potential ${}^{\mathsf{q}}\mathcal{E}$ is defined by ${}^{\mathsf{q}}\mathcal{E} = \mathcal{E}_{cd}\Omega^c\Omega^d$. Differentiating this with respect to θ^A gives $D_A{}^{\mathsf{q}}\mathcal{E} = 2\Omega^c_A\mathcal{E}_{cd}\Omega^d$. In view of the identity $\Omega_c\Omega^c_A = 0$, we may write this as $D_A{}^{\mathsf{q}}\mathcal{E} = 2\Omega^a_A(\delta_a{}^c - \Omega_a\Omega^c)\mathcal{E}_{cd}\Omega^d$. Referring once more to Table I, we see that this is $D_A{}^{\mathsf{q}}\mathcal{E} = 2\Omega^a_A{}^{\mathsf{q}}\mathcal{E}_a$ and we conclude that

$$ {}^{\mathsf{q}}\mathcal{E}_A = \tfrac{1}{2}D_A{}^{\mathsf{q}}\mathcal{E}. \tag{4.13}$$

Acting on ${}^{\mathsf{q}}\mathcal{E}$ with two derivative operators gives $D_AD_B{}^{\mathsf{q}}\mathcal{E} = 2\mathcal{E}_{cd}\Omega^c_A\Omega^d_B + 2\mathcal{E}_{cd}\Omega^c\Omega^d_{AB}$. Using Eq. (4.8) produces $D_AD_B{}^{\mathsf{q}}\mathcal{E} = 2\mathcal{E}_{cd}\Omega^c_A\Omega^d_B - 2\mathcal{E}_{cd}\Omega^c\Omega^d\Omega_{AB} = 2\mathcal{E}_{cd}\Omega^c_A\Omega^d_B - 2\Omega_{AB}{}^{\mathsf{q}}\mathcal{E}$. We write this in the form $(D_AD_B + 3\Omega_{AB}){}^{\mathsf{q}}\mathcal{E} = 2\mathcal{E}_{cd}\Omega^c_A\Omega^d_B + \Omega_{AB}{}^{\mathsf{q}}\mathcal{E} = 2\mathcal{E}_{cd}\Omega^a_A\Omega^b_B(\delta_a{}^c - \Omega_a\Omega^c)(\delta_b{}^d - \Omega_b\Omega^d)\mathcal{E}_{cd} + \Omega^a_A\Omega^b_B(\delta_{ab} - \Omega_a\Omega_b){}^{\mathsf{q}}\mathcal{E}$, after involving Eq. (4.4). Consulting Table I once more, we see that the right-hand side is equal to $\Omega^a_A\Omega^b_B{}^{\mathsf{q}}\mathcal{E}_{ab} =: {}^{\mathsf{q}}\mathcal{E}_{AB}$ and we conclude that

$$ {}^{\mathsf{q}}\mathcal{E}_{AB} = (D_AD_B + 3\Omega_{AB}){}^{\mathsf{q}}\mathcal{E}. \tag{4.14}$$

We observe that this tensor is trace-free, because $\Omega^{AB}{}^{\mathsf{q}}\mathcal{E}_{AB} = \Omega^{AB}\Omega^a_A\Omega^b_B{}^{\mathsf{q}}\mathcal{E}_{ab} = (\delta^{ab} - \Omega^a\Omega^b){}^{\mathsf{q}}\mathcal{E}_{ab} = 0$, after involving Eq. (4.6), and due to the fact that ${}^{\mathsf{q}}\mathcal{E}_{ab}$ is transverse and trace-free (in the Cartesian sense). The equation $(\Omega^{AB}D_AD_B + 6){}^{\mathsf{q}}\mathcal{E} = 0$, which we obtain from Eq. (4.14), reveals that ${}^{\mathsf{q}}\mathcal{E}(w, \theta^A)$ is a spherical-harmonic function of degree $l = 2$.

We define a magnetic potential ${}^{\mathsf{q}}\mathcal{B} := \mathcal{B}_{cd}\Omega^c\Omega^d$ and differentiate it with respect to θ^B, giving $D_B{}^{\mathsf{q}}\mathcal{B} = 2\Omega^c_B\mathcal{B}_{cd}\Omega^d$. We next multiply this by the Levi-Civita tensor of Eq. (4.7) and get $-\varepsilon_A{}^BD_B{}^{\mathsf{q}}\mathcal{B} = -2\varepsilon_{apq}\Omega^a_A\Omega^{pB}\Omega^q\Omega^c_B\mathcal{B}_{cd}\Omega^d$. Using Eq. (4.6) and the antisymmetry property of the permutation symbol, this is $-\varepsilon_A{}^BD_B{}^{\mathsf{q}}\mathcal{B} = -2\Omega^a_A\varepsilon_{apq}\Omega^q\mathcal{B}^p_d\Omega^d = 2\Omega^a_A{}^{\mathsf{q}}\mathcal{B}_a$, and we conclude that

$$ {}^{\mathsf{q}}\mathcal{B}_A = -\tfrac{1}{2}\varepsilon_A{}^BD_B{}^{\mathsf{q}}\mathcal{B}, \qquad {}^{\mathsf{q}}\mathcal{B} := \mathcal{B}_{cd}\Omega^c\Omega^d. \tag{4.15}$$

Acting on ${}^{\mathsf{q}}\mathcal{B}$ with two derivative operators and multiplying by the Levi-Civita tensor gives $-\varepsilon_A{}^CD_BD_C{}^{\mathsf{q}}\mathcal{B} = 2\Omega^a_A\varepsilon_{apq}\Omega^p\mathcal{B}^q{}_b\Omega^b_B + 2\varepsilon_{AB}{}^{\mathsf{q}}\mathcal{B}$. Symmetrizing with respect to A and B and consulting Table I yields $-(\varepsilon_A{}^CD_B + \varepsilon_B{}^CD_A)D_C{}^{\mathsf{q}}\mathcal{B} = 2\Omega^a_A\Omega^b_B{}^{\mathsf{q}}\mathcal{B}_{ab}$, and we conclude that

$$ {}^{\mathsf{q}}\mathcal{B}_{AB} = -\tfrac{1}{2}(\varepsilon_A{}^CD_B + \varepsilon_B{}^CD_A)D_C{}^{\mathsf{q}}\mathcal{B}. \tag{4.16}$$

This tensor is trace-free, because $\Omega^{AB}{}^{\mathsf{q}}\mathcal{B}_{AB} = -\varepsilon^{BC}D_BD_C{}^{\mathsf{q}}\mathcal{B} = 0$ by virtue of the symmetry of $D_BD_C{}^{\mathsf{q}}\mathcal{B}$ and the antisymmetry of the Levi-Civita tensor.

Similar results can be obtained for the matter potentials. Differentiating ${}^{\mathsf{d}}j := j_a\Omega^a$ produces $D_A{}^{\mathsf{d}}j = j_a\Omega^a_A = j_c(\delta_a{}^c - \Omega_a\Omega^c)\Omega^a_A = \Omega^a_A{}^{\mathsf{d}}j_a$, and we conclude that

$$ {}^{\mathsf{d}}j_A = D_A{}^{\mathsf{d}}j. \tag{4.17}$$

Finally,

$$ {}^{\mathsf{q}}S_A = \tfrac{1}{2}D_A{}^{\mathsf{q}}S \tag{4.18}$$

follows after a calculation similar to the one leading to Eq. (4.13).

C. Decomposition in spherical harmonics

The results obtained in the preceding subsection indicate that the tidal and matter potentials that appear in the metric of Eqs. (4.9)–(4.12) can all be obtained by covariant differentiation of the scalar potentials ${}^{\mathsf{q}}\mathcal{E}$, ${}^{\mathsf{q}}\mathcal{B}$, ${}^{\mathsf{d}}j$, and ${}^{\mathsf{q}}S$. These are functions of the null coordinate w and the dependence on the angles θ^A appears in the factors $\Omega^a(\theta^A)$.

This angular dependence can be made more explicit by involving spherical-harmonic functions. Let

$$ Y^{1\mathsf{m}} = \{Y^{1,0}, Y^{1,1c}, Y^{1,1s}\} \tag{4.19}$$

be a set of real, unnormalized, spherical-harmonic functions of degree $l = 1$. And let

$$ Y^{2\mathsf{m}} = \{Y^{2,0}, Y^{2,1c}, Y^{2,1s}, Y^{2,2c}, Y^{2,2s}\} \tag{4.20}$$

be a set of real, unnormalized, spherical-harmonic functions of degree $l = 2$. The abstract index m describes the dependence of the spherical harmonics on the angle ϕ; the numerical part of the label refers to the azimuthal index m, and the letter indicates whether the function is proportional to $\cos(m\phi)$ or $\sin(m\phi)$. Explicit expressions are listed in Tables II and III.

We decompose the scalar potentials according to

$$ {}^{\mathsf{q}}\mathcal{E}(w, \theta^A) = \sum_{\mathsf{m}}{}^{\mathsf{q}}\mathcal{E}_{\mathsf{m}}(w)Y^{2\mathsf{m}}(\theta^A), \tag{4.21}$$

$$ {}^{\mathsf{q}}\mathcal{B}(w, \theta^A) = \sum_{\mathsf{m}}{}^{\mathsf{q}}\mathcal{B}_{\mathsf{m}}(w)Y^{2\mathsf{m}}(\theta^A), \tag{4.22}$$

$$ {}^{\mathsf{d}}j(w, \theta^A) = \sum_{\mathsf{m}}{}^{\mathsf{d}}j_{\mathsf{m}}(w)Y^{1\mathsf{m}}(\theta^A), \tag{4.23}$$

$$ {}^{\mathsf{q}}S(w, \theta^A) = \sum_{\mathsf{m}}{}^{\mathsf{q}}S_{\mathsf{m}}(w)Y^{2\mathsf{m}}(\theta^A), \tag{4.24}$$

in terms of their harmonic components ${}^{\mathsf{q}}\mathcal{E}_{\mathsf{m}}$, ${}^{\mathsf{q}}\mathcal{B}_{\mathsf{m}}$, ${}^{\mathsf{d}}j_{\mathsf{m}}$, and ${}^{\mathsf{q}}S_{\mathsf{m}}$. These are in a one-to-one correspondence with the frame tensors $\mathcal{E}_{ab}(w)$, $\mathcal{B}_{ab}(w)$, $j_a(w)$, and $S_{ab}(w)$; the relationships are displayed in Table IV.

The derivatives of the scalar potentials will be expressed in terms of derivatives of the spherical-harmonic functions. The vectorial harmonics are

$$ Y^{l\mathsf{m}}_A = D_AY^{l\mathsf{m}}, \qquad X^{l\mathsf{m}}_A = -\varepsilon_A{}^BD_BY^{l\mathsf{m}}, \tag{4.25}$$

and the tensorial harmonics are

$$ Y^{l\mathsf{m}}\Omega_{AB}, \qquad Y^{l\mathsf{m}}_{AB} = [D_AD_B + \tfrac{1}{2}l(l+1)\Omega_{AB}]Y^{l\mathsf{m}} \tag{4.26}$$

and

$$ X^{l\mathsf{m}}_{AB} = -\tfrac{1}{2}(\varepsilon_A{}^CD_B + \varepsilon_B{}^CD_A)D_CY^{l\mathsf{m}}. \tag{4.27}$$

Apart from notation and normalization, these definitions agree with those of Regge and Wheeler [19]. We note that the tensorial harmonics Y^{lm}_{AB} and X^{lm}_{AB} are symmetric and trace-free.

The decompositions of the vectorial and tensorial potentials in terms of vectorial and tensorial harmonics are displayed in Table IV. They are obtained by substituting Eqs. (4.21)–(4.24) into Eqs. (4.13)–(4.18).

The most explicit form for the metric tensor is obtained after substituting the spherical-harmonic decompositions of Table IV, along with the spherical-harmonic functions listed in Tables II and III, into Eqs. (4.9)–(4.12). This leads to long expressions, but in practical applications it may happen that only a few frame components among $\mathcal{E}_{ab}$, $\mathcal{B}_{ab}$, ρ, j_a, S_{ab}, and T are nonzero; in such cases only a few harmonic components among ${}^{\mathsf{q}}\mathcal{E}_{\mathsf{m}}$, ${}^{\mathsf{q}}\mathcal{B}_{\mathsf{m}}$, ${}^{\mathsf{d}}j_{\mathsf{m}}$, and ${}^{\mathsf{q}}S_{\mathsf{m}}$ will contribute to the metric, and the expressions will simplify. We shall encounter such cases in the next section.

V. APPLICATIONS

A. Comoving observer in a spatially flat cosmology

To illustrate how the formalism works, we first consider the world line of a comoving observer in a cosmological spacetime. The global metric is

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2), \tag{5.1}$$

where $a(t)$ is an arbitrary scale factor; for simplicity we take the cosmology to be spatially flat. This application was already presented in Ref. [1], but we generalize it here from the retarded coordinates considered there to light-cone coordinates of both types (retarded and advanced). Furthermore, the decomposition of the energy-momentum tensor into irreducible parts was not accomplished in the earlier paper, and this decomposition adds insight to our earlier results.

Without loss of generality we take our observer to be at the spatial origin of the global coordinate system ($x = y = z = 0$), and his velocity vector is given by

$$u^\mu = (1, 0, 0, 0) \tag{5.2}$$

in the ordering (t, x, y, z) of the cosmological coordinates. This vector satisfies the geodesic equation, and t is proper time for the observer. We wish to transform the metric of Eq. (5.1) to light-cone coordinates (w, r, θ^A) centered on the world line of this observer.

To do so we must first construct a triad of orthonormal spatial vectors e^μ_a. A simple choice is

$$e^\mu_1 = (0, a^{-1}, 0, 0), \qquad e^\mu_2 = (0, 0, a^{-1}, 0), \qquad e^\mu_3 = (0, 0, 0, a^{-1}); \tag{5.3}$$

these vectors are all parallel transported on γ.

According to Eq. (3.8) and the fact that the Weyl tensor of the spacetime vanishes, we have

$$\mathcal{E}_{ab} = \mathcal{B}_{ab} = 0. \tag{5.4}$$

And according to Eq. (3.9) and a simple computation, we have $j_a = S_{ab} = 0$ and

$$\rho = \frac{3}{8\pi}(\dot{a}/a)^2, \qquad T = -\frac{3}{8\pi}[2\ddot{a}/a + (\dot{a}/a)^2]. \tag{5.5}$$

Here the scale factor is expressed in terms of $w \equiv$ [proper time on γ] by simply making the functional substitution $a(t) \to a(w)$; overdots indicate differentiation with respect to w. Recall that ρ is the mass-energy density measured by the observer, and that $\frac{1}{3}T$ is the measured pressure of the cosmological fluid.

The vanishing of $\mathcal{E}_{ab}$, $\mathcal{B}_{ab}$, j_a, and S_{ab} implies that the metric is spherically symmetric around γ (this does not come as a surprise). After substituting Eqs. (5.4) and (5.5) into Eqs. (4.9)–(4.12), a short calculation reveals that the metric components are given by

$$g_{ww} = -1 + r^2(\ddot{a}/a) + O(r^3), \tag{5.6}$$

$$g_{wr} = \eta, \tag{5.7}$$

$$g_{wA} = O(r^4), \tag{5.8}$$

$$g_{AB} = r^2\Omega_{AB}\{1 + \tfrac{1}{3}r^2[\ddot{a}/a - (\dot{a}/a)^2] + O(r^3)\}. \tag{5.9}$$

We recall that the scale factor and its derivatives are functions of the null coordinate w. When the scale factor behaves as a power law, $a(t) \propto t^\alpha$ with α a constant, we have $\ddot{a}/a = -\alpha(1-\alpha)/w^2$ and $\ddot{a}/a - (\dot{a}/a)^2 = -\alpha/w^2$. When instead the scale factor behaves as an exponential, $a(t) \propto e^{Ht}$ with H a constant, we have $\ddot{a}/a = H^2$ and $\ddot{a}/a - (\dot{a}/a)^2 = 0$.

B. Static observer in Melvin's magnetic universe

Melvin's magnetic universe [13–15] is a static, cylindrically symmetric spacetime that is filled with a magnetic field held together by gravity. The exact solution to the Einstein-Maxwell equations that describes this situation consists of the metric

$$ds^2 = \Lambda^2(-dt^2 + d\bar{\rho}^2 + dz^2) + \Lambda^{-2}\bar{\rho}^2 d\varphi^2 \tag{5.10}$$

and the vector potential

$$A^\alpha = \tfrac{1}{2}B\Lambda\varphi^\alpha, \tag{5.11}$$

where $\varphi^\alpha = \partial x^\alpha/\partial\varphi$ is the spacetime's azimuthal Killing vector. We have introduced

$$\Lambda := 1 + \tfrac{1}{4}B^2\bar{\rho}^2, \tag{5.12}$$

and the constant B measures the strength of the magnetic field. The metric and the vector potential are expressed in cylindrical coordinates $(t, \bar{\rho}, z, \varphi)$.

The metric of Eq. (5.10) can be decomposed in terms of a tetrad of orthonormal vectors. We introduce a

"Cartesian" frame described by

$$e_0^\alpha := (\Lambda^{-1}, 0, 0, 0), \tag{5.13}$$

$$e_1^\alpha := (0, \Lambda^{-1}\cos\varphi, 0, -\Lambda\bar{\rho}^{-1}\sin\varphi), \tag{5.14}$$

$$e_2^\alpha := (0, \Lambda^{-1}\sin\varphi, 0, \Lambda\bar{\rho}^{-1}\cos\varphi), \tag{5.15}$$

$$e_3^\alpha := (0, 0, \Lambda^{-1}, 0). \tag{5.16}$$

It is easy to check that the inverse metric can be expressed as $g^{\alpha\beta} = -e_0^\alpha e_0^\beta + e_1^\alpha e_1^\beta + e_2^\alpha e_2^\beta + e_3^\alpha e_3^\beta$. It is also easy to check that, in this tetrad, the electromagnetic field tensor has

$$B_3 := F_{12} := F_{\alpha\beta}e_1^\alpha e_2^\beta = \frac{B}{\Lambda^2} \tag{5.17}$$

as its only nonvanishing component.

We wish to consider a static observer in Melvin's magnetic universe. To ensure that this observer moves on a world line γ that is a geodesic of the spacetime, we place him on the axis of symmetry at $\bar{\rho} = 0$. The observer has a velocity vector given by $u^\alpha = e_0^\alpha(\bar{\rho} = 0)$, and $e_a^\alpha(\bar{\rho} = 0)$ is a triad of parallel-transported vectors on γ.

A straightforward computation reveals that the metric of Eq. (5.10) comes with a Weyl tensor whose nonvanishing electric components are

$$\mathcal{E}_{11} = \mathcal{E}_{22} = \tfrac{1}{2}B^2, \qquad \mathcal{E}_{33} = -B^2. \tag{5.18}$$

The magnetic part of the Weyl tensor vanishes: $\mathcal{B}_{ab} = 0$. A computation of the energy-momentum tensor (either from the metric or from the electromagnetic field tensor) reveals that

$$\rho = \frac{B^2}{8\pi}, \tag{5.19}$$

$$S_{11} = S_{22} = \frac{B^2}{12\pi}, \qquad S_{33} = -\frac{B^2}{6\pi}, \tag{5.20}$$

and

$$T = \frac{B^2}{8\pi}, \tag{5.21}$$

while $j_a = 0$. (Recall that $T := \delta^{ab}T_{ab}$ is the three-dimensional trace of the energy-momentum tensor; the four-dimensional trace is ${}^4T = T - \rho$, and it vanishes by virtue of the conformal invariance of Maxwell's equations.) These relations imply that ${}^{\mathsf{q}}\mathcal{E}_0 = \frac{1}{2}B^2$ and ${}^{\mathsf{q}}S_0 = B^2/(12\pi)$ are the only nonvanishing harmonic components of the tidal and matter potentials.

Making the substitutions from Eqs. (5.18)–(5.21), Table III and IV into Eqs. (4.9)–(4.12), we find that the metric components in light-cone coordinates are

$$g_{ww} = -1 - \tfrac{1}{2}B^2r^2\sin^2\theta + O(r^3), \tag{5.22}$$

$$g_{wr} = \eta, \tag{5.23}$$

$$g_{w\theta} = -\tfrac{1}{3}\eta B^2 r^3 \sin\theta\cos\theta + O(r^4), \tag{5.24}$$

$$g_{\theta\theta} = r^2 + \tfrac{1}{6}B^2r^4\sin^2\theta + O(r^5), \tag{5.25}$$

$$g_{\phi\phi} = r^2\sin^2\theta - \tfrac{5}{6}B^2r^4\sin^4\theta + O(r^5). \tag{5.26}$$

As expected, the metric is axially symmetric, but the full cylindrical symmetry of the spacetime is not revealed by the light-cone coordinates.

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