Macroscopic effects of the quantum trace anomaly

Emil Mottola*
Theoretical Division, T-8 Los Alamos National Laboratory M.S. B285 Los Alamos, New Mexico 87545, USA

Ruslan Vaulin†
Department of Physics Florida Atlantic University 777 Glades Road, Boca Raton, Florida 33431, USA
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The low energy effective action of gravity in any even dimension generally acquires nonlocal terms associated with the trace anomaly, generated by the quantum fluctuations of massless fields. The local auxiliary field description of this effective action in four dimensions requires two additional scalar fields, not contained in classical general relativity, which remain relevant at macroscopic distance scales. The auxiliary scalar fields depend upon boundary conditions for their complete specification, and therefore carry global information about the geometry and macroscopic quantum state of the gravitational field. The scalar potentials also provide coordinate invariant order parameters describing the conformal behavior and divergences of the stress tensor on event horizons. We compute the stress tensor due to the anomaly in terms of its auxiliary scalar potentials in a number of concrete examples, including the Rindler wedge, the Schwarzschild geometry, and de Sitter spacetime. In all of these cases, a small number of classical order parameters completely determine the divergent behaviors allowed on the horizon, and yield qualitatively correct global approximations to the renormalized expectation value of the quantum stress tensor.

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I. INTRODUCTION

Gravitation is a macroscopic phenomenon, governing the behavior of matter at scales from a few cm. up to the largest scales of cosmology, 10^{28} cm. On the other hand quantum phenomena are associated generally with microscopic distances, on the order of atomic scales, 10^{-8} cm. or smaller. The physical and mathematical frameworks used to describe physics at these disparate scales are quite different as well, and at first sight appear even to be irreconcilable. In general relativity gravitation is founded on the equivalence principle, which finds its mathematical expression in differential equations which transform covariantly under a local change of frame at every spacetime point. In classical physics this idealization of spacetime as a local differentiable structure is quite natural. Yet because of quantum fluctuations it is very likely that the classical description of spacetime as a smooth pseudo-Riemannian manifold fails at very short distance scales, of the order of the Planck length, $L_{\rm Pl} = \sqrt{\hbar G/c^3} \sim 10^{-33}$ cm.

Perhaps less widely appreciated is that quantum theory also raises questions about the purely local, classical description of the gravitational interactions of matter, even at macroscopic distances. This is because of the peculiarly quantum phenomena of phase coherence and entanglement, which may be present on any scale, given the right conditions. Indeed macroscopic quantum states are encountered in virtually all branches of physics on a very wide variety of scales. Some of the better known examples occur in nonrelativistic many-body and condensed matter systems, such as Bose-Einstein condensation, superfluidity, and superconductivity. These quantum coherence effects due to the wavelike properties of matter have their close analogs in relativistic field theories as well, in spontaneous symmetry breaking by the Higgs mechanism in electroweak theory, and chiral quark pair condensation in low energy QCD. In each of these cases the ground state or "vacuum" is a macroscopic quantum state, described at very long wavelengths by a nonvanishing quasiclassical order parameter, in a low energy effective field theory.

The strictly classical theory of general relativity would seem to preclude any incorporation of macroscopic coherence and entanglement effects of quantum matter, on any scale. Taking account of the quantum wavelike properties of matter, and its propensity to form phase correlated states over macroscopic distance scales at low enough temperatures and/or high enough densities requires at the least a semiclassical treatment of the effective stress-energy tensor source for Einstein's equations. The corresponding one-loop effective action for gravity can be determined by the methods of effective field theory (EFT). Our principal purpose in this paper is to demonstrate that nonlocal macroscopic coherence effects are contained in the low energy EFT of gravity, provided that Einstein's classical theory is supplemented by the contributions of the one-loop quantum trace anomaly of massless fields.

The low energy EFT of gravity is determined by the same general principles as in other contexts [1], namely, by an expansion in powers of derivatives of local terms consistent with symmetry. Short distance effects are parameterized by the coefficients of local operators in the effective action, with higher order terms suppressed by inverse powers of an ultraviolet cutoff scale M. The effective

*Email: emil@lanl.gov
†Email: vaulin@physics.fau.edu

theory need not be renormalizable, as indeed Einstein's theory is not, but is expected nonetheless to be quite insensitive to the details of the underlying microscopic degrees of freedom, because of decoupling [1]. It is the decoupling of short distance degrees of freedom from the macroscopic physics that makes EFT techniques so widely applicable, and which we assume applies also to gravity.

As a covariant metric theory with a symmetry dictated by the equivalence principle, namely, invariance under general coordinate transformations, general relativity may be regarded as just such a local EFT, truncated at second order in derivatives of the metric field $g_{ab}(x)$ [2]. When quantum matter is considered, the stress tensor becomes an operator whose expectation value acts as a source for the Einstein equations in the semiclassical limit. Fourth order terms in derivatives of the metric are necessary to absorb divergences in the expectation value of the stress tensor in curved spacetime, but the effects of such higher derivative *local* terms in the gravitational effective action are suppressed at distance scales $L \gg L_{\rm Pl}$ in the low energy EFT limit. Hence surveying only local curvature terms, it is often tacitly assumed that Einstein's theory contains all the low energy macroscopic degrees of freedom of gravity, and that general relativity cannot be modified at macroscopic distance scales, much greater than $L_{\rm Pl}$, without violating general coordinate invariance and/or EFT principles. However, this presumption should be reexamined in the presence of quantum anomalies.

When a classical symmetry is broken by a quantum anomaly, the naive decoupling of short and long distance physics assumed by an expansion in local operators with ascending inverse powers of M fails. In this situation even the low energy symmetries of the effective theory are changed by the presence of the anomaly, and some remnant of the ultraviolet physics survives in the low energy description. An anomaly can have significant effects in the low energy EFT because it is not suppressed by any large energy cutoff scale, surviving even in the limit $M \to \infty$. Any explicit breaking of the symmetry in the classical Lagrangian serves only to mask the effects of the anomaly, but in the right circumstances the effects of the nonlocal anomaly may still dominate the local terms. A well known example is the chiral anomaly in QCD with massless quarks, whose effects are unsuppressed by any inverse power of the EFT ultraviolet cutoff scale $M \sim \Lambda_{\rm QCD}$. Although the quark masses are nonzero, and chiral symmetry is only approximate in nature, the chiral anomaly gives the dominant contribution to the low energy decay amplitude of $\pi^0 \to 2\gamma$ in the standard model [3,4], a contribution that is missed entirely by a local EFT expansion in pion fields. Instead the existence of the chiral anomaly requires the explicit addition to the local effective action of a *nonlocal* term in four physical dimensions to account for its effects [1,5].

Although when an anomaly is present, naive decoupling between the short and long distance degrees of freedom fails, it does so in a well-defined way, with a coefficient that depends only on the quantum numbers of the underlying microscopic theory. In fact, since the chiral anomaly depends on the color charge assignments of the short distance quark degrees of freedom, the measured low energy decay width of $\pi^0 \to 2\gamma$ affords a clean, nontrivial test of the underlying microscopic quantum theory of QCD with three colors of fractionally charged quarks [1,4,6]. The bridge between short and long distance physics which anomalies provide is the basis for the anomaly matching conditions [7].

In curved space an anomaly closely related to the chiral anomaly also appears in massless quantum field theory [8,9]. This conformal or trace anomaly provides us with additional infrared relevant terms that do not decouple in the limit $M_{\rm Pl}^{-1} = L_{\rm Pl} \to 0$, and which should be added to the Einstein-Hilbert action of classical relativity, to complete the EFT of low energy gravity.

That the trace anomaly terms are necessary for the low energy completion of the EFT of gravity may be seen from the classification of possible terms in the gravitational effective action according to their behavior under global Weyl rescalings [10]. The terms in the classical Einstein-Hilbert action scale with positive powers ($\sim L^4$ and $\sim L^2$) under rescaling of distance, and are clearly relevant operators of the low energy description. The nonlocal anomalous terms scale logarithmically ($\sim \log L$) with distance under Weyl rescalings. Unlike local higher derivative terms in the effective action, which are either marginal or scale with negative powers of L under global Weyl rescalings of the metric, the anomalous terms should not be discarded in the low energy, large distance limit. The addition of the anomaly term(s) to the low energy effective action of gravity amounts to a nontrivial infrared modification of general relativity, fully consistent with both quantum theory and the equivalence principle [10].

The anomalous terms in the effective action lead to corresponding additional terms in the stress tensor and equations of motion of the low energy EFT of gravity. These terms are most conveniently expressed in terms of two auxiliary scalar fields which permit the nonlocal effective action of the trace anomaly and its variations to be cast into local form. The additional scalar degrees of freedom in these terms take account of macroscopic effects of quantum matter in gravitational fields, which are not contained in the purely classical, local metric description of Einstein's theory.

In this paper we give explicit formulae for the stress tensor due to the trace anomaly of massless fields to be added to the classical Einstein equations in the low energy EFT limit, and expose its macroscopic effects in a number of familiar geometries. The most interesting effects are associated with geometries that have horizons, boundaries, or nontrivial global topologies. For example, the two new auxiliary fields in the effective action generically diverge as the apparent horizon of Rindler, Schwarzschild, and de Sitter spacetimes is approached. This behavior of the scalar auxiliary fields gives a coordinate invariant semiclassical order parameter description of possible divergences of stress tensors of quantum field theories on spacetime horizons, enabling an understanding of such divergences in terms of the global properties of the geometry. The Casimir energy density between two conducting plates is an example of a bounded space without horizons or curvature which also admits a simple description in terms of the scalar auxiliary fields. The various examples taken together suggest that the degrees of freedom contained in the scalar potentials induced by the trace anomaly should be regarded as semiclassical macroscopic order parameter fields (condensates), whose nonvanishing values are connected with nontrivial boundary conditions, horizons, or the topology of spacetime.

The paper is organized as follows. In the next section we review the auxiliary field form of the effective action of the trace anomaly in two dimensions, recovering results for the stress tensor of conformal fields in the two-dimensional Rindler wedge, Schwarzschild and de Sitter metrics. In Section III we discuss the general form of the effective action in four dimensions, and give the additional terms in the stress tensor in terms of the auxiliary fields, as well as the conserved Noether current associated with it. In Sec. IV we consider the effects of these additional terms and the new local scalar degrees of freedom of low energy gravity they represent in flat, Rindler and the conformally flat de Sitter spacetime. In Sec. V we discuss the approximate conformal symmetry which pertains in the vicinity of any static, spherically symmetric Killing horizon and give the explicit form of the auxiliary fields in the Schwarzschild case. We characterize the possible singularities of the stress tensor on the Schwarzschild horizon, and find approximate stress tensors for the Hartle-Hawking, Boulware, and Unruh states, comparing our results with numerical results for $\langle T_a{}^b\rangle$ of massless fields of various spins. We conclude in Sec. VI with a summary of our results and their implications for quantum effects in gravity. The appendix contains the explicit formulae for the components of the stress energy in static spherically symmetric spacetimes, upon which the Schwarzschild and de Sitter results are based.

II. EFFECTIVE ACTION AND STRESS TENSOR IN TWO DIMENSIONS

The effective action for the trace anomaly in any even dimension is nonlocal when expressed in terms only of the metric g_{ab}. It can be made local by the introduction of scalar auxiliary field(s). To illustrate this procedure and introduce the general framework which we will use extensively in four dimensions, we consider first the somewhat simpler case of two physical dimensions. In $d = 2$ the trace anomaly takes the simple form [9],

$$\langle T_a{}^a\rangle = \frac{N}{24\pi}R, \qquad (d = 2) \tag{2.1}$$

where $N = N_S + N_F$ is the total number of massless fields, either scalar (N_S) or fermionic (N_F). The fact that the anomalous trace is independent of the quantum state of the matter field(s), and dependent only upon the geometry through the local Ricci scalar R suggests that it should be regarded as a geometric effect. However, no local coordinate invariant action exists whose metric variation leads to (2.1).

A nonlocal action corresponding to (2.1) can be found by introducing the conformal parameterization of the metric,

$$g_{ab} = e^{2\sigma}\bar{g}_{ab}, \tag{2.2}$$

and noticing that the scalar curvature densities of the two metrics g_{ab} and $\bar{g}_{ab}$ are related by

$$R\sqrt{-g} = \bar{R}\sqrt{-\bar{g}} - 2\sqrt{-\bar{g}}\,\bar{\Box}\sigma, \qquad (d = 2) \tag{2.3}$$

a linear relation in σ in two (and only two) dimensions. Multiplying (2.1) by $\sqrt{-g}$, using (2.3) and noting that $\sqrt{-g}\langle T_a{}^a\rangle$ defines the conformal variation, $\delta\Gamma^{(2)}/\delta\sigma$ of an effective action $\Gamma^{(2)}$, we conclude that the σ dependence of $\Gamma^{(2)}$ can be at most quadratic in σ. Hence the Wess-Zumino effective action in two dimensions, $\Gamma^{(2)}_{\rm WZ}$ is

$$\Gamma^{(2)}_{\rm WZ}[\bar{g};\sigma] = \frac{N}{24\pi}\int d^2x\sqrt{-\bar{g}}(-\sigma\bar{\Box}\sigma + \bar{R}\sigma). \tag{2.4}$$

This action functional of the base metric $\bar{g}_{ab}$ and the Weyl shift parameter σ may be regarded as a one-form representative of the cohomology of the local Weyl group in two dimensions [10]. This means that $\Gamma^{(2)}_{\rm WZ}[\bar{g};\sigma]$ is closed under the coboundary (i.e. antisymmetrized) composition of Weyl shifts,

$$\Delta_{\sigma_2}\circ\Gamma^{(2)}_{\rm WZ}[\bar{g};\sigma_1] = \Gamma^{(2)}_{\rm WZ}[\bar{g}e^{2\sigma_1};\sigma_2] - \Gamma^{(2)}_{\rm WZ}[\bar{g}e^{2\sigma_2};\sigma_1] + \Gamma^{(2)}_{\rm WZ}[\bar{g};\sigma_1] - \Gamma^{(2)}_{\rm WZ}[\bar{g};\sigma_2] = 0. \tag{2.5}$$

but is nonexact, in the sense that $\Gamma^{(2)}_{\rm WZ}$ cannot itself be written as the coboundary shift of a local action functional. The relation (2.5) following directly from the hermiticity of $\Box$ is exactly the Wess-Zumino consistency condition for $\Gamma^{(2)}_{\rm WZ}[\bar{g};\sigma]$ [5,11].

Although $\Gamma^{(2)}_{\rm WZ}$ cannot be written as a coboundary shift Δ_σ of a local single-valued covariant scalar functional of the metric $\bar{g}_{ab}$, it is straightforward to find a *nonlocal* scalar functional $S_{\rm anom}[g]$ such that

$$\Gamma^{(2)}_{\rm WZ}[\bar{g};\sigma] = \Delta_\sigma\circ S^{(2)}_{\rm anom}[\bar{g}] \equiv S^{(2)}_{\rm anom}[g] - S^{(2)}_{\rm anom}[\bar{g}]. \tag{2.6}$$

Indeed by solving (2.3) formally for σ, and using the fact that $\sqrt{-g}\Box = \sqrt{-\bar{g}}\,\bar{\Box}$ is conformally invariant in two (and only two) dimensions, we find that $\Gamma^{(2)}_{\rm WZ}$ can be written as a Weyl shift (2.6) with

$$S^{(2)}_{\rm anom}[g] = \frac{Q^2}{16\pi} \times \int d^2x\sqrt{-g}\int d^2x'\sqrt{-g'}R(x)\Box^{-1}(x,x')R(x'), \tag{2.7}$$

and $\Box^{-1}(x,x')$ denoting the Green's function inverse of the scalar differential operator $\Box$. The parameter Q^2 is $-N/6$ if only matter fields in a fixed spacetime metric are considered. It becomes $(25-N)/6$ if account is taken of the contributions of the metric fluctuations themselves in addition to those of the N matter fields, thus effectively replacing N by $N-25$ [12]. In the general case, the coefficient Q^2 is arbitrary, and can be treated as simply an additional free parameter of the low energy effective action, to be fixed by experiment.

The anomalous effective action (2.7) is a scalar under coordinate transformations and therefore fully covariant and geometric in character. However since it involves the Green's function $\Box^{-1}(x,x')$, which requires boundary conditions for its unique specification, it is quite nonlocal, and dependent upon more than just the local curvature invariants of spacetime. The nonlocal and non-single-valued functional of the metric, $S^{(2)}_{\rm anom}$ may be expressed in a local form by the standard method of introducing auxiliary fields. In the case of (2.7) a single scalar auxiliary field, φ satisfying

$$-\Box\varphi = R \tag{2.8}$$

is sufficient. Indeed, varying

$$S^{(2)}_{\rm anom}[g;\varphi] \equiv \frac{Q^2}{16\pi}\int d^2x\sqrt{-g}(\nabla_a\varphi\nabla^a\varphi - 2R\varphi) \tag{2.9}$$

with respect to φ gives the Eq. of motion (2.8) for the auxiliary field, which when solved (formally) by $\varphi = -\Box^{-1}R$ and substituted back into $S^{(2)}_{\rm anom}[g;\varphi]$ returns the nonlocal form of the anomalous action (2.7), up to a surface term. The nonlocal information in addition to the local geometry which was contained previously in the specification of the Green's function $\Box^{-1}(x,x')$ now resides in the auxiliary local field $\varphi(x)$, and the freedom to add to it homogeneous solutions of (2.8).

The variation of (2.9) with respect to the metric yields a stress-energy tensor,

$$T^{(2)}_{ab}[g;\varphi] \equiv -\frac{2}{\sqrt{-g}}\frac{\delta S^{(2)}_{\rm anom}[g;\varphi]}{\delta g^{ab}} = \frac{Q^2}{4\pi}\Big[-\nabla_a\nabla_b\varphi + g_{ab}\Box\varphi - \frac{1}{2}(\nabla_a\varphi)(\nabla_b\varphi) + \frac{1}{4}g_{ab}(\nabla_c\varphi)(\nabla^c\varphi)\Big], \tag{2.10}$$

which is covariantly conserved, by use of (2.8) and the vanishing of the Einstein tensor, $G_{ab} = R_{ab} - Rg_{ab}/2 = 0$ in two (and only two) dimensions. The *classical* trace of the stress tensor,

$$g^{ab}T^{(2)}_{ab}[g;\varphi] = \frac{Q^2}{4\pi}\Box\varphi = -\frac{Q^2}{4\pi}R \tag{2.11}$$

reproduces the *quantum* trace anomaly in a general classical background (with Q^2 proportional to $\hbar$). Hence (2.9) is exactly the local auxiliary field form of the effective action which should be added to the action for two-dimensional gravity to take the trace anomaly of massless quantum fields into account.

The full effective action of two-dimensional gravity is

$$S^{(2)}_{\rm eff}[g;\varphi] = S^{(2)}_{\rm anom}[g;\varphi] + S^{(2)}_{\rm cl}[g], \tag{2.12}$$

where $S^{(2)}_{\rm cl}$ is the local classical Einstein-Hilbert action,

$$S^{(2)}_{\rm cl}[g] = \gamma\int d^2x\sqrt{-g}R + \lambda\int d^2x\sqrt{-g}, \tag{2.13}$$

which are the only terms we would have written down according to the usual EFT approach, expanding in strictly local terms up to and including dimension two which transform as scalars under general coordinate transformations. Under global rescalings of the metric (2.2) with $\sigma = \sigma_0$, a spacetime constant, the volume term scales like $e^{2\sigma_0}$ and its coefficient λ has positive mass dimension two. Hence it is clearly a relevant term in the effective action at large distances. The integral of the Ricci scalar is independent of σ_0, with a coefficient γ that is dimensionless, while the integrals of all higher local curvature invariants scale with negative powers of e^{σ_0}, multiplied by coefficients having negative mass dimensions. These terms are neglected in (2.13) since they are strictly irrelevant in the low energy EFT limit.

The anomalous term (2.7), or (2.9) scales linearly with σ_0, i.e. logarithmically with distance. Even if it is not included at the classical level it will be generated by the one-loop effects of massless fields, which do not decouple at any scale. Hence it should be retained in the full low energy effective action (2.12) of two-dimensional gravity. Moreover, since the integral of R is a topological invariant in two dimensions, the classical action (2.13) contains no propagating degrees of freedom whatsoever, and it is $S_{\rm anom}$ which contains the only kinetic terms of the low energy EFT. In the local auxiliary field form (2.9), it is clear that $S_{\rm anom}$ describes an additional scalar degree of freedom φ, not contained in the classical action $S^{(2)}_{\rm cl}$. This is reflected also in the shift of the central charge from $N-26$, which would be expected from the contribution of conformal matter plus ghosts by one unit to $N-25$.

Extensive study of the stress tensor and its correlators arising from this effective action established that the two-dimensional trace anomaly gives rise to a modification or gravitational "dressing" of critical exponents in conformal field theories at second order critical points [12]. Since critical exponents in a second order phase transition depend only upon fluctuations at the largest allowed infrared scale, this dressing is clearly an infrared effect, independent of any ultraviolet cutoff. These dressed exponents are evidence of the infrared fluctuations of the additional scalar degree of freedom φ which are quite absent in the classical action (2.13). The appearance of the gravitational dressing exponents and the anomalous effective action (2.7) itself have been confirmed in the large volume scaling limit of two-dimensional simplicial lattice simulations in the dynamical triangulation approach [13,14].

The formal similarity between (2.4) and (2.9) suggests that the introduction of the local auxiliary field φ has simply undone the steps leading from (2.4) to (2.7), which eliminated the conformal factor σ. By comparing (2.3) and (2.8), we observe that the metric

$$\tilde{g}_{ab} = e^{-\varphi} g_{ab} \tag{2.14}$$

is a metric with zero scalar curvature, $\tilde{R} = 0$, conformally related to the physical metric g_{ab}. Hence $\varphi(x)$ does parameterize the local Weyl transformations of the metric along the same fiber as $\sigma(x)$ in the conformal parameterization (2.2). The difference between the two representations is that whereas $\bar{g}_{ab}$ was regarded as a fixed base metric related to the physical metric g_{ab} by (2.2), the possibility of adding homogeneous solutions to (2.8) shows that φ and hence $\tilde{g}_{ab}$ defined by (2.14) are not unique, and hence no $\tilde{g}_{ab}$ plays any privileged role over any other. The only physical metric is g_{ab}. The nonuniqueness of φ allows for additional degrees of freedom at boundaries or coordinate singularities of g_{ab}, which need not be associated with any smooth $\tilde{g}_{ab}$. The completely covariant local form of the action (2.9) makes it clear that these effects, associated with the boundary conditions on the spacetime scalar φ satisfying (2.8) are not coordinate artifacts which can be removed by reparameterizations of coordinates in the physical spacetime g_{ab}.

Both the auxiliary field equation of motion (2.8) and stress tensor (2.10) are left invariant by the constant shift,

$$\varphi \to \varphi + \varphi_0. \tag{2.15}$$

The underlying reason for this shift symmetry is that the integral,

$$\chi = \frac{1}{4\pi}\int d^2x\sqrt{-g}R \tag{2.16}$$

is a topological invariant in two dimensions. For Euclidean signature metrics $\chi = \chi_E$ is the Euler number of the manifold. This implies that in two (and only two) dimensions $\sqrt{-g}R$ can be expressed as a total derivative, or equivalently,

$$R = \frac{1}{\sqrt{-g}}\partial_a(\sqrt{-g}\Omega^a) \equiv \nabla_a\Omega^a, \qquad (d = 2). \tag{2.17}$$

The topological current Ω^a, analogous to the Chern-Simons current for the gauge theory chiral anomaly [15,16], depends upon the choice of gauge or coordinate frame and is nonunique, although its line integral around a closed contour,

$$\oint_{\partial V}\sqrt{-g}\Omega^a n_a ds = \oint_{\partial V}\sqrt{-g}\Omega^a \epsilon_{ab} dx^b = \int_V \sqrt{-g}R d^2x \tag{2.18}$$

is a coordinate invariant scalar quantity. Here we have used the notation, $n_a = \epsilon_{ab}\frac{dx^b}{ds}$ for the normal to the boundary ∂V of the two-dimensional region V and ϵ_{ab} is the standard alternating tensor in two dimensions (with $\epsilon_{12} = +1$). The existence of Ω^a satisfying (2.17) implies that the action (2.9) may be rewritten in the alternative form,

$$S_{\rm anom,2}[g;\varphi] = \frac{Q^2}{16\pi}\int_V d^2x\sqrt{-g}(\nabla^a\varphi + 2\Omega^a)\nabla_a\varphi - \frac{Q^2}{8\pi}\oint_{\partial V}\sqrt{-g}\varphi\Omega^a n_a ds, \tag{2.19}$$

leaving (2.8), (2.10), and (2.11) unchanged. Since up to the last surface term the anomalous action depends on φ only through its derivatives, there is a frame dependent Noether current corresponding to the global Weyl rescaling (2.15),

$$J^a = \nabla^a\varphi + \Omega^a, \tag{2.20}$$

which is covariantly conserved,

$$\nabla_a J^a \equiv \frac{1}{\sqrt{-g}}\partial_a(\sqrt{-g}J^a) = \Box\varphi + R = 0, \tag{2.21}$$

by virtue of (2.8) and (2.17).

Within the framework of the low energy effective action (2.12) all fields may be treated classically, with $\hbar$ contained implicitly only in the c-number coefficients γ, λ, and Q^2. In this (semi-)classical framework, the auxiliary field φ is a classical scalar potential which contains information about the macroscopic quantum state and topological boundary effects. The geometric information $\varphi(x)$ contains cannot be reduced to local geometric invariants constructed from the curvature tensor, its derivatives and contractions at x. Instead the massless correlation function $\Box^{-1}(x,x')$, which grows logarithmically with the invariant distance between the spacetime points x and x', and the existence of a conserved global charge, corresponding to the Noether current J^a imply that φ is a long-range field depending upon the global properties of spacetime. It is this long-range behavior of φ that enables the effective action (2.12) to incorporate macroscopic quantum coherence effects, quite absent from the purely local classical geometric action $S^{(2)}_{\rm cl}$. Since the effective action contains $S^{(2)}_{\rm anom}$, these nonlocal macroscopic quantum coherence effects are incorporated in the low energy EFT of two-dimensional gravity in a natural way.

The long-range macroscopic effects and the physical meaning of the Noether charge are best illustrated by means of a few simple examples. Consider first flat spacetime in Rindler coordinates,

$$ds^2 = -dt^2 + dx^2 = -\rho^2 d\eta^2 + d\rho^2, \tag{2.22}$$

with $x = \rho \cosh\eta$, $t = \rho \sinh\eta$. The Killing vector ∂_η generates Lorentz boosts and this vector field becomes singular on the light cone, $\rho^2 = x^2 - t^2 = 0$. If the state of the system is boost invariant then the semiclassical auxiliary field can be assumed to be independent of η. With $\varphi = \varphi(\rho)$, (2.8) has the solution (for finite nonzero ρ),

$$\varphi = 2q \ln\left(\frac{\rho}{\rho_0}\right), \tag{2.23}$$

with q and ρ_0 arbitrary constants.

If $q \neq 0$ the singularity of φ at $\rho = 0$ gives a delta function contribution to the scalar curvature at the origin of the Euclidean metric obtained by replacing in (2.22), $\eta \to i\theta$, i.e.,

$$\Box\varphi = 4\pi q \frac{\delta(\rho)}{\rho}, \tag{2.24}$$

so that (2.23) should properly be considered a solution of (2.8) only in the distributional sense. If the point at $\rho^{-1} = 0$ is included in the manifold then there is also a delta function contribution to (2.24) at infinity. In the Euclidean signature metric, the operator $\Box$ is the usual two-dimensional Laplacian and (2.24) is Laplace's equation for the electrostatic potential with a point charge of magnitude $-q$ at the origin, and $+q$ at infinity. The contribution at $\rho = 0$ is obtained by integrating over a Gaussian surface containing the origin, i.e.

$$q = \frac{1}{4\pi} \oint_{\partial V} J^a d\Sigma_a. \tag{2.25}$$

A nonzero value of q corresponds to a topological defect on the Minkowski light cone, with the metric $e^{-\varphi} ds^2$, conformal to the original metric (2.22) possessing a conical singularity at $\rho = 0$ for generic $q \neq 0$.

Substituting (2.23) into the stress tensor (2.10), and dropping henceforth the superscript $^{(2)}$ on $T_a^{b(2)}$, we find in the (η, ρ) coordinates,

$$T_a{}^b = \frac{Q^2}{4\pi} \frac{(2-q)q}{\rho^2} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.26}$$

The case $q = 0$ corresponds to the Minkowski vacuum with no singularity on the light cone, $t = \pm x$. However when $q = 1$,

$$e^{-\varphi}|_{q=1} ds^2 = \rho_0^2\left(-d\eta^2 + \frac{d\rho^2}{\rho^2}\right) = \rho_0^2(-d\eta^2 + d\xi^2), \tag{2.27}$$

with $\xi = \ln(\rho/\rho_0)$, is again flat. The vanishing Euler number of this flat metric may be regarded as resulting from the cancellation of the -1 Euler charge at the origin from (2.8), (2.16), and (2.24) with the $+1$ Euler number of a circular flat disc with boundary ∂V at large but finite radius, as the boundary is taken to infinity in the original metric. In the conformal metric (2.27) in (η, ξ) coordinates, for $q = 1$ both singularities at $\rho = 0$ and $\rho = \infty$ are removed to infinity and we again obtain a flat spacetime with no singularities and no boundaries. The original flat spacetime (2.22) has been conformally mapped to another flat spacetime with a different global topology. Indeed the Euclidean signature metric $d\theta^2 + d\xi^2$ with θ 2π-periodic is the metric of $R \times S^1$, with punctures at $\xi = \pm\infty$. The Euler charge of ± 1 unit at $\xi = \pm\infty$ is also a consequence of general theorems relating the Euler number to the number of fixed points of the Killing field $K = \frac{\partial}{\partial\theta}$, where the metric (2.22) (or its inverse) becomes singular in the original (ρ, η) coordinates [17].

When $q = 2$ the metric, $e^{-\varphi} ds^2$ becomes equivalent to the original flat metric (2.22) by the change of variables $\rho \to \rho_0^2/\rho$ which turns the spacetime inside out, exchanging the coordinate singularities at $\rho = 0$ and $\rho = \infty$. Since the new metric is equivalent to the original metric up to a (singular) coordinate transformation, the stress tensor (2.26) again vanishes for $q = 2$. The invariance of the physical stress tensor (2.26) under the transformation $q \to 2 - q$ is evidence for a topological two-fold degenerate vacuum in two-dimensional gravity.

For a single free scalar field $Q^2 = -1/6$ and at the value of $q = 1$, (2.26) gives

$$T_a{}^b|_{\rm FR} = -\frac{1}{24\pi\rho^2} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{2.28}$$

in the (η, ρ) coordinates. This is exactly the expectation value of the stress tensor of a quantum scalar field in the boost invariant Fulling-Rindler state [18]. The preceding discussion shows that the divergence of $T_a{}^b$ in the Fulling-Rindler state is associated with a unit topological defect on the Minkowski light cone, contained in the auxiliary potential φ, corresponding to a puncture at $\rho = 0$ (equivalently, $\rho = \infty$) in the analytically continued Euclidean signature metric. A singular conformal transformation (2.23) with $q = 1$ is required to transform the Minkowski vacuum to the Fulling-Rindler state. Whereas the Hamiltonian defined with respect to the global Minkowski time coordinate t is bounded from below and possesses a well-defined vacuum state, the boost generator corresponding to the Killing field ∂_η changes sign and is unbounded from below. These global properties distinguishing the Fulling-Rindler state from the Minkowski vacuum are reflected in the topological defect (2.24) and divergence of the stress-energy tensor (2.28) on the light cone.

A second example of macroscopic effects of topology of the trace anomaly is provided by two-dimensional static metrics of the form,

$$ds^2 = -f dt^2 + \frac{dr^2}{f} = f(-dt^2 + dr^{*2}), \tag{2.29}$$

where $f = f(r)$ and $dr^* = dr/f$. This class of metrics include the two-dimensional Schwarzschild metric, with $f(r) = 1 - 2M/r$ and de Sitter metric in static coordinates, with $f(r) = 1 - H^2 r^2$. The Killing field $K = \partial_t$ is timelike for $f > 0$, and the scalar curvature corresponding to (2.29) is $R = -f''$. Hence in this case a particular inhomogeneous solution to (2.8) is $\varphi = \ln(-K^a K_a) = \ln f$. The conformally transformed line element $e^{-\varphi} ds^2 = -dt^2 + dr^{*2}$ is then the flat optical metric [19].

The components of the stress tensor (2.10) due to the anomaly are given by

$$T_{r^*}{}^{r^*} = \frac{N}{24\pi}\left\{\frac{\varphi_{,r^* r^*}}{f} - \frac{f'}{2f}\varphi_{,r^*} + \frac{1}{4f}(\varphi_{,r^*}{}^2 + \dot{\varphi}^2) + R\right\}, \tag{2.30a}$$

$$T_t{}^t = \frac{N}{24\pi}\left\{-\frac{\ddot{\varphi}}{f} + \frac{f'}{2f}\varphi_{,r^*} - \frac{1}{4f}(\varphi_{,r^*}{}^2 + \dot{\varphi}^2) + R\right\}, \quad \text{and} \tag{2.30b}$$

$$T_t{}^{r^*} = \frac{N}{24\pi f}\left\{\dot{\varphi}_{,r^*} - \frac{f'}{2}\dot{\varphi} + \frac{1}{2}\dot{\varphi}\varphi_{,r^*}\right\}. \tag{2.30c}$$

The off-diagonal flux component is generally time dependent unless we restrict φ to at most linear functions of time. With this restriction, the general solution of (2.8) leading to a stationary stress tensor in the Schwarzschild case is

$$\varphi = c_0 + \frac{q}{2M} r^* + \frac{p}{2M} t + \ln f, \tag{2.31}$$

where c_0, q, and p are constants. As in the Rindler case this is a solution to (2.8) in the distributional sense, since $\Box\varphi$ contains delta function singularities. With $p = 0$ the strength of the delta function at the horizon, mapped to a point on the Euclidean section $t \to i\tau$ for real τ, is $4\pi(q + 1)$. For $q = -1$ this horizon singularity is cancelled, although that value leads to a divergence in (2.31) as $r \to \infty$.

The stress-energy tensor for N scalar (or fermion) fields in the Schwarzschild case with $f = 1 - 2M/r$ is

$$T_t{}^t = \frac{N}{24\pi}\left\{-\frac{1}{4f}\left(\frac{p^2 + q^2}{4M^2} - \frac{4M^2}{r^4}\right) + \frac{4M}{r^3}\right\}, \tag{2.32a}$$

$$T_t{}^{r^*} = \frac{N}{192\pi M^2}\frac{pq}{f}, \tag{2.32b}$$

$$T_{r^*}{}^{r^*} = \frac{N}{96\pi f}\left(\frac{p^2 + q^2}{4M^2} - \frac{4M^2}{r^4}\right). \tag{2.32c}$$

This general form of the stress tensor for stationary states in the two-dimensional Schwarzschild metric was obtained in Ref. [20], from considerations of conservation, stationarity, and the trace anomaly. The identification of the arbitrary constants K and Q in Eqs. (3.4) of that work with $-pq/192\pi$ and $[(p - q)^2 - 1]/384\pi$ respectively of the present article shows that the stress tensor obtained from the anomaly, (2.10) with (2.31) coincides with that obtained in [20].

The stress tensor (2.32) is generally singular at $r = 2M$, diverging as f^{-1} as $f \to 0$. The condition for finiteness on the future horizon is

$$\left|\frac{1}{f}(T_t{}^t - T_{r^*}{}^{r^*} + 2T_t{}^{r^*})\right| < \infty \quad \text{as} \quad r \to 2M. \tag{2.33}$$

Substituting (2.32) into this condition and expanding about $r = 2M$ gives the condition

$$(q - p)^2 = 1, \qquad \text{(finiteness on future horizon).} \tag{2.34}$$

Finiteness on the past horizon would imply $(q + p)^2 = 1$. Thus, finiteness on both horizons implies $p^2 + q^2 = 1$, $pq = 0$ and zero flux. Since (2.34) cannot be satisfied with both q and p equal to zero, the solution (2.31) must diverge as either r^* or t goes to infinity. Note that this is completely unlike the previous Fulling-Rindler case in which a finite (in fact, vanishing) stress tensor on the horizon is obtained with $\varphi = 0$,which has both zero charge on the light cone, *and* no divergence at infinity.

A time-independent state with zero flux satisfying (2.34) is the two-dimensional Hartle-Hawking state with either p or q equal to zero, and the other equal to ± 1. The stress tensor for this state is given by

$$T_t{}^t|_{\rm HH} = \frac{N}{24\pi}\left\{\frac{1}{4f}\left(\frac{4M^2}{r^4} - \frac{1}{4M^2}\right) + \frac{4M}{r^3}\right\}, \tag{2.35a}$$

$$T_t{}^{r^*}|_{\rm HH} = 0, \tag{2.35b}$$

$$T_{r^*}{}^{r^*}|_{\rm HH} = \frac{N}{96\pi f}\left(\frac{1}{4M^2} - \frac{4M^2}{r^4}\right). \tag{2.35c}$$

The degeneracy of the stress tensor for two different values of p or q is similar to the behavior obtained in the Rindler case, (2.26). As the horizon $r \to 2M$ is approached,

$$T_a{}^b|_{\rm HH} = \frac{\pi N T_H^2}{6}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad r \to 2M, \tag{2.36}$$

where $T_H = (8\pi M)^{-1}$ is the Hawking temperature. Thus the stress tensor approaches that of a two-dimensional perfect gas at $T = T_H$ near the horizon in the Hartle-Hawking state. The same value is also obtained in the limit $r \to \infty$, showing that this state has infinite total energy.

Another state obeying the finiteness condition (2.33) on the future horizon is the Unruh state with $p = -q = 1/2$. Its time inverse, given by $p = q = -1/2$ is finite on the past horizon. These states have nonzero flux $T_t{}^{r^*}$ either outwardly or inwardly directed from the horizon extending to infinity. The nonzero q in these states reflects singularities, i.e. sources or sinks of the current (2.20) at both $r = 0$ and $r = \infty$.

Inspection of (2.32) shows that the only state which has vanishing stress tensor as $r \to \infty$ is the Boulware state with $p = q = 0$, which does not satisfy the finiteness condition (2.33) at the horizon. Unlike the previous Fulling-Rindler example in which finiteness at both the horizon and infinity is achieved by one choice of $\varphi = 0$, in the Schwarzschild metric (2.29) because of its different topology, one is forced to choose between finiteness on the horizon and falloff at infinity. In the Boulware state with $p = q = 0$,

$$T_a{}^b|_B \to -\frac{\pi}{6}\frac{T_H^2}{f}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad r \to 2M, \tag{2.37}$$

as $r \to 2M$ and $f \to 0$, corresponding to the *negative* of the stress tensor of a thermal distribution of massless particles at the blue-shifted local temperature $T_{\rm loc} = T_H/\sqrt{f}$.

Comparing (2.37) with (2.28) we observe that the behavior of the stress tensor in the Boulware state in two-dimensional Schwarzschild spacetime is *locally* the same as that of the Fulling-Rindler state in flat spacetime near the corresponding horizons, and indeed with the correspondence, $\rho \leftrightarrow 4M\sqrt{f}$, the behavior near the horizons in the two cases may be mapped onto each other. However the two situations are quite different *globally*. The Euler number of flat space is zero, and no singular behavior of φ is required on either the light cone $t = \pm x$ or infinity. The Fulling-Rindler state can be obtained only by requiring a nonzero topological charge $q = 1$ in the auxiliary field, which leads to φ of (2.23) diverging at both large and small ρ.

On the other hand the scalar curvature $R = -f''$ is nonzero in the Schwarzschild case and the Euler number of the corresponding nonsingular Euclidean signature metric is $+1$. This nontrivial topology of the background spacetime requires an inhomogeneous solution to (2.8), such as $\varphi = \ln f$ in (2.31). The divergence on the horizon of this particular solution to the inhomogeneous equation can be cancelled by homogeneous solutions with nonzero p and/or $q = -1$ satisfying (2.34), but only at the price of a nonzero energy density, and nontrivial topological charge at infinity. Conversely the solution of (2.31) with $c_0 = p = q = 0$ which vanishes at infinity and produces a transformed metric $e^{-\varphi}ds^2$ which is flat there, necessarily has a diverging stress tensor on the horizon. Thus while particular solutions with particular boundary conditions for the auxiliary field can be chosen, the impossibility of producing a solution of (2.31) which is well-behaved at *both* the horizon and infinity is a global property of the Schwarzschild spacetime (2.29), which is quite distinct from flat spacetime in Rindler coordinates (2.22). The stress tensor of the anomaly and the auxiliary scalar φ captures these global effects of the macroscopic quantum state in terms of the classical solutions of (2.8).

Similar conclusions obtain in the two dimensional de Sitter spacetime (2.29) with $f = 1 - H^2r^2$. The general solution of (2.8) leading to a stationary stress tensor is

$$\varphi = c_0 + 2qHr^* + 2pHt + \ln f. \tag{2.38}$$

The components of the stress tensor in the general stationary state labeled by (p, q) are

$$T_t{}^t = \frac{NH^2}{24\pi}\left\{-\frac{1}{f}(p^2 + q^2 - H^2r^2) + 2\right\}, \tag{2.39a}$$

$$T_{r^*}{}^{r^*} = \frac{NH^2}{24\pi f}(p^2 + q^2 - H^2r^2). \tag{2.39b}$$

$$T_t{}^{r^*} = \frac{NH^2}{12\pi}\frac{pq}{f}. \tag{2.39c}$$

The generic stress tensor again diverges on the horizon where $f = 1 - H^2r^2 = 0$, despite the finiteness of the scalar curvature $R = -f'' = 2H^2$ there. The condition that the stress tensor be finite on the future horizon is again $(q - p)^2 = 1$. The divergence of the stress tensor can be cancelled only by the addition of a specific homogeneous solution to (2.8). States which are regular on both the past and future horizon have $p^2 + q^2 = 1$, $pq = 0$ and zero flux. These conditions give the Bunch-Davies state with the nonvanishing de Sitter invariant stress tensor, $T_t{}^t = T_{r^*}{}^{r^*} = NH^2/24\pi$. The analog of the Boulware state with $p = q = 0$ diverges as $f \to 0$ exactly as in (2.37) with $T_H = H/2\pi$.

In all the two-dimensional examples considered, the parameter q may be viewed as an effective topological charge carried by the auxiliary field of the trace anomaly. The classical solutions of this order parameter field generally depend on global coordinate invariant quantities, such as $\ln(-K^aK_a)$ which diverges on a horizon where a timelike Killing field K^a becomes null, notwithstanding the finiteness of the local curvature there. Thus the scalar auxiliary field provides a coordinate invariant characterization of the local divergences of the stress energy on the horizon, in macroscopic quantum states defined globally in the spacetime.

III. EFFECTIVE ACTION AND STRESS TENSOR IN FOUR DIMENSIONS

The low energy effective action for gravity in four dimensions contains first of all, the local terms constructed from the Riemann curvature tensor and its derivatives and contractions up to and including dimension four. This includes the usual Einstein-Hilbert action of general relativity,

$$S_{\rm EH}[g] = \frac{1}{16\pi G}\int d^4x\sqrt{-g}(R - 2\Lambda) \tag{3.1}$$

as well as the spacetime integrals of the fourth order curvature invariants,

$$S^{(4)}_{\rm local}[g] = -\frac{1}{2}\int\sqrt{-g}(\alpha C_{abcd}C^{abcd} + \beta R^2)d^4x, \tag{3.2}$$

with arbitrary dimensionless coefficients α and β. There are two additional fourth order invariants, namely $E = {}^*R_{abcd}{}^*R^{abcd}$ and $\Box R$, which could be added to (3.2) as well, but as they are total derivatives yielding only a surface term and no local variation, we omit them. All the possible local terms in the effective action may be written as the sum,

$$S_{\rm local}[g] = \frac{1}{16\pi G}\int d^4x\sqrt{-g}(R - 2\Lambda) + S^{(4)}_{\rm local} + \sum_{n=3}^{\infty} S^{(2n)}_{\rm local}. \tag{3.3}$$

with the terms in the sum with $n \geq 3$ composed of integrals of local curvature invariants with dimension $2n \geq 6$, and suppressed by M^{-2n+4} at energies $E \ll M$. Here M is the ultraviolet cutoff scale of the low energy effective theory which we may take to be of order $M_{\rm Pl}$. The higher derivative terms with $n \geq 3$ are irrelevant operators in the infrared, scaling with negative powers of e^{σ_0} under global rescalings of the metric, as in (2.2), and may be neglected at macroscopic distance scales. On the other hand the two terms in the Einstein-Hilbert action $n = 0, 1$ scale as $e^{4\sigma_0}$ or $e^{2\sigma_0}$ respectively, and are clearly relevant in the infrared. The fourth order terms in (3.2) are neutral under such global rescalings, and marginal. The running of their respective couplings α and β determine whether $C_{abcd}C^{abcd}$ and R^2 are marginally relevant or marginally irrelevant in the infrared.

The exact quantum effective action also contains nonlocal terms in general. All possible terms in the effective action (local or not) can be classified according to how they respond to global Weyl rescalings of the metric. If the nonlocal terms are noninvariant under global rescalings, then they scale either positively or negatively under

$$g_{ab} \to e^{2\sigma_0} g_{ab}. \tag{3.4}$$

If m^{-1} is some fixed length scale associated with the nonlocality, arising, for example, by the integrating out of fluctuations of fields with mass m, then at much larger macroscopic distances ($mL \gg 1$) the nonlocal terms in the effective action become approximately local. The terms which scale with positive powers of e^{σ_0} are constrained by general covariance to be of the same form as the $n = 0, 1$ Einstein-Hilbert terms in $S_{\rm local}$, (3.1). Terms which scale negatively with e^{σ_0} become negligibly small as $mL \gg 1$ and are infrared irrelevant at macroscopic distances. This is the expected decoupling of short distance degrees of freedom in an effective field theory description, which are verified in detailed calculations of loops in massive field theories in curved space. The only possibility for contributions to the effective field theory of gravity at macroscopic distances, which are not contained in the local expansion of (3.3) arise from fluctuations not associated with any finite length scale, i.e. $m = 0$. These are the nonlocal contributions to the low energy EFT which include those associated with the anomaly.

Classical fields satisfying wave equations with zero mass, which are invariant under conformal transformations of the spacetime metric, $g_{ab} \to e^{2\sigma} g_{ab}$ have stress tensors with zero classical trace, $T_a{}^a = 0$. Because the corresponding quantum theory requires a UV regulator, classical conformal invariance cannot be maintained at the quantum level. The trace of the stress tensor is generally nonzero when $\hbar \neq 0$, and any UV regulator which preserves the covariant conservation of $T_a{}^b$ (a necessary requirement of any theory respecting general coordinate invariance) yields an expectation value of the quantum stress tensor with the nonzero trace [8,9],

$$\langle T_a{}^a\rangle = bF + b'\left(E - \frac{2}{3}\Box R\right) + b''\Box R + \sum_i \beta_i H_i, \tag{3.5}$$

in a general four dimensional curved spacetime. This is the four dimensional analog of (2.1) in two dimensions. In Eq. (3.5) we employ the notation,

$$E \equiv {}^*R_{abcd}{}^*R^{abcd} = R_{abcd}R^{abcd} - 4R_{ab}R^{ab} + R^2 \tag{3.6a}$$

and

$$F \equiv C_{abcd}C^{abcd} = R_{abcd}R^{abcd} - 2R_{ab}R^{ab} + \frac{R^2}{3}. \tag{3.6b}$$

with R_{abcd} the Riemann curvature tensor, ${}^*R_{abcd} = \varepsilon_{abef}R^{ef}{}_{cd}/2$ its dual, and C_{abcd} the Weyl conformal tensor. The coefficients b, b' and b'' are dimensionless parameters multiplied by $\hbar$. Additional terms denoted by the sum $\sum_i \beta_i H_i$ in (3.5) may also appear in the general form of the trace anomaly, if the massless field in question couples to additional long-range gauge fields. Thus in the case of massless fermions coupled to a background gauge field, the invariant $H_1 = {\rm tr}(F_{ab}F^{ab})$ appears in (3.5) with a coefficient β_1 determined by the beta function of the relevant gauge coupling. We shall concentrate on the universal gravitational terms in the anomaly in most of what follows, returning to consider the effect of the $\sum_i \beta_i H_i$ terms at the end of this section.

The form of (3.5) and coefficients b and b' do not depend on the state in which the expectation value of the stress tensor is computed. Instead they are determined only by the number of massless fields [9],

$$b = \frac{1}{120(4\pi)^2}(N_S + 6N_F + 12N_V), \tag{3.7a}$$

$$b' = -\frac{1}{360(4\pi)^2}(N_S + 11N_F + 62N_V), \tag{3.7b}$$

with (N_S, N_F, N_V) the number of fields of spin $(0, \frac{1}{2}, 1)$ respectively and we have taken $\hbar = 1$.

Three local fourth order curvature invariants E, F, and $\Box R$ appear in the trace of the stress tensor, but only the first two (b and b') terms of (3.5) cannot be derived from a local effective action of the metric alone. If these terms could be derived from a local gravitational action we could simply make the necessary finite redefinition of the corresponding local counterterms to remove them from the trace, in which case the trace would no longer be nonzero or anomalous. This redefinition of a local counterterm (namely, the R^2 term in the effective action) is possible only with respect to the third b'' coefficient in (3.5), which is therefore regularization dependent and not part of the true anomaly. Only the nonlocal effective action corresponding to the b and b' terms in (3.5) lead to the possibility of effects that extend over arbitrarily large, macroscopic distances, unsuppressed by any ultraviolet cutoff scale. The distinction of the two kinds of terms in the effective action (local or not) is emphasized in the cohomological approach to the trace anomaly [10].

The number of massless fields of each spin, N_S, N_F, N_V is a property of the low energy effective description of matter, having no direct connection with physics at the ultrashort Planck scale. Indeed massless fields fluctuate at all distance scales and do not decouple in the far infrared. As in the case of the chiral anomaly with massless quarks, the b and b' terms in the trace anomaly were calculated originally by techniques usually associated with UV regularization (such as dimensional regularization, point splitting or heat kernel techniques) [9]. However just as in the case of the chiral anomaly in QCD, (3.5) and (3.7) can have significant effects in the far infrared as well.

To find the WZ effective action corresponding to the b and b' terms in (3.5), introduce as in two dimensions the conformal parameterization (2.14), and compute

$$\sqrt{-g}F = \sqrt{-\bar{g}}\bar{F}, \tag{3.8a}$$

$$\sqrt{-g}(E - \tfrac{2}{3}\Box R) = \sqrt{-\bar{g}}(\bar{E} - \tfrac{2}{3}\bar{\Box}\bar{R}) + 4\sqrt{-\bar{g}}\bar{\Delta}_4\sigma, \tag{3.8b}$$

whose σ dependence is no more than linear. The fourth order differential operator appearing in this expression is [10,21,22]

$$\Delta_4 \equiv \Box^2 + 2R^{ab}\nabla_a\nabla_b - \tfrac{2}{3}R\Box + \tfrac{1}{3}(\nabla^a R)\nabla_a, \tag{3.9}$$

which is the unique fourth order scalar operator that is conformally covariant, viz.

$$\sqrt{-g}\Delta_4 = \sqrt{-\bar{g}}\bar{\Delta}_4, \tag{3.10}$$

for arbitrary smooth $\sigma(x)$ in four (and only four) dimensions. Thus multiplying (3.5) by $\sqrt{-g}$ and recognizing that the result is the σ variation of an effective action Γ_{WZ}, we find immediately that this effective action is

$$\Gamma_{WZ}[\bar{g};\sigma] = b\int d^4x\sqrt{-\bar{g}}\bar{F}\sigma + b'\int d^4x\sqrt{-\bar{g}}\Big\{\Big(\bar{E} - \frac{2}{3}\bar{\Box}\bar{R}\Big)\sigma + 2\sigma\bar{\Delta}_4\sigma\Big\}, \tag{3.11}$$

up to terms independent of σ. This Wess-Zumino action is a one-form representative of the nontrivial cohomology of the local Weyl group and satisfies (2.5) just as its two-dimensional analog $\Gamma^{(2)}_{WZ}$ does. Explicitly, by solving (3.8b) formally for σ and substituting the result in (3.11) we obtain

$$\Gamma_{WZ}[\bar{g};\sigma] = \Delta_\sigma \circ S_{\rm anom}[\bar{g}] \equiv S_{\rm anom}[g = e^{2\sigma}\bar{g}] - S_{\rm anom}[\bar{g}], \tag{3.12}$$

with

$$S_{\rm anom}[g] = \frac{1}{8}\int d^4x\sqrt{-g}\int d^4x'\sqrt{-g'}\Big(E - \frac{2}{3}\Box R\Big)_x \times \Delta_4^{-1}(x,x')\Big[2bF + b'\Big(E - \frac{2}{3}\Box R\Big)\Big]_{x'} \tag{3.13}$$

and $\Delta_4^{-1}(x,x')$ denoting the Green's function inverse of the fourth order differential operator defined by (3.9). From the foregoing construction it is clear that if there are additional Weyl invariant terms in the anomaly (3.5) they should be included in the $S_{\rm anom}$ by making the replacement $bF \to bF + \sum_i \beta_i H_i$ in the last square bracket of Eq. (3.13).

Because of (3.12) the anomalous effective action is clearly fixed only up to an arbitrary (local or nonlocal) Weyl invariant functional $S_{\rm inv}[g]$ of the metric, obeying

$$\Delta_\sigma \circ S_{\rm inv}[g] = S_{\rm inv}[e^{2\sigma}g] - S_{\rm inv}[g] = 0. \tag{3.14}$$

Hence the general nonlocal form of the *exact* effective action of gravity must be of the form, $S_{\rm anom} + S_{\rm inv}$, and behavior under the local Weyl group enables us to classify all terms in the exact effective action into three parts, viz.

$$S_{\rm exact} = S_{\rm local} + S_{\rm anom} + S_{\rm inv}, \tag{3.15}$$

given by (3.3), (3.13), and (3.14) respectively. This completely general decomposition of the exact effective action based on the cohomology of the Weyl group has been confirmed by explicit one-loop calculations in quantum field theory, expanding around flat space up to third order in the metric perturbations [23]. In particular these calculations show that nonlocal terms in $S_{\rm exact}$ involving $\ln(\Box)$, which had been speculated in Refs. [8] to be necessary for the global Weyl anomaly, are contained in fact in the conformally invariant piece $S_{\rm inv}$. The global Weyl anomaly under the constant rescaling of the metric (3.4) is contained in Γ_{WZ} and $S_{\rm anom}$ (by construction), simply as a special case of the local anomaly with $\sigma = \sigma_0$, a spacetime constant, and neither Γ_{WZ} nor $S_{\rm anom}$ as defined by Eqs. (3.11) and (3.13) respectively contain any $\ln(\Box)$ term.

As in two dimensions, the anomalous term (3.11) or (3.13) scales linearly with σ_0, i.e. logarithmically with the distance scale. The fluctuations generated by $S_{\rm anom}$ also define a nonperturbative Gaussian infrared fixed point, with conformal field theory anomalous dimensions analogous to the two-dimensional case [22,24]. This is possible only because new low energy degrees of freedom are contained in $S_{\rm anom}$ which can fluctuate independently of the local metric degrees of freedom in $S_{\rm EH}$. Thus the effective action of the anomaly $S_{\rm anom}$ should be retained in the EFT of low energy gravity, which is specified then by the first two strictly relevant local terms of the classical Einstein-Hilbert action (3.1), and the logarithmic, but nonlocal $S_{\rm anom}$, i.e.

$$S_{\rm eff}[g] = S_{\rm EH}[g] + S_{\rm anom}[g] \tag{3.16}$$

contains all the infrared relevant terms in low energy gravity for $E \ll M_{\rm Pl}$.

The low energy (Wilson) effective action (3.16), in which infrared irrelevant terms are systematically neglected in the renormalization group program of critical phenomena is to be contrasted with the exact (field theoretic) effective action of (3.15), in which the effects of all scales are included in principle, at least in the approximation in which spacetime can be treated as a continuous manifold. Ordinarily, i.e. absent anomalies, the Wilson effective action should contain only *local* infrared relevant terms consistent with symmetry [25]. However, like the anomalous effective action generated by the chiral anomaly in QCD, the nonlocal $S_{\rm anom}$ must be included in the low energy EFT to account for the anomalous Ward identities, even in the zero momentum limit, and indeed logarithmic scaling with distance indicates that $S_{\rm anom}$ is an infrared relevant term. Also even if no massless matter fields are assumed, the quantum fluctuations of the metric itself will generate a term of the same form as $S_{\rm anom}$ [11]. Therefore the classification of relevant and irrelevant local operators in EFT should be extended to allow also the inclusion of the logarithmic, nonlocal terms required by the presence of quantum anomalies.

The variations of the two (b and b') terms in $S_{\rm anom}$ give rise to two new conserved tensors in the equations of low energy gravity. To find their explicit form and exhibit the new scalar degrees of freedom they contain, it is convenient as in the two-dimensional case to rewrite the nonlocal action $S_{\rm anom}$ in local form by introducing auxiliary fields. Two scalar auxiliary fields satisfying

$$\Delta_4 \varphi = \tfrac{1}{2}\left(E - \tfrac{2}{3}\Box R\right), \tag{3.17a}$$
$$\Delta_4 \psi = \tfrac{1}{2} F, \tag{3.17b}$$

respectively may be introduced, corresponding to the two nontrivial cocycles of the b and b' terms in the anomaly [10]. It is then easy to see that

$$S_{\rm anom}[g;\varphi,\psi] = \frac{b'}{2}\int d^4x\sqrt{-g}\left\{-\varphi\Delta_4\varphi + \left(E - \frac{2}{3}\Box R\right)\varphi\right\} + \frac{b}{2}\int d^4x\sqrt{-g}\left\{-\varphi\Delta_4\psi - \psi\Delta_4\varphi + F\varphi + \left(E - \frac{2}{3}\Box R\right)\psi\right\} \tag{3.18}$$

is the desired local form of the anomalous action (3.13) [26]. Indeed the variation of (3.18) with respect to the auxiliary fields φ and ψ yields their Eqs. of motion (3.17), which may be solved for φ and ψ by introducing the Green's function $\Delta_4^{-1}(x,x')$. Substituting this formal solution for the auxiliary fields into (3.18) returns (3.13). The local auxiliary field form (3.18) is the most useful and explicitly contains two new scalar fields satisfying the massless fourth order wave Eqs. (3.17) with fourth order curvature invariants as sources. The freedom to add homogeneous solutions to φ and ψ corresponds to the freedom to define different Green's functions inverses $\Delta_4^{-1}(x,x')$ in (3.13). The auxilliary scalar fields are new local massless degrees of freedom of four dimesnional gravity, not contained in the Einstein-Hilbert action.

As in the two-dimensional case, the four dimensional anomalous action has an invariance, in this case $\psi \to \psi + \psi_0$, with the corresponding Noether current given by:

$$J^a = \nabla^a\Box\varphi + 2\left(R^{ab} - \frac{R}{3}g^{ab}\right)\nabla_b\varphi - \frac{1}{2}\Omega^a + \frac{1}{3}\nabla^a R, \tag{3.19}$$

where Ω^a is the topological current whose divergence,

$$\nabla_a\Omega^a = E = R_{abcd}R^{abcd} - 4R_{ab}R^{ab} + R^2 \tag{3.20}$$

is the Euler-Gauss-Bonnet integrand in four dimensions. The current J^a is conserved,

$$\nabla_a J^a = \Delta_4\varphi - \frac{E}{2} + \frac{1}{3}\Box R = 0 \tag{3.21}$$

by (3.17a), and the conserved charge may be taken to be

$$q = \frac{1}{16\pi^2}\int_\Sigma J^a d\Sigma_a \tag{3.22}$$

with Σ a spacelike Cauchy surface on which initial data are specified or

$$q = \frac{1}{16\pi^2}\oint_{\partial V} J^a d\Sigma_a \tag{3.23}$$

with ∂V the boundary of a Euclidean four volume V. The normalization of the Noether charge is chosen so that $q = 1$ corresponds to a delta function source of unit strength in the four dimensional Euler number,

$$\chi = \frac{1}{32\pi^2}\int d^4x\sqrt{-g}E. \tag{3.24}$$

The constant shift in the other auxiliary field, i.e. $\varphi \to \varphi + \varphi_0$ generates a global rescaling of the metric, and is not a symmetry of the action (3.18) unless $E - 2\Box R/3 = F = 0$. It is worth remarking that this enhanced conformal symmetry applies to solutions of the vacuum Einstein Eqs. if and only if the cosmological term $\Lambda = 0$. That the effective action of the anomaly may provide a mechanism to relax the cosmological term to zero was first proposed in Ref. [22].

By using the definition of Δ_4 and integrating by parts, we may express the anomalous action also in the form,

$$S_{\rm anom} = b' S^{(E)}_{\rm anom} + b S^{(F)}_{\rm anom}, \qquad (3.25)$$

with

$$S^{(E)}_{\rm anom} \equiv \frac{1}{2}\int d^4x\sqrt{-g}\Big\{-(\Box\varphi)^2 + 2\Big(R^{ab} - \frac{R}{3}g^{ab}\Big)(\nabla_a\varphi) \times (\nabla_b\varphi) + \Big(E - \frac{2}{3}\Box R\Big)\varphi\Big\};$$
$$S^{(F)}_{\rm anom} \equiv \int d^4x\sqrt{-g}\Big\{-(\Box\varphi)(\Box\psi) + 2\Big(R^{ab} - \frac{R}{3}g^{ab}\Big) \times (\nabla_a\varphi)(\nabla_b\psi) + \frac{1}{2}F\varphi + \frac{1}{2}\Big(E - \frac{2}{3}\Box R\Big)\psi\Big\} \qquad (3.26)$$

It is this final local auxiliary field form of the effective action which is to be added to classical Einstein-Hilbert action to obtain the effective action of low energy gravity in (3.16). We note that in this form the simple shift of the auxiliary field φ by a spacetime constant,

$$\varphi \rightarrow \varphi + 2\sigma_0 \qquad (3.27)$$

yields the entire dependence of $S_{\rm anom}$ on the global Weyl rescalings (3.4), *viz.*

$$S_{\rm anom}[g;\varphi,\psi] \rightarrow S_{\rm anom}[e^{2\sigma_0}g;\varphi + 2\sigma_0,\psi] = S_{\rm anom}[g;\varphi,\psi] + \sigma_0\int d^4x\sqrt{-g} \times \Big[bF + b'\Big(E - \frac{2}{3}\Box R\Big)\Big], \qquad (3.28)$$

owing to the strict invariance of the terms quadratic in the auxiliary fields under (3.4) and (3.8). Thus the auxiliary field form of the anomalous action (3.26) contains the same information about the global Weyl anomaly and large distance scaling as $\Gamma_{\rm WZ}$.

We note next that the actions $S^{(E)}$ and $S^{(F)}$ in (3.26) are very similar. Each is composed of terms at most quadratic and linear in the auxiliary fields φ and ψ. Defining the general quadratic action,

$$A[g;\varphi,\psi] \equiv \int d^4x\sqrt{-g}\Big\{-(\Box\varphi)(\Box\psi) + 2\Big(R^{ab} - \frac{R}{3}g^{ab}\Big)(\nabla_a\varphi)(\nabla_b\psi)\Big\} \qquad (3.29)$$

and the two linear actions,

$$B[g;\varphi] \equiv \frac{1}{2}\int d^4x\sqrt{-g}\Big(E - \frac{2}{3}\Box R\Big)\varphi; \qquad (3.30)$$

$$C[g;\varphi] \equiv \frac{1}{2}\int d^4x\sqrt{-g}F\varphi, \qquad (3.31)$$

as functionals of the metric and the two auxiliary scalar fields φ and ψ, we can express (3.26) in the form,

$$S_{\rm anom}[g;\varphi,\psi] = b' S^{(E)}_{\rm anom}[g;\varphi] + b S^{(F)}_{\rm anom}[g;\varphi,\psi], \qquad (3.32a)$$
$$S^{(E)}_{\rm anom}[g;\varphi] = \tfrac{1}{2}A[g;\varphi,\varphi] + B[g;\varphi], \qquad (3.32b)$$
$$S^{(F)}_{\rm anom}[g;\varphi,\psi] = A[g;\varphi,\psi] + B[g;\psi] + C[g;\varphi]. \qquad (3.32c)$$

This implies that the covariantly conserved stress tensors derived from the E and F terms in the effective action, namely,

$$E_{ab} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta S^{(E)}_{\rm anom}}{\delta g^{ab}}, \qquad (3.33a)$$
$$F_{ab} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta S^{(F)}_{\rm anom}}{\delta g^{ab}}. \qquad (3.33b)$$

can be obtained from one fundamental tensor quadratic in the scalar auxiliary fields,

$$A_{ab}[g;\varphi,\psi] \equiv -\frac{2}{\sqrt{-g}}\frac{\delta A[g;\varphi,\psi]}{\delta g^{ab}} = -2(\nabla_{(a}\varphi)(\nabla_{b)}\Box\psi) - 2(\nabla_{(a}\psi)(\nabla_{b)}\Box\varphi) + 2\nabla^c[(\nabla_c\varphi)(\nabla_a\nabla_b\psi) + (\nabla_c\psi)(\nabla_a\nabla_b\varphi)] - \frac{4}{3}\nabla_a\nabla_b[(\nabla_c\varphi)(\nabla^c\psi)] + \frac{4}{3}R_{ab}(\nabla_c\varphi)(\nabla^c\psi) - 4R^c{}_{(a}[(\nabla_{b)}\varphi)(\nabla_c\psi) + (\nabla_{b)}\psi)(\nabla_c\varphi)] + \frac{4}{3}R(\nabla_{(a}\varphi)(\nabla_{b)}\psi) + \frac{1}{3}g_{ab}\{-3(\Box\varphi)(\Box\psi) + \Box[(\nabla_c\varphi)(\nabla^c\psi)] + 2(3R^{cd} - Rg^{cd})(\nabla_c\varphi)(\nabla_d\psi)\}, \qquad (3.34)$$

which is traceless,

$$A_a{}^a[g;\varphi,\psi] = 0, \qquad (3.35)$$

for arbitrary φ and ψ, and two fundamental tensors linear in the scalar auxiliary fields,

$$B_{ab}[g;\varphi] \equiv -\frac{2}{\sqrt{-g}}\frac{\delta B[g;\varphi]}{\delta g^{ab}} = -\frac{2}{3}\nabla_a\nabla_b\Box\varphi - 4C_a{}^c{}_b{}^d\nabla_c\nabla_d\varphi - 4R^c_{(a}\nabla_{b)}\nabla_c\varphi + \frac{8}{3}R_{ab}\Box\varphi + \frac{4}{3}R\nabla_a\nabla_b\varphi - \frac{2}{3}(\nabla_{(a}R)\nabla_{b)}\varphi + \frac{1}{3}g_{ab}\{2\Box^2\varphi + 6R^{cd}\nabla_c\nabla_d\varphi - 4R\Box\varphi + (\nabla^c R)\nabla_c\varphi\}, \tag{3.36}$$

and

$$C_{ab}[g;\varphi] \equiv -\frac{2}{\sqrt{-g}}\frac{\delta C[g;\varphi]}{\delta g^{ab}} = -4\nabla_c\nabla_d(C_{(a}{}^c{}_{b)}{}^d\varphi) - 2C^{cd}_{ab}R_{cd}\varphi R^{cd}\varphi. \tag{3.37}$$

The latter of these is also traceless, whereas the B_{ab} tensor has a nonzero trace whose value depends on which auxiliary field is substituted, i.e.

$$B_a{}^a[g;\varphi] = 2\Delta_4\varphi = E - \tfrac{2}{3}\Box R, \tag{3.38a}$$
$$B_a{}^a[g;\psi] = 2\Delta_4\psi = F. \tag{3.38b}$$

From (3.25), (3.26), (3.32), (3.34), (3.36), and (3.37) we have

$$T_{ab}^{(\mathrm{anom})} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta S_{\mathrm{anom}}}{\delta g^{ab}} = b' E_{ab} + bF_{ab} \tag{3.39}$$

with

$$E_{ab} = \tfrac{1}{2}A_{ab}[g;\varphi,\varphi] + B_{ab}[g;\varphi] \quad \text{and} \tag{3.40a}$$
$$F_{ab} = A_{ab}[g;\varphi,\psi] + B_{ab}[g;\psi] + C_{ab}[g;\varphi]. \tag{3.40b}$$

Explicitly,

$$E_{ab} = -2(\nabla_{(a}\varphi)(\nabla_{b)}\Box\varphi) + 2\nabla^c[(\nabla_c\varphi)(\nabla_a\nabla_b\varphi)] - \tfrac{2}{3}\nabla_a\nabla_b[(\nabla_c\varphi)(\nabla^c\varphi)] + \tfrac{2}{3}R_{ab}(\nabla_c\varphi)(\nabla^c\varphi) - 4R^c{}_{(a}(\nabla_{b)}\varphi)(\nabla_c\varphi) + \tfrac{2}{3}R(\nabla_a\varphi)(\nabla_b\varphi) + \tfrac{1}{6}g_{ab}\{-3(\Box\varphi)^2 + \Box[(\nabla_c\varphi)(\nabla^c\varphi)] + 2(3R^{cd} - Rg^{cd})(\nabla_c\varphi)(\nabla_d\varphi)\} - \tfrac{2}{3}\nabla_a\nabla_b\Box\varphi - 4C_a{}^c{}_b{}^d\nabla_c\nabla_d\varphi - 4R^c_{(a}\nabla_{b)}\nabla_c\varphi + \tfrac{8}{3}R_{ab}\Box\varphi + \tfrac{4}{3}R\nabla_a\nabla_b\varphi - \tfrac{2}{3}(\nabla_{(a}R)\nabla_{b)}\varphi + \tfrac{1}{3}g_{ab}\{2\Box^2\varphi + 6R^{cd}\nabla_c\nabla_d\varphi - 4R\Box\varphi + (\nabla^c R)\nabla_c\varphi\}, \tag{3.41}$$

and

$$F_{ab} = -2(\nabla_{(a}\varphi)(\nabla_{b)}\Box\psi) - 2(\nabla_{(a}\psi)(\nabla_{b)}\Box\varphi) + 2\nabla^c[(\nabla_c\varphi)(\nabla_a\nabla_b\psi) + (\nabla_c\psi)(\nabla_a\nabla_b\varphi)] - \tfrac{4}{3}\nabla_a\nabla_b[(\nabla_c\varphi)(\nabla^c\psi)] + \tfrac{4}{3}R_{ab}(\nabla_c\varphi)(\nabla^c\psi) - 4R^c{}_{(a}[(\nabla_{b)}\varphi)(\nabla_c\psi) + (\nabla_{b)}\psi)(\nabla_c\varphi)] + \tfrac{4}{3}R(\nabla_{(a}\varphi)(\nabla_{b)}\psi) + \tfrac{1}{3}g_{ab}\{-3(\Box\varphi)(\Box\psi) + \Box[(\nabla_c\varphi)(\nabla^c\psi)] + 2(3R^{cd} - Rg^{cd})(\nabla_c\varphi)(\nabla_d\psi)\} - 4\nabla_c\nabla_d(C_{(a}{}^c{}_{b)}{}^d\varphi) - 2C^{cd}_{ab}R_{cd}\varphi - \tfrac{2}{3}\nabla_a\nabla_b\Box\psi - 4C_a{}^c{}_b{}^d\nabla_c\nabla_d\psi - 4R^c_{(a}(\nabla_{b)}\nabla_c\psi) + \tfrac{8}{3}R_{ab}\Box\psi + \tfrac{4}{3}R\nabla_a\nabla_b\psi - \tfrac{2}{3}(\nabla_{(a}R)\nabla_{b)}\psi + \tfrac{1}{3}g_{ab}\{2\Box^2\psi + 6R^{cd}\nabla_c\nabla_d\psi - 4R\Box\psi + (\nabla^c R)(\nabla_c\psi)\}. \tag{3.42}$$

Each of these two tensors are individually conserved and they have the local traces,

$$E_a{}^a = 2\Delta_4\varphi = E - \tfrac{2}{3}\Box R, \tag{3.43a}$$
$$F_a{}^a = 2\Delta_4\psi = F = C_{abcd}C^{abcd}, \tag{3.43b}$$

corresponding to the two terms, respectively, in the trace anomaly in four dimensions (with $\beta_i = 0$). Results for the E_{ab} tensor in terms of the decomposition (2.2), and the F_{ab} tensor in a slightly different notation were presented previously in Refs. [10,27], and partial results for E_{ab} appear also in [26].

It is straightforward to generalize the above development to include the effects of any additional Weyl invariant terms in the trace anomaly, transforming under (2.2) by

$$\sqrt{-g}H_i = \sqrt{\bar{g}}\bar{H}_i. \tag{3.44}$$

By making the replacement $F \to F + \sum_i \beta_i H_i/b$ in (3.17a), (3.18), and (3.26), we observe that $C[g;\varphi]$ should undergo the replacement,

$$C[g;\varphi] \to C[g;\varphi] + \sum_i \frac{\beta_i}{2b}\int d^4x\sqrt{-g}H_i\varphi, \tag{3.45}$$

with the corresponding additional terms in C_{ab} and F_{ab} in (3.37), (3.40b), (3.42), and (3.43b) to take account of any additional contributions to the trace anomaly. Hence no additional scalar auxiliary fields beyond φ and ψ are required in the general case.

As in two dimensions, S_{anom} in four dimensions contains massless scalar degrees of freedom which remain relevant at low energies and macroscopic distances, and must be retained in addition to the usual EFT local derivative expansion in order to take account of the nondecoupling of massless fields. In contrast, the Weyl invariant terms in

(3.2) or $S_{\rm inv}$ do not contain additional degrees of freedom of the gravitational field which are relevant at energies far below the Planck scale $M_{\rm Pl}$, and are either marginally or strictly irrelevant in the infrared.

In a given fixed spacetime background the solutions of the fourth order differential Eqs. (3.17) for the scalar auxiliary fields φ and ψ may be found, and the results substituted into the stress tensors E_{ab} and F_{ab}. The freedom to add homogeneous solutions of (3.17) to any given inhomogeneous solution corresponds to the freedom to change the Weyl invariant part of the effective action and corresponding traceless parts of the stress tensors, without altering their local traces (3.43). The traceless terms in the stress tensor thus depend on the state of the underlying quantum field(s) which can be fixed by specifying boundary conditions on the auxiliary fields. When the spacetime is conformally flat or approximately so, the Weyl invariant action $S_{\rm inv}$ can be taken to vanish, as it does in the conformally related flat spacetime, and the low energy effective action (3.16) should become a good approximation to the exact effective action (3.15). In order to test this hypothesis we evaluate the corresponding anomalous stress tensor (3.39) in conformally flat spacetimes for a number of different macroscopic states.

IV. STRESS-ENERGY TENSOR IN CONFORMALLY FLAT SPACETIMES

A. Flat spacetime

In flat Minkowski spacetime, $E = F = \Box R = 0$, and the auxiliary fields satisfy the homogeneous wave eqs.,

$$\Delta_4|_{\rm flat}\varphi = \Box^2\varphi = 0 = \Box^2\psi. \tag{4.1}$$

For the trivial solutions $\varphi = \psi = 0$ (or an arbitrary spacetime constant), the stress tensor vanishes identically. This corresponds to the ordinary Lorentz invariant vacuum state, with vanishing renormalized $\langle T_a{}^b\rangle$. Hence flat spacetime will continue to satisfy the Einstein equations modified by the trace anomaly terms (with vanishing cosmological term).

In flat spacetime the wave Eq. (4.1) has plane wave solutions obeying the massless dispersion relation, $\omega_{\mathbf{k}} = \pm|\mathbf{k}|$. Since the wave operator here, $\Box^2$ is fourth order, there are two sets of massless scalar modes for *each* auxiliary field, φ and ψ. These are new *local* scalar degrees of freedom for the gravitational field not present in classical general relativity, and give rise to scalar gravitational waves. These scalar gravitational waves of the augmented effective theory have as their sources the fourth order curvature invariants, E, F and $\Box R$ which are quite small for most astrophysical sources, excepting perhaps those of cosmological origin. Hence in the flat space limit the scalar degrees of freedom decouple completely. Even in the presence of matter, they couple only indirectly through the effects of spacetime curvature, and only very weakly at that. Localized sources act as only very weak generators of scalar gravitational radiation in the φ and ψ fields, and primordial scalar gravitational radiation, even if present, would be difficult to detect, except indirectly through the gravitational effects of the stress energy (3.39). The effects on the production of scalar gravitational waves from the coupling of ψ to gauge fields through the H_i terms in the trace (3.5) is also possible, and merits a separate investigation.

For nonvanishing φ the tensor E_{ab} in flat space becomes

$$\begin{aligned} E_{ab}|_{\rm flat} = &-2(\nabla_{(a}\varphi)(\nabla_{b)}\Box\varphi) + 2(\Box\varphi)(\nabla_a\nabla_b\varphi) \\ &+ \tfrac{2}{3}(\nabla_c\varphi)(\nabla^c\nabla_a\nabla_b\varphi) - \tfrac{4}{3}(\nabla_a\nabla_c\varphi)(\nabla_b\nabla^c\varphi) \\ &+ \tfrac{1}{6}g_{ab}\{-3(\Box\varphi)^2 + \Box(\nabla_c\varphi\nabla^c\varphi)\} \\ &- \tfrac{2}{3}\nabla_a\nabla_b\Box\varphi, \end{aligned} \tag{4.2}$$

in general curvilinear coordinates. We note that although the energy density for the scalar auxiliary fields is not positive definite, this does not lead to any instability at macroscopic distances in flat space, because of the decoupling of the auxiliary fields in the flat space limit. In Ref. [27] the physical state space of the conformal scalar fluctuations of $S_{\rm anom}$ (proportional to φ), once decoupled from metric fluctuations, was shown to have positive norm, and be free of any tachyon or ghost modes. When $S_{\rm anom}$ is considered together with the classical Einstein-Hilbert terms, because of the higher derivatives in the auxiliary field stress tensor, instability in flat space appears only at ultrashort Planck scale wavelengths [28], i.e. at the order of the cutoff scale and outside the range of validity of the low energy EFT approach. At the Planck scale the local expansion of the effective action in powers of $E^2/M_{\rm Pl}^2$ clearly breaks down. For $E \ll M_{\rm pl}$ where the low energy effective action (3.16) can be applied, the auxiliary fields are coupled very weakly and indirectly to matter and propagate as very nearly noninteracting free field scalar excitations. This shows that empty flat space is stable to the effects of $S_{\rm anom}$ on macroscopic distance scales, and more subtle signatures of the presence of $S_{\rm anom}$ in nearly flat spacetimes must be sought.

As a simple example of how the stress tensor (4.2) may be used to evaluate macroscopic semiclassical effects of quantum fields in flat space, consider the special solution to (4.1),

$$\varphi = \frac{c_1}{2}\frac{z^2}{a^2} \tag{4.3}$$

for some constant a with dimensions of length and c_1 dimensionless. With a similar ansatz for $\psi = c_2z^2/2a^2$ the total stress tensor of the anomaly (3.39) becomes

$$T_{ab}^{(\rm anom)} = \frac{C}{3a^4}{\rm diag}(-1, 1, 1, -3), \tag{4.4}$$

with $C = c_1^2b' + 2bc_2^2$. This is the form of the stress tensor of the Casimir effect in region between two infinite parallel plates a distance a apart in the z direction. The constants

c_1, c_2 and C depend on the boundary conditions which the quantum field(s) obey on the plates, and parameterize the discontinuities in the components of the stress tensor there, but the generic form of the stress tensor appropriate for the geometry of two parallel conducting plates follows from the simple ansatz (4.3) for the classical auxiliary fields together with (3.39). Thus the low energy effective action of the trace anomaly in *curved* space correctly determines the form of the *traceless* components of the macroscopic quantum stress tensor for the parallel plate geometry in *flat* space.

If instead of (4.3) we choose

$$\varphi = \frac{c_1}{2} T^2 t^2 \tag{4.5}$$

with $\psi = c_2 T^2 t^2/2$ the stress tensor (3.39) becomes

$$T_{ab}^{(\mathrm{anom})} = C\frac{T^4}{6}\mathrm{diag}(-3, 1, 1, 1), \tag{4.6}$$

with $C = -c_1^2 b' + 2bc_2^2$ which is the form of the stress tensor for massless radiation at temperature T, which is again traceless in flat space. The same form for the anomalous stress tensor is also obtained by considering the general static spherically symmetric ansatz, $\varphi = \varphi(r)$. Then

$$\varphi = \frac{c_{-1}}{r} + c_0 + c_1 r + c_2 r^2, \tag{4.7}$$

with $c_{-1} = 0$ to ensure a regular stress tensor at the origin. We note that although a thermal state is a mixed state for the radiation field, it is a macroscopic pure coherent state for the gravitational auxiliary potentials at the semiclassical level, where fluctuations around the mean stress tensor $\langle T_{ab}\rangle = T_{ab}^{(\mathrm{anom})}$ are neglected.

For the four dimensional Rindler metric,

$$ds^2 = -\rho^2 d\eta^2 + d\rho^2 + dy^2 + dz^2 \tag{4.8}$$

and $\varphi = \varphi(\rho)$, $\Delta_4\varphi = \Box^2\varphi = 0$ has the nontrivial solutions $\rho^2\ln\rho$, ρ^2 and $\ln\rho$. The first two solutions lead to stress tensors which are nonvanishing at infinity similar to the thermal state, while the $\ln\rho$ solution for φ and ψ gives

$$T_a{}^b = \frac{C}{\rho^4}\mathrm{diag}(-3, 1, 1, 1). \tag{4.9}$$

Adding a term linear in η to the solution for $\varphi(\rho)$ gives a stress tensor of the same form as (4.9) with a different constant C. In either case the form of the stress tensor (4.9) is that of the Fulling-Rindler state in four dimensions [29]. Because of (2.24), the solution $\varphi = c_1'\ln(\rho/\rho_0)$ gives rise to a singularity in $\Box\varphi$ at $\rho = 0$ proportional to $\delta(\rho)/\rho$, and therefore is a solution only in the distributional sense, as in the two-dimensional case. The higher order derivatives in the Noether current in four dimensions produce higher order singularities at $\rho = 0$, and the conformally transformed metric $e^{-\varphi}ds^2$ is singular at $\rho = 0$ for any $c_1' \neq 0$.

B. de Sitter spacetime

In conformally flat spacetimes with $g_{ab} = e^{2\sigma}\eta_{ab}$, one can choose $\varphi = 2\sigma$ and $\psi = 0$ to obtain the stress tensor of the state conformally transformed from the Minkowski vacuum. In this state F_{ab} vanishes, and φ can be eliminated completely in terms of the Ricci tensor with the result [10,30,31],

$$\begin{aligned} E_{ab} &= -2\,{}^{(3)}H_{ab} - \tfrac{1}{9}\,{}^{(1)}H_{ab} \\ &= \tfrac{2}{9}\nabla_a\nabla_b R + 2R_a{}^c R_{bc} - \tfrac{14}{9}RR_{ab} \\ &\quad + g_{ab}(-\tfrac{2}{9}\Box R - R_{cd}R^{cd} + \tfrac{5}{9}R^2). \end{aligned} \tag{4.10}$$

Thus all nonlocal dependence on boundary conditions of the auxiliary fields φ and ψ vanishes in conformally flat spacetimes for the state conformally mapped from the Minkowski vacuum. In the special case of maximally $O(4, 1)$ symmetric de Sitter spacetime, $R_{ab} = 3H^2 g_{ab}$ with $R = 12H^2$ a constant, the tensor ${}^{(1)}H_{ab}$ vanishes identically and

$$E_{ab}|_{dS} = -2\,{}^{(3)}H_{ab}|_{dS} = 6H^4 g_{ab}. \tag{4.11}$$

Hence we obtain immediately the expectation value of the stress tensor of a massless conformal field of any spin in the Bunch-Davies state in de Sitter spacetime,

$$\begin{aligned} T_{ab}|_{\mathrm{BD},dS} &= 6b'H^4 g_{ab} \\ &= -\frac{H^4}{960\pi^2} g_{ab}(N_s + 11N_f + 62N_v), \end{aligned} \tag{4.12}$$

which is determined completely by the trace anomaly.

It is also possible to consider states which are not maximally $O(4, 1)$ symmetric. For example, if de Sitter spacetime is expressed in the spatially flat coordinates,

$$ds^2|_{dS} = -d\tau^2 + e^{2H\tau}d\vec{x}^2 \tag{4.13}$$

and $\varphi = \varphi(\tau)$, we obtain from (3.39) the stress energy in spatially homogeneous, isotropic states. Since in this maximally symmetric spacetime the Δ_4 operator factorizes,

$$\Delta_4|_{dS}\varphi = (\Box - 2H^2)(\Box\varphi) = 12H^4. \tag{4.14}$$

it is straightforward to show that the general solution to this equation with $\varphi = \varphi(\tau)$ is

$$\varphi(\tau) = 2H\tau + c_0 + c_{-1}e^{-H\tau} + c_{-2}e^{-2H\tau} + c_{-3}e^{-3H\tau}. \tag{4.15}$$

The first (inhomogeneous) term can be understood geometrically from the fact that the conformal transformation,

$$e^{-\varphi_{\mathrm{BD}}}ds^2|_{dS} = e^{-2H\tau}ds^2|_{dS} = -d\eta^2 + d\vec{x}^2 \tag{4.16}$$

brings de Sitter spacetime to flat spacetime, with $\eta = -H^{-1}e^{-H\tau}$ the conformal time. This value of $\varphi = \varphi_{\mathrm{BD}}(\tau) \equiv 2H\tau$ is exactly the one that gives the Bunch-Davies stress energy (4.12) when substituted into (3.41). When the full solution for $\varphi(\tau)$ of (4.15) is substituted into

(3.41) we obtain additional terms in the stress tensor which are not de Sitter invariant, but which fall off at large τ, as $e^{-4H\tau}$. The stress tensor of this time behavior is traceless and corresponds to the redshift of massless modes with the equation of state, $p = \rho/3$. This is a special case of the detailed analysis of the stress tensor expectation value of a field of arbitrary mass in homogeneous, isotropic states in de Sitter spacetime of Ref. [31]. It is straightforward to generalize these considerations to spatially homogeneous and isotropic states in arbitrary conformally flat cosmological spacetimes with Robertson-Walker scale factor $a(\eta)$. Because of the reduction of the auxiliary field stress tensor E_{ab} to the local geometric form (4.10) in conformally flat spacetimes, it can have no significant effects in states with Robertson-Walker symmetries when the curvature is much less than the Planck scale.

States of lower symmetry in de Sitter spacetime may be found by considering static coordinates,

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2, \tag{4.17}$$

where

$$f(r)|_{dS} = 1 - H^2 r^2. \tag{4.18}$$

If we insert the ansatz $\varphi = \varphi(r)$ in (4.14), the general $O(3)$ spherically symmetric solution regular at the origin is easily found:

$$\varphi(r)|_{dS} = \ln(1 - H^2 r^2) + c_0 + \frac{q}{2}\ln\left(\frac{1 - Hr}{1 + Hr}\right) + \frac{2c_H - 2 - q}{2Hr}\ln\left(\frac{1 - Hr}{1 + Hr}\right). \tag{4.19}$$

A possible homogeneous solution proportional to $1/r$ has been discarded, since it is singular at the origin. An arbitrary linear time dependence $2Hpt$ could also be added to $\varphi(r)$, i.e. $\varphi(r) \rightarrow \varphi(r, t) = \varphi(r) + 2Hpt$. The particular solution,

$$\varphi_{\rm BD}(r, t) = \ln(1 - H^2 r^2) + 2Ht = 2H\tau. \tag{4.20}$$

is simply the previous solution for the Bunch-Davies state we found in the homogeneous flat coordinates (4.13). The priveledged role of this fully $O(4, 1)$ invariant state may be understood in the static coordinates by recognizing that the line element (4.17) may be put into the alternate forms,

$$\begin{aligned} ds^2 &= f\left[-dt^2 + dr^{*2} + \frac{r^2}{f}d\Omega^2\right] \\ &= f[-dt^2 + \ell^2(d\chi^2 + \sinh^2\chi d\Omega^2)] \\ &= \frac{\ell^2 f}{\rho^2}[-d\rho^2 + \rho^2(d\chi^2 + \sinh^2\chi d\Omega^2)] \\ &= \frac{f}{\rho^2}(-d\eta^2 + d\vec{x}^2) = e^{\varphi_{\rm BD}}(ds^2)_{\rm flat}. \end{aligned} \tag{4.21}$$

by means of the successive changes of variables, $\ell = H^{-1}$,

$$\begin{aligned} \chi &= Hr^* = H\int_0^r \frac{dr}{f(r)} = \frac{1}{2}\ln\left(\frac{1 + Hr}{1 - Hr}\right), \\ t &= -\ell\ln\rho, \qquad \eta = \ell\rho\cosh\chi, \\ \vec{x} &= \ell\rho\sinh\chi(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta). \end{aligned} \tag{4.22}$$

Thus de Sitter spacetime in static coordinates is conformally related to flat spacetime by the conformal transformation $e^{\varphi_{\rm BD}}$ with $\varphi_{\rm BD}$ given by (4.20) as before. The benefit of carrying this out explicitly in the static coordinates is that the steps in (4.21) can be repeated essentially unchanged in the vicinity of *any* static Killing horizon, without reliance on the special higher symmetries of de Sitter spacetime. Then

$$\varphi_h(r, t) = \ln f(r) \pm \frac{2}{\ell}t \tag{4.23}$$

is the value of the auxiliary field which conformally transforms the local neighborhood of the horizon to flat spacetime. In de Sitter space this leading order conformal transformation mapping the vicinity of the horizon to flat space is exact, c.f. (4.21).

In the general $O(3)$ symmetric state in de Sitter spacetime the radial component of the Noether current,

$$J^r = \nabla^r\Box\varphi - 2H^2\nabla^r\varphi - \frac{1}{2}\Omega^r = \frac{2Hq}{r^2}, \tag{4.24}$$

in the frame in which $\Omega^r = 8H^4 r$. Thus the parameter q coincides with the definition (3.22) and if nonzero gives rise to a delta function singularity for $\Delta_4\varphi$ at the origin $r = 0$. It is also the strength of the puncture configurations (also called "spikes") found in [32].

In the general $O(3)$ symmetric state centered about the origin of the coordinates $r = 0$, the stress tensor is generally dependent on r. In fact, it generally diverges as the observer horizon $r = H^{-1}$ is approached. From (4.19) we observe that

$$\varphi(r)|_{dS} \rightarrow \left[c_H + \left(c_H - 1 - \frac{q}{2}\right)(1 - Hr) + \ldots\right] \times \ln\left(\frac{1 - Hr}{2}\right) + \mathcal{O}(1 - Hr), \tag{4.25}$$

as $Hr \rightarrow 1$, so that the integration constant c_H controls the most singular behavior at the observer horizon $r = H^{-1}$.

The second auxiliary field ψ satisfies the homogeneous Eq., $\Delta_4\psi = 0$, which has the general spherically symmetric solution linear in t,

$$\psi(r, t)|_{dS} = d_0 + 2Hp't + \frac{q'}{2}\ln\left(\frac{1 - Hr}{1 + Hr}\right) + \frac{2d_H - q'}{2Hr}\ln\left(\frac{1 - Hr}{1 + Hr}\right). \tag{4.26}$$

Note that the constant d_H enters this expression differently than c_H enters the corresponding Eq. (4.19), due to the inhomogeneous term in (4.14), which is absent from the ψ

equation. Since the anomalous stress tensor is independent of c_0 and d_0, it depends on the six parameters (c_H, d_H, q, q', p, p') in the general stationary $O(3)$ invariant state. The simplest way to insure no r^{-2} or r^{-1} singularity of the stress tensor at the origin is to choose $q = q' = 0$. With $q = q' = 0$ there are no sources or sinks at the origin and the zero flux condition,

$$T_t^{\ r} = -\frac{4H^2}{r^2}(bpq' + bp'q + b'pq) = 0 \tag{4.27}$$

is satisfied automatically, for any p and p'. Because of the subleading logarithmic behavior (4.25) of φ on the observer horizon at $r = H^{-1}$, there will $\ln^2(1 - Hr)$ and $\ln(1 - Hr)$ divergences in the other components of the stress tensor (3.39) in general. These divergences do not appear however when $q = q' = 0$ and $c_H = 1$. All the divergences of the stress energy at both the origin and the observer horizon are cancelled if the four conditions,

$$2b(d_H + pp') = b'(1 - p^2), \tag{4.28a}$$

$$c_H = 1, \quad \text{and} \tag{4.28b}$$

$$q = q' = 0 \tag{4.28c}$$

are satisfied. These are satisfied by the Bunch-Davies state with $p = \pm 1$ and $d_H = p' = 0$. In fact, with conditions (4.28) the tensor F_{ab} of (3.42) vanishes identically, the ψ field drops out entirely, and the full anomalous stress-energy tensor is given by (3.41), which takes the Bunch-Davies form (4.12).

If the first two of the four conditions (4.28) are relaxed, then the stress energy remains finite at the origin, but becomes divergent at $r = H^{-1}$. This is the generic case. It includes, in particular, the analog of the static Boulware vacuum [33] in de Sitter space. There is no analog of the Unruh state [34] in de Sitter spacetime, since by continuity a flux through the future or past observer horizon at $r = H^{-1}$ would require a source or sink of flux at the origin $r = 0$, a possibility we have excluded by (4.27) above.

V. NEAR HORIZON CONFORMAL SYMMETRY AND STRESS TENSOR

The fact that the trace anomaly can account for even the tracefree parts of the stress tensor exactly in two dimensions and conformally flat spacetimes in higher dimensions is due to the fact that $S_{\rm inv}$ may be neglected in these cases. In these circumstances the low energy effective action (3.16) becomes a good approximation to the exact effective action (3.15). However, as has been noted by several authors [35], spacetimes with a static Killing horizon are locally conformally related to flat spacetime in the vicinity of the horizon. Because of this near horizon conformal symmetry we expect the anomalous action (3.13) to generate a stress tensor (3.39) which is a good approximation to the quantum stress tensor near an arbitrary Killing horizon.

To exhibit the conformal geometry of a static Killing horizon consider the general static spherically symmetric spacetime of the form, (4.17) where the function $f(r)$ possesses a simple zero at $r = r_+$, i.e.,

$$f(r) = \pm\frac{2}{\ell}(r - r_+) + \mathcal{O}(r - r_+)^2. \tag{5.1}$$

Defining r^* in the usual way,

$$r^* = \int^r \frac{dr}{f(r)} \simeq \pm\frac{\ell}{2}\ln f + \ldots, \tag{5.2}$$

where the ellipsis denotes regular terms as $f \to 0$, and

$$\chi = -\frac{1}{2}\ln\left(\frac{\ell^2 f}{4r_+^2}\right) = \mp\frac{r^*}{\ell} + \ldots, \tag{5.3}$$

we find that near $r = r_+$,

$$\frac{r^2}{f(r)} \simeq \frac{\ell^2}{4}e^{2\chi} \simeq \ell^2\sinh^2\chi. \tag{5.4}$$

Hence we may repeat the steps (4.21) and (4.22) of the previous de Sitter case for an arbitrary spherically symmetric static Killing horizon, in the vicinity of the horizon. The length scale ℓ is also related to the surface gravity of the horizon by $\ell^{-1} = \kappa$. Thus we have the conformal mapping (4.21) of the near horizon metric (4.17) to the flat metric with $\varphi \simeq \varphi_h(r, t)$ given by (4.23). Since this conformal transformation is singular at $r = r_+$, the exact effective action (3.15) is dominated by its anomalous contribution $S_{\rm anom}$, and the stress energy is dominated by (3.39) in any state for which φ and the corresponding $T_a^{\ b}$ diverges on the horizon. Hence the near horizon conformal symmetry (4.21) may be used to calculate the exact behavior of the divergent terms of the quantum stress-energy tensor from (3.39) in any spherically symmetric static spacetime with a Killing horizon. We illustrate this for the most important case of Schwarzschild spacetime.

A. Schwarzschild spacetime

In the four dimensional Schwarzschild geometry, $f(r) = 1 - \frac{2M}{r} \simeq r/2M - 1 + \mathcal{O}(r - 2M)^2$. Hence $r_+ = 2M$ and $\ell = 4M$. The conformal transformation that maps the near horizon geometry to flat spacetime is therefore

$$\varphi_h(r, t) = \ln f \pm \frac{t}{2M}. \tag{5.5}$$

Since Schwarzschild spacetime is not globally conformally flat the conformal transformation of the near horizon geometry will receive subleading corrections to (5.5). To find these subleading terms we require the general solution to (3.17). Noting that Schwarzschild spacetime is Ricci flat, we have

$$F|_S = E|_S = R_{abcd}R^{abcd}|_S = \nabla_a\Omega^a = \frac{48M^2}{r^6}. \tag{5.6}$$

A particular solution of either of the inhomogeneous Eqs. (3.17) is given then by $\bar{\varphi}(r)$, with

$$\frac{d\bar{\varphi}}{dr}\bigg|_S = -\frac{4M}{3r^2 f}\ln\left(\frac{r}{2M}\right) - \frac{1}{2M}\left(1+\frac{4M}{r}\right). \tag{5.7}$$

Notice that this particular inhomogeneous solution to (3.17) is regular as $r \to 2M$. The general solution of (3.17) for $\varphi = \varphi(r)$ away from the singular points $r = (0, 2M, \infty)$ is easily found and may be expressed in the form [26],

$$\begin{aligned}\frac{d\varphi}{dr}\bigg|_S &= \frac{d\bar{\varphi}}{dr}\bigg|_S + \frac{2Mc_H}{r^2 f} + \frac{q-2}{4M^2 r^2 f}\int_{2M}^r dr r^2 \ln f \\ &\quad + \frac{c_\infty}{2M}\left(\frac{r}{2M}+1+\frac{2M}{r}\right) \\ &= \frac{q-2}{6M}\left(\frac{r}{2M}+1+\frac{2M}{r}\right)\ln\left(1-\frac{2M}{r}\right) \\ &\quad - \frac{q}{6r}\left[\frac{4M}{r-2M}\ln\left(\frac{r}{2M}\right)+\frac{r}{2M}+3\right] - \frac{1}{3M} - \frac{1}{r} \\ &\quad + \frac{2Mc_H}{r(r-2M)} + \frac{c_\infty}{2M}\left(\frac{r}{2M}+1+\frac{2M}{r}\right)\end{aligned} \tag{5.8}$$

in terms of the three dimensionless constants of integration, c_H, c_∞, and q. This expression has the limits,

$$\begin{aligned}\frac{d\varphi}{dr}\bigg|_S &\to \frac{c_H}{r-2M} + \frac{q-2}{2M}\ln\left(\frac{r}{2M}-1\right) \\ &\quad - \frac{1}{2M}\left(3c_\infty - c_H - q - \frac{5}{3}\right) + \ldots, \qquad r \to 2M,\end{aligned} \tag{5.9a}$$

$$\begin{aligned}\frac{d\varphi}{dr}\bigg|_S &\to \frac{c_\infty r}{4M^2} + \frac{2c_\infty - q}{4M} + \frac{c_\infty}{r} - \frac{2M}{3r^2}q\ln\left(\frac{r}{2M}\right) \\ &\quad + \frac{2M}{r^2}\left[c_H - \frac{7}{18}(q-2)\right] + \ldots, \qquad r \to \infty.\end{aligned} \tag{5.9b}$$

Hence c_H controls the leading behavior as r approaches the horizon, while c_∞ controls the leading behavior as $r \to \infty$, which is the same as in flat space. The leading behavior at the horizon is determined by the homogeneous solution to (3.17), $c_H \ln f = c_H \ln(-K^a K_a)$ where $K = \partial_t$ is the timelike Killing field of the Schwarzschild geometry for $r > 2M$. To the general spherically symmetric static solution (5.8) we may add also a term linear in t, i.e. we replace $\varphi(r)$ by

$$\varphi(r,t) = \varphi(r) + \frac{p}{2M}t. \tag{5.10}$$

Linear time dependence in the auxiliary fields is the only allowed time dependence that leads to a time-independent stress energy. We see that the conformal transformation to flat space near the horizon (5.5) corresponds to the particular choice $c_H = \pm p = 1$, leaving the subdominant terms in (5.9a) parameterized by q and c_∞ undetermined.

The topological charge q may be identified from

$$\nabla^r \Box\varphi = -\frac{8M^2}{r^5} + \frac{q}{2Mr^2}, \tag{5.11}$$

so that q is given by (3.23) in the Euclidean signature Schwarzschild geometry in the frame in which $\Omega^r = -16M^2/r^5$ and $\Omega^t = 0$. A nonzero q gives $\Delta_4\varphi$ a delta function singularity at the origin. Nonzero c_H or c_∞ produce higher order singular distributions in $\Delta_4\varphi$ at the horizon or infinity, respectively. We note that unlike in flat space there are logarithmic terms in $\varphi(r)$ and no value of q which can eliminate the subleading logarithmic behavior of φ' at both $r = 2M$ and $r = \infty$.

Since the equation for the second auxiliary field ψ is identical to that for φ, its solution for $\psi = \psi(r,t)$ is of the same form as (5.8) and (5.10) with four new integration constants, d_H, d_∞, q', and p' replacing c_H, c_∞, q, and p in $\varphi(r,t)$. Adding terms with any higher powers of t or more complicated t dependence produces a time dependent stress-energy tensor. Inspection of the stress tensor terms in (3.39) also shows that it does not depend on either a constant φ_0 or ψ_0 but only the derivatives of both auxiliary fields in Ricci flat metrics such as Schwarzschild spacetime. For that reason we do not need an additional integration constant for either of the fourth order differential Eqs. (3.17).

With the general spherically symmetric solution for $\varphi(r,t)$ and $\psi(r,t)$, we can proceed to compute the stress-energy tensor in a stationary, spherically symmetric quantum state. For example the Boulware state may be characterized as that state which approaches the flat space vacuum as rapidly as possible as $r \to \infty$ [33]. In the flat space limit this means that the allowed r^2 and r behavior in the auxiliary fields (r and constant behavior in their first derivatives) must be set to zero. Inspection of the asymptotic form (5.9b) shows that this is achieved by requiring

$$c_\infty = d_\infty = 0 \quad \text{and} \tag{5.12a}$$

$$q = q' = 0. \qquad \text{(Boulware)} \tag{5.12b}$$

If we set $p = p' = 0$ as well, in order to have a static ansatz for the Boulware state, then the remaining two constants c_H and d_H are free parameters of the auxiliary fields, and lead to a stress energy which generically has the divergent behaviors,

$$s^{-2}, \quad s^{-1}, \quad \ln^2 s \text{ and } \ln s, \text{ as } s \equiv \frac{r-2M}{M} \to 0, \tag{5.13}$$

on the horizon. It is not possible to cancel all four of these divergent behaviors with the two remaining free parameters. As in the two-dimensional case, it is not possible to have auxiliary fields and stress energies which both fall off at infinity and are regular on the horizon. This macroscopic effect of the trace anomaly is a result of the nontrivial global topology of the Schwarzschild metric, notwithstanding the smallness of local curvature invariants at the horizon.

The linear time dependence (5.10) is consistent with the singular behaviors (5.13) of the stress energy on the horizon. Since the stress energy in the Boulware state diverges as s^{-2} in any case, we can match this leading divergence in the anomalous stress tensor by adjusting c_H and d_H appropriately. One then has a one parameter fit to the numerical data of [36]. The results of this fit of (3.39) with c_H and d_H treated as free parameters are illustrated in Fig. 1–3, for all three nonzero components of the stress-tensor expectation value of a massless, conformally coupled scalar field in the Boulware state. The best fit values plotted were obtained with $c_H = -\frac{7}{20}$ and $d_H = \frac{55}{84}$.

For comparison purposes, we have plotted also the analytic approximation of Page, Brown, and Ottewill [37–39] (dashed curves in Fig. 1–3). We observe that the two parameter fit with the anomalous stress tensor in terms of the auxiliary φ and ψ fields is more accurate than the approximation of Refs. [37–39] for the Boulware state. In the latter case the stress tensor is approximated by making a special conformal transformation $\sigma = \frac{1}{2}\ln f = \frac{1}{2} \times \ln(-K^a K_a)$ to the optical metric, obtained from the Schwarzschild line element (4.17) by dividing by $f(r) = 1 - \frac{2M}{r}$. This procedure was also motivated by the form of the trace anomaly, in that for ultrastatic metrics with the timelike Killing field K^a, the optical metric conformally related to it has vanishing trace anomaly [38]. However, unlike the general form of the anomalous stress tensor in (3.39), which involves solving linear equations for two independent auxiliary fields, the approximation of Refs. [37–39] is applicable only to special cases, such as Schwarzschild geometry which are both ultrastatic and Ricci flat, and admit a particular conformal transformation to the static optical metric. In contrast the approach based on the auxiliary field anomalous effective action does not require any special properties of the background spacetime, and because of the near horizon conformal symmetry gives the general form of the divergences of the stress energy on the horizon. An important practical difference between these approximations and that considered here is that they yield simple algebraic approximations to the stress energy, whereas (3.39) generally contains logarithmic terms as well. Evidently the general form of the effective stress-energy tensor due to the anomaly in terms of two auxiliary fields and the freedom to add homogeneous solutions to the linear Eqs. (3.17) allows for a better approximation to the exact stress tensor in the Boulware state than that of Refs. [37–39] near the Schwarzschild horizon.

The stress energy diverges on the horizon in an entire family of states for generic values of the eight auxiliary field parameters $(c_H, q, c_\infty, p; d_H, q', d_\infty, p')$, in addition to the Boulware state. Hence in the general allowed parameter space of spherically symmetric macroscopic states, horizon divergences of the stress energy are quite generic,

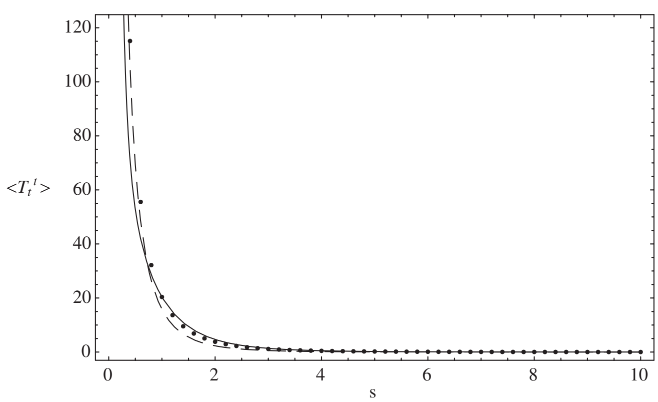

FIG. 1. The expectation value $\langle T_t{}^t \rangle$ of a conformal scalar field in the Boulware state in Schwarzschild spacetime, as a function of $s = \frac{r-2M}{M}$ in units of $\pi^2 T_H^4/90$. The solid curve is Eq. (3.39) with (5.12) and $c_H = -\frac{7}{20}$, $d_H = \frac{55}{84}$, the dashed curve is the analytic approximation of [38], and the points are the numerical results of [36].

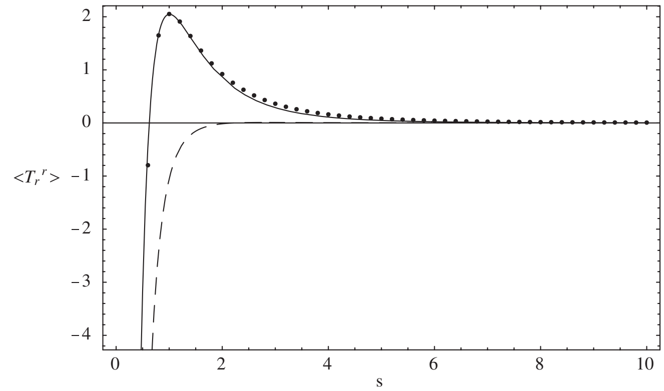

FIG. 2. The radial pressure $\langle T_r{}^r \rangle$ of a conformal scalar field in the Boulware state in Schwarzschild spacetime. The axes and solid and dashed curves and points are as in Fig. 1.

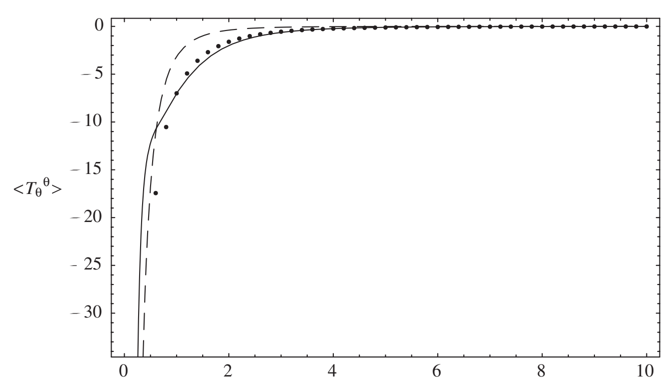

FIG. 3. The tangential pressure $\langle T_\theta{}^\theta \rangle$ of a conformal scalar field in the Boulware state in Schwarzschild spacetime. The axes and solid and dashed curves and points are as in Fig. 1.

and not resticted to the Boulware state. On the other hand, the condition (2.33) that the stress energy on the horizon be finite gives four conditions on these eight parameters, in order to cancel the four possible divergences listed in (5.13). Referring to the explicit form of the stress-energy components in static spherically symmetric spacetimes given in the appendix, these four conditions can be satisfied in several different ways. The simplest possibility for a finite stress tensor on the horizon with the minimal number of conditions on the auxiliary field parameters is:

$$(2b + b')c_H^2 + p(2bp' + b'p) = 0, \qquad (s^{-2}), \tag{5.14a}$$

$$(b + b')c_H = bd_H, \qquad (s^{-1}), \tag{5.14b}$$

$$q = q' = 2, \qquad (\ln^2 s \quad \text{and} \quad \ln s). \tag{5.14c}$$

It is noteworthy that this cancellation requires a nonzero topological charge $q = 2$, much as in the two-dimensional Schwarzschild case, and again unlike the Fulling-Rindler case in either two or four dimensions. The $q = 2$ condition may be understood from the fact that it cancels the logarithmic divergence of $\ln f$ in the second form of (5.8), leaving however a logarithmic singularity and nontrivial topological charge at both the origin and infinity.

Also noteworthy is that the auxiliary field configurations which lead to diverging stress tensors on the horizon have *finite action* (3.25). This is clear from the fact that

$$\Box\varphi \to \frac{q-2}{4M^2}\ln\left(\frac{r}{2M} - 1\right) + \text{const} \quad \text{as} \quad r \to 2M. \tag{5.15}$$

Hence both $(\Box\varphi)^2$ and $(\Box\varphi)(\Box\psi)$ are proportional to $\ln^2(r/2M - 1)$, which is integrable with respect to the measure $r^2 dr$ as $r \to 2M$, and the integrals in (3.26) converge at the horizon for the general solution (5.10) of the auxiliary field equations. This is not the case at infinity, unless the four constants c_∞, d_∞, q and q' are all required to vanish. From (5.8) in this case the auxiliary fields fall off at large distances $r \gg 2M$ similarly to the scalar potential of Newtonian gravity, or Brans-Dicke theory [40]. Because of the very different way they couple to spacetime curvature, the auxiliary fields of $S_{\rm anom}$ give only a very weak long-range interaction between massive bodies, that falls off very rapidly with distance (at least as fast as r^{-7}). Thus from consideration of the general form of the anomalous action and auxiliary fields in the vicinity of the Schwarzschild horizon and their falloff at infinity, we find no *a priori* justification for excluding generic states with stress-energy tensors that grow without bound as the horizon is approached. In such states the backreaction of the stress energy on the geometry will be substantial in this region and lead to qualitatively new macroscopic effects on the horizon scale.

If instead of requiring rapid falloff at infinity we require regularity of the stress energy on the horizon, i.e. conditions (5.14), we obtain a four parameter restriction of the eight parameters in the auxiliary fields of the form (5.8) and (5.10). These are the conditions appropriate to the Hartle-Hawking-Israel and Unruh states [34,41]. Because of the linear time dependence (5.10) we also obtain a no-zero flux $\langle T_t{}^r\rangle$ in general. For the Hartle-Hawking state the vanishing of this flux gives a fifth condition, viz.

$$b(qp' + q'p) + b'pq = 0, \qquad (\text{zero flux}) \tag{5.16}$$

on the parameters of the auxiliary fields. Of the remaining three parameters two can be fit by the finite values of $\langle T_t{}^t\rangle$ on the horizon and at infinity, respectively. This leaves one final parameter free to adjust for a best fit of the form of all three components of the stress-energy tensor at intermediate points. Alternatively, the conformal transformation (5.5) could be used to fix the leading order behavior of the auxiliary field φ near the horizon, i.e. $c_H = 1 = \pm p$. Then (5.16) fixes p' which with $q = q'$ becomes redundant with the first of the conditions (5.14), and (5.14b) fixes d_H. Hence only c_∞ and d_∞ are left undetermined for a finite stress energy, and no free parameters at all are left if the values of $\langle T_t{}^t\rangle$ on the horizon and at infinity are fixed. Using the first method with one free parameter, the results for the $\langle T_t{}^t\rangle$ component of the scalar, Dirac spin $\frac{1}{2}$ and electromagnetic fields are shown in Fig. 4–6 (solid curves), for typical values of the parameters. The data points plotted in Figs. 4 and 5 are the numerical results of the direct quantum field theory calculations for spin 0 and $\frac{1}{2}$ of Refs. [42,43] respectively. In the spin 1 case the dashed curve of Fig. 6 represents the numerical results of Ref. [44] in an accurate analytic polynomial fit to their data provided by the same authors.

We note that the energy density in the Hartle-Hawking state computed from the general form of the anomaly is not in especially good agreement with the direct numerical calculations from the underlying quantum field theory. This is in accordance with the general discussion of the

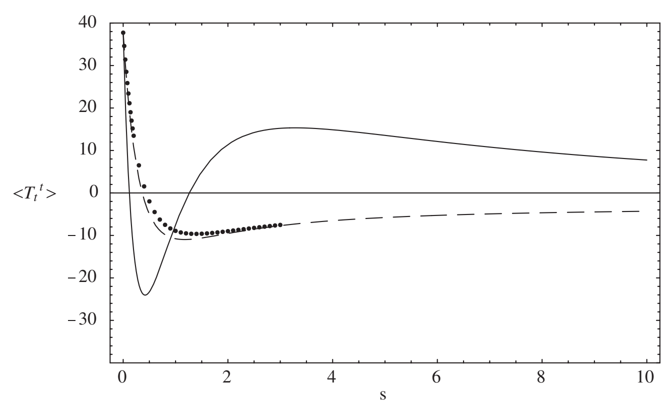

FIG. 4. The expectation value $\langle T_t{}^t\rangle$ of a conformal scalar field in the Hartle-Hawking state in Schwarzschild spacetime as a function of $s = \frac{r-2M}{M}$ in units of $\pi^2 T_H^4/90$. The solid curve is Eq. (3.39) with $c_\infty = 0.035$, $c_H = -1.5144$, $p = 1.5144$, $q = q' = 2$, $d_\infty = 1.0262$, $d_H = 1.0096$, $p' = -1.0096$, the dashed curve is the analytic approximation of [45] and the data points are the numerical results of [42].

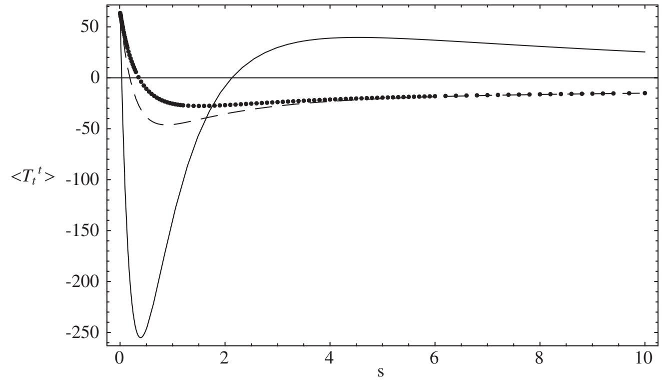

FIG. 5. The expectation value $\langle T_t{}^t \rangle$ of a massless Dirac field in the Hartle-Hawking state in Schwarzschild spacetime. The solid curve is Eq. (3.39) with $c_\infty = 0.035$, $c_H = -1.571 = -p$, $q = q' = 2$, $d_\infty = 0.6059$, $d_H = 0.6109 = -p'$, the dashed curve is the analytic approximation of [45] and the data points are the numerical results of [43].

exact and low energy effective actions of the previous section. In states which have no divergences on the horizon there is no particular reason to neglect the Weyl invariant terms S_{inv} in the exact effective action, and the stress-energy tensor it produces would be expected to be comparable in magnitude to that from S_{anom}. In this case of bounded stress tensors, the contributions from both S_{anom} and S_{inv} are both of order M^{-4} and negligibly small on macroscopic scales. However in states such as the previous Boulware example, the diverging behavior of the stress tensor near the horizon is captured accurately by the terms in (3.39) arising from the anomaly, which have the same generically diverging behaviors as the quantum field theory expectation value $\langle T_a{}^b \rangle$. Even in states for which the expectation value remains regular on the horizon, the auxiliary field anomalous stress energy can be matched accurately to the finite value of $\langle T_a{}^b \rangle$ on the horizon with suitable choice of integration constants in (5.8).

For comparison plotted also in Figs. 4–6 is the analytic approximation to $\langle T_t{}^t \rangle$ of Frolov and Zel'nikov (FZ) [45]. This latter approximation gives an improvement over that of [37–39] since it permits adjustment of the value of the stress energy on the horizon to the numerical data, rather than fixing it to an incorrect value. Once that is done, the FZ approximation automatically gives the correct asymptotic value of the stress energy in the Hartle-Hawking state at infinity, but contains no additional free parameters. For the spin $\frac{1}{2}$ case the FZ approximation is a significant improvement over that of PBO which disagrees with the numerics even in the sign of $\langle T_t{}^t \rangle$ on the horizon. Although it is a noticeably better fit to the numerical data than (3.39), the FZ approximation, like that of PBO, relies on the existence of a timelike Killing field and both are therefore highly specialized to certain spacetime backgrounds such as the Schwarzschild black hole. In contrast the stress-energy tensor (3.39) arising from S_{anom} in the low energy EFT does not require the existence of a timelike Killing field, and can be computed in principle for arbitrary spacetimes.

For the time asymmetric Unruh state the zero flux condition (5.16) is replaced by the properly normalized finite flux condition

$$\langle T_t{}^r \rangle|_U = -\frac{L}{4\pi r^2}, \tag{5.17}$$

where the luminosity, L is given by

$$L = \frac{\pi}{M^2}(bqp' + bq'p + b'pq) \tag{5.18}$$

in terms of the auxiliary field parameters. Otherwise one can proceed as in the previous Hawking-Hartle state by fixing two of the three remaining auxiliary field parameters by the values of $\langle T_t{}^t \rangle$ on the future horizon and at infinity

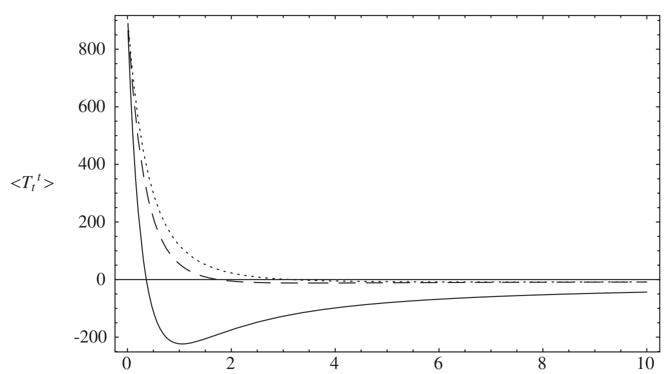

FIG. 6. The expectation value $\langle T_t^t \rangle$ of the electromagnetic field in the Hartle-Hawking state in Schwarzschild spacetime. The solid curve is Eq. (3.39) with $c_\infty = -0.0237$, $c_H = 1.0584 = p$, $q = q' = 2$, $d_\infty = -0.2716$, $d_H = 0.7644 = p'$, the dashed curve is the analytic approximation of [45] and the dotted curve is the accurate analytic fit of [44] to the numerical results of the same authors.

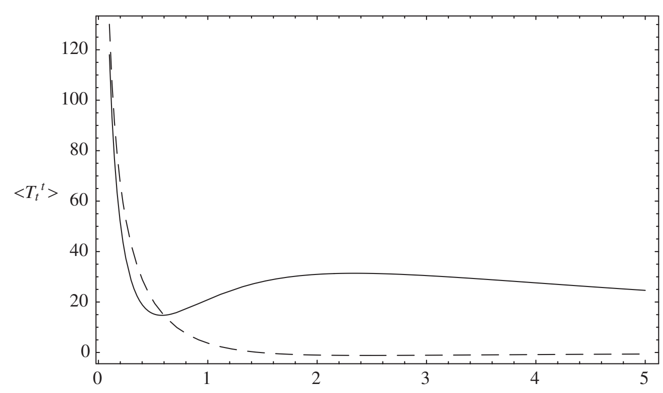

FIG. 7. The expectation value $\langle T_t{}^t \rangle$ of a massless, conformal field in the Unruh state in Schwarzschild spacetime. The solid curve is Eq. (3.39) with $c_\infty = 0$, $c_H = -1.6500$, $p = 1.9192$, $q = q' = 2$, $d_\infty = 1.4441$, $d_H = 1.3244$, $p' = -1.0552$, and the dashed curve is the polynomial approximation of [49] which is an accurate fit to the numerical results of [47].

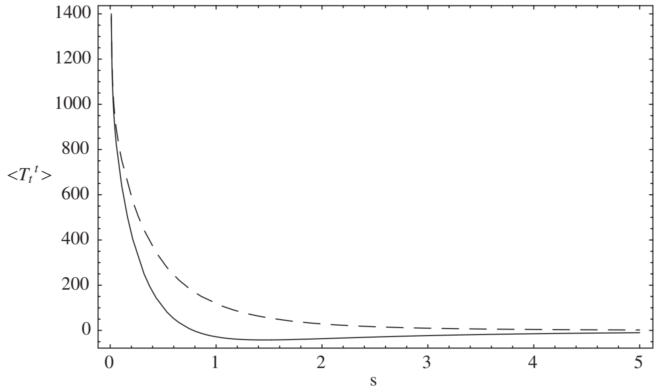

FIG. 8. The expectation value $\langle T_t{}^t \rangle$ of the electromagnetic field in the Unruh state in Schwarzschild spacetime. The solid curve is Eq. (3.39) with $c_\infty = 0$, $c_H = 0.5594$, $p = -0.4984$, $q = q' = 2$, $d_\infty = -0.2595$, $d_H = 0.4125$, $p' = -.3514$, and the dashed curve is the polynomial approximation of [50] which is an accurate fit to the numerical results of [48].

which are known numerically for the scalar and electromagnetic fields in the Unruh state from the work of Refs. [46–48]. This leaves again one parameter free to fit. The results of that fit for the conformally coupled scalar and electromagnetic fields in the Unruh state are shown in Figs. 7 and 8 (solid lines). The dashed curves are accurate polynomial fits of [49,50] to the numerical data of [47,48] for the spin 0 and 1 fields, respectively. There is at present no numerical calculation of the stress-energy tensor of the massless Dirac field in the Unruh state with which to compare.

B. Other spacetimes with horizons

The foregoing analysis of the anomalous action and stress tensor in the Schwarzschild background may be extended to other spacetimes with metrics of the form (4.17). The electrically charged Reisner-Nordstrom black hole for which $f(r) = 1 - 2M/r + e^2/r^2$ is an example in which Eqs. (5.1), (5.2), (5.3), and (5.4) may also be applied. The extreme Reisner-Nordstrom geometry in which $e = M$ is particularly interesting. In that case the two zeroes of $f(r)$ at $r_\pm$ coincide and f vanishes quadratically. The possible singularities of the stress energy on the extreme Reissner-Nordstrom horizon are correspondingly more severe, behaving like $f^{-2} \sim s^{-4}$. Nevertheless a power series expansion near $s = 0$ shows that it is possible to obtain a regular stress energy on the horizon with the anomalous tensors even in the extreme Reisner-Nordstrom case with appropriate choice of auxiliary field integration constants, in agreement with the direct numerical results of [43]. This shows that the anomaly indeced effective action can reproduce the correct behaviors of the quantum stress tensor also in this case, in contrast to the approximation of [37–39] which predicts a divergent stress energy on the horizon of an extreme Reisner-Nordstrom black hole as the only possibility. As it is not possible to solve the auxiliary field equations analytically in the black hole geometries when $e \neq 0$, and a numerical solution is required, we postpone a full discussion of these results to a future publication [51].

Finally, this study of the anomalous effective action, auxiliary fields, and stress energy could be extended to rotating Kerr black hole geometries. The previous analytical approximations of [37–39,45] do not apply to this case. However there is no problem of principle in computing the stress tensor (3.39) in stationary spacetimes possessing only axial symmetry. Because of the lower symmetry, solutions of the auxiliary field Eqs. (3.17) which are functions of both r and θ can be sought. Hence the solutions characterizing macroscopic stationary quantum states with axial symmetry are considerably richer than the Schwarzschild case. However, a simple spherically symmetric homogeneous solution of the second order equation $\Box\varphi = 0$ which generalizes the $\ln f$ solution found in the the Schwarzschild and de Sitter cases is

$$\varphi(r) \propto \ln\left(\frac{r - r_+}{r - r_-}\right), \tag{5.19}$$

for

$$r_\pm = M \pm \sqrt{M^2 - a^2}, \tag{5.20}$$

with a the angular momentum per unit mass of the Kerr solution. Since (5.19) diverges logarithmically on the horizon at $r = r_+$, similar conformal behavior of the stress tensor as $r \to r_+$ is expected as that found in the Schwarzschild case, which clearly merits a separate investigation.

VI. CONCLUSIONS

The suggestion that general relativity may be regarded as a low energy effective field theory, comparable to the Fermi theory of weak decays or the chiral EFT of pions in nuclei has been made previously [2]. In this paper we have considered the EFT framework for gravity in arbitrary curved spacetimes, unrestricted by the weak field expansion around flat space considered in [2]. The validity of the EFT approach requires only that the local curvature invariants be small compared to the Planck scale.

The EFT expansion in inverse powers of the high energy cutoff scale of order $M_{\rm Pl}$ requires a systematic classification of relevant and irrelevant operators in the general coordinate invariant effective action. According to this classification, based on the behavior under global Weyl rescalings [10], the classical Einstein-Hilbert action should be supplemented with certain well-defined nonlocal terms associated with the trace anomaly of massless fields. These additional terms are nonlocal in terms of the metric, c.f. (3.13) but because of their logarithmic scaling with distance, they are nevertheless relevant in the infrared. The anomalous terms are unsuppressed by any inverse power of the ultraviolet cutoff scale $M_{\rm Pl}$, and do not decouple for $E \ll M_{\rm Pl}$. Hence they are required to complete the Wilson effective action (3.16) for gravitational physics at macroscopic distances $L \gg L_{\rm Pl}$. Their addition to the classical Einstein-Hilbert action constitutes a nontrivial infrared modification of general relativity, which is completely consistent with the equivalence principle.

One may ask if there are any other modifications of classical general relativity at low energies that are consistent with general covariance and EFT principles. The complete classification of the terms in (3.15) into just three categories means that all possible infrared relevant terms in the low energy EFT, which are not contained in $S_{\rm local}$ of (3.3) must fall into $S_{\rm anom}$, i.e. they must correspond to nontrivial cocycles of the local Weyl group. The Weyl invariant terms, $S_{\rm inv}$ in the exact effective action (3.15) are by definition insensitive to rescaling of the metric at large distances. Hence these (generally quite nonlocal) terms do not give rise to infrared relevant terms in the Wilson effective action for low energy gravity.

Furthermore, the form of the nontrivial cocycles in $S_{\rm anom}$ is severely restricted by the locality and general covariance of quantum field theory. The ultraviolet divergences in the stress-energy tensor of quantum fields in curved spacetime are purely local. It is these divergences when renormalized consistently with covariant conservation of the local operator $T_{ab}(x)$ that give rise to the purely local form of the trace anomaly. Since all the local gauge invariant terms with mass dimension four matched to the physical dimension of spacetime are easily cataloged, the only nontrivial terms in $S_{\rm anom}$ at low energies which can arise from short distance renormalization effects are exactly those generated by the known form of the local trace anomaly (3.5). Decoupling fails in local quantum field theory only in the very narrow and well-defined way dictated by local anomalies, and these uniquely determine the nonlocal additions to the effective action, up to any contributions from $S_{\rm inv}$, which in any case have negligibly small effect on very large distance scales. The form of the effective action $S_{\rm anom}$ at macroscopic distances $L \gg L_{\rm Pl}$ is not expected to change substantially even if the condition of strict locality of the underlying quantum theory is relaxed or replaced eventually by a more fundamental, microscopic description of gravitational interactions at very large mass scales of order $M_{\rm Pl}$. If this were not the case, then the classical Einstein theory could be overwhelmed by all sorts of nonlocal quantum corrections from unknown microscopic physics, and would lose all predictive power for macroscopic gravitational phenomena. Instead, under the defining assumptions of general covariance and the EFT hypothesis of decoupling of physics associated with massive degrees of freedom, the infrared modification of Einstein's theory specified by (3.16), (3.25), and (3.26) is tightly constrained and becomes essentially unique.

The nonlocal anomalous action may be brought into the local form, by the introduction two scalar auxiliary fields, φ and ψ, one for each of the nontrivial cocycles of the Weyl group. These auxiliary scalar fields satisfy the fourth order massless wave Eqs. (3.17), and are only very weakly coupled to ordinary matter and radiation. In fact, in the flat space limit they become noninteracting fields at length scales $L \gg L_{\rm pl}$, and for that reason may have easily escaped direct detection. The auxiliary fields are new local scalar degrees of freedom of low energy gravity, not contained in classical general relativity.

The anomalous terms in the effective action generate two new conserved tensors (3.41) and (3.42), local in the auxiliary fields, which act as a source for Einstein's equations *in vacuo*. Thus the presence of the additional terms in the EFT of gravity may be detectable indirectly through their effects on the geometry of spacetime. Since they arise from the long-range effects of the fluctuations of massless fields, and require boundary conditions for their complete specification, these additional terms carry information about the global macroscopic effects of quantum matter not contained in the classical Einstein theory. In the approximation where the auxiliary fields are treated as classical potentials, they may be viewed as scalar order parameter fields or *condensates*, incorporating the effects of macroscopic quantum coherence and entanglement naturally into low energy gravity.

Because of the conformal properties of spacetime horizons, the anomalous terms in the effective action and stress energy become important in their vicinity, characterizing the allowed behaviors of the stress tensor there. Near an horizon the most important terms in the effective action (3.16) are those which scale positively in the ultraviolet short distance limit under conformal rescalings, while nevertheless keeping local curvature invariants fixed and small compared to the Planck scale (in order for the EFT approach to remain valid). Only the $S_{\rm anom}$ term in (3.16) has this property, scaling logarithmically in both the ultraviolet and infrared limits. Because the anomalous terms depend nonlocally on the geometry through the two auxiliary scalar fields φ and ψ, they can be significant near horizons even for very much sub-Planckian local curvatures, where the EFT description should continue to apply. Indeed although local curvature invariants may remain small on horizons, the auxiliary scalar fields depend in general on coordinate invariant global quantities such as $\ln(-K^a K_a)$, which diverge on the horizon when the spacetime possesses a timelike Killing field K^a which becomes null there. Whether or not this divergence actually occurs in the stress energy of a given state is a matter of boundary conditions fixed globally over the entire spacetime. We investigated these macroscopic topological effects in both two and four dimensions. The existence of a topological current (2.20) or (3.19), and associated Noether charge are characteristic of global effects of the quantum state. The generic singular behavior of the stress energy on horizons is thereby characterized by spacetime invariant scalar order parameters in the low energy EFT, and the possible divergence of the stress-energy tensor on horizons acquires a coordinate invariant description in terms of the auxiliary scalar fields φ and ψ.

In the semiclassical EFT framework where both the metric and auxiliary scalars are treated as classical fields, Eqs. (3.17) may be solved classically and provide a new method to calculate the approximate quantum stress tensor of conformal matter by classical means, bypassing the standard but cumbersome procedure of field quantization and renormalization of the quantum stress tensor. When states for which the stress energy does not diverge on the horizon are considered, approximations such as those in [37–39,45], developed specifically for static backgrounds with a timelike Killing field generally yield a more accurate approximation to the expectation value of the stress-energy tensor. We conclude that in states with a stress energy that remains finite on the horizon, the low energy Wilson effective action (3.16) is not a particularly good approximation to the exact quantum effective action (3.15). There is no reason why the $S_{\rm inv}$ terms in the exact effective action should be small compared to the $S_{\rm anom}$ terms in such states. However, in such states all macroscopic quantum effects remain strictly bounded (and negligibly small) in any case.

On the other hand when the scalar potentials diverge on the horizon, the associated terms in the stress energy yield a good approximation to the vacuum expectation value of the stress-energy tensor in its vicinity. Compared to previous analytic approximations, (3.39) with (3.17) has the considerable advantage of being completely general, and giving all possible allowed singular behaviors of the stress tensor on horizons, regardless of special symmetries of the background spacetime. Even when the stress energies diverge, and the effects of the modifications to the classical theory may become quite significant, they are characterized by a finite effective action, suggesting that such states should not be excluded *a priori* from physical consideration.

The study of the effective stress-energy tensor (3.39) in several special fixed spacetime backgrounds presented here clearly only begins a systematic treatment of their many possible effects. In addition to extension to other backgrounds, particularly those with horizons, and a study of the scalar gravitational waves and long-range effects the scalar potentials may generate, a rich variety of new solutions are to be expected when the T_{ab} of Eqs. (3.39) is used as fully dynamical sources for Einstein's equations. How these effects can be used to test the new theory against astrophysical and/or cosmological observations is a most important issue remaining for future investigation. In the case of divergent behavior of the stress energy near the classical horizon, a significant deviation of the geometry from the vacuum solutions of the purely classical Einstein equations should be expected. This raises the distinct possibility of macroscopic quantum effects associated with the trace anomaly qualitatively changing the global structure of both classical black hole and cosmological solutions [52]. The existence of an abelian conserved Noether charge and macroscopic scalar condensates in the EFT of gravity invites comparison with the conserved number current of nonrelativistic many-body systems at the microscopic level [53].

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APPENDIX: STRESS ENERGY IN STATIC, SPHERICAL, VACUUM SPACETIMES

We give in this appendix the explicit forms for the diagonal components of the two conserved tensors $E_a{}^b$ and $F_a{}^b$, c.f. Eqs. (3.41) and (3.42), for static, spherically symmetric spacetimes with line elements of the form (4.17). The auxiliary fields are assumed to be of the form,

$$\varphi = \varphi(r) + 2p\kappa t, \tag{A1a}$$

$$\psi = \psi(r) + 2p'\kappa t, \tag{A1b}$$

with κ the surface gravity at the horizon where f vanishes. All components of the two tensors were generated by MathTensor programs [54], which were separately checked for satisfying the conservation and trace conditions (3.43).

The time components of the two tensors so obtained are

$$E_t{}^t = \frac{f^2}{6}(2\varphi'''\varphi' - \varphi''^2) + \frac{4f}{3}\left(f' - \frac{f}{r}\right)\varphi''\varphi' + \frac{1}{3}\left(2ff'' + f'^2 - \frac{2f(1+2f)}{r^2}\right)\varphi'^2 + \frac{2p^2\kappa^2 f'^2}{f^2} - \frac{8p^2\kappa^2}{3f}\left(f'' + \frac{f'}{r} + \frac{1-f}{r^2}\right) - \frac{ff'}{3}\varphi''' - \frac{2}{9}\left[\frac{ff''}{2} + 3f'^2 + \frac{8ff'}{r} + \frac{14f(1-f)}{r^2}\right]\varphi'' - \frac{2}{9}\left[\frac{ff'''}{2} + 2f'f'' + \frac{9ff'' + 8f'^2}{r} + \frac{(5-17f)f'}{r^2} + \frac{6f(1-f)}{r^3}\right]\varphi' + \frac{4}{9}\left[\frac{ff'''' + f'f'''}{2} + \frac{2f'f'' + 3ff'''}{r} + \frac{f''(4f-3) + 2f'^2}{r^2} + \frac{2f'(1-2f)}{r^3} - \frac{2f(1-f)}{r^4}\right], \tag{A2}$$

and

$$F_t{}^t = \frac{f^2}{3}(\varphi'''\psi' + \varphi'\psi''' - \varphi''\psi'') + \frac{4f}{3}\left(f' - \frac{f}{r}\right)(\varphi''\psi' + \varphi'\psi'') + \frac{2}{3}\left[2ff'' + f'^2 - \frac{2f(1+2f)}{r^2}\right]\varphi'\psi' + \frac{4pp'\kappa^2 f'^2}{f^2}$$
$$- \frac{16pp'\kappa^2}{3f}\left(f'' + \frac{f'}{r} + \frac{1-f}{r^2}\right) - \frac{ff'}{3}\psi''' - \frac{2}{9}\left[\frac{ff''}{2} + 3f'^2 + \frac{8ff'}{r} + \frac{14f(1-f)}{r^2}\right]\psi'' - \frac{2}{9}\left[\frac{ff'''}{2} + 2f'f''\right.$$
$$\left. + \frac{9ff'' + 8f'^2}{r} + \frac{(5-17f)f'}{r^2} + \frac{6f(1-f)}{r^3}\right]\psi' + \frac{4f}{3}\left[\frac{f''}{2} - \frac{f'}{r} - \frac{1-f}{r^2}\right]\varphi'' + \frac{2}{3}\left[2ff''' + \frac{f'f''}{2} + \frac{ff'' - f'^2}{r}\right.$$
$$\left. - \frac{f'(1+f)}{r^2} - \frac{2f(1-f)}{r^3}\right]\varphi' + \frac{2}{3}\left[ff'''' + \frac{f'f'''}{2} - \frac{f''^2}{4} + \frac{f'f'' + 3ff'''}{r} - \frac{ff'' + f'^2}{r^2} + \frac{2ff'}{r^3} + \frac{1-f^2}{r^4}\right]\varphi$$
$$+ \frac{4}{9}\left[\frac{f''^2}{4} - \frac{f'f''}{r} + \frac{f'^2 - f'' + ff''}{r^2} + \frac{2f'(1-f)}{r^3} + \frac{(1-f)^2}{r^4}\right], \tag{A3}$$

where primes denote differentiation with respect to r.

The radial pressures of the two stress tensors are

$$E_r{}^r = -f^2\varphi'\varphi''' + \frac{f^2}{2}\varphi''^2 - \frac{4f}{3}\left(f' + \frac{f}{r}\right)\varphi'\varphi'' + \frac{1}{3}\left[-2ff'' + f'^2 - \frac{8ff'}{r} + \frac{2f(1+2f)}{r^2}\right]\varphi'^2 - \frac{2\kappa^2p^2f'^2}{3f^2}$$
$$+ \frac{8\kappa^2p^2}{3rf}\left(f' + \frac{1-f}{r}\right) + \frac{f}{3}\left(f' + \frac{4f}{r}\right)\varphi''' + \frac{1}{3}\left(-ff'' + 2f'^2 + \frac{8ff'}{r} + \frac{8f^2}{r^2}\right)\varphi''$$
$$+ \left[\frac{ff'''}{3} + \frac{2ff''}{r} + \frac{2(3f-1)}{r^2}\left(f' - \frac{2}{3}\frac{f}{r}\right)\right]\varphi', \tag{A4}$$

and

$$F_r{}^r = -f^2(\varphi'\psi''' + \psi'\varphi''' - \varphi''\psi'') - \frac{4f}{3}\left(f' + \frac{f}{r}\right)(\varphi'\psi'' + \psi'\varphi'') - \frac{4\kappa^2pp'f'^2}{3f^2} + \frac{16\kappa^2pp'}{3rf}\left(f' + \frac{1-f}{r}\right)$$
$$+ \frac{2}{3}\left[f'^2 - 2ff'' - \frac{8ff'}{r} + \frac{2f(1+2f)}{r^2}\right]\varphi'\psi' + \frac{f}{3}\left(f' + \frac{4f}{r}\right)\psi''' + \frac{1}{3}\left(-ff'' + 2f'^2 + \frac{8ff'}{r} + \frac{8f^2}{r^2}\right)\psi''$$
$$+ \left[\frac{ff'''}{3} + \frac{2ff''}{r} + \frac{2(3f-1)f'}{r^2} + \frac{4f(1-3f)}{3r^3}\right]\psi' + \frac{1}{3}\left[f'f'' - \frac{2(ff'' + f'^2)}{r} + \frac{2(3f-1)f'}{r^2} + \frac{4f(1-f)}{r^3}\right]\varphi'$$
$$+ \frac{1}{3}\left[f'f''' - \frac{f''^2}{2} + \frac{2(f'f'' - ff''')}{r} - \frac{2(ff'' + f'^2)}{r^2} + \frac{4ff'}{r^3} + \frac{2(1-f^2)}{r^4}\right]\varphi. \tag{A5}$$

Finally the tangential pressures of the two tensors are

$$E_\theta{}^\theta = \frac{f^2}{3}\left(\varphi'\varphi''' - \frac{\varphi''^2}{2}\right) + \frac{4f^2}{3r}\varphi'\varphi'' + \frac{f'}{3}\left(-f' + \frac{4f}{r}\right)\varphi'^2 + \frac{2\kappa^2p^2}{3f^2}(2ff'' - f'^2) - \frac{2f^2}{3r}\varphi''' + \frac{2f}{9}\left(f'' - \frac{2f'}{r}\right.$$
$$\left. + \frac{7-13f}{r^2}\right)\varphi'' + \frac{2}{9}\left[-\frac{ff'''}{2} + f'f'' + \frac{4f'^2}{r} + \frac{(7-22f)f'}{r^2} + \frac{6f^2}{r^3}\right]\varphi' + \frac{4}{9}\left[\frac{ff'''' + f'f'''}{2} + \frac{3ff''' + 2f'f''}{r}\right.$$
$$\left. + \frac{4ff'' - 3f'' + 2f'^2}{r^2} + \frac{2f'(1-2f)}{r^3} + \frac{2f(f-1)}{r^4}\right], \tag{A6}$$

and

$$F_\theta{}^\theta = \frac{f^2}{3}(\varphi'\psi''' + \psi'\varphi''' - \varphi''\psi'') + \frac{4f^2}{3r}(\psi'\varphi'' + \varphi'\psi'') + \frac{2f'}{3}\left(-f' + \frac{4f}{r}\right)\varphi'\psi' + \frac{4\kappa^2 pp'}{3f^2}(2ff'' - f'^2) - \frac{2f^2}{3r}\psi''' + \frac{2f}{9}\left(f'' - \frac{2f'}{r} + \frac{7-13f}{r^2}\right)\psi'' + \frac{2}{9}\left[-\frac{ff'''}{2} + f'f'' + \frac{4f'^2}{r} + \frac{(7-22f)f'}{r^2} + \frac{6f^2}{r^3}\right]\psi' + \frac{f}{3}\left[-f'' + \frac{2f'}{r} + \frac{2(1-f)}{r^2}\right]\varphi'' + \frac{2}{3}\left[-ff''' - \frac{f'f''}{2} + \frac{f'^2}{r} + \frac{(1-f)f'}{r^2}\right]\varphi' + \frac{2}{3}\left[-\frac{(ff'''' + f'f''')}{2} + \frac{f''^2}{4} - \frac{(ff''' + f'f'')}{r} + \frac{ff'' + f'^2}{r^2} - \frac{2ff'}{r^3} + \frac{f^2-1}{r^4}\right]\varphi + \frac{4}{9}\left[\frac{f''^2}{4} - \frac{f'f''}{r} + \frac{(f-1)f'' + f'^2}{r^2} + \frac{2(1-f)f'}{r^3} + \frac{(f-1)^2}{r^4}\right]. \tag{A7}$$

The flux components $E_t{}^r$ and $F_t{}^r$ are determined by the conservation equation

$$\frac{\partial}{\partial r}T_t{}^r + \frac{2}{r}T_t{}^r = 0 \tag{A8}$$

to be proportional to $1/r^2$ with a proportionality constant given by the luminosity L as in (5.17) of the text. For the general surface gravity κ, $T_t{}^r = -L/4\pi r^2$ with

$$L = 16\pi\kappa^2(bpq' + bp'q + b'pq). \tag{A9}$$

[1] See e.g. J. F. Donoghue, E. Golowich, and B. R. Holstein, *Dynamics of the Standard Model* (Cambridge University Press, Cambridge, England, 1992), Chap. IV, Secs. VI-5, VII-3, and references therein.
[2] J. F. Donoghue, Phys. Rev. D **50**, 3874 (1994).
[3] S. L. Adler, Phys. Rev. **177**, 2426 (1969); J. S. Bell and R. Jackiw, Nuovo Cimento A **60**, 47 (1969).
[4] W. A. Bardeen, H. Fritzsch, and M. Gell-Mann, in *Scale and Conformal Symmetry in Hadron Physics*, edited by R. Gatto (Wiley, New York, 1973).
[5] J. Wess and B. Zumino, Phys. Lett. B **37**, 95 (1971); E. Witten, Nucl. Phys. **B156**, 269 (1979); **B223**, 422 (1983); W. A. Bardeen and B. Zumino, Nucl. Phys. **B244**, 421 (1984).
[6] T.-P. Cheng and L.-F. Li, *Gauge Theory of Elementary Particle Physics* (Oxford University Press, Oxford, England, 1984), Sec. 6.2, pp. 173–182; S. Weinberg, *The Quantum Theory of Fields, Vol. II* (Cambridge University Press, Cambridge, England, 1995), Sec. 22.2, pp. 362–370.
[7] G. 't Hooft, in *Recent Developments in Gauge Theories*, 1979 NATO Cargese Institute Lectures (Plenum, New York, 1980); See also S. Weinberg, Ref. [6], Sec. 22.5, pp. 389–396.
[8] D. Capper and M. Duff, Nuovo Cimento A **23**, 173 (1974); Phys. Lett. A **53**, 361 (1975); S. Deser, M. Duff, and C. J. Isham, Nucl. Phys. **B111**, 45 (1976); M. Duff, Nucl. Phys. **B125**, 334 (1977); **B234**, 269 (1984); Classical Quantum Gravity **11**, 1387 (1994).
[9] N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, England, 1982), and references therein.
[10] P. O. Mazur and E. Mottola, Phys. Rev. D **64**, 104022 (2001).
[11] I. Antoniadis, P. O. Mazur, and E. Mottola, Nucl. Phys. **B388**, 627 (1992).
[12] V. G. Knizhnik, A. M. Polyakov, and A. B. Zamolodchikov, Mod. Phys. Lett. A **3**, 819 (1988); F. David, Mod. Phys. Lett. A **3**, 1651 (1988); J. Distler and H. Kawai, Nucl. Phys. **B321**, 509 (1989).
[13] See e.g. J. Ambjørn, B. Durhuus, and T. Jónsson, *Quantum Geometry: A Statistical Field Theory Approach* (Cambridge University Press, Cambridge, England, 1997); J. Ambjørn, M. Carfora, and A. Marzuoli, *The Geometry of Dynamical Triangulations* (Springer, Berlin, 1997).
[14] S. Catterall and E. Mottola, Phys. Lett. B **467**, 29 (1999); Nucl. Phys. B, Proc. Suppl. **83**, 748 (2000).
[15] S. S. Chern and J. Simons, Ann. Math. **99**, 48 (1974).
[16] T. Eguchi, P. B. Gilkey, and A. J. Hanson, Phys. Rep. **66**, 213 (1980).
[17] R. Bott, Mich. Math. J. **14**, 231 (1967); M. F. Atiyah and I. M. Singer, Ann. Math. **87**, 546 (1968); M. F. Atiyah and R. Bott, Ann. Math. **88**, 451 (1968).
[18] S. A. Fulling, Phys. Rev. D **7**, 2850 (1973).
[19] G. W. Gibbons and M. J. Perry, Proc. R. Soc. A **358**, 467 (1978); J. S. Dowker and G. Kennedy, J. Phys. A **11**, 895 (1978).
[20] S. M. Christensen and S. A. Fulling, Phys. Rev. D **15**, 2088 (1977).
[21] R. J. Riegert, Phys. Lett. B **134**, 56 (1984); E. S. Fradkin and A. A. Tseytlin, Phys. Lett. B **134**, 187 (1984).
[22] I. Antoniadis and E. Mottola, Phys. Rev. D **45**, 2013 (1992).
[23] A. O. Barvinsky, A. G. Mirzabekian, and V. V. Zhytnikov, gr-qc/9510037.
[24] S. D. Odintsov, Z. Phys. C **54**, 531 (1992); I. Antoniadis and S. D. Odintsov, Phys. Lett. B **343**, 76 (1995).
[25] K. G. Wilson, Rev. Mod. Phys. **47**, 773 (1975).
[26] R. Balbinot, A. Fabbri, and I. L. Shapiro, Phys. Rev. Lett. **83**, 1494 (1999); Nucl. Phys. **B559**, 301 (1999).
[27] I. Antoniadis, P. O. Mazur, and E. Mottola, Phys. Rev. D

55, 4756; **55**, 4770 (1997).
[28] P.R. Anderson, C. Molina-Paris, and E. Mottola, Phys. Rev. D **67**, 024036 (2003).
[29] P. Candelas and D. Deutsch, Proc. R. Soc. A **354**, 79 (1977).
[30] L.S. Brown and J.P. Cassidy, Phys. Rev. D **16**, 1712 (1977); P. Candelas and J.S. Dowker, Phys. Rev. D **19**, 2902 (1979).
[31] P.R. Anderson, W. Eaker, S. Habib, C. Molina-Paris, and E. Mottola, Phys. Rev. D **62**, 124019 (2000).
[32] I. Antoniadis, P.O. Mazur, and E. Mottola, Phys. Lett. B **394**, 49 (1997).
[33] D.G. Boulware, Phys. Rev. D **11**, 1404 (1975); **13**, 2169 (1976).
[34] W.G. Unruh, Phys. Rev. D **14**, 870 (1976).
[35] A.O. Barvinsky, V.P. Frolov, and A.I. Zel'nikov, Phys. Rev. D **51**, 1741 (1995); I. Sachs and S.N. Solodukhin, Phys. Rev. D **64**, 124023 (2001).
[36] B.P. Jensen, J.G. Mc Laughlin, and A.C. Ottewill, Phys. Rev. D **45**, 3002 (1992).
[37] D.N. Page, Phys. Rev. D **25**, 1499 (1982).
[38] M.R. Brown and A.C. Ottewill, Phys. Rev. D **31**, 2514 (1985); **34**, 1776 (1986).
[39] M.R. Brown, A.C. Ottewill, and D.N. Page, Phys. Rev. D **33**, 2840 (1986).
[40] C. Brans and R.H. Dicke, Phys. Rev. **124**, 925 (1961).
[41] J.B. Hartle and S.W. Hawking, Phys. Rev. D **13**, 2188 (1976); W. Israel, Phys. Lett. A **57**, 107 (1976).
[42] K.W. Howard, Phys. Rev. D **30**, 2532 (1984).
[43] E.D. Carlson, W.H. Hirsch, B. Obermayer, P.R. Anderson, and P.B. Groves, Phys. Rev. Lett. **91**, 051301 (2003).
[44] B.P. Jensen and A.C. Ottewill, Phys. Rev. D **39**, 1130 (1989).
[45] V.P. Frolov and A.I. Zel'nikov, Phys. Rev. D **35**, 3031 (1987).
[46] D.N. Page, Phys. Rev. D **13**, 198 (1976).
[47] T. Elster, Phys. Lett. A **94**, 205 (1983).
[48] B.P. Jensen, J.G. Mc Laughlin, and A.C. Ottewill, Phys. Rev. D **43**, 4142 (1991).
[49] M. Visser, Phys. Rev. D **56**, 936 (1997).
[50] J. Matyjasek, Phys. Rev. D **55**, 809 (1997); **59**, 044002 (1999).
[51] P.R. Anderson, E. Mottola, and R. Vaulin (unpublished).
[52] P.O. Mazur and E. Mottola, Proc. Natl. Acad. Sci. U.S.A. **101**, 9545 (2004).
[53] P.O. Mazur, hep-th/9712208.
[54] S.M. Christensen and L. Parker, *MathTensor: A System for Doing Tensor Analysis by Computer* (Addison-Wesley, Reading, Massachusetts, 1994).