Black holes in pure Lovelock gravities

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Lovelock gravity is a fascinating extension of general relativity, whose action consists of dimensionally extended Euler densities. Compared to other higher order derivative gravity theories, Lovelock gravity is attractive since it has a lot of remarkable features such as the fact that there are no more than second order derivatives with respect to the metric in its equations of motion, and that the theory is free of ghosts. Recently, in the study of black strings and black branes in Lovelock gravity, a special class of Lovelock gravity is considered, which is named pure Lovelock gravity, where only one Euler density term exists. In this paper we study black hole solutions in the special class of Lovelock gravity and associated thermodynamic properties. Some interesting features are found, which are quite different from the corresponding ones in general relativity.

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I. INTRODUCTION

Over the past years there has been a lot of interest in black holes in higher derivative gravity theories. This is partly due to the anti-de Sitter space/conformal field theory (AdS/CFT) correspondence, where the higher derivative terms can be regarded as the corrections of large N expansion in the dual conformal field theory, and partly due to the brane world scenario, where TeV black holes are expected to be produced in colliders. Thus it is natural to study the effects of higher derivative curvature terms (see for example [1] and references therein). Among the higher derivative gravity theories, so-called Lovelock gravity [2] is rather special. The Lagrangian of Lovelock gravity consists of the dimensionally extended Euler densities

$$\mathcal{L} = \sum_{n=0}^{m} c_n \mathcal{L}_n, \tag{1.1}$$

where c_n are arbitrary constants and $\mathcal{L}_n$ are the Euler densities of a $2n$-dimensional manifold,

$$\mathcal{L}_n = \frac{1}{2^n} \delta^{a_1 b_1 \cdots a_n b_n}_{c_1 d_1 \cdots c_n d_n} R^{c_1 d_1}{}_{a_1 b1} \cdots R^{c_n d_n}{}_{a_n b_n}. \tag{1.2}$$

The generalized delta function is totally antisymmetric in both sets of indices. $\mathcal{L}_0$ represents the identity, so the constant c_0 is just the cosmological constant. $\mathcal{L}_1$ gives us the usual curvature scalar term, while $\mathcal{L}_2$ is just the Gauss-Bonnet term. Usually, in order for the Einstein general relativity to be recovered in the low energy limit, the constant c_1 must be positive. For simplicity one may take $c_1 = 1$. Since the action of Lovelock gravity is the sum of the dimensionally extended Euler densities, it is found that there are no more than second order derivatives with respect to the metric in its equations of motion. Furthermore, Lovelock gravity is shown to be free of ghosts when expanded on a flat space, evading any problems with unitarity [3,4]. It is also known that these terms arise with positive coefficients as higher order corrections in superstring theories, and their implications for cosmology have been studied [5].

In the literature, the so-called Gauss-Bonnet gravity, containing the first three terms in (1.1), has been intensively discussed. The spherically symmetric black hole solutions in Gauss-Bonnet gravity have been found in [3,6] and discussed in [7], and topological nontrivial black holes have been studied in [8]. The Gauss-Bonnet black holes in de Sitter space have been discussed separately in [9]. See also [10] for some other extensions including perturbative AdS black hole solutions in gravity theories with second order curvature corrections. In addition, the references in [11] have investigated the holographic properties associated with Gauss-Bonnet theory.

With many terms, the Lagrangian (1.1) is complicated. But the static, spherically symmetric black hole solutions can indeed be found [6] by solving for a real root of a polynomial equation for the metric function of the solution. Such black hole solutions have been generalized to the case with nontrivial horizon topology in [12]. Since there are m ($m = [(d-1)/2]$; $[N]$ denotes the integral part of the number N) coefficients c_n in (1.1), it is quite difficult to analyze the black hole solution and to extract physical information from the solution. In [13] a set of special coefficients has been chosen so that the metric function has a simple expression. In odd dimensions the action is the Chern-Simons form for the AdS group, and in even dimensions it is the Euler density constructed with the Lorentz part of the AdS curvature tensor. Thus the m Lovelock coefficients are reduced to two independent parameters: a cosmological constant and a gravitational constant.

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Rewrite the Lagrangian (1.1) in the form [13]

$$\mathcal{I} = \kappa \sum_{n=0}^{m} \alpha_n \mathcal{I}_n, \tag{1.3}$$

where

$$\mathcal{I}_n = \int \varepsilon_{a_1\cdots a_d} R^{a_1a_2} \wedge \cdots \wedge R^{a_{2n-1}a_{2n}} \wedge e^{a_{2n+1}} \wedge \cdots e^{a_d}. \tag{1.4}$$

These coefficients α_n are given by

$$\alpha_n = \begin{cases} \frac{1}{d-2n}\binom{m-1}{n} l^{-d+2n} & \text{for } d = 2m-1, \\ \binom{m}{n} l^{-d+2n} & \text{for } d = 2m, \end{cases} \tag{1.5}$$

where l is a length and κ in (1.3) is another parameter. The static, spherically symmetric, black hole solution in the theory (1.3) has a very simple form,

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega^2_{d-2}, \tag{1.6}$$

where

$$f(r) = \begin{cases} 1 - (2M/r)^{1/(m-1)} + (r/l)^2 & \text{for } d = 2m, \\ 1 - (M+1)^{1/(m-1)} + (r/l)^2 & \text{for } d = 2m-1, \end{cases} \tag{1.7}$$

where M is an integration constant, interpreted as the mass of the black hole solution. The nontrivial topological black holes in this gravity have been studied in [14]. On the other hand, the authors of [15] have chosen a set of coefficients so that gravity theory has a unique AdS vacuum with a fixed cosmological constant and the theory is labeled by an integer i (in [15] the integer is denoted by k; in this paper, however, the symbol k will be used for another purpose). In that case, the black hole solution also has a simple expression. The coefficients chosen in [15] are

$$\alpha_n = \begin{cases} \frac{l^{2(n-i)}}{d-2n}\binom{i}{n}, & n \le i, \\ 0, & n > i, \end{cases} \tag{1.8}$$

where the integer i is in the range $1 \le i \le [(d-1)/2]$. In that theory the black hole solution has the form (1.6), but with

$$f(r) = 1 + \frac{r^2}{l^2} - \sigma\left(\frac{C_1}{r^{d-2i-1}}\right)^{1/i}, \tag{1.9}$$

where C_1 is an integration constant related to the mass of the black hole solution, and $\sigma = (\pm 1)^{i+1}$. We can see from the solution that when $i = m-1$, the solution (1.9) reduces to (1.7) in even dimensions.

The black strings and black branes are generalized configurations of black holes; they are some extended objects covered by the event horizon in transverse directions of these extended objects. These black configurations play a key role in establishing the AdS/CFT correspondence. In the vacuum Einstein gravity, it is easy to construct black string and black brane solutions by simply adding some Ricci flat directions to a Schwarzschild black hole solution or its rotating generalization. However, it turns out not to be trivial to find the black string and black brane solutions in Lovelock gravity. It was first noticed in [16] that such a simple method does not work for Gauss-Bonnet gravity; instead some numerical approaches have to be adopted [17]. More recently it has been independently realized by Kastor and Mann [18] and Giribet *et al.* [19] that, to construct some simple black string and black brane solutions in Lovelock gravity, the so-called pure Lovelock gravity has to be invoked, in particular, in the case of the asymptotically flat black string and black brane solutions. Simply speaking, the action of pure Lovelock gravity is just the one (1.1), but only one of those coefficients does not vanish.

It is well known that some thermodynamic properties of black string and black brane solutions can be obtained by studying thermodynamics of the corresponding black holes, which come from the dimensional reductions along the isometric directions of black string and black brane solutions. In this paper we will therefore study black hole solutions in pure Lovelock gravity. On the other hand, due to the characteristic role of black holes in quantum gravity, studying black hole solutions in pure Lovelock gravity might be of interest in its own right, for example, in order to find the difference of black hole solutions in general relativity and in pure Lovelock gravity. We notice that the black hole solution has been studied in Weyl conformal gravity [20].

In the Lagrangian (1.1), the cosmological constant term $\mathcal{L}_0$ appears as an independent term. In this paper, we will discuss black hole solutions in the theory with only a Euler density term (1.1) plus the cosmological constant term. Since the case of $n = 1$ is just the Einstein general relativity, we will mainly discuss the case with $n > 1$.

The organization of the paper is as follows. In the next section we will present the black hole solution in pure Lovelock gravity and study associated thermodynamic properties. In Secs. II A, II B, and II C, we discuss the cases with vanishing, positive, and negative cosmological constants, respectively. The conclusions and discussions are given in Sec. III.

II. BLACK HOLE SOLUTIONS IN PURE LOVELOCK GRAVITY

Consider the gravity theory whose Lagrangian consists of the cosmological constant term $\mathcal{L}_0$ and the Euler density term $\mathcal{L}_n$ with $1 \le n \le m$,

$$\mathcal{L} = c_0 + c_n \mathcal{L}_n. \tag{2.1}$$

Then the equations of motion are $\mathcal{G}_{ab} = 0$, where [18]

$$\mathcal{G}^{a}_{b} = c_0\delta^a_b + c_n\delta^{ac_1\cdots c_nd_1\cdots d_n}_{be_1\cdots e_kf_1\cdots f_n}R^{e_1f_1}{}_{c_1d_1}\cdots R^{e_nf_n}{}_{c_nd_n}. \quad (2.2)$$

Assume that the metric is of the form

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2d\Sigma^2_{d-2}, \quad (2.3)$$

where $d\Sigma^2_{d-2}$ is the line element for a $(d-2)$-dimensional Einstein manifold with constant curvature scalar $(d-2)\times(d-3)k$. Here k is a constant, and, without loss of generality, one may take $k=0$ or ± 1. For the theory (2.1), the solution can be expressed by [6,12]

$$f(r) = k - r^2F(r), \quad (2.4)$$

where $F(r)$ is determined by the equation

$$\hat{c}_0 + \hat{c}_nF^n(r) = \frac{16\pi GM}{(d-2)\Sigma_{d-2}r^{d-1}} \quad (2.5)$$

where G is the Newtonian constant in d dimensions, and Σ_{d-2} is the volume of the $(d-2)$-dimensional Einstein constant curvature manifold. M is an integration constant, which is, in fact, the mass of the solution according to the Hamiltonian method [13,14]. In addition, we have

$$\hat{c}_0 = \frac{c_0}{(d-1)(d-2)}, \qquad \hat{c}_1 = 1, \qquad \hat{c}_n = c_n\Pi^{2m}_{j=3}(d-j), \qquad \text{for } n>1. \quad (2.6)$$

Note that the parameter c_n has the dimension $[\text{length}]^{2n-2}$. Since only one parameter $\hat{c}_n$ appears in (2.5), except for the cosmological constant $\hat{c}_0$, we may normalize the parameter $c_n(>0)$ so that one has $\hat{c}_n = \alpha^{2n-2}$ for simplicity, where α is a length scale. Here we have assumed that $c_n>0$, as in general relativity and higher derivative terms in superstring theories. Furthermore, we set $\hat{c}_0 = -1/l^2$ and then find that the solution has the form

$$F(r) = \begin{cases} \pm\frac{1}{\alpha^{2-2/n}}\left(\frac{16\pi GM}{(d-2)\Sigma_{d-2}r^{d-1}} + \frac{1}{l^2}\right)^{1/n} & \text{for } n = \text{even}, \\ \frac{\text{sign}(x)}{\alpha^{2-2/n}}\left|\frac{16\pi GM}{(d-2)\Sigma_{d-2}r^{d-1}} + \frac{1}{l^2}\right|^{1/n} & \text{for } n = \text{odd}, \end{cases} \quad (2.7)$$

where $x \equiv 16\pi GM/[(d-2)\Sigma_{d-2}r^{d-1}] + 1/l^2$. As expected, when $n=1$, the solution is just the one describing a Schwarzschild black hole in AdS (dS) space. When $n<(d-1)/2$, except the singularity at $r=0$, the solution (2.4) with (2.7) has a potential singularity at $x=0$. To see this, let us calculate the Riemann tensor squared for the metric (2.3):

$$R_{abcd}R^{abcd} = (f''(r))^2 + \frac{2(d-2)}{r^2}(f'(r))^2 + \frac{2(d-2)(d-3)}{r^4}(k-f(r))^2, \quad (2.8)$$

where a prime denotes the derivative with respect to r. It is easy to see that the Riemann tensor squared diverges if $x=0$ at some point r_x. Therefore, we always consider the region $r>r_x$ in what follows, if this singularity exists.

When n is even, the solution becomes $F(r) = \pm(1/\alpha^{2-2/n})(1/l^2)^{1/n}$ for $M\to 0$. Therefore, in order to have a well-defined vacuum solution, the cosmological constant l^2 must be positive, otherwise the theory is not well defined. That is, in this case, pure Lovelock gravity (2.1) must have a positive cosmological constant (note that the cosmological constant usually appears in the Lagrangian like $\mathcal{L} = -2\Lambda + \cdots$). On the other hand, when n is odd, in order to have a well-defined vacuum solution ($M\to 0$), the sign of x must be the same as the one of l^2. In addition, when $l^2=0$, the solution reduces to

$$F(r) = \begin{cases} \pm\frac{1}{\alpha^{2-2/n}}\left(\frac{16\pi GM}{(d-2)\Sigma_{d-2}r^{d-1}}\right)^{1/n} & \text{for } n = \text{even}, \\ \frac{\text{sign}(M)}{\alpha^{2-2/n}}\left|\frac{16\pi GM}{(d-2)\Sigma_{d-2}r^{d-1}}\right|^{1/n} & \text{for } n = \text{odd}, \end{cases} \quad (2.9)$$

where M must be positive when n is even. In the following subsections we will discuss the cases $1/l^2=0$, $1/l^2>0$, and $1/l^2<0$, respectively.

A. The case with a vanishing cosmological constant $1/l^2 = 0$

In this subsection, we consider the case without the cosmological constant, namely, the case $1/l^2=0$. A related solution has also been considered in [21]. In this case, we see from (2.9) that the solution is asymptotically flat for $n<(d-1)/2$. When $n=(d-1)/2$, the solution exists only in odd dimensions and describes a topological defect. The metric function $f(r)$ is constant:

$$f(r) = \begin{cases} k \mp \frac{1}{\alpha^{2-2/n}}\left(\frac{16\pi GM}{(d-2)\Sigma_{d-2}}\right)^{1/n} & \text{for } n = \text{even}, \\ k - \frac{\text{sign}(M)}{\alpha^{2-2/n}}\left|\frac{16\pi GM}{(d-2)\Sigma_{d-2}}\right|^{1/n} & \text{for } n = \text{odd}. \end{cases} \quad (2.10)$$

Note that, although the metric function f is a constant in this case, the spacetime is locally flat only when $d=3$, and, when $d>3$, some scalar invariants diverge at the origin as can be seen from (2.8). When $k=1$, we see from the solution that the solid deficit angle is negative in the plus branch in (2.10) for even n and $M<0$ for odd n, while the solution just corresponds to coordinate rescalings when $k=0$ and -1, since there are no fixed periods for coordinates of the Einstein manifolds in these two cases.

When $n<(d-1)/2$, the solution describes a naked singularity or black hole,

$$f(r) = \begin{cases} k \mp \frac{r^2}{\alpha^{2-2/n}}\left(\frac{16\pi GM}{(d-2)\Sigma_{d-2}r^{d-1}}\right)^{1/n} & \text{for } n = \text{even}, \\ k - \frac{\text{sign}(M)r^2}{\alpha^{2-2/n}}\left|\frac{16\pi GM}{(d-2)\Sigma_{d-2}r^{d-1}}\right|^{1/n} & \text{for } n = \text{odd}. \end{cases} \tag{2.11}$$

When $k = 0$, there is always a naked singularity in the solution. When $k = -1$, a naked singularity with a cosmological horizon appears in the plus branch in (2.11) for even n, and $M < 0$ for odd n; the naked singularity has no cosmological horizon for the minus branch, and $M > 0$ for odd n. When $k = 1$, the naked singularity appears in the plus branch in (2.11) for even n, and $M < 0$ for odd n. The naked singularity is of little physical interest. We therefore turn to the black hole solution. The black hole horizon appears for positive mass $M > 0$ in both cases. Thus we can uniformly rewrite the black hole solution as

$$f(r) = 1 - r^2\left(\frac{16\pi GM\alpha^{2-2n}}{(d-2)\Sigma_{d-2}r^{d-1}}\right)^{1/n}. \tag{2.12}$$

We notice that this solution is just the asymptotically flat limit discussed in the first reference of [15]. The black hole has a horizon at $r = r_+$,

$$r_+ = \left(\frac{16\pi GM\alpha^{2-2n}}{(d-2)\Sigma_{d-2}}\right)^{1/(d-2n-1)}. \tag{2.13}$$

The black hole has a Hawking temperature, which can be obtained by continuing the black hole solution (2.3) to its Euclidean sector, and requiring the absence of conical singularity at the black hole horizon, which leads to a fixed period of the Euclidean time, namely, the inverse Hawking temperature of the black hole. It is given by

$$T = \frac{d-2n-1}{4\pi n}\frac{1}{r_+}. \tag{2.14}$$

The black hole has an associated entropy with the horizon. In Einstein general relativity, the black hole entropy obeys the so-called horizon area formula. But in the higher derivative gravity theories, this is no longer true. The black hole as a thermodynamic system must obey the first law of thermodynamics. Therefore, we can use the first law to obtain the black hole entropy in pure Lovelock gravity, as we did in the case of general Lovelock gravity [12]. According to the first law

$$dM = \text{TdS}, \tag{2.15}$$

the black hole entropy can be obtained from the following integration:

$$S = \int_0^M T^{-1}dM = \int_0^{r_+} T^{-1}\left(\frac{\partial M}{\partial r_+}\right)dr_+. \tag{2.16}$$

Here we have assumed that the black hole entropy vanishes as the horizon radius goes to zero. Substituting (2.13) and (2.14) into (2.16), we find

$$S = \frac{(d-2)\Sigma_{d-2}}{4G}\frac{n\alpha^{2n-2}}{d-2n}r_+^{d-2n}. \tag{2.17}$$

We see that the entropy obeys the area formula only when $n = 1$, namely, the case of Einstein general relativity. It is easy to show that this entropy of the black hole can also be obtained by using the entropy formula for black holes in Lovelock gravity [22]. Compared to the Schwarzschild black hole, the black hole with $n > 1$ in pure Lovelock gravity has smaller entropy. But, like the Schwarzschild black hole, the black hole in pure Lovelock gravity always has a negative heat capacity,

$$C \equiv \frac{\partial M}{\partial T} = -\frac{(d-2)n\Sigma_{d-2}\alpha^{2n-2}}{4G}r_+^{d-2n}, \tag{2.18}$$

which indicates the thermodynamic instability of the black hole. Since the black hole solution (2.12) is just the one for an asymptotically flat limit in [15], the thermodynamic properties we obtained above are completely the same as those found in [15].

B. The case with a positive cosmological constant $l^2 > 0$

When $l^2 > 0$, the solutions are written as

$$f(r) = \begin{cases} k \mp \frac{r^2}{\alpha^{2-2/n}}\left(\frac{16\pi GM}{(d-2)\Sigma_{d-2}r^{d-1}} + \frac{1}{l^2}\right)^{1/n} & \text{for } n = \text{even}, \\ k - \frac{r^2}{\alpha^{2-2/n}}\left|\frac{16\pi GM}{(d-2)\Sigma_{d-2}r^{d-1}} + \frac{1}{l^2}\right|^{1/n} & \text{for } n = \text{odd}. \end{cases} \tag{2.19}$$

In this case, this solution is asymptotically dS or AdS. When $n = (d-1)/2$, the solution reduces to

$$f(r) = \begin{cases} k \mp \frac{1}{\alpha^{2-2/n}}\left(\frac{16\pi GM}{(d-2)\Sigma_{d-2}} + \frac{r^{2n}}{l^2}\right)^{1/n} & \text{for } n = \text{even}, \\ k - \frac{1}{\alpha^{2-2/n}}\left|\frac{16\pi GM}{(d-2)\Sigma_{d-2}} + \frac{r^{2n}}{l^2}\right|^{1/n} & \text{for } n = \text{odd}. \end{cases} \tag{2.20}$$

Clearly this is the topological defect solution in pure Lovelock gravity with a positive cosmological constant (2.1). We see that the topological defect solution in the case $n > 1$ is quite different from the case of $n = 1$.

When $n < (d-1)/2$, which we discuss in what follows, the solution (2.19) describes a naked singularity or black hole again.

(1) When $k = 1$, we see that the plus branch in (2.19) for even n describes a naked singularity. The solution is asymptotically AdS, although the cosmological constant l^2 is positive. For other cases with $M > 0$, the solution (2.20) describes a black hole. Thus, for any n the black hole solution can be written as

$$f(r) = 1 - r^2\left(\frac{16\pi GM\alpha^{2-2n}}{(d-2)\Sigma_{d-2}r^{d-1}} + \frac{\alpha^{2-2n}}{l^2}\right)^{1/n}. \tag{2.21}$$

In the limit of $r \to \infty$ or $M = 0$, the solution reduces to

$$f(r) = 1 - r^2(\alpha^{2-2n}/l^2)^{1/n}. \tag{2.22}$$

This is a dS solution with dS radius $r_c = (l/\alpha^{1-n})^{1/n}$. Therefore, the solution (2.21) is asymptotically dS and it describes a black hole in dS space. This solution is very similar to the Schwarzschild solution in dS space: in the small r limit, the first term in the parentheses in (2.21) dominates, while the second term dominates in the large r limit. We therefore expect that both the black hole and cosmological horizons appear in a suitable parameter space (see Fig. 1). Both horizons satisfy the equation $f(r) = 0$: the smaller root denotes the black hole horizon r_+ while the larger one r_c corresponds to the cosmological horizon.

The black hole mass M can be expressed by the black hole horizon r_+ as

$$M = \frac{(d-2)\Sigma_{d-2} r_+^{d-2n-1}}{16\pi G \alpha^{2-2n}}\left(1 - \frac{\alpha^{2-2n} r_+^{2n}}{l^2}\right), \tag{2.23}$$

or in terms of the cosmological horizon r_c as

$$M = \frac{(d-2)\Sigma_{d-2} r_c^{d-2n-1}}{16\pi G \alpha^{2-2n}}\left(1 - \frac{\alpha^{2-2n} r_c^{2n}}{l^2}\right). \tag{2.24}$$

Note that both the black hole and cosmological horizons are always less than $(l/\alpha^{1-n})^{1/n}$ as $M \neq 0$. As in the asymptotically flat case, the Hawking temperatures associated with the black hole and cosmological horizons can be obtained. They are

$$T_{+,c} = \pm \frac{d-2n-1}{4\pi n r_{+,c}}\left(1 - \frac{d-1}{d-2n-1}\frac{\alpha^{2-2n} r_{+,c}^{2n}}{l^2}\right). \tag{2.25}$$

When the temperature vanishes, the black hole and cosmological horizons coincide with each other. In that case, the black hole has a horizon radius r_n,

$$r_n = \left(\frac{d-2n-1}{d-1}\frac{l^2}{\alpha^{2-2n}}\right)^{1/2n}. \tag{2.26}$$

This is the maximal black hole in pure Lovelock gravity with a positive cosmological constant, and it is the counterpart of the Nariai black hole in this pure Lovelock gravity. When the mass of the solution is beyond the value M_n,

$$M > M_n \equiv \frac{n(d-2)\Sigma_{d-2} r_n^{d-2n-1}}{8(d-1)\pi G \alpha^{2-2n}}, \tag{2.27}$$

the solution (2.21) describes a naked singularity.

As for the entropy associated with horizons, it is easy to check that both the entropies of the black hole and cosmological horizons still have the form (2.17), and they obey the first law of thermodynamics,

$$dM = T_+ dS, \qquad -dM = T_c dS_c, \tag{2.28}$$

respectively, where S_c is obtained by replacing r_+ in (2.17) with r_c. This further verifies that the (black hole and cosmological) horizon entropy is a function of the horizon geometry only; the cosmological constant does not appear explicitly in the expressions of the horizon entropy. In addition, the negative sign in the first law of the cosmological horizon is due to the fact that when M increases the cosmological horizon shrinks and therefore the entropy S_c decreases [23]. The heat capacities of the black hole and cosmological horizons are

$$C_{+,c} = \left(\frac{\partial M}{\partial T_{+,c}}\right) = \mp \frac{n^2\pi(d-2)\Sigma_{d-2} T_{+,c} r_{+,c}^{d-2n+1}}{(d-2n-1)G\alpha^{2-2n}} \times \left(1 + \frac{(d-1)(2n-1)}{d-2n-1}\frac{\alpha^{2-2n} r_{+,c}^{2n}}{l^2}\right)^{-1}. \tag{2.29}$$

While the heat capacity of the black hole horizon is always negative, the heat capacity associated with the cosmological horizon is positive if one views the energy E of the cosmological horizon as $E = -M$, as in [23]. This indicates that the black hole horizon is thermodynamically unstable while the cosmological horizon is stable as in the case of the Schwarzschild black hole in dS space [9].

Note that, when $M < 0$ in (2.21), the black hole horizon disappears; instead the solution describes a singularity at $x = 0$ covered by a cosmological horizon, which is still determined by $f(r) = 0$.

(1) When $k = 0$, it is easy to see from (2.19) that the solution always describes a naked singularity.

(2) When $k = -1$, it is a naked singularity solution again for the minus branch in (2.19) for even n and for odd n. However, for the plus branch in the case of even n, the solution becomes

$$f(r) = -1 + \frac{r^2}{\alpha^{2-2/n}}\left(\frac{16\pi G M}{(d-2)\Sigma_{d-2} r^{d-1}} + \frac{1}{l^2}\right)^{1/n}. \tag{2.30}$$

This solution is asymptotically AdS although the cosmological constant in the theory (2.1) is positive. Clearly, the black hole horizon will appear in this case.

(i) When $M > 0$, the solution has properties as follows. In Fig. 2 we plot the metric function $f(r)$

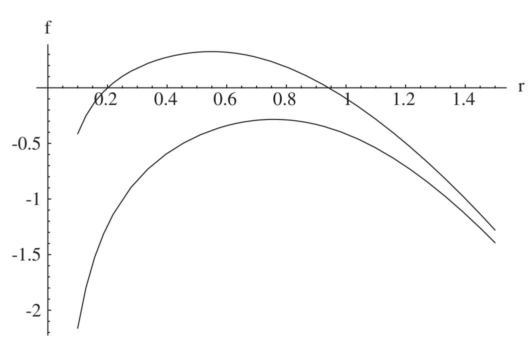

FIG. 1. The metric function $f(r)$ (2.21) versus the radius. The upper curve indicates that both the black hole and cosmological horizons exist, while the lower curve is a naked singularity solution.

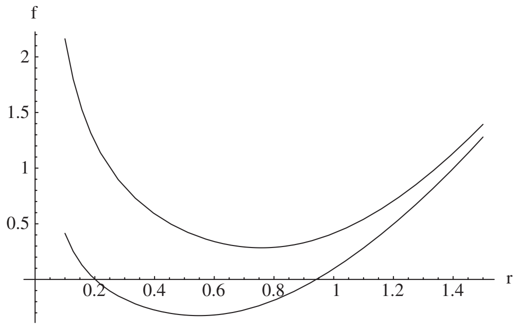

FIG. 2. The metric function $f(r)$ (2.30) versus the radius. The upper curve for a larger M indicates that this solution has a naked singularity, while there are two black hole horizons for the lower curve.

versus the radius. From the figure we can see that, in a suitable parameter space, the solution can have two horizons (one outer horizon and one inner horizon), while as M becomes large enough, the black hole horizon disappears and the solution contains a naked singularity. The black hole mass M can be expressed in terms of the outer horizon r_+,

$$M = \frac{(d-2)\Sigma_{d-2} r_+^{d-1}}{16\pi G}\left(\frac{\alpha^{2n-2}}{r_+^{2n}} - \frac{1}{l^2}\right). \tag{2.31}$$

The Hawking temperature of the black hole is

$$T = \frac{d-2n-1}{4\pi n r_+}\left(-1 + \frac{d-1}{d-2n-1}\frac{\alpha^{2-2n} r_+^{2n}}{l^2}\right). \tag{2.32}$$

When the temperature vanishes, the black hole has a horizon $r_{\min}$,

$$r_{\min} \equiv \left(\frac{d-2n-1}{d-1}\frac{l^2}{\alpha^{2-2n}}\right)^{1/2n}. \tag{2.33}$$

(ii) When $M = 0$, the solution (2.30) still has a black hole (massless black hole) horizon

$$r_{\max} = (l/\alpha^{1-n})^{1/n}. \tag{2.34}$$

Therefore, we conclude that the horizon (2.33) is the minimal one, while the horizon (2.34) is the maximal one if $M > 0$. Here let us mention that, when $M = 0$, the solution (2.30) has only a horizon $r_{\max}$; when M increases, two horizons appears; and when M reaches

$$M_{\max} = \frac{(d-2)n\Sigma_{d-2} r_{\min}^{d-2n-1}\alpha^{2n-2}}{8(d-1)\pi G}, \tag{2.35}$$

two horizons coincide with each other, beyond which the solution gets a naked singularity. Using the first law of black hole thermodynamics, $dM = \mathrm{TdS}$, we obtain the entropy of the black hole,

$$S = C_0 - \frac{(d-2)\Sigma_{d-2}}{4G}\frac{n}{d-2n}\alpha^{2n-2} r_+^{d-2n}, \tag{2.36}$$

where C_0 is an integration constant. If one takes C_0 to be zero, one is led to a negative entropy. In fact, the minus sign arises due to the fact that, when the mass M increases, the horizon radius decreases. This implies that this black hole has a negative heat capacity; it is thermodynamically unstable.

(iii) When $M < 0$, we can rewrite the solution (2.30) as

$$f(r) = -1 + \frac{r^2}{\alpha^{2-2/n}}\left(-\frac{16\pi G M'}{(d-2)\Sigma_{d-2} r^{d-1}} + \frac{1}{l^2}\right)^{1/n}, \tag{2.37}$$

where $M'(= -M) > 0$ is used. In this case, we have only one black hole horizon r_+, which is always larger than $(l/\alpha^{1-n})^{1/n}$ for $M' > 0$. Namely, now the massless black hole (2.34) becomes a minimal one. For this black hole, a new singularity appears at $x = 0$, that is, $r_x^{d-1} = 16\pi G M' l^2/(d-2)\Sigma_{d-2}$. Note that the singularity is always covered by the black hole horizon r_+. For this black hole, we have the Hawking temperature

$$T = \frac{d-2n-1}{4\pi n r_+}\left(\frac{d-1}{d-2n-1}\frac{r_+^{2n}}{l^2\alpha^{2n-2}} - 1\right), \tag{2.38}$$

and a positive entropy with

$$S = \frac{(d-2)\Sigma_{d-2}}{4G}\frac{n}{d-2n}\alpha^{2n-2} r_+^{d-2n}. \tag{2.39}$$

We see from (2.38) that the Hawking temperature increases when r_+ grows. Therefore, this black hole is thermodynamically stable and has a positive heat capacity.

From the above analysis, we can see that, if one takes the solution as the form (2.37) with any sign of M', everything goes well and nothing strange appears: When $M' > 0$, the solution has only one horizon larger than $(l/\alpha^{1-n})^{1/n}$, and the temperature and entropy are given by (2.38) and (2.39), respectively; when $M' = 0$, this is just the massless black hole with horizon radius $(l/\alpha^{1-n})^{1/n}$; when $M' < 0$, the black hole solution will have two horizons, and the two horizons coincide with each other when the temperature (2.38) vanishes. In that case, the black hole has a minimal horizon (2.33); its mass is negative,

$$M_{\min} = -\frac{(d-2)n\Sigma_{d-2} r_{\min}^{d-2n-1}\alpha^{2n-2}}{8(d-1)\pi G},$$

beyond which the solution describes a naked singularity. This is nothing but the counterpart of the negative mass hyperbolic black holes in pure Lovelock gravity (for negative mass hyperbolic black holes in general relativity see, for example, some references in [8,10]).

C. The case with a negative cosmological constant $l^2 < 0$

Now we turn to the case with a negative cosmological constant $l^2 < 0$, namely, $\hat{c}_0 > 0$. In this case, we can see from (2.5) that, when n is even, there is no physical solution for $F(r)$ unless $\hat{c}_n < 0$, even for the case with $M > 0$ in (2.7), because, in the latter case, there is no well-defined vacuum solution as mentioned above. On the other hand, when n is odd, the sign of x is also negative (to have a well-defined vacuum solution again). Combining (2.4) and (2.7), the solution can be written down as

$$f(r) = k + \frac{r^2}{\alpha^{2-2/n}}\left(-\frac{16\pi GM}{(d-2)\Sigma_{d-2}r^{d-1}} + \frac{1}{|l|^2}\right)^{1/n}, \tag{2.40}$$

where $M > 0$. This solution is asymptotically AdS. It is easy to see that, when $k = 1$ or $k = 0$, the solution describes a naked singularity, while when $k = -1$, the solution is the same as the one in (2.37) if $M > 0$, and the one in (2.30) if $M < 0$. Namely, in the case of $k = -1$, a black hole with a negative constant curvature horizon appears in a suitable parameter space. We will not repeat the analysis here.

III. CONCLUSION AND DISCUSSION

We studied black hole solutions in pure Lovelock gravity with a cosmological constant. The Lagrangian of pure Lovelock gravity is a Euler density for a certain spacetime dimension. Such a theory naturally arises in the construction of black string and black brane solutions in general Lovelock gravity [18,19]. In the case without the cosmological constant, the solution we found is either a topological defect solution (2.10) or a black hole solution (2.12), otherwise it describes a naked singularity. The black hole thermodynamics was analyzed, and we found that it has similar properties to a Schwarzschild black hole. When the cosmological constant is positive, we found a black hole solution (2.21), which is asymptotically dS. The black hole solution, again, has similar properties to a Schwarzschild black hole in dS space. Interestingly enough, in this case, we also found an asymptotically AdS black hole solution (2.30), which has a negative constant curvature horizon. For a negative cosmological constant, we have not found any solution of physical interest if the number n is even. However, when n is odd, we again found the asymptotically AdS black hole solution with a negative constant curvature horizon.

It is well known that, in general relativity, black holes in AdS space can have positive, zero, or negative constant curvature horizons, namely, the cases of $k = 1$, 0, and -1. In pure Lovelock gravity, however, we have seen that only $k = -1$ black holes are allowed to appear in the asymptotically AdS space.

Finally we mention that a Maxwell field can be added to the Lagrangian (2.1). In this case, a static, spherically symmetric solution (2.3) can be determined by solving the following equation [12,24]:

$$\hat{c}_0 + \hat{c}_n F^n(r) = \frac{16\pi GM}{(d-2)\Sigma_{d-2}r^{d-1}} - \frac{q^2}{r^{2(d-2)}}, \tag{3.1}$$

where q is another integration constant, which is related to the electric charge of the solution. As in the case without the charge, we can also discuss the causal structure of the charged solution and associated thermodynamic properties.

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