

Eccentricity content of binary black hole initial data

Emanuele Berti,^{*} Sai Iyer,[†] and Clifford M. Will[‡]

McDonnell Center for the Space Sciences, Department of Physics, Washington University, St. Louis, Missouri 63130, USA

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Using a post-Newtonian diagnostic tool developed by Mora and Will, we examine numerically generated quasiequilibrium initial data sets that have been used in recently successful numerical evolutions of binary black holes through plunge, merger and ringdown. We show that a small but significant orbital eccentricity is required to match post-Newtonian and quasiequilibrium calculations. If this proves to be a real eccentricity, it could affect the fine details of the subsequent numerical evolutions and the predicted gravitational waveforms.

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I. INTRODUCTION

Recent breakthroughs in numerical relativity have made it possible to evolve Einstein's equations for binary black holes (BBH) stably for several orbits, including the plunge, merger and ringdown phases, and to generate intriguingly robust gravitational waveforms [1–5]. The starting point of these evolutions is a set of initial data, obtained from the initial-value equations of general relativity, intended to represent two black holes in circular orbital motion; this is the expected end product of long-term binary evolution under the circularizing and damping effects of gravitational radiation emission.

In earlier work [6,7] we developed an approach, based on the post-Newtonian approximation, designed to study and elucidate the physical content of these initial data sets, and showed that, in order to match post-Newtonian theory with some data sets [8,9], a small but significant orbital eccentricity was required. In this paper we apply this post-Newtonian diagnostic to the initial data used in recent BBH evolutions, and find that they also require an orbital eccentricity. In particular we examine the corotating and non-spinning initial data computed by Cook and collaborators [10,11] and the nonspinning “puncture” initial data of Tichy and Brüggmann and of the “Lazarus” group [12,13]. If this residual and unintended noncircularity is real, it may affect the detailed structure of the numerically generated gravitational waveforms.

The plan of the paper is as follows. In Sec. II we summarize the Post-Newtonian diagnostic equations for BBH derived in [7]. In Sec. III we apply the diagnostic to the BBH quasiequilibrium configurations of [10–13]. Sec. IV presents conclusions.

II. POST-NEWTONIAN DIAGNOSTIC FOR BINARY BLACK HOLES

Consider a binary system of black holes of irreducible masses m_1 and m_2 , and rotational angular velocities ω_1 and

ω_2 , with $m_{\text{irr}} = m_1 + m_2$ and $\eta = m_1 m_2 / m_{\text{irr}}^2$ defining the total irreducible mass and reduced mass parameter, respectively, ($0 < \eta \leq 1/4$). Following [6,7] we define an eccentricity e and a quantity $\zeta \equiv m_{\text{irr}}/p$ related to the semilatus rectum p of the orbit, according to:

$$e \equiv \frac{\sqrt{\Omega_p} - \sqrt{\Omega_a}}{\sqrt{\Omega_p} + \sqrt{\Omega_a}}, \quad (1)$$

$$\zeta \equiv \frac{m_{\text{irr}}}{p} \equiv \left(\frac{\sqrt{m_{\text{irr}} \Omega_p} + \sqrt{m_{\text{irr}} \Omega_a}}{2} \right)^{4/3},$$

where Ω_p is the value of the orbital angular frequency Ω where it passes through a local maximum (pericenter), and Ω_a is the value of Ω where it passes through the *next* local minimum (apocenter). These quantities reduce exactly to their Newtonian counterparts in the small orbital frequency (Newtonian) limit, and are gauge invariant through first post-Newtonian order, among other advantages [6,7].

We want to compare with quasiequilibrium configurations of equal-mass BH-BH binaries, so we set $m_1 = m_2$ and $\eta = 1/4$. For corotating binaries we also set $\omega_1 = \omega_2 \equiv \omega = \Omega$, while for nonspinning binaries we have $\omega_1 = \omega_2 = 0$. We exploit the fact that there exist exact formulae for the energy and spin of isolated Kerr black holes in terms of the irreducible mass, $M = M_{\text{irr}}/[1 - 4(M_{\text{irr}}\omega)^2]^{1/2}$, $S = 4M_{\text{irr}}^3\omega/[1 - 4(M_{\text{irr}}\omega)^2]^{1/2}$. The total binding energy and angular momentum of the system are then given by

$$E_b = E_{\text{Self}} + E_{\text{Orb}} + E_{\text{N,Corr}} + E_{\text{Spin}}, \quad (2)$$

$$J = S + J_{\text{Orb}} + J_{\text{N,Corr}} + J_{\text{Spin}},$$

where

^{*}Electronic address: berti@wugrav.wustl.edu

[†]Electronic address: sai@physics.wustl.edu

[‡]Electronic address: cmw@wuphys.wustl.edu

$$E_{\text{Self}} = m_{\text{irr}} \left[\frac{1}{2} (m_{\text{irr}} \omega)^2 + \frac{3}{8} (m_{\text{irr}} \omega)^4 + \dots \right], \quad (3a)$$

$$S = m_{\text{irr}}^3 \omega \left[1 + \frac{1}{2} (m_{\text{irr}} \omega)^2 + \frac{3}{8} (m_{\text{irr}} \omega)^4 + \dots \right], \quad (3b)$$

$$E_{\text{Orb}} = -\frac{1}{8} m_{\text{irr}} (1 - e^2) \zeta \left[1 - \frac{1}{48} (37 - e^2) \zeta - \frac{1}{384} (1069 - 718e^2 + 57e^4) \zeta^2 + \left(\frac{1}{331776} (1427365 - 434775e^2 + 110127e^4 - 3133e^6) - \frac{41\pi^2}{384} (5 - e^2) \right) \zeta^3 \right], \quad (3c)$$

$$J_{\text{Orb}} = \frac{1}{4} m_{\text{irr}}^2 \frac{1}{\sqrt{\zeta}} \left[1 + \frac{1}{24} (37 - e^2) \zeta + \frac{1}{384} (1069 + 450e^2 - 55e^4) \zeta^2 - \left(\frac{1}{82944} (285473 - 271419e^2 - 93e^4 - 713e^6) - \frac{41\pi^2}{96} (1 + e^2) \right) \zeta^3 \right], \quad (3d)$$

$$E_{\text{N,Corr}} = -\frac{5}{48} m_{\text{irr}} (1 - e^2) (m_{\text{irr}} \omega)^2 \zeta, \quad (3e)$$

$$J_{\text{N,Corr}} = \frac{5}{24} m_{\text{irr}}^2 (m_{\text{irr}} \omega)^2 / \sqrt{\zeta}, \quad (3f)$$

$$E_{\text{Spin}} = -\frac{1}{12} m_{\text{irr}} (1 - e^2) (7 - 2e^2) (m_{\text{irr}} \omega) \zeta^{5/2}, \quad (3g)$$

$$J_{\text{Spin}} = -\frac{5}{24} m_{\text{irr}}^2 (7 + e^2) (m_{\text{irr}} \omega) \zeta. \quad (3h)$$

The orbital (“Orb”) contributions are expressed in an ADM-like (Arnowitt-Deser-Misner) gauge and are valid to third post-Newtonian (3PN) order (see [7] for a discussion of different gauges in the diagnostic). In Eqs. (3a) and (3b), we have expanded the Kerr formulae for M and S in powers of $m_{\text{irr}} \omega$, assumed to be small compared to unity, keeping as many terms as needed to reach a precision comparable to our 3PN orbital formulae, and have subtracted m_{irr} in order to obtain the binding energy. The

“N,Corr” terms come from converting the individual *total* masses that appear in the Newtonian orbital energy to irreducible masses and their corrections due to spin, and the “Spin” terms are spin-orbit effects. For black hole binaries, tidal and spin-spin effects can be shown to be negligible [7]. To obtain E_b and J at a turning point as functions of Ω , we substitute $\zeta = (m_{\text{irr}} \Omega_a)^{2/3} / (1 - e)^{4/3}$ or $\zeta = (m_{\text{irr}} \Omega_p)^{2/3} / (1 + e)^{4/3}$ for apocenter or pericenter, respectively. When E_b , J , ω and Ω are suitably scaled by m_{irr} , there remains only one free parameter, the eccentricity of the orbit. This approach was used in [6] to compare with the numerical quasiequilibrium solutions of Grandclément *et al.* [9], and it was found that a substantially better fit to the numerical data was obtained for nonzero values of e , of the order of 0.03, with the system at apocenter, than for $e = 0$. We now apply this diagnostic to other data sets that have recently played an important role in BBH evolutions.

III. DIAGNOSIS OF BBH INITIAL DATA SETS

Cook and collaborators have developed initial data sets for quasiequilibrium BBH, allowing for both corotation and zero spin, in a series of papers [10,11,14–18]. Using the thin-sandwich approach, combined with “excision” boundary conditions for the black holes adapted for treating spin, they considered systems possessing a “helical Killing vector”, $\partial/\partial t + \Omega \partial/\partial \phi$, meant to represent a circular orbit, one that is stationary in a frame rotating with angular velocity Ω . Additionally, they impose the condition that the Komar mass, a mass defined for stationary systems, equal the ADM mass, an invariant mass measured at spatial infinity. It is believed that this condition helps ensure that the orbit is truly circular. We apply our diagnostic to two data sets, taken from Refs. [10,11], respectively. For nonspinning BH, the second data set used a more accurate prescription for setting the BH spins to zero; in the earlier data, the black holes were not truly non-rotating. We take the data from Tables IV (corotating) and V (nonspinning) of [10] and of [11], and plot E_b/m_{irr} and J/m_{irr}^2 vs $m_{\text{irr}} \Omega$. Figure 1 shows the compari-

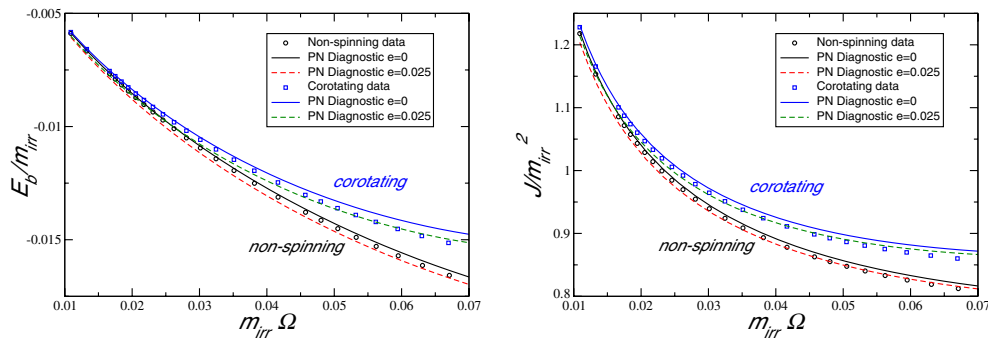


FIG. 1 (color online). Binding energy (left) and angular momentum (right) for the corotating and nonspinning BH-BH initial data from [11]. Circles (squares) are nonspinning (corotating) data from [11], the solid lines are circular PN diagnostics with $e = 0$, the dashed lines are eccentric PN diagnostics with $e = 0.025$.

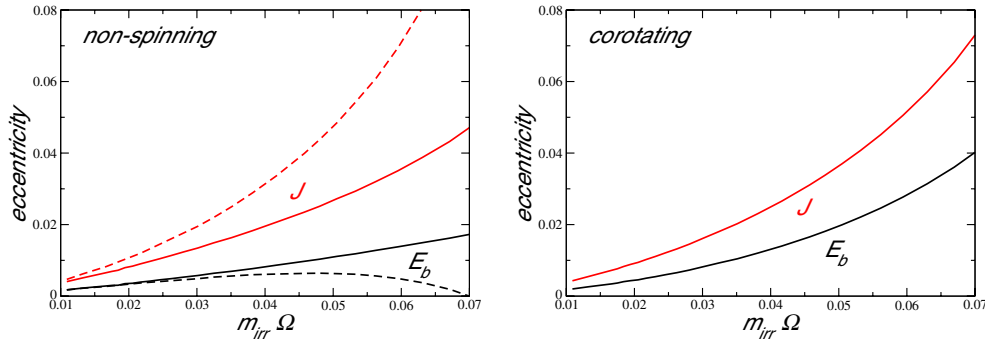


FIG. 2 (color online). Eccentricity required to match data points for nonspinning (left) and corotating data (right). Black lines are solutions for the energy, red lines are solutions for the angular momentum. In the nonspinning case, continuous lines refer to the Caudill *et al.* data [11], dashed lines to the earlier Cook-Pfeiffer data [10].

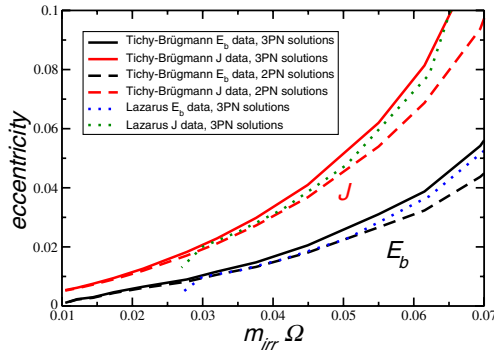


FIG. 3 (color online). Eccentricity solutions for puncture initial data. Solid and dashed lines are solutions for the Tichy-Brügmann puncture data using 3PN and 2PN diagnostics, respectively. Dotted lines are from 3PN solutions for the Lazarus puncture data.

son between the data of Caudill *et al.* [11] and our diagnostic, plotted for $e = 0$ and for $e = 0.025$ (with our definitions, positive values of e correspond to the system

TABLE I. Least-squares fit of the eccentricity for the corotating (corot) and nonspinning (nospin) data by Caudill *et al.*, for the Tichy-Brügmann puncture data (TB) and for the Lazarus data (Lazarus). The integer N is the number of data points used for the fit. We fit by a cubic polynomial $e = \sum_{k=0}^3 e_k (m_{\text{irr}} \Omega)^k$. $\Delta e^{\text{max}} = \max[(e - e^{\text{num}})/e^{\text{num}}]$ is the maximum percentage error of the fit with respect to the numerical data e^{num} .

Fit of E_b						
	N	$10^3 \times e_0$	e_1	e_2	e_3	Δe^{max}
corot	25	-0.73418894	0.23870832	-0.37207668	74.523304	-3.758
nospin	25	-0.11438527	0.15748053	1.2258277	0.83081005	5.751
TB	12	-5.4145414	0.75157074	-14.093449	224.07983	-11.64
Lazarus	5	-2.1760344	0.60151690	-13.280282	221.40238	1.244
Fit of J						
	N	$10^3 \times e_0$	e_1	e_2	e_3	Δe^{max}
corot	25	-1.8164595	0.56024586	-3.2850593	147.08294	-4.452
nospin	25	-0.99496055	0.45656217	-1.0622433	60.466300	-3.088
TB	12	-9.4590074	1.6687984	-40.943406	624.96446	-20.98
Lazarus	5	-16.332921	2.0037212	-43.661018	577.56427	1.161

being at apocenter). Figure 2 shows the eccentricity required to match each data point from both [10,11]. In the improved data set of Caudill *et al.* for the nonspinning case, the apparent eccentricity in the fits to J is reduced, and the functional behavior of e with $m_{\text{irr}} \Omega$ is now the same (monotonically increasing) in both the nonspinning and corotating cases. For the corotating case, there is essentially no difference in the fits between the two data sets. Furthermore, as in earlier comparisons [6], there is a systematic difference between the eccentricity required to match the binding energy and that required to match the angular momentum.

Another approach to initial quasiequilibrium data for BBH evolutions is the puncture method, in which the conformal factor of the conformally flat spatial slices is written in terms of a Newtonian-like potential $m_A/|\mathbf{x} - \mathbf{x}_A|$ (a puncture) for each body. This approach can also be made to incorporate the helical Killing vector and Komar-ADM mass equality assumptions [19]. Tichy and Brügmann [12] and Baker *et al.* [13] (the Lazarus project) have used this approach to generate initial data for nonspinning binary black holes in quasicircular orbits. In Fig. 3 we show the eccentricity required to match these data sets; again the eccentricity is small but significant, and again different between E_b and J . We also show explicitly the eccentricity required to match the Tichy-Brügmann data using a 2PN diagnostic, obtained by truncating the 3PN terms in the orbital expressions; the differences are comparable to the differences between the two numerical data sets, and are not large enough to account for the difference between the E_b and J curves. Table I lists the coefficients of a cubic fitting function to the eccentricity required to match E_b and J for different initial data sets.

IV. CONCLUSIONS

We have shown, using a post-Newtonian diagnostic tool, that initial data sets for binary black hole mergers may actually represent slightly eccentric orbits. Several remarks are called for. First, there is evidence from the dynamical evolutions of some of these initial data sets that the orbits

are slightly eccentric. For example, in the recent evolution of several BBH orbits through merger and ringdown by Baker *et al.* [3] using the Lazarus initial data, an oscillatory behavior of the separation of the black holes can be seen in their Figure 9. Similar oscillations were seen in the binary neutron star evolutions of [20,21], although the results there were very sensitive to grid resolution and size of the computational domain.

On the other hand, we continue to be puzzled by the difference in values of e inferred from fits to E_b and J . This difference was also seen in fits of the diagnostic to data from the Meudon group [6], and could be cited as a defect of the PN approximation. However, this difference occurs systematically even at the smallest values of $m_{\text{irr}}\Omega$, where relativistic corrections are quite small.

We want to emphasize that the eccentricity we are discussing here is not related to the mismatch between an initial quasicircular orbit (with $\dot{r} = \ddot{r} = 0$ by construction) and the reality of a pre-existing inspiral (with $\dot{r} \approx -16(m/r)^3/5$), since the initial data sets know nothing about radiation reaction. That eccentricity, which would be induced on an evolution from a perfectly circular initial orbit, has been discussed in detail by Miller [22]. Depending on the starting point of the evolution, the induced eccentricity from this effect could be as large as 0.03.

Irrespective of the origin of the eccentricity, Miller pointed out that the result could be a substantial decrease in detection signal-to-noise when a numerically generated,

eccentric waveform template is matched against a “true” waveform generated by a quasicircular inspiral of a real BBH (see, for example, Figures 7 and 9 of [22]).

If eccentricity *is* an issue and cannot be removed or reduced by tuning the initial data sets, one could ask whether it could be damped away naturally by numerically evolving several orbits leading up to the onset of plunge, around $m_{\text{irr}}\Omega \sim 0.1$. Using Eqs. (2.34) of [7], which give the evolution of our orbit elements e and ζ under radiation reaction, it is straightforward to show, at 2.5PN order and in the small eccentricity limit, that the number of orbits N required to reduce the eccentricity by a factor $X = e_f/e_i$ by the time the orbit reaches a final angular velocity Ω_f is given by $N = (X^{-30/19} - 1)/64\pi\eta(m_{\text{irr}}\Omega_f)^{5/3}$. For $\eta = 1/4$ and $m_{\text{irr}}\Omega_f = 0.1$, this gives 34 orbits for a reduction by 1/10, and 11 orbits for a reduction by 1/5. Suppressing eccentricity this way is likely to be a challenge without additional breakthroughs in numerical relativity.

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