Interplay of the chiral and large N_c limits in πN scattering

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Light-quark hadronic physics admits two useful systematic expansions, the chiral and $1/N_c$ expansions. Their respective limits do not commute, making such cases where both expansions may be considered to be especially interesting. We first study πN scattering lengths, showing that (as expected for such soft-pion quantities) the chiral expansion converges more rapidly than the $1/N_c$ expansion, although the latter nevertheless continues to hold. We also study the Adler-Weisberger and Goldberger-Miyazawa-Oehme sum rules of πN scattering, finding that both fail if the large N_c limit is taken prior to the chiral limit.

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I. INTRODUCTION

The $1/N_c$ expansion of QCD has proven to be a useful qualitative and semiquantitative approach to hadronic physics. Recently, a number of model-independent properties of meson-baryon scattering and their associated baryon resonances have been derived [1–5] using the contracted $SU(2N_f)$ symmetry emergent from QCD in the large N_c limit [6]. This general approach verifies that results [7–10] previously derived in the context of Skyrme-type models are in fact general results of large N_c QCD. In assessing the usefulness of this approach it is relevant to recall that in nature $N_c = 3$, and depending upon the observable $1/N_c$ corrections, can be quite large: indeed, large enough in some cases to render the expansion useless. Accordingly, it is important to try to understand *which* types of observables are particularly likely to suffer from large $1/N_c$ corrections.

In this context let us note that QCD allows another useful expansion, namely, about its chiral limit [11]. The chiral expansion is essentially an expansion in $\{m_\pi, q\}/\Lambda$, where q is the relevant momentum transfer in the process and Λ is a typical hadronic scale (such as $4\pi f_\pi$). The power counting of the chiral expansion limits its validity to processes with low momentum transfer.

This paper focuses on the interplay of the chiral and large N_c limits for processes involving πN scattering. Several *a priori* reasons suggest that this interplay should be important in understanding at least one regime in which the $1/N_c$ might be expected to break down badly, the soft-pion limit. First, each expansion is taken around a different limit, and the convergence need not be uniform: The ordering of limits can matter. When this occurs, at least one of the expansions *must* break down. Second, the large N_c

consistency conditions used to derive the model-independent results are based upon unitarity in πN scattering. For a typical pion momentum and energy of $O(N_c^0)$, unitarity in πN scattering imposes a constraint: The scattering amplitude cannot grow without bound and hence cannot be $O(N_c^1)$ at large N_c [6]. However, consider such scattering near the soft limit of zero momentum and zero pion mass. In that regime unitarity imposes no constraints; zero-momentum scattering can have an arbitrarily large amplitude without violating unitarity. However, this is precisely the regime of relevance to chiral physics. Thus, nothing protects the convergence of the $1/N_c$ expansion in the chiral regime.

This ability of the interplay between chiral and $1/N_c$ limits to shed light on the convergence of the $1/N_c$ expansion is provided by a number of examples in which the two expansions are known not to commute. In particular, it known that the coefficient of the leading term nonanalytic in m_π^2 ($\propto m_q^1$) dependence differs if one if one takes the chiral limit before the large N_c limit or after [12]. For example, the nucleon mass has a term proportional to $m_\pi^3 \propto m_q^{3/2}$ as its leading nonanalytic contribution. Simply expanding around the the chiral limit, one finds

$$m_N^{\text{LNA}} = -\frac{3g_A^2}{32f_\pi^2 m_\pi^2}, \qquad (1)$$

where LNA indicates leading nonanalytic contribution. Taking N_c large prior to the chiral limit, one finds

$$m_N^{\text{LNA}} = -\frac{9g_A^2}{32f_\pi^2 m_\pi^2}, \qquad (2)$$

which differs from Eq. (1) by a factor of 3. This behavior is ubiquitous; it is seen in all static nucleon quantities.

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It is not hard to uncover the origin of this difference in the special role played by the Δ at large N_c. Recall that $\Delta \equiv m_\Delta - m_N$ scales as $1/N_c$, implying that the Δ is degenerate with the nucleon at large N_c. For small N_c values (including 3) one has $m_\pi < \Delta$, and in this region one can compute using a standard chiral expansion with pion and nucleon intermediate states alone to obtain the leading nonanalytic dependence. However, as $\Delta \to 0$ the regime of validity of the standard chiral expansion goes to zero. Had one started with a large N_c expansion, then the Δ would be taken as degenerate with the nucleon, and pion loops with Δ intermediate states would contribute to the nonanalytic behavior, yielding a different result.

While this behavior is indeed ubiquitous, it is generally hidden from view. Note that there is no direct way to isolate from data the leading nonanalytic behaviors of static quantities. The only direct way to access them is via lattice simulations of the relevant quantities for varying quark mass values. In contrast, the quantities associated with πN scattering studied in this paper are directly obtainable via experiment.

We focus here on two issues. The first is a description of the constants that parametrize the threshold behavior (i.e., the scattering lengths). Their chiral behavior has been known for decades [13]; however, their relationship with large N_c has not received the attention it deserves. For convenience we use the formalism of Ref. [13], which uses

a chiral Lagrangian formalism including an explicit Δ, facilitating our comparison with the large N_c limit. The second issue concerns sum rules relating low-energy or static quantities to integrals of πN scattering; we focus on the Adler-Weisberger and Goldberger-Miyazawa-Oehme sum rules.

In Sec. II we compute the $1/N_c$ expansions of the S and P waves for πN scattering, while Sec. III studies the scattering lengths in the $1/N_c$ and chiral expansions. The famous πN scattering sum rules are considered in Sec. IV, and in Sec. V we conclude.

II. πN SCATTERING IN THE $1/N_c$ EXPANSION

In Ref. [2] the current authors derived a master expression for linear relations among πN scattering amplitudes, including subleading corrections in the $1/N_c$ expansion. The method is very straightforward: As has been known since the 1980s [14], the leading-order $[O(N_c^0)]$ amplitudes satisfy the rule $I_t = J_t$. But it is equally true [15] that amplitudes with $|I_t - J_t| = n$ are at most $O(1/N_c^n)$. Therefore, all terms of $O(1/N_c)$ not simply proportional to the leading-order $\mathcal{J} \equiv I_t = J_t$ terms have either $x \equiv I_t = J_t - 1$ or $y \equiv I_t = J_t + 1$. Once this is taken into account, a distinct $O(N_c^0)$ reduced amplitude s, determined by the underlying QCD dynamics, appears for each structure:

$$S_{LL'RR'I_sJ_s} = \sum_{\mathcal{J}} \begin{bmatrix} 1 & R' & I_s \\ R & 1 & \mathcal{J} \end{bmatrix} \begin{bmatrix} L' & R' & J_s \\ R & L & \mathcal{J} \end{bmatrix} s^t_{\mathcal{J}LL'} - \frac{1}{N_c} \sum_x \begin{bmatrix} 1 & R' & I_s \\ R & 1 & x \end{bmatrix} \begin{bmatrix} L' & R' & J_s \\ R & L & x+1 \end{bmatrix} s^{t(+)}_{xLL'}$$

$$- \frac{1}{N_c} \sum_y \begin{bmatrix} 1 & R' & I_s \\ R & 1 & y \end{bmatrix} \begin{bmatrix} L' & R' & J_s \\ R & L & y-1 \end{bmatrix} s^{t(-)}_{yLL'} + O(1/N_c^2). \tag{3}$$

This expression represents a spinless, nonstrange isovector meson (π) scattering from a nonstrange baryon of $I = J = R$ (for N_c large, baryons in this ground-state band, including the N and Δ, are stable), through the Lth partial wave into an intermediate state of quantum numbers I_s, J_s. Primes indicate final-state quantum numbers. In fact, the quantum numbers of the mesons can be generalized [3–5,14,16], and the $I_t = J_t$ rule also holds for 3-flavor processes [5]. The square-bracketed quantities in this expression are trivial redefinitions of the standard $6j$ symbols, called $[6j]$ symbols in Ref. [2], and are used to compactify the notation:

$$\begin{bmatrix} a & b & e \\ c & d & f \end{bmatrix} \equiv \frac{(-1)^{-(b+d+e+f)}}{([a][b][c][d])^{1/4}} \begin{Bmatrix} a & b & e \\ c & d & f \end{Bmatrix}, \tag{4}$$

where $[a] \equiv 2a + 1$ denotes a multiplicity factor. In particular, the $[6j]$ symbols retain all the usual triangle rules of their $6j$ counterparts.

Restricting to the case of πN scattering ($R = R' = \frac{1}{2}$), $\mathcal{J}$ can only be 0 or 1, x can only be 0, and y can only be 1; indeed, no new structures arise at $O(1/N_c^2)$, although the reduced amplitudes s themselves have $O(1/N_c)$ and higher-order corrections. Of course, arbitrarily high partial waves are permitted since L, L', and J_s have no upper bound, but since we are interested in soft pions with the characteristic threshold behavior k^{2L}, only the lowest partial waves are of interest. For this paper we limit to S- and P-wave amplitudes. The latter are especially interesting because Δ intermediate states appear there as resonant poles and contribute prominently; indeed, we shall see that one cannot obtain the correct large N_c scaling of P-wave scattering lengths unless cancellations between intermediate N and Δ states occur. This result is strongly reminiscent of the consistency condition approach [6] of imposing order-by-order unitarity in powers of $1/N_c$ in meson-baryon scattering.

Using Eq. (3), the expansions for the S and P waves read

$$S_{11} = s_{000}^t - \frac{1}{N_c} \frac{2}{\sqrt{6}} s_{100}^{t(-)}, \qquad S_{31} = s_{000}^t + \frac{1}{N_c} \frac{1}{\sqrt{6}} s_{100}^{t(-)},$$

$$P_{11} = \left(s_{011}^t + \frac{2}{3} s_{111}^t \right) - \frac{1}{N_c} \frac{2}{\sqrt{6}} (s_{011}^{t(+)} + s_{111}^{t(-)}),$$

$$P_{31} = \left(s_{011}^t - \frac{1}{3} s_{111}^t \right) - \frac{1}{N_c} \frac{2}{\sqrt{6}} \left(s_{011}^{t(+)} - \frac{1}{2} s_{111}^{t(-)} \right),$$

$$P_{13} = \left(s_{011}^t - \frac{1}{3} s_{111}^t \right) + \frac{1}{N_c} \frac{2}{\sqrt{6}} \left(\frac{1}{2} s_{011}^{t(+)} - s_{111}^{t(-)} \right), \qquad (5)$$

$$P_{33} = \left(s_{011}^t + \frac{1}{6} s_{111}^t \right) - \frac{1}{N_c} \frac{1}{\sqrt{6}} (s_{011}^{t(+)} + s_{111}^{t(-)}).$$

Inverting these equations yields

$$s_{000}^t = \frac{1}{3} (S_{11} + 2S_{31}), \qquad \frac{s_{100}^{t(-)}}{N_c} = -\sqrt{6} \left[\frac{1}{3} (S_{11} - S_{31}) \right], \qquad (6)$$

$$s_{011}^t = \frac{1}{3} \left\{ \left[\frac{1}{3} (P_{11} + 2P_{31}) \right] + 2 \left[\frac{1}{3} (P_{13} + 2P_{33}) \right] \right\},$$

$$s_{111}^t = 2 \left\{ \left[\frac{1}{3} (P_{11} - P_{31}) \right] - \left[\frac{1}{3} (P_{13} - P_{33}) \right] \right\},$$

$$\frac{s_{011}^{t(+)}}{N_c} = -\sqrt{\frac{2}{3}} \left\{ \left[\frac{1}{3} (P_{11} + 2P_{31}) \right] - \left[\frac{1}{3} (P_{13} + 2P_{33}) \right] \right\}, \qquad (7)$$

$$\frac{s_{111}^{t(-)}}{N_c} = -\sqrt{\frac{2}{3}} \left\{ \left[\frac{1}{3} (P_{11} - P_{31}) \right] + 2 \left[\frac{1}{3} (P_{13} - P_{33}) \right] \right\}.$$

As mentioned above, no new structures arise at $O(1/N_c^2)$; therefore, Eqs. (6) and (7) are *exact*. The $1/N_c$ expansion predicts each reduced amplitude s to be $O(N_c^0)$ [although, again, each s has its own $O(1/N_c)$ corrections]. We see that the second of Eq. (6) and the last two of Eq. (7) must be $O(1/N_c)$ at all energies. While performing such a calculation at arbitrary energy would require a mastery of non-perturbative QCD, it is nevertheless possible to compute the amplitudes for soft pions using the technology of the chiral Lagrangian; this is the task we consider in the next section.

III. SCATTERING LENGTHS

The calculation of S- and P-wave scattering lengths of πN scattering, performed by Peccei nearly four decades ago [13] was one of the very first calculations carried out using the chiral Lagrangian formalism. Scattering lengths refer to behavior of the amplitudes in the soft-pion limit; the scattering length corresponding to the L_{IJ} partial-wave amplitude is defined the coefficient of its threshold ($\propto k^{2L}$) behavior, where k is the c.m. momentum of the π. In addition, it is conventional to multiply by m_π^{2L+1} to obtain a dimensionless quantity.

The lowest-order (tree-level) results for the S- and P-wave scattering amplitudes, including both N and Δ

intermediate states, are presented in Appendix A, Eqs. (A6)–(A10), as are the relations between the couplings used in Ref. [13] and the usual couplings $g_{\pi NN}$ and $g_{\pi N\Delta}$, and the large N_c scalings of all relevant quantities. It is our task in this section to expand the scattering amplitude expressions given in Appendix A in $1/N_c$, and demonstrate that the scalings found for the full partial-wave amplitudes in Eqs. (6) and (7) are supported by their threshold behaviors. In particular, using the notation introduced in Eq. (A5), we wish to show that

$$a_S^+, \qquad a_{P_{1/2}}^+ + 2a_{P_{3/2}}^+, \qquad a_{P_{1/2}}^- - a_{P_{3/2}}^- = O(N_c^0),$$

$$a_S^-, \qquad a_{P_{1/2}}^+ - a_{P_{3/2}}^+, \qquad a_{P_{1/2}}^- + 2a_{P_{3/2}}^- = O(1/N_c). \qquad (8)$$

In this way, one may explore the nature of the combined chiral and large N_c limits.

Let us begin by considering the chiral limit ($m_\pi \to 0$) independent of the $1/N_c$ limit. Including the factors of m_π that make the scattering amplitudes dimensionless, the generic scaling is $O(m_\pi^2)$. The exceptions, as one may check using Eqs. (A6)–(A10), are the combinations

$$a_S^+, \qquad a_{P_{1/2}}^+ + 2a_{P_{3/2}}^+, \qquad a_{P_{1/2}}^- - a_{P_{3/2}}^-,$$

$$a_{P_{3/2}}^- + a_{P_{3/2}}^+ = O(m_\pi^3). \qquad (9)$$

The result $a_S^+ = O(m_\pi^3)$ gives the celebrated Weinberg-Tomozawa (WT) relation [17],

$$a_S^{I=1/2} = -2a_S^{I=3/2} = \frac{g_V^2}{4\pi f_\pi^2} \frac{m_\pi^2}{1 + m_\pi/M} + O\left(\frac{m_\pi^3}{M^3}\right), \qquad (10)$$

where $M \equiv m_N$, and the ratio $-2a_S^{I=3/2}/a_S^{I=1/2}$ experimentally equals unity to within 5% [18,19].

Now we present the amplitudes in the $1/N_c$ expansion. The expressions are presented twice: first imposing just the well-known N-Δ large N_c mass degeneracy

$$\Delta \equiv m_\Delta - m_N = O(1/N_c), \qquad (11)$$

and then imposing Eq. (A2), $g_{\pi N\Delta}/g_{\pi NN} = \frac{3}{2} \times [1 + O(1/N_c^2)]$. The purpose is to show the necessity of including the Δ degree of freedom in some of the P-wave (but not S-wave) results to obtain the correct $1/N_c$ counting.

Starting with the S waves,

$$a_S^+ = -\frac{m_\pi^3}{16\pi M^3} \frac{g_{\pi NN}^2 + 3g_{\pi N\Delta}^2}{1 + m_\pi/M} + O\left(\frac{1}{N_c^2}\right)$$

$$= -\frac{31 m_\pi^3 g_{\pi NN}^2}{64\pi M^3 (1 + m_\pi/M)} + O\left(\frac{1}{N_c^2}\right), \qquad (12)$$

The factor $1 + m_\pi/M$ is retained because it is a source of known $O(1/N_c)$ corrections. The exhibited term scales as N_c^0. Next,

$$a_S^- = \frac{m_\pi^2}{32\pi(1+m_\pi/M)}\left[\frac{4g_V^2}{f_\pi^2} + \frac{m_\pi^2}{M^4}(g_{\pi NN}^2 + g_{\pi N\Delta}^2)\right]$$
$$+ O\left(\frac{1}{N_c^3}\right)$$
$$= \frac{m_\pi^2}{128\pi(1+m_\pi/M)}\left[\frac{16g_V^2}{f_\pi^2} + \frac{13m_\pi^2}{M^4}g_{\pi NN}^2\right] + O\left(\frac{1}{N_c^3}\right),$$

$$(13)$$

where the exhibited terms scale as $1/N_c$. We see that the leading term in a strict chiral expansion [the g_V^2 term in Eq. (13)], which gives rise to Eq. (10), is actually subleading in the $1/N_c$ expansion and comparable to another term in that expansion; this is an excellent illustration of the noncommutativity of the chiral and large N_c limits. On the other hand, we confirm the formal results $a_S^+ = O(N_c^0)$ and $a_S^- = O(1/N_c)$.

Before proceeding, a comment is in order. While the general phenomenon of the noncommutativity of the large N_c and chiral limits is quite reminiscent of the leading nonanalytic chiral behavior of static nucleon properties, the underlying *mechanism* in this case is rather different. In the static nucleon case, the key issue is the role of the Δ. In the case of a_S^-, it is the quantum numbers in the t channel; its leading chiral behavior comes from a t-channel exchange with quantum numbers that are scalar-isovector (i.e., like the time component of the ρ meson), which are suppressed at large N_c [6].

Which expansion works better experimentally? Since we are merely keeping track of $1/N_c$ factors but calculating in the soft-pion limit, it stands to reason that results based on the chiral limit should be much more accurate in this case (as we saw for the WT relation). In order to quantify this belief, one may numerically compute the leading coefficients in Eqs. (12) and (13). Despite the former being formally $O(N_c^1)$ larger than the latter, up to a sign the two are numerically almost precisely equal (∓ 0.081, respectively), surely a coincidence. The strict $1/N_c$ expansion correctly predicts a_S^- but grossly overestimates a_S^+; experimentally, $a_S^- = +0.08676$, $a_S^+ = -0.00135$ [19].

The expansions for the P-wave scattering lengths are

$$a_{P_{1/2}}^+ + 2a_{P_{3/2}}^+ = \frac{g_{\pi N\Delta}^2 m_\pi}{144\pi M^3(1+m_\pi/M)}[16M\Delta - 17m_\pi^2]$$
$$+ O\left(\frac{1}{N_c}\right),$$

$$(14)$$

$$a_{P_{1/2}}^- - a_{P_{3/2}}^- = \frac{g_{\pi N\Delta}^2 m_\pi}{288\pi M^3(1+m_\pi/M)}[-8M\Delta + 7m_\pi^2]$$
$$+ O\left(\frac{1}{N_c}\right),$$

$$(15)$$

which are both $O(N_c^0)$ [and $O(m_\pi^3)$; see below]. The numerical comparisons are $+0.221$ vs $+0.543 + O(1/N_c)$

and $+0.0695$ vs $-0.138 + O(1/N_c)$, respectively. The agreement is apparently poor, and demonstrates that the strict leading-order result does not dominate; clearly, the $O(1/N_c)$ corrections are substantial, as including even a naive estimate of their magnitude $[O(1/3)]$ indicates: The $1/N_c$ expansion here is numerically true, but not very predictive.

Superficially, one finds the other two combinations to be $O(N_c^0)$ as well:

$$a_{P_{1/2}}^- + 2a_{P_{3/2}}^- = -\frac{m_\pi^2(9g_{\pi NN}^2 - 4g_{\pi N\Delta}^2)}{72\pi M^2} + O\left(\frac{1}{N_c}\right),$$

$$(16)$$

$$a_{P_{1/2}}^+ - a_{P_{3/2}}^+ = -\frac{m_\pi^2(9g_{\pi NN}^2 - 4g_{\pi N\Delta}^2)}{72\pi M^2} + O\left(\frac{1}{N_c}\right), \quad (17)$$

but using the relation Eq. (A2) between $g_{\pi NN}$ and $g_{\pi N\Delta}$, in fact the combinations turn out to be

$$a_{P_{1/2}}^- + 2a_{P_{3/2}}^- = \frac{g_{\pi NN}^2}{64\pi M^4(1+m_\pi/M)}$$
$$\times \left\{8M\left[Mm_\pi^2\left(\frac{4g_{\pi N\Delta}^2}{9g_{\pi NN}^2} - 1\right) - m_\pi^2\Delta\right.\right.$$
$$\left.\left. + M\Delta^2\right] - m_\pi^4\right\} + O\left(\frac{1}{N_c^2}\right),$$

$$(18)$$

$$a_{P_{1/2}}^+ - a_{P_{3/2}}^+ = \frac{g_{\pi NN}^2}{64\pi M^4(1+m_\pi/M)}$$
$$\times \left\{8M\left[Mm_\pi^2\left(\frac{4g_{\pi N\Delta}^2}{9g_{\pi NN}^2} - 1\right) - m_\pi^2\Delta\right.\right.$$
$$\left.\left. + M\Delta^2\right] + m_\pi^4\right\} + O\left(\frac{1}{N_c^2}\right).$$

$$(19)$$

Using the scaling rules previously noted, one observes that the term explicitly given is actually $O(1/N_c)$ because each term inside the braces is $O(N_c^0)$. The scaling rules demanded by Eq. (8) are satisfied. In this case, the numerical comparisons are -0.153 vs $+0.570 + O(1/N_c)$ and -0.178 vs $+0.571 + O(1/N_c)$, respectively. Again, the agreement is unimpressive—after all, not even the signs are correct—but one does not require excessively large $O(1/N_c)$ corrections to bring them into accord.

Note moreover that the expansions given in Eqs. (14)–(19) curiously appear to violate the m_π^2 and m_π^3 scaling described in and above Eq. (9). This is yet another manifestation of the noncommutativity of the chiral and large N_c limits, and specifically originates in the question of whether $\delta \equiv m_\pi/\Delta \to 0$ (chiral limit) or ∞ (large N_c limit). As a purely phenomenological matter $\delta \approx 1/2$,

and one can consider a hybrid limit with $m_\pi, 1/N_c \to 0$ but δ finite. An excellent illustration of this fact is given by the unique combination that is both $O(m_\pi^3)$ [see Eq. (9)] and $O(1/N_c)$ [see Eq. (8)], which as one notes is the difference

$$a_{P_{1/2}}^- + 2a_{P_{3/2}}^- - a_{P_{1/2}}^+ + a_{P_{3/2}}^+ \equiv a_{P13} - a_{P31}$$

$$= \frac{m_\pi^3}{16\pi M(1 + m_\pi/M)}\left\{\frac{g_V^2}{f_\pi^2} - \frac{m_\pi}{M^3}\left[g_{\pi NN}^2 - \frac{2}{9}\left(\frac{3-\delta}{1+\delta}\right)g_{\pi N\Delta}^2\right]\right\} + O\left(\frac{m_\pi^3}{N_c^3}\right) + O\left(\frac{m_\pi^4}{N_c^2}\right). \quad (20)$$

The g_V term is $O(m_\pi^3/N_c^2)$ and the axial terms are $O(m_\pi^4/N_c)$, and are now treated as comparable. The non-commutativity is clearly seen in the coefficient multiplying $g_{\pi N\Delta}^2$: The factor in parentheses is $+3$ in the chiral limit and -1 in the large N_c limit; the latter appears when taking Eq. (18) minus Eq. (19). The numerical comparison in this case is $+0.0256$ to $+0.0035$ (strict large N_c limit), $+0.0066$ (strict chiral limit), $+0.0056$ (physical value of δ; note that this value is numerically much closer to the chiral than large N_c limit). Again, the central value is not spot-on, since the corrections as indicated in Eq. (20) easily account for the difference, but note that the experimental value of $+0.0256$ is numerically much smaller than either of the other P-wave $O(m_\pi^3)$ combinations, Eqs. (14) and (15), an indication that the extra $1/N_c$ suppression is still significant even in the soft-pion case. One sees that the chiral limit dominates these quantities, but that $1/N_c$ suppressions still persist.

IV. PION SCATTERING SUM RULES

In this section we discuss the interplay of the chiral and large N_c limits in pion scattering sum rules. We first discuss the Adler-Weisberger (AW) sum rule [20] and then the Goldberger-Miyazawa-Oehme (GMO) sum rule [21], both of which raise interesting physical issues.

The interplay of chiral dynamics and large N_c QCD for the AW sum rule is qualitatively different from that of the WT relation. The WT relation is an approximate relation that becomes increasingly exact as the chiral limit is approached. In contrast, the AW sum rule is exact; it depends only on current algebra, certain assumptions about the analytic structure of the theory (in particular, on the convergence of the appropriate dispersion relation), and the absence of πN bound states. Since the AW relation is a sum rule, it requires information about pion scattering at all energies and not merely for soft pions. In this sense it is not a "chiral" relation. However, as will become clear in this section, an understanding of the relation at large N_c depends upon some subtle issues in chiral physics.

The AW sum rule reads

$$g_A^2 - 1 = \frac{2f_\pi^2}{\pi}\int_{m_\pi}^\infty \frac{d\omega}{\omega}\sqrt{\omega^2 - m_\pi^2}(\sigma_{\pi^+ p} - \sigma_{\pi^- p}), \quad (21)$$

of Eq. (18) and (19). Indeed, this combination turns out to be simply the threshold difference between P_{13} and P_{31}. Working in the hybrid expansion, one finds

where $\sigma_{\pi^\pm p}$ is the total cross section for $\pi^\pm$ scattering off a proton as a function of π c.m. energy ω.

As noted by a number of groups [22], this relation superficially contradicts Witten's large N_c scaling rules [23]. The apparent contradiction arises since

$$g_A^2 = O(N_c^2), \qquad f_\pi^2 = O(N_c^1), \qquad \sigma_{\pi^\pm p} = O(N_c^0), \quad (22)$$

which seems to imply that the left-hand side of the relation scales as N_c^2 while the right-hand side scales as N_c^1.

The resolution of this issue, as discussed in detail by Broniowski [24], is the special role played by the Δ resonance. Since the Δ is a partner of the nucleon in the contracted $SU(2N_f)$ symmetry, it behaves differently than other resonances; its mass is anomalously low compared to typical resonances, since $\Delta \equiv m_\Delta - m_N = O(1/N_c)$ rather than $O(N_c^0)$ as for a typical baryonic excited state, and has an anomalously strong coupling to the πN channel: $g_{\pi N\Delta} = O(N_c^{1/2})$ rather than the typical $O(N_c^0)$. The contribution of the Δ is anomalously large, $O(N_c^2)$, and it dominates the AW sum-rule at both leading order and next-to-leading order in a $1/N_c$ expansion.

The Δ contribution to the sum rule can be computed [24] in the chiral limit and in a $1/N_c$ expansion. It is tractable because the Δ is narrow: Its width scales as $1/N_c^2$ in the chiral limit. Using a narrow-width approximation, the Δ contributes an amount $(g_A^*)^2$ to the right-hand side of the sum rule, where g_A^* is given by

$$g_A^* \equiv \frac{2g_{\pi N\Delta}}{3g_{\pi NN}}g_A + O(1/N_c) = g_A + O(1/N_c), \quad (23)$$

where the Goldberger-Treiman (GT) relation (A3) has been used to relate pion couplings to g_A. The second equality follows from contracted $SU(2N_f)$ symmetry, which fixes the ratio $g_{\pi N\Delta}/g_{\pi NN}$ to be $\frac{3}{2}$ at leading and next-to-leading order in $1/N_c$. This allows the AW sum rule to be rewritten in the form

$$(g_A^2 - g_A^{*2}) - 1 = \frac{2f_\pi^2}{\pi}\int_{m_\pi}^\infty \frac{d\omega}{\omega}\sqrt{\omega^2 - m_\pi^2}(\bar\sigma_{\pi^+ p} - \bar\sigma_{\pi^- p}), \quad (24)$$

where $\bar{\sigma}_{\pi^+ p}$ indicates cross sections with the Δ contribution removed. The quantity $(g_A^2 - g_A^{*2})$ is $O(N_c^0)$, so that the left-hand side is no longer characteristically larger than the right-hand side. Indeed, the right-hand side receives contributions from $\bar{\sigma}_{\pi^+ p}$ and $\bar{\sigma}_{\pi^- p}$, each of which is $O(N_c^1)$ (due to the f_π^2). Evidently, these two contributions must cancel to leading order in $1/N_c$. An analysis of why this happens appears in Ref. [24].

We prefer to recast this analysis slightly, using the language of Refs. [1–5]. To begin with, one should note that the total πN cross section can be related to the imaginary part of a forward scattering amplitude via the optical theorem. Moreover, by isospin invariance $\bar{\sigma}_{\pi^- p} = \bar{\sigma}_{\pi^+ n}$, while by rotational invariance the total cross section (and hence the forward amplitude) must be independent of the spin state of the nucleon. From the perspective of the nucleon, the scattering amplitude can be represented by some operator that depends on the nucleon spin and isospin and parametrically on the momentum transfer. Thus $(\bar{\sigma}_{\pi^+ p} - \bar{\sigma}_{\pi^- p})$ acts on the space of nucleon states as a scalar-isovector operator. However, contracted $SU(2N_f)$ symmetry implies the $I = J$ rule [15], which states that matrix elements of operators characterized by an isospin I and angular momentum J in nucleon states scale according to $N_c^{-|I-J|}$ times the generic N_c scaling obtained via Witten's N_c counting (which, for scattering amplitudes, gives N_c^0). This in turn implies

$$(\bar{\sigma}_{\pi^+ p} - \bar{\sigma}_{\pi^- p}) = O(1/N_c), \tag{25}$$

which is the required cancellation. Note that this cancellation occurs at the level of the integrand and not the integral.

Reference [24] also considered the right-hand side in terms of contributions of baryon resonances, and noted that it is not obvious one may legitimately describe the right-hand side in terms of baryon resonances since their widths are $O(N_c^0)$. However, if one were to describe the right-hand in terms of baryon resonances, one would find that for a cancellation to occur at all energies (as it does in the integral), the masses of classes of different resonances must be (nearly) degenerate and their relative couplings must be proportional. That is, large N_c consistency rules must provide group-theoretic constraints on the masses and couplings of the resonances. It is interesting to observe that that that formalism of Refs. [1–5] provides precisely such constraints for baryon resonances. At large N_c they fall into degenerate multiplets labeled by an emergent quantum number K, and the relative couplings are fixed by $SU(2N_f)$ Clebsch-Gordan factors. Of course, this is hardly surprising: The underlying physical input of the formalism of Refs. [1–5] is the $I = J$ rule for the t-channel amplitudes; the constraints emerge when one recasts the physics into the s channel.

The preceding analysis of this problem in Ref. [24] is quite elegant and resolves the fundamental issues.

However this analysis raises an interesting issue we wish to address here, concerning the interplay of chiral dynamics and the large N_c limit. It seems evident that this interplay is important, given the role of a chiral relation (the GT relation) in deriving Eq. (23). Indeed, as noted above, Eq. (23) is strictly only valid in the chiral limit; an incisive question is what happens away from this limit. The problem has an interesting feature, in that the AW sum rule is exact: It has no chiral corrections. Thus the matching of the two sides does not depend upon the precise size of the explicit chiral symmetry breaking; similarly the N_c matching of the two sides should continue to hold as one varies m_π.

In this context, the interplay of the chiral and large N_c limits is crucial. If one takes the large N_c limit prior to taking the chiral limit, the Δ drops below πN threshold and becomes stable, contrary to the situation in the physical world. If the ordering of limits is taken the other way, the Δ remains above threshold and accessible via πN scattering. The distinction between the two cases is critical since it determines the region of validity of the AW sum rule in Eq. (21). Recall that its derivation requires the πN scattering states to form a complete set; thus, if there are πN bound states the relation ceases to be valid. Thus if one takes the large N_c limit prior to the chiral limit, the AW sum rule is no longer valid, and for obvious reason: The Δ contributes to the sum rule (indeed, dominates it) regardless of whether or not it is above threshold. However, the form of Eq. (21) is such that it contributes only if it is a scattering state. Thus, Eq. (21) as written holds only if the Δ is above threshold.

Suppose we restrict our attention to the regime of validity of Eq. (21), i.e., the region of an unstable Δ. Then, as one approaches the large N_c limit, of necessity one is approaching the chiral limit:

$$m_\pi < \Delta = O(1/N_c). \tag{26}$$

This in turn implies that the size of chiral corrections are, of necessity, bounded. In particular, let us consider chiral corrections to Eqs. (23) and (24). These arise due to chiral corrections to the GT relation, which essentially give a form factor correction in the pion coupling from $q^2 = 0$ to $q^2 = m_\pi^2$; as such, it is of relative $O(m_\pi^2/\Lambda^2)$, where Λ is a typical hadronic scale characterizing the form factor. In the regime of validity of Eq. (21), such a term is bounded parametrically to be less than of relative $O(1/N_c^2)$. Since $g_A^* = O(N_c^1)$, this means that errors induced by chiral corrections are bounded parametrically to be of $O(1/N_c)$ or less. However, there are already $1/N_c$ corrections of this order in Eq. (23). Since corrections of this scale are consistent with (indeed, lead to) the N_c scaling of Eq. (24), it is apparent that chiral corrections cannot spoil the N_c counting consistency, provided the system is in the regime of validity of the AW sum rule.

Next let us briefly consider the GMO sum rule. This relation connects a threshold quantity, a_S^-, to an integral over πN scattering. A useful representation is given in Ref. [13]:

$$4\pi\left(1 + \frac{m_\pi}{M}\right)a_S^- = \frac{2g_{\pi NN}^2 m_\pi^2}{4M^2 - m_\pi^2}$$
$$- \frac{m_\pi^2}{\pi}\int_{m_\pi^2}^\infty \frac{d\omega}{\sqrt{\omega^2 - m_\pi^2}}(\sigma_{\pi^+ p} - \sigma_{\pi^- p}),$$

$$(27)$$

where the first term on the right-hand side represents the contribution of the nucleon Born terms.

As noted in Ref. [24], many of the issues associated with the large N_c behavior of the GMO parallel the AW sum rule. Like the AW relation, there is an apparent discrepancy between the N_c scaling of the two sides. The right-hand side naively scales as N_c^1 due to the Born terms, while the left-hand side scales as $1/N_c$ from the scaling of a_S^-. Fortunately, the Δ contribution to the scattering cancels off the nucleon Born term at both leading order and the first subleading order in $1/N_c$, yielding consistency in N_c counting.

As in the case of the AW sum rule, the GMO sum rule is exact provided there are no bound states in the πN channel, which constrains the system to approach the chiral limit as fast as the large N_c limit in order to lie in the regime of validity of the sum rule. This issue is less important in the case of the GMO sum rule than for the AW sum rule since the sum rule does not rely on current algebra, and hence the scaling of the GT discrepancy with m_π plays no role in the derivation of the sum rule.

Similarly, the chiral behavior of the GMO sum rule is straightforward. The left-hand side, as seen in the last section, is expected to be $O(m_\pi^2)$, which matches the factor on the right-hand side.

However, the large N_c and chiral behavior holds a surprise in the chiral dependence of contributions proportional to $g_{\pi NN}^2$. In Eq. (27) this term contributes at leading chiral order [$O(m_\pi^2)$]. In contrast, in Eq. (13) the term proportional $g_{\pi NN}^2$ is of order $O(m_\pi^4)$. Clearly, this is possible only if the scattering term in the sum rule cancels out the dominant term proportional to $g_{\pi NN}^2$, yielding a term at a higher chiral order. The scattering terms in the sum rule must "know" about the value of $g_{\pi NN}^2$ and conspire to cancel the leading chiral behavior of the Born term. This is a purely chiral phenomenon and has nothing to do with N_c. The way that this comes about seems rather mysterious. However, the $1/N_c$ expansion provides some insights about how it can occur. The dominant term proportional to $g_{\pi NN}^2$ in the GMO sum rule is $O(N_c^1)$, two orders larger than the total. Accordingly it is canceled by the Δ contribution as it must be to ensure large N_c consistency. Thus, the large N_c limit provides a natural mecha-

nism by which the Born term may be canceled: exactly as is needed to obtain consistency in chiral counting.

V. CONCLUSIONS

We have shown that the chiral and $1/N_c$ expansions may be considered simultaneously, although one must take special care to accommodate the noncommutativity of their limits associated with the parameter $\delta \equiv m_\pi/(m_\Delta - m_N)$.

The scattering lengths, which are intrinsically soft-pion quantities, naturally obey a more rapidly convergent chiral expansion than $1/N_c$ expansion; nevertheless, a strict $1/N_c$ expansion is still meaningful, with numerically large but parametrically natural $O(1/N_c)$ corrections. We have also seen that a combination of scattering amplitudes ($a_{P13} - a_{P31}$) at the same order of m_π but suppressed by $1/N_c$ is indeed experimentally smaller than those without the $1/N_c$ suppression. The $1/N_c$ expansion continues to work even in the hostile environment of soft-pion physics.

The Adler-Weisberger and Goldberger-Miyazawa-Oehme sum rules have an especially interesting structure in terms of the chiral and $1/N_c$ expansions. In particular, the Δ contribution must be separated out of the cross section integrals by hand before the proper counting in each expansion is manifest. If one takes the $1/N_c$ limit prior to the chiral limit, then the Δ becomes a stable state degenerate with the nucleon, and, in particular, is no longer a πN scattering state that appears in the integrals, invalidating the usual forms of the sum rules.

APPENDIX: πN SCATTERING AMPLITUDE EXPRESSIONS IN CHIRAL PERTURBATION THEORY

For convenience of the reader, we present here the expressions for πN scattering from Ref. [13]. The calculation there was presented at lowest (tree) order, and included π, N, ρ, and Δ degrees of freedom; however, Ref. [13] conveniently describes how to eliminate from those expressions (in the modern terminology, integrate out) ρ degrees of freedom in favor of π only.

Three π-baryon couplings appear in Ref. [13]: f and f_0 for πNN in Eq. (6), and h for $\pi N\Delta$ in Eq. (17). In perhaps more familiar notation,

$$f = -\frac{g_{\pi NN}m_\pi}{2m_N}, \qquad f_0 = \frac{g_V m_\pi}{2f_\pi}, \qquad h = \frac{g_{\pi N\Delta}m_\pi}{4\sqrt{2}m_N},$$

$$(A1)$$

where numerically $g_{\pi NN} \simeq 13.5$ and $f_\pi \simeq 93$ MeV, and $g_V = 1$ because it represents the polar vector current coupling via the chirally-covariant derivative in the πNN kinetic energy term. $g_{\pi N\Delta}$ can be extracted from the $\Delta \to \pi N$ width (and turns out experimentally to be about 20); however, in the spirit of the $1/N_c$ expansion, the N and Δ lie in the same multiplet and the ratio $g_{\pi N\Delta}/g_{\pi NN}$ has a fixed value. Indeed, if one uses the group theory treating N and Δ as bound states of N_c u and d quarks, one finds [25]

$$\frac{g_{\pi N\Delta}}{g_{\pi NN}} = \frac{3}{2}\frac{\sqrt{(N_c - 1)(N_c + 5)}}{N_c + 2} = \frac{3}{2} + O\left(\frac{1}{N_c^2}\right). \quad (A2)$$

The absence of an $O(1/N_c)$ correction is known also from the contracted spin-flavor symmetry that relates the N and Δ at large N_c [6]. As for $g_{\pi NN}$, it may be expressed using the Goldberger-Treiman relation [26]

$$g_{\pi NN} = g_A \frac{m_N}{f_\pi}, \quad (A3)$$

up to chiral corrections. Since $g_A = O(N_c^1)$ (experimentally 1.26, it equals $(N_c + 2)/3$ in the quark picture [25]) while $m_N = O(N_c^1)$ and $f_\pi = O(N_c^{1/2})$, we have $g_{\pi NN}, g_{\pi N\Delta} = O(N_c^{3/2})$, and therefore $f, h = O(N_c^{1/2})$ and $f_0 = O(N_c^0)$. It follows that

$$\frac{h^2}{f^2} = \frac{9}{32} + O\left(\frac{1}{N_c^2}\right), \qquad \frac{f_0^2}{f^2} = O\left(\frac{1}{N_c^2}\right). \quad (A4)$$

Similarly, as is well known, the $O(N_c^1)$ Δ mass m_Δ (M^* in Ref. [13]) is split from m_N (M in Ref. [13]) only at $O(1/N_c)$, while $m_\pi = O(N_c^0)$.

The results of Ref. [13] are presented in terms of isospin-even and -odd combinations of the $I = \frac{1}{2}$ and $\frac{3}{2}$ amplitudes $\mathcal{A}$ for each partial wave [Eqs. (20) and (21)]:

$$\begin{aligned}
\mathcal{A}^+ &= \tfrac{1}{3}(\mathcal{A}_{1/2} + 2\mathcal{A}_{3/2}), \\
\mathcal{A}^- &= \tfrac{1}{3}(\mathcal{A}_{1/2} - \mathcal{A}_{3/2}).
\end{aligned} \quad (A5)$$

Comparing to Eqs. (6) and (7) above, we see that $\mathcal{A}^+$ and $\mathcal{A}^-$ correspond neatly to $I_t = 0$ and 1, respectively.

According to the definitions of Ref. [13], Eq. (25), the scattering lengths are strictly speaking not actually lengths (or volumes, for P waves), but are made dimensionless by virtue of including appropriate powers of m_π. The S-wave scattering lengths a_S (i.e., the amplitudes in the soft-pion limit) have only $J = \frac{1}{2}$, while P-wave scattering lengths a_P (i.e., the first derivative of the amplitudes with respect to the square of the π c.m. momentum in the soft-pion limit) allow both $J = \frac{1}{2}$ and $\frac{3}{2}$.

We now present the expressions of Ref. [13] for $a_S^\pm$, $a_{P_{1/2}}^\pm$, and $a_{P_{3/2}}^\pm$, where the final subscript indicates the value of J. From that article's Eqs. (30), (38), (33), and (39), respectively,

$$a_S^- = \frac{1}{4\pi(1 + m_\pi/M)}\left(\frac{2m_\pi^2}{4M^2 - m_\pi^2}f^2 + 2f_0^2 + \frac{4m_\pi^2}{M^{*2}}h^2\right), \quad (A6)$$

$$a_S^+ = \frac{1}{4\pi(1 + m_\pi/M)}\left[-\frac{4Mm_\pi}{4M^2 - m_\pi^2}f^2 - \frac{8(2M^* + M)m_\pi}{M^{*2}}h^2\right], \quad (A7)$$

$$a_{P_{3/2}}^- = \frac{1}{36\pi(1 + m_\pi/M)}\Bigg\{-\frac{6}{(1 - m_\pi/2M)^2}f^2 + \left(-\frac{16m_\pi}{M^* - (M + m_\pi)} + \frac{m_\pi}{M^{*2} - (M - m_\pi)^2}\right.$$
$$\times \left.\left[8M^* + 4M - 12m_\pi + \frac{8}{3M^*}(-M^2 + 6Mm_\pi + 3m_\pi^2) + \frac{4}{3M^{*2}}(M^3 + 11M^2m_\pi - 9Mm_\pi^2 - 3m_\pi^3)\right]\right)h^2\Bigg\}, \quad (A8)$$

$$a_{P_{3/2}}^+ = \frac{1}{36\pi(1 + m_\pi/M)}\Bigg\{\frac{6}{(1 - m_\pi/2M)^2}f^2 + \left(\frac{32m_\pi}{M^* - (M + m_\pi)} + \frac{2m_\pi}{M^{*2} - (M - m_\pi)^2}\right.$$
$$\times \left.\left[8M^* + 4M - 12m_\pi + \frac{8}{3M^*}(-M^2 + 6Mm_\pi + 3m_\pi^2) + \frac{4}{3M^{*2}}(M^3 + 11M^2m_\pi - 9Mm_\pi^2 - 3m_\pi^3)\right]\right)h^2\Bigg\}, \quad (A9)$$

while from Eq. (35), applied to not only $-$ but also $+$ amplitudes, we have

$$a_{P_{1/2}}^\pm = a_{P_{3/2}}^\pm - \frac{m_\pi^2}{4M^2}a_S^\pm - \frac{m_\pi}{16\pi M}R^{(\pm)}, \quad (A10)$$

where [Eqs. (34) and (40) of Ref. [13], respectively]

$$R^{(-)} = -\frac{4m_\pi^2}{4M^2 - m_\pi^2} f^2 - 4f_0^2 + \left\{ -\frac{\frac{4}{3}(M^{*2} - M^2 - m_\pi^2)}{[(M^* + M)^2 - m_\pi^2][(M^* - M)^2 - m_\pi^2]} \right.$$

$$\times \left[6m_\pi^2 + M(8M^* + 12M) + \frac{8M}{3M^*}(M^2 - 3m_\pi^2) - \frac{2}{3M^{*2}}(2M^4 - M^2 m_\pi^2 + 9m_\pi^4) \right]$$

$$\left. + \frac{\frac{32}{3}M^2 m_\pi^2}{[(M^* + M)^2 - m_\pi^2][(M^* - M)^2 - m_\pi^2]} \left[1 + \frac{4M}{3M^*} - \frac{1}{3M^{*2}}(M^2 - 6m_\pi^2) \right] \right\} h^2, \tag{A11}$$

$$R^{(+)} = \frac{32M^3}{m_\pi(4M^2 - m_\pi^2)} f^2 + \left\{ \left[\frac{44}{3}M^2 - 2M^{*2} + \frac{16MM^*}{3} + 4m_\pi^2 + \frac{16M}{3M^*}(M^2 - m_\pi^2) - \frac{2(M^2 - m_\pi^2)^2}{M^{*2}} \right] \right.$$

$$\left. \times \frac{\frac{16}{3}Mm_\pi}{[(M^* + M)^2 - m_\pi^2][(M^* - M)^2 - m_\pi^2]} \right\} h^2. \tag{A12}$$

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