Tests of two-body Dirac equation wave functions in the decays of quarkonium and positronium into two photons

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Two-body Dirac equations of constraint dynamics provide a covariant framework to investigate the problem of highly relativistic quarks in meson bound states. This formalism eliminates automatically the problems of relative time and energy, leading to a covariant three dimensional formalism with the same number of degrees of freedom as appears in the corresponding nonrelativistic problem. It provides bound state wave equations with the simplicity of the nonrelativistic Schrödinger equation. Here we begin important tests of the relativistic 16 component wave function solutions obtained in a recent work on meson spectroscopy, extending a method developed previously for positronium decay into two photons. Preliminary to this we examine the positronium decay in the $^3P_{0,2}$ states as well as the $^1S_0$. The two-gamma quarkonium decays that we investigate are for the η_c, η_c', χ_{c0}, χ_{c2}, π^0, π_2, a_2, and f_2' mesons. Our results for the four charmonium states compare well with those from other quark models and show the particular importance of including all components of the wave function as well as strong and c.m. energy dependent potential effects on the norm and amplitude. The results for the π^0, although off the experimental rate by 13%, are much closer than the usual expectations from a potential model. We conclude that the two-body Dirac equations lead to wave functions which provide good descriptions of the two-gamma decay amplitude and can be used with some confidence for other purposes.

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I. INTRODUCTION

Relativistic treatments of the two-body problem arise in many problems in particle and nuclear physics. Relativistic effects are important for composite systems with light quarks, in systems with large coupling strength, and in reactions of these composite objects. In recent years, there is much interest in the dissociation and the recombinations of the J/ψ particle in hadron matter or in the quark-gluon plasma [1]. Reactions of the form

$$J/\psi + \pi \leftrightarrow D + \bar{D}^* \tag{1.1}$$

provide useful information on the suppression or the enhancement of J/ψ in high-energy heavy-ion collisions and are relevant to the use of heavy quarkonium as a diagnostic tool for the quark-gluon plasma [2].

Previously, Wong, Barnes, and Swanson studied the above reactions using a nonrelativistic model of the reacting composite objects including pions [1,3]. While the results have been calibrated with the $\pi\pi$ scattering phase shifts for the $I = 2$ S-wave channel, the use of the nonrelativistic formalism for pions with light constituents may be subject to question. One should examine the reaction process using a well tested relativistic formalism. The two-body Dirac equations (TBDE) of constraint dynamics has had successful applications to relativistic two-body bound states in QED [4,5], QCD [6,7], and two-body nucleon-nucleon scattering [8,9]. But its relativistic extension [10] of the nonrelativistic four-body scattering formalism of Barnes and Swanson [11,12] involves untested assumptions beyond the standard constraint formalism. The reaction process is sensitive to the spatial distribution of the reacting objects. It is thus important to have a sensitive test of the wave functions obtained in [7].

We perform this test in this paper by examining the application of the relativistic constraint formalism in the description of decays of mesons into two photons. In the next section we present a brief review of the constraint formalism as it applies to quark-antiquark bound states. Part of the purpose of this review section is to outline some of the numerous tests made so far on the formalism. We give the Pauli forms of the two-body Dirac equations of constraint dynamics that we used in [7] to describe the entire meson spectrum (exceptions being light quark isoscalars such as the η, ω, η' and their orbital and radial excitations). We review those aspects of the formalism which give one confidence in the accurate accounts for all bound states from the excited states of bottomonium to

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the pion. We also list those aspects particularly related to the pion and its Goldstone boson behavior. A perturbative treatment of the TBDE formalism for the QED bound state spectra gives the standard perturbative results. However, unlike other formalisms that purport to account for the entire meson spectrum, the constraint formalism has been shown to produce these same standard perturbative results in nonperturbative and numerical treatments of QED [5]. We emphasize the importance of this by showing the correlation between that agreement for the singlet and triplet positronium systems and the π and ρ states, including the Goldstone behavior of the former.

This connection to QED brings us in Sec. III to our treatment of the 2γ decays of positronium and quarkonium. In constraint dynamics, the two-body Dirac equations lead to an analytic solution of the singlet states of positronium. For the singlet ground state the wave function is mildly singular. Standard formalisms [13] will fail with wave functions that are singular at the origin,

$$\Gamma(e^+e^- \to 2\gamma) = \sigma_{\rm tot} v_{e^+} |\psi(0)|^2. \tag{1.2}$$

Independent treatments by Crater [14] and Ackleh and Barnes [15] develop related (but distinctly different) approaches for folding in the effect of the Yukawa fermion exchange mass, giving a smearing of the singularity over the corresponding Compton wave length. We give a brief review of the first of these approaches and how we extend it to include the effects of the full 16 component two-body Dirac wave function. This extension does not have any significant effect on the $^1S_0$ positronium decay rate. However, the effects on the decays of the more relativistic quark-antiquark systems is significant.

We include in Sec. III technical aspects in which we establish in the context of a 4×4 matrix wave function, more natural for use in the decay formalism of a particle-antiparticle system than the 16 component form, the relation between the sector of the full wave function used in the Pauli form of the bound state equations and the remaining sectors necessary for a complete description of the decay. We review our $^1S_0$ positronium decay results as well as those of our constraint approach for $^3P_0$ and $^3P_2$ positronium decay. Finally we present the results for the decay rates of the η_c, η_c', χ_{c0}, χ_{c2}, π^0, π_2, a_2, f_2' mesons. We conclude in Sec. IV with a discussion of our results and a comparison with other approaches.

II. CONSTRAINT DYNAMICS AND MESON BOUND STATES

A. Constraint dynamics for two classical spinless particles

Here we give a brief review of the highlights of the constraint approach serving also to introduce notations. Although Sazdjian has shown that the bound state equations of constraint dynamics are to be viewed as "quantum mechanical transforms" of the Bethe-Salpeter equation [16–18] the constraint approach to the two-body problem has its origins in classical relativistic physics [19–24]. Our review here is base on [21,25]. Two free spinless particles are described by the mass-shell constraints

$$\mathcal{H}_1^0 \equiv p_1^2 + m_1^2 \approx 0, \qquad \mathcal{H}_2^0 \equiv p_2^2 + m_2^2 \approx 0. \tag{2.1}$$

We introduce Poincare' invariant world scalar interactions (to display most simply the basic ideas) by

$$\begin{aligned} &m_1 \to m_1 + S_1(x, p_1, p_2) \equiv M_1(x, p_1, p_2), \\ &m_2 \to m_2 + S_2(x, p_1, p_2) \equiv M_2(x, p_1, p_2), \\ &x = x_1 - x_2. \end{aligned} \tag{2.2}$$

Kinematical constraints then become dynamical mass-shell constraints:

$$\begin{aligned} \mathcal{H}_i^0 &= p_i^2 + m_i^2 \to p_i^2 + M_i^2 \equiv \mathcal{H}_i \\ &\equiv p_i^2 + m_i^2 + \Phi_i(x, p_1, p_2). \end{aligned} \tag{2.3}$$

Each constraint must be conserved, implying that the two constraints must be compatible

$$\begin{aligned} 0 \approx \{\mathcal{H}_1, \mathcal{H}_2\} &= -(p_1 + p_2) \cdot \frac{\partial}{\partial x}(\Phi_2 + \Phi_1) \\ &\quad - (p_1 - p_2) \cdot \frac{\partial}{\partial x}(\Phi_2 - \Phi_1) + \{\Phi_1, \Phi_2\}. \end{aligned} \tag{2.4}$$

Its simplest solution is

$$\Phi_1 = \Phi_2 = \Phi(x_\perp, p_1, p_2) \equiv \Phi_w, \tag{2.5}$$

and requires abandoning $x = x_1 - x_2$ in favor of

$$\begin{aligned} &x_{12\perp}^\mu = (\eta^{\mu\nu} + \hat{P}^\mu \hat{P}^\nu)(x_1 - x_2)_\nu \equiv \eta_\perp^{\mu\nu}(x_1 - x_2)_\nu, \\ &\hat{P}^\mu \equiv \frac{P^\mu}{\sqrt{-P^2}}; \qquad P^\mu = p_1^\mu + p_2^\mu; \qquad x_{12\perp} \cdot \hat{P} = 0. \end{aligned} \tag{2.6}$$

Thus we have a "third law" condition (2.5) of action and reaction plus a restriction on how the quasipotential Φ_w may depend on relative separation. The invariant r defined below is the interparticle separation in the center of momentum (c.m.) frame $\hat{P} = (1, \mathbf{0})$

$$r \equiv \sqrt{x_\perp^2} = \sqrt{\mathbf{r}^2} \text{ in c.m. frame } \hat{P} = (1, \mathbf{0}), \tag{2.7}$$

since the time component of $x_\perp$ is zero in that frame. Relative time is thus controlled in a covariant way. Assume the two invariants M_i, $i = 1, 2$ are simply functions of r and the c.m. energy

$$w = \sqrt{-P^2}. \tag{2.8}$$

The invariant potentials M_i are not independent. The third law condition implies they are related by

$$M_1^2 - M_2^2 = m_1^2 - m_2^2. \tag{2.9}$$

Hence there is only one independent invariant function controlling the scalar interaction which we designate by

$$S(r), \tag{2.10}$$

the underlying scalar interaction. Alternatively, the third law allows us to recast the mass potentials into hyperbolic function solutions depending on a single invariant function L,

$$\begin{aligned} M_1 &= m_1 \cosh L(S(r)) + m_2 \sinh L(S(r)), \\ M_2 &= m_2 \cosh L(S(r)) + m_1 \sinh L(S(r)). \end{aligned} \tag{2.11}$$

Subtracting the constraints gives us a complimentary covariant restriction [to Eq. (2.7)] on the relative energy

$$\begin{aligned} \mathcal{H}_1 - \mathcal{H}_2 &= p_1^2 + M_1^2 - p_2^2 - M_2^2 \\ &= p_1^2 + m_1^2 - p_2^2 - m_2^2 = 2P \cdot p \approx 0, \end{aligned} \tag{2.12}$$

with relative momentum

$$\begin{gathered} p^\mu = \frac{(\varepsilon_2 p_1^\mu - \varepsilon_1 p_2^\mu)}{w}, \\ \varepsilon_1 + \varepsilon_2 = w, \qquad \varepsilon_1 - \varepsilon_2 = \frac{(m_1^2 - m_2^2)}{w}, \\ \varepsilon_i = \text{c.m. energy of particle } i. \end{gathered} \tag{2.13}$$

The relative momentum is canonically conjugate to $x_\perp$,

$$\{x_\perp^\mu, p^\nu\} = \eta_\perp^{\mu\nu}. \tag{2.14}$$

The other combination of our constraints is the primary dynamical equation

$$\mathcal{H} \equiv \frac{(\varepsilon_2 \mathcal{H}_1 + \varepsilon_1 \mathcal{H}_2)}{w} = p_\perp^2 + \Phi_w - b^2(w) \approx 0, \tag{2.15}$$

and incorporates exact two-body kinematics with

$$\begin{aligned} b^2(w) &= \frac{(w^4 - 2w^2(m_1^2 + m_2^2) + (m_1^2 - m_2^2)^2)}{4w^2} \\ &= \varepsilon_w^2 - m_w^2, \end{aligned} \tag{2.16}$$

and

$$m_w = \frac{m_1 m_2}{w}, \qquad \varepsilon_w = \frac{(w^2 - m_1^2 - m_2^2)}{2w}, \tag{2.17}$$

defined as the mass and energy of the fictitious particle of relative motion. Under quantization all of the constraints become equations the wave functions must satisfy.

B. Two-body Dirac equations

The constraint formalism embodies spin in a system of two coupled, compatible Dirac equations on a single wave function. For particles interacting through world vector and scalar interactions the TBDE take this general minimal-coupling form

$$\begin{aligned} \mathcal{S}_1 \psi &\equiv \gamma_{51}(\gamma_1 \cdot (p_1 - \tilde{A}_1) + m_1 + \tilde{S}_1)\psi = 0, \\ \mathcal{S}_2 \psi &\equiv \gamma_{52}(\gamma_2 \cdot (p_2 - \tilde{A}_2) + m_2 + \tilde{S}_2)\psi = 0. \end{aligned} \tag{2.18}$$

The wave function has 16 components

$$\psi = [\psi_1, \psi_2, \psi_3, \psi_4], \tag{2.19}$$

in which each ψ_i is a four component Pauli spinor for two spin-one-half particles. The two equations are compatible:

$$[\mathcal{S}_1, \mathcal{S}_2]\psi = 0. \tag{2.20}$$

This is a result of the presence of spin supersymmetries [25,26], in addition to the relativistic third law, and covariant restrictions on the relative time and energy appearing in the spinless case. There is automatic incorporation of correct spin-dependent recoil terms [27],

$$\begin{aligned} \tilde{A}_i^\mu &= \tilde{A}_i^\mu(A(r), p_\perp, \hat{P}, w, \gamma_1, \gamma_2), \\ \tilde{S}_i &= \tilde{S}_i(S(r), A(r), p_\perp, \hat{P}, w, \gamma_1, \gamma_2). \end{aligned} \tag{2.21}$$

This two-body formalism has many advantages over the traditional Bethe-Salpeter equation and its numerous three dimensional truncations. One is its simplicity. A Pauli reduction and scale transformation brings our equations to this covariant Schrödinger-like form

$$(p^2 + \Phi_w(\sigma_1, \sigma_2, p_\perp, A(r), S(r)))\psi = b^2(w)\psi. \tag{2.22}$$

1. Schrödinger-like form of the two-body Dirac equations

From classical [28] or quantum field theories [18] for separate scalar and vector interactions one can show that the spin independent part of the quasipotential Φ_w involves the difference of squares of the invariant mass and energy potentials (M_i and E_i respectively)

$$\begin{aligned} &M_i^2 = m_i^2 + 2m_w S + S^2; \qquad E_i^2 = \varepsilon_i^2 - 2\varepsilon_w A + A^2 \\ &M_i^2 - E_i^2 = 2m_w S + S^2 + 2\varepsilon_w A - A^2 - b^2(w). \end{aligned} \tag{2.23}$$

"Squaring" the TBDE (2.18) yields a Schrödinger-like equation [5] for the upper-upper ψ_1 component

$$\begin{aligned} \{p^2 &+ 2m_w S + S^2 + 2\varepsilon_w A - A^2 + \Phi_D i\hat{r} \cdot p + \Phi_{D'} \\ &+ \Phi_{SO1} L \cdot \sigma_1 + \Phi_{SO2} L \cdot \sigma_2 + \Phi_{SS} \sigma_1 \cdot \sigma_2 \\ &+ \Phi_T S_T\}\psi_1 + \{\Phi'_{SS} \sigma_1 \cdot \sigma_2 + \Phi'_T S_T\}\psi_4 = b^2(w)\psi_1, \end{aligned} \tag{2.24}$$

coupled to a Schrödinger-like wave equation for the lower-lower component ψ_4 [29]

$$\{p^2 + 2m_w S + S^2 + 2\varepsilon_w A - A^2 + \tilde{\Phi}_D i\hat{r}\cdot p + \tilde{\Phi}_{D'} + \tilde{\Phi}_{SO1} L\cdot\sigma_1 + \tilde{\Phi}_{SO2} L\cdot\sigma_2 + \tilde{\Phi}_{SS}\sigma_1\cdot\sigma_2 + \tilde{\Phi}_T S_T\}\psi_4 + \{\tilde{\Phi}'_{SS}\sigma_1\cdot\sigma_2 + \tilde{\Phi}'_T S_T\}\psi_1 = b^2(w)\psi_4. \tag{2.25}$$

These equations can be solved nonperturbatively for QED ($S = 0$) or quark model calculations since everyone of the quasipotentials terms Φ_i (including the Darwin pieces Φ_D) is quantum mechanically well defined (less singular than $-1/4r^2$).

2. Nonperturbative solutions of the two-body Dirac equations

For two-body Dirac equations of constraint dynamics applied to QED we have

$$A(r) = -\frac{\alpha}{r}. \tag{2.26}$$

For singlet positronium system we can obtain an exact solution [4] for the total c.m. energy w

$$w = m\sqrt{2 + 2\bigg/\sqrt{1+\alpha^2\bigg/\left(n+\sqrt{\left(l+\frac{1}{2}\right)^2-\alpha^2}-l-\frac{1}{2}\right)^2}} = 2m - m\alpha^2/4n^2 - m\alpha^4/2n^3(2l+1) + 11/64m\alpha^4/n^4 + O(\alpha^6), \tag{2.27}$$

that agrees through order α^4 with the standard spectrum found by perturbative treatments of the Darwin and spin-dependent terms in our Pauli form. Numerical triplet state calculations agree equally well with perturbative QED [5].

Many of the standard approaches to QED bound states have been applied in QCD in nonperturbative numerical calculations of the meson spectra without first testing them nonperturbatively in QED. Sommerer *et al.* [30] have shown that the Blankenbecler-Sugar equation and the Gross equations fail this test. This indicates danger in applying such three dimensional truncations of the Bethe-Salpeter equation to quark models, for if failure occurs in their applications to QED how can similar non-perturbative (i.e. numerical) approaches based on the same truncations (but with QCD kernels) give results that are trustworthy representations of the physics for meson spectroscopy?

C. Two-body Dirac equations for meson spectroscopy—The Adler-Piran potential

We obtain a constraint version of the naive quark model for mesons by employing a covariant adaptation of a static quark potential due to Adler and Piran [31]. From an effective nonlinear field theory derived from QCD they obtain

$$V_{AP}(r) = \Lambda(U(\Lambda r) + U_0)(= A + S). \tag{2.28}$$

The original V_{AP} is nonrelativistic, and appears in our equations in that limit as the sum of world vector and scalar potentials with

$$\Lambda U(\Lambda r \ll 1) \sim \frac{1}{r \ln\Lambda r}, \quad V_{AP}(r) = \Lambda\left[c_1\Lambda r + c_2\log(\Lambda r) + \frac{c_3}{\sqrt{\Lambda r}} + \frac{c_4}{\Lambda r} + c_5\right], \quad \Lambda r > 2. \tag{2.29}$$

The explicit form for $V_{AP}(r)$ at all distances is given in [31,32].

1. Relativistic naive quark model

We reinterpret the static V_{AP} covariantly by replacing the nonrelativistic r by $\sqrt{x_\perp^2} \equiv r$, and parceling out the static potential V_{AP} into the invariant functions $A(r)$ and $S(r)$ [7] as follows:

$$A = \exp(-\beta r)\left[V_{AP} - \frac{c_4}{r}\right] + \frac{c_4}{r} + \frac{e_1 e_2}{r}, \qquad S = V_{AP} + \frac{e_1 e_2}{r} - A. \tag{2.30}$$

(The constants c_1, c_2, c_3, c_4 are fixed by the Adler-Piran formalism while e_1, e_2 are the quark and antiquark electric charges). Thus at short distances the potential is strictly vector while at long distances the vector portion is strictly Coulombic with the confining portion at long distance (including subdominant portions) strictly scalar. Once A and S have been determined, so are all the accompanying spin-dependent interactions

$$\Phi_i = \Phi_i(\sigma_1, \sigma_2, p_\perp, A(r), S(r)); \qquad i = D, D', SO1, SO2, SS, T, \ldots. \tag{2.31}$$

Our bound state results are quite accurate, from the heaviest bottomonium states to the pion. They compare quite favorably with the results of Godfrey and Isgur [33], but with only two parametric functions (A, S) as opposed to the six or so used in their approach. In Table I we reproduce a portion of the entire spectrum given in [7]. The quark masses and potential parameters are given by $m_u \sim 55$ MeV, $m_c \sim 1.5$ GeV, $m_d \sim 58$ MeV, $\Lambda = 0.216$ GeV, and $\Lambda U_0 = 1.865$ GeV.

2. Positronium and the pion

Positronium numerical spectral predictions of the constraint approach for hyperfine splittings are inadequate if we ignore coupling to the small (including the lower-lower one ψ_4) components of the wave function [5]. In Table II, N_c refers to the number of coupled equations, which for the fully coupled system is two for the singlet and four for the triplet states [5] (We are not including the effects of the annihilation diagram for the triplet states). Units are in eV.

As seen in the table, only the fully coupled system of equations (lower-lower and upper-upper for the singlet, the same in addition to tensor coupling for the triplet) produces accurate results to the require precision.

The corresponding good $\pi - \rho$ splitting obtained in [7] is spoiled if the we ignore these couplings, leading to $m_\pi \sim$ 850 MeV, $m_\rho \sim 1060$ MeV. The same relativistic structure in the constraint equations responsible for the success of the Sommerfeld-Balmer formula for positronium spin-singlet states appears to be important for bringing the pion mass down to its observed value.

3. Goldstone behavior of the pion

As a bonus, we find [6,7,32] that the pion is a Goldstone boson in the sense that

$$m_\pi(m_q \to 0) \to 0, \tag{2.32}$$

while the ρ and excited π have finite mass in this limit. However, if the TBDE for the pion is truncated so that the coupling to the lower-lower component is dropped, then the pion loses its Goldstone boson behavior. Its mass no longer decreases toward zero with vanishing quark mass. This and the $\pi - \rho$ result above support our contention that the pion does not need to be treated in a special way insofar as the binding mechanism is concerned. The light pion mass as well as its Goldstone behavior is a natural outgrowth of the covariant two-body Dirac formalism. We now see how this model for the pion and other mesons holds up for a different probe, that of 2γ decays.

TABLE I. Selected portions of meson spectrum.

Meson	Exp (GeV)	($\pm$ MeV)	Theory
η_c: $c\bar{c}1^1S_0$	2.980	(2.1)	2.978
ψ: $c\bar{c}1^3S_1$	3.097	(0.0)	3.129
χ_0: $c\bar{c}1^1P_1$	3.526	(0.2)	3.520
χ_0: $c\bar{c}1^3P_0$	3.415	(1.0)	3.407
χ_1: $c\bar{c}1^3P_1$	3.510	(0.1)	3.507
χ_2: $c\bar{c}1^3P_2$	3.556	(0.1)	3.549
η_c: $c\bar{c}2^1S_0$	3.594	(5.0)	3.610
ψ: $c\bar{c}2^3S_1$	3.686	(0.1)	3.688
ψ: $c\bar{c}1^3D_1$	3.770	(2.5)	3.808
ψ: $c\bar{c}3^3S_1$	4.040	(10.0)	4.081
ψ: $c\bar{c}2^3D_1$	4.159	(20.0)	4.157
ψ: $c\bar{c}3^3D_1$	4.415	(6.0)	4.454
π: $u\bar{d}1^1S_0$	0.140	(0.0)	0.144
ρ: $u\bar{d}1^3S_1$	0.767	(1.2)	0.792
b_1: $u\bar{d}1^1P_1$	1.231	(10.0)	1.392
a_0: $u\bar{d}1^3P_0$	1.450	(40.0)	1.491
a_1: $u\bar{d}1^3P_1$	1.230	(40.0)	1.568
a_2: $u\bar{d}1^3P_2$	1.318	(0.7)	1.310
π: $u\bar{d}2^1S_0$	1.300	(100.0)	1.536
ρ: $u\bar{d}2^3S_1$	1.465	(25.0)	1.775
π_2: $u\bar{d}1^1D_2$	1.670	(20.0)	1.870
ρ: $u\bar{d}1^3D_1$	1.700	(20.0)	1.986
ρ_3: $u\bar{d}1^3D_3$	1.691	(5.0)	1.710

TABLE II. Nonperturbative (numerical) positronium spectral results

l	s	j	n	N_c	Perturbative (eV)	Numerical (eV)	Diff$/\mu\alpha^4$
0	0	0	1	1	−6.8033256279	−6.8032861579	5.45E-02
0	0	0	1	2	−6.8033256279	−6.8033256719	−6.08E-05
0	1	1	1	1	−6.8028426132	−6.8028074990	−0.84E-02
0	1	1	1	2	−6.8028426132	−6.8028082195	−4.75E-02
0	1	1	1	2	−6.8028426132	−6.8028239499	−2.58E-02
0	1	1	1	4	−6.8028426132	−6.8028426636	−6.97E-05

III. TWO-GAMMA DECAY AMPLITUDES FOR POSITRONIUM AND QUARKONIUM

Our treatment of decays in the sections below are for general angular momentum states but for illustrative purposes we begin by considering a treatment of singlet positronium or quarkonium systems. They can be viewed as bosons given by the state vector

$$|{}^1S_0\rangle = \frac{1}{\sqrt{2}} \int d^3p \tilde{\psi}(\mathbf{p})(b_{\mathbf{p}}^{\dagger 1} d_{-\mathbf{p}}^{\dagger 2} - b_{\mathbf{p}}^{\dagger 2} d_{-\mathbf{p}}^{\dagger 1})|0\rangle. \tag{3.1}$$

Both the electron and positron (or quark and antiquark) are off shell but on energy shell. The amplitudes for the annihilation of a quark-antiquark pair into two photons are given by the Feynman diagrams in Fig. 1.

The singlet amplitude for annihilation of a free e^+e^- pair with momenta p_+ and p_- into two photons with polarizations ϵ^{α_1}, ϵ^{α_2} and momenta $k_1 = (w/2, \mathbf{k})$, $k_2 = (w/2, -\mathbf{k})$ is

$$M_{\alpha\beta} = \frac{e^2}{(2\pi)^3 w\sqrt{2}} \Big\{ \bar{v}^{(s_+)}(p_+) \Big[\gamma \cdot \epsilon^{(\alpha_1)} \frac{m - \gamma \cdot (p_- - k_1)}{(p_- - k_1)^2 + m^2 - i0} \times \gamma \cdot \epsilon^{(\alpha_2)} + \gamma \cdot \epsilon^{(\alpha_2)} \frac{m - \gamma \cdot (p_- - k_2)}{(p_- - k_2)^2 + m^2 - i0} \gamma \cdot \epsilon^{(\alpha_1)} \Big] \times u^{(s_-)}(p_-) - (s_+ \Leftrightarrow s_-) \Big\}. \tag{3.2}$$

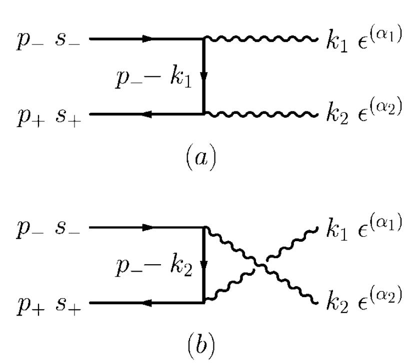

FIG. 1. Feynman diagrams for the annihilation of a quark-antiquak pair into two photons.

For positronium or quarkonium, we replace this decay amplitude by

$$M_{\alpha\beta} \rightarrow \int d^3p \tilde{\psi}_{^1S_0}(\mathbf{p}) M_{\alpha\beta} \equiv \frac{1}{(2\pi)^3 w} \mathcal{M}_{^1S_0\rightarrow 2\gamma}. \quad (3.3)$$

Unlike free amplitudes, the fermion spinors and momenta in $\mathcal{M}_{^1S_0\rightarrow 2\gamma}$ are not on mass shell but on energy shell.

A. Sixteen component two-gamma decay formalism

The amplitude in Eq. (3.3) above is of the form (in c.m.)

$$\mathcal{M}_{X\rightarrow 2\gamma} = \int d^3p \psi(\mathbf{p}) \frac{1}{\sqrt{2}} [\bar{v}^{(s_+)}(-\mathbf{p}) \boldsymbol{\Gamma}(\mathbf{p}, \mathbf{k}) u^{(s_-)}(\mathbf{p}) - \bar{v}^{(s_-)}(-\mathbf{p}) \boldsymbol{\Gamma}(\mathbf{p}, \mathbf{k}) u^{(s_+)}(\mathbf{p})] = \frac{1}{\sqrt{2}} \int d^3p \psi(\mathbf{p}) \,\mathrm{Tr} \boldsymbol{\Gamma}(\mathbf{p}, \mathbf{k}) [u^{(s_-)}(\mathbf{p}) \bar{v}^{(s_+)}(-\mathbf{p}) - u^{(s_+)}(\mathbf{p}) \bar{v}^{(s_-)}(-\mathbf{p})], \quad (3.4)$$

in which

$$\boldsymbol{\Gamma}(\mathbf{p}, \mathbf{k}) = e^2 \left[\boldsymbol{\gamma} \cdot \boldsymbol{\epsilon}^{(\alpha_1)} \frac{m - \gamma \cdot (p - k)}{(\mathbf{p} - \mathbf{k})^2 + m^2} \boldsymbol{\gamma} \cdot \boldsymbol{\epsilon}^{(\alpha_2)} + \boldsymbol{\gamma} \cdot \boldsymbol{\epsilon}^{(\alpha_2)} \frac{m - \boldsymbol{\gamma} \cdot (\mathbf{p} + \mathbf{k})}{(\mathbf{p} + \mathbf{k})^2 + m^2} \boldsymbol{\gamma} \cdot \boldsymbol{\epsilon}^{(\alpha_1)} \right]. \quad (3.5)$$

We replace this decay amplitude for general angular momentum states by

$$\int d^3p \,\mathrm{Tr} \boldsymbol{\Gamma}(\mathbf{p}, \mathbf{k}) \psi(\mathbf{p}), \quad (3.6)$$

where $\psi(\mathbf{p})$ is our bound state wave function in matrix form in an arbitrary angular momentum state. Thus we are expanding our investigation from $^1S_0$ states to general ^1L_l and $^3L_{l\pm 1}$ states. In the case of $^1S_0$ states what we are doing amounts to replacing the matrix wave function $\psi(\mathbf{p})[u^{(s_-)}(\mathbf{p})\bar{v}^{(s_+)}(-\mathbf{p}) - u^{(s_+)}(\mathbf{p})\bar{v}^{(s_-)}(-\mathbf{p})]$ having a spin structure governed by free (but off mass-shell) Dirac spinors by the matrix wave function $\psi(\mathbf{p})$ which, unlike the solution constructed from free spinors, is a solution of the full interacting set of two-body Dirac equations. Similar comments apply for the other angular momentum states. In terms of the Fourier transformed matrix wave function $\psi(\mathbf{r})$ [defined below in Eq. (3.24) and (3.28)],

$$\int d^3p \,\mathrm{Tr} \boldsymbol{\Gamma}(\mathbf{p}, \mathbf{k}) \psi(\mathbf{p}) = \int d^3r \,\mathrm{Tr} \left[\psi(\mathbf{r}) \times \int d^3p \frac{\exp(-i\mathbf{p} \cdot \mathbf{r})}{(2\pi)^{3/2}} \boldsymbol{\Gamma}(\mathbf{p}, \mathbf{k}) \right]. \quad (3.7)$$

Performing the transforms of each term [34] of (3.5) gives

$$\mathcal{M}_{X\rightarrow 2\gamma} = \int d^3r \exp(-i\mathbf{k} \cdot \mathbf{r}) \,\mathrm{Tr} \left[\psi(\mathbf{r}) \times \int d^3\mathbf{p} \frac{\exp(-i\mathbf{p} \cdot \mathbf{r})}{(2\pi)^{3/2}} \boldsymbol{\Gamma}(\mathbf{p}, \mathbf{k}) \right] = e^2 \sqrt{\frac{\pi}{2}} \int d^3r \,\mathrm{Tr} \left\{ \psi(\mathbf{r}) \left[\exp(-i\mathbf{k} \cdot \mathbf{r}) \times \boldsymbol{\gamma} \cdot \boldsymbol{\epsilon}^{(\alpha_1)} (m - i\boldsymbol{\gamma} \cdot \nabla) \frac{\exp(-mr)}{r} \boldsymbol{\gamma} \cdot \boldsymbol{\epsilon}^{(\alpha_2)} + \exp(i\mathbf{k} \cdot \mathbf{r}) \boldsymbol{\gamma} \cdot \boldsymbol{\epsilon}^{(\alpha_2)} (m - i \cdot \nabla) \times \frac{\exp(-mr)}{r} \boldsymbol{\gamma} \cdot \boldsymbol{\epsilon}^{(\alpha_1)} \right] \right\}. \quad (3.8)$$

This decay amplitude generalizes the configuration space form previously given in [14] to one depending on the full 4×4 matrix wave function.

Relativistic wave functions will often display mild singularities at the origin. For example the ground state solution corresponding to Eq. (2.27) is

$$\psi(r) = \frac{(m\alpha)^{3/2}}{\sqrt{4\pi\Gamma(2 + 2\sqrt{1/4 - \alpha^2})}} (rm\alpha)^{(1/2 + \sqrt{1/4 - \alpha^2})} \times \exp(-\alpha mr/2). \quad (3.9)$$

The mild singularity at the origin appearing in this equation is rendered harmless by the smearing action of the Yukawa distribution that comes from folding the effects of the decay amplitude with that of the wave function [14].

1. 4×4 *matrix form of solutions of the two-body Dirac equations*

To accommodate the structure of the TBDE to the above decay amplitude we explicitly construct the 4×4 matrix wave function solution $\psi(\mathbf{r})$ of the equation. First we observe that one can write Eqs. (2.18) in terms of mass and energy potentials and their derivatives analogous to what is done in the case of two-spinless particles [5,6,8,35]. In analogy to the solution (2.11) we gave to the third law condition in the spinless case we define

$$M_1 = m_1 \cosh L(S, A) + m_2 \sinh L(S, A), \quad M_2 = m_2 \cosh L(S, A) + m_1 \sinh L(S, A), \quad (3.10)$$

$$E_1 = \varepsilon_1 \cosh \mathcal{G}(A) - \varepsilon_2 \sinh \mathcal{G}(A), \quad E_2 = \varepsilon_2 \cosh \mathcal{G}(A) - \varepsilon_1 \sinh \mathcal{G}(A). \quad (3.11)$$

In terms of these functions the coupled two-body Dirac equations in an arbitrary frame have the form $\mathcal{S}_i \psi = 0$ in which

$$\mathcal{S}_1 = \exp(\mathcal{G})\beta_1\Sigma_1\cdot\mathcal{P}_1 + E_1\beta_1\gamma_{51} + M_1\gamma_{51} - \exp(\mathcal{G})\frac{i}{2}\Sigma_2\cdot\partial(\mathcal{G}\beta_1 + L\beta_2)\gamma_{51}\gamma_{52},$$
$$\mathcal{S}_2 = -\exp(\mathcal{G})\beta_2\Sigma_2\cdot\mathcal{P}_2 + E_2\beta_2\gamma_{52} + M_2\gamma_{52} + \exp(\mathcal{G})\frac{i}{2}\Sigma_1\cdot\partial(\mathcal{G}\beta_2 + L\beta_1)\gamma_{51}\gamma_{52}, \tag{3.12}$$

with

$$\mathcal{P}_i \equiv p - \frac{i}{2}\Sigma_i\cdot\partial\mathcal{G}\Sigma_i. \tag{3.13}$$

The gamma matrices have block forms given in Appendix A.

If we use the combinations $\phi_\pm = \psi_1 \pm \psi_4$ and $\chi_\pm = \psi_2 \pm \psi_3$, then unlike Eqs. (2.24) and (2.25), the corresponding Schrödinger-like equations decouple [6,7,35]. We obtain [7]

$$\Big[p^2 + 2m_wS + S^2 + 2\varepsilon_wA - A^2 - \frac{1}{2}\nabla^2\mathcal{G} + \frac{3}{4}\mathcal{G}'^2 - (\mathcal{G}' + L')^2 + \mathcal{G}'F' - \frac{L\cdot(\sigma_1+\sigma_2)}{r}F' + L\cdot(\sigma_1-\sigma_2)l' + iq'L\cdot(\sigma_1\times\sigma_2) + 2F'i\hat{r}\cdot p + iK'(\sigma_1\cdot\hat{r}\sigma_2\cdot p + \sigma_2\cdot\hat{r}\sigma_1\cdot p) + \sigma_1\cdot\sigma_2\Big(\frac{1}{2}\nabla^2\mathcal{G} + \frac{1}{2r}L' - \frac{1}{2}\mathcal{G}'^2 - \mathcal{G}'F'\Big) + \sigma_1\cdot\hat{r}\sigma_2\cdot\hat{r}\Big(\frac{1}{2}\nabla^2L - \Big(\frac{3}{2r} + F'\Big)\Big)\Big]\phi_+ = b^2(w)\phi_+, \tag{3.14}$$

where the prime symbol stands for d/dr. We have used the abbreviations

$$F = \frac{1}{2}\log\mathcal{D} - \mathcal{G}, \qquad \mathcal{D} = E_2M_1 + E_1M_2,$$
$$K = \frac{(\mathcal{G}+L)}{2}, \qquad l'(r) = -\frac{1}{2r}\frac{E_2M_2 - E_1M_1}{E_2M_1 + E_1M_2}(L-\mathcal{G})',$$
$$q'(r) = \frac{1}{2r}\frac{E_1M_2 - E_2M_1}{E_2M_1 + E_1M_2}(L-\mathcal{G})'. \tag{3.15}$$

We work in the c.m. frame in which $\hat{P} = (1, \mathbf{0})$ and $\hat{r} = (0, \hat{\mathbf{r}})$. Once we find the four component solutions ϕ_+ to this equation we can obtain the other 12 components ϕ_-, $\chi_\pm$. In Appendix B of [36] we find from Eq. (3.12)

$$\chi_+ = \frac{\exp(\mathcal{G})}{\mathcal{D}}\Big\{M_2\Big[\boldsymbol{\sigma}_1\cdot\mathbf{p} - \frac{i}{2}\boldsymbol{\sigma}_2\cdot\boldsymbol{\nabla}(-\mathcal{G} - L + \mathcal{G}\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)\Big] - M_1\Big[\boldsymbol{\sigma}_2\cdot\mathbf{p} - \frac{i}{2}\boldsymbol{\sigma}_1\cdot\boldsymbol{\nabla}(-\mathcal{G} - L + \mathcal{G}\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)\Big]\Big\}\phi_+, \tag{3.16}$$

and similarly

$$\chi_- = -\frac{\exp(\mathcal{G})}{\mathcal{D}}\Big\{E_2\Big[\boldsymbol{\sigma}_1\cdot\mathbf{p} - \frac{i}{2}\boldsymbol{\sigma}_2\cdot\boldsymbol{\nabla}(-\mathcal{G} - L + \mathcal{G}\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)\Big] + E_1\Big[\boldsymbol{\sigma}_2\cdot\mathbf{p} - \frac{i}{2}\boldsymbol{\sigma}_1\cdot\boldsymbol{\nabla}(-\mathcal{G} - L + \mathcal{G}\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)\Big]\Big\}\phi_+, \tag{3.17}$$

and

$$\phi_- = \frac{(E_2E_1 + M_2M_1)}{\mathcal{D}}\phi_+ - \frac{1}{2\mathcal{D}}\Big[(E_2D_1^{-+} - E_1D_2^{-+})\frac{1}{\mathcal{D}}\times(M_2D_1^{++} - M_1D_2^{++}) - (M_2D_1^{--} + M_1D_2^{--})\times\frac{1}{\mathcal{D}}(E_2D_1^{++} + E_1D_2^{++})\Big]\phi_+, \tag{3.18}$$

in which

$$D_1^{++} = \exp(\mathcal{G})\Big[\boldsymbol{\sigma}_1\cdot\mathbf{p} - \frac{i}{2}\boldsymbol{\sigma}_2\cdot\boldsymbol{\nabla}(-\mathcal{G} - L + \mathcal{G}\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)\Big],$$
$$D_1^{-+} = \exp(\mathcal{G})\Big[\boldsymbol{\sigma}_1\cdot\mathbf{p} + \frac{i}{2}\boldsymbol{\sigma}_2\cdot\boldsymbol{\nabla}(\mathcal{G} - L - \mathcal{G}\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)\Big],$$
$$D_1^{--} = \exp(\mathcal{G})\Big[\boldsymbol{\sigma}_1\cdot\mathbf{p} + \frac{i}{2}\boldsymbol{\sigma}_2\cdot\boldsymbol{\nabla}(\mathcal{G} + L - \mathcal{G}\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)\Big], \tag{3.19a}$$

and

$$D_2^{++} = \exp(\mathcal{G})\Big[\boldsymbol{\sigma}_2\cdot\mathbf{p} - \frac{i}{2}\boldsymbol{\sigma}_1\cdot\boldsymbol{\nabla}(\mathcal{G} - L + \mathcal{G}\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)\Big],$$
$$D_2^{-+} = \exp(\mathcal{G})\Big[\boldsymbol{\sigma}_2\cdot\mathbf{p} + \frac{i}{2}\boldsymbol{\sigma}_1\cdot\boldsymbol{\nabla}(\mathcal{G} - L - \mathcal{G}\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)\Big],$$
$$D_2^{--} = \exp(\mathcal{G})\Big[\boldsymbol{\sigma}_2\cdot\mathbf{p} + \frac{i}{2}\boldsymbol{\sigma}_1\cdot\boldsymbol{\nabla}(\mathcal{G} + L - \mathcal{G}\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)\Big]. \tag{3.19b}$$

We then further define four component wave functions $\psi_\pm$, $\eta_\pm$ related to the above by [8]

$$\begin{aligned}\phi_\pm &= \exp(F + K\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}})\psi_\pm \\ &= \exp F(\cosh K + \sinh K\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}})\psi_\pm, \\ \chi_\pm &= \exp(F + K\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}})\eta_\pm \\ &= \exp F(\cosh K + \sinh K\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}})\eta_\pm.\end{aligned} \tag{3.20}$$

In this case the decoupled form of the Schrödinger-like equation for ψ_+ has the convenient property that the coefficients of the first order relative momentum terms $2F'i\hat{\mathbf{r}}\cdot\mathbf{p} + iK'(\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\mathbf{p} + \sigma_2\cdot\hat{\mathbf{r}}\sigma_1\cdot\mathbf{p})$ in Eq. (3.14) vanish. We obtain [8]

$$\Big\{\mathbf{p}^2+2m_wS+S^2+2\varepsilon_wA-A^2-\frac{2F'(\cosh 2K-1)}{r}+2F'^2+2K'^2+\frac{2K'\sinh 2K}{r}-\boldsymbol{\nabla}^2F-F'^2-K'^2$$
$$-\frac{2(\cosh 2K-1)}{r^2}+m(r)+\mathbf{L}\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\Big[-\frac{F'}{r}-\frac{F'(\cosh 2K-1)}{r}-\frac{(\cosh 2K-1)}{r^2}+\frac{K'\sinh 2K}{r}\Big]$$
$$+\mathbf{L}\cdot(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)(l'\cosh 2K-q'\sinh 2K)+i\mathbf{L}\cdot\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2(q'\cosh 2K+l'\sinh 2K)+\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}\mathbf{L}\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)$$
$$\times\Big(-\frac{K'(\cosh 2K-1)}{r}+\frac{\sinh 2K}{r^2}-\frac{K'}{r}+\frac{F'\sinh 2K}{r}\Big)+\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2\Big[k(r)-\frac{F'\sinh 2K}{2r}-\frac{F'(\cosh 2K-1)}{r}$$
$$+\frac{K'\sinh 2K}{r}+\frac{K'(\cosh 2K-1)}{r}+\frac{\sinh 2K}{r^2}-\frac{(\cosh 2K-1)}{r^2}\Big]+\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}\Big[n(r)+\frac{3F'\sinh 2K}{r}$$
$$+\frac{F'(\cosh 2K-1)}{r}+2F'K'-\frac{K'\sinh 2K}{r}-\frac{3K'(\cosh 2K-1)}{r}-\boldsymbol{\nabla}^2K+\frac{3\sinh 2K}{r^2}+\frac{(\cosh 2K-1)}{r^2}\Big]\Big\}\psi_+=b^2\psi_+, \tag{3.21}$$

in which

$$k(r)=\frac{1}{2}\nabla^2\mathcal{G}-\frac{1}{2}\mathcal{G}'^2-\frac{1}{2}\mathcal{G}'K'-\frac{1}{2}\frac{\mathcal{G}'}{r}+\frac{K'}{r},$$
$$n(r)=\nabla^2K-\frac{1}{2}\nabla^2\mathcal{G}-2K'F'+\mathcal{G}'F'-\frac{3}{2r}\mathcal{G}', \tag{3.22}$$
$$m(r)=-\frac{1}{2}\nabla^2\mathcal{G}+\frac{3}{4}\mathcal{G}'^2+\mathcal{G}'F'-K'^2,$$

For equal mass singlet states, the hyperbolic terms cancel. The spin-orbit difference terms in general produce spin mixing.

2. Matrix form of the wave functions

We now construct the 4×4 matrix forms of the wave functions (appropriate for a spin-one-half particle-antiparticle system) from the 16 component forms (appropriate for system of two spin-one-half particles). We begin by writing the 16 component spinor wave function as

$$\psi=\psi_1\begin{pmatrix}1\\0\\0\\0\end{pmatrix}+\psi_2\begin{pmatrix}0\\1\\0\\0\end{pmatrix}+\psi_3\begin{pmatrix}0\\0\\1\\0\end{pmatrix}+\psi_4\begin{pmatrix}0\\0\\0\\1\end{pmatrix} \tag{3.23}$$

$$=\frac{\phi_+}{2}\Big[\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix}+\begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix}\Big]+\frac{\phi_-}{2}\Big[\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix}-\begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix}\Big]+\frac{\chi_+}{2}\Big[\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix}+\begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix}\Big]+\frac{\chi_-}{2}\Big[\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix}-\begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix}\Big]. \tag{3.24}$$

The spinors ψ_i as well as $\phi_\pm=\psi_1\pm\psi_4$, $\chi_\pm=\psi_2\pm\psi_3$ are themselves four component Pauli spinors (upon which σ_{1i}, σ_{2i} operate). The conversion from 16 component spinor wave functions to four by four matrix wave functions now can be carried out in a two-step process. First, as in [35,37], the "energy" or q space column vector direct products are converted to 4×4 matrices as follows (recall the factor of $i\alpha_y$ plus the transpose operation changes particle spinor into antiparticle spinor)

$$\Psi_{(1)}\otimes\Psi_{(2)}\rightarrow\Psi_{(1)}\Psi_{(2)}^Ti\alpha_2=\Psi_{(1)}\Psi_{(2)}^Ti\sigma_2\otimes q_1, \tag{3.25}$$

in which σ_0, σ_i, q_0, q_i; $i=1,2,3$ are the 2×2 unit and three Pauli matrices in commuting spaces (spin and energy space)

$$\sigma_i\sigma_j=\delta_{ij}\sigma_0+i\varepsilon_{ijk}\sigma_k,\qquad q_iq_j=\delta_{ij}q_0+i\varepsilon_{ijk}q_k, \tag{3.26}$$

and whose direct products form the Dirac matrices. Second, the $\phi_\pm$, $\chi_\pm$ four component Pauli spinors are converted to 2×2 matrices in σ and q space by

$$\phi_\pm\rightarrow\phi_\pm=(\phi_{\pm0}\sigma_0+\boldsymbol{\phi}_\pm\cdot\boldsymbol{\sigma}),$$
$$\chi_\pm\rightarrow\chi_\pm=(\chi_{\pm0}\sigma_0+\boldsymbol{\chi}_\pm\cdot\boldsymbol{\sigma}). \tag{3.27}$$

Together, the 4×4 matrix wave function in σ, q space is

$$\Big(\frac{\phi_+}{2}\otimes q_0+\frac{\phi_-}{2}\otimes q_3+\frac{\chi_+}{2}\otimes q_1+\frac{\chi_-}{2}\otimes iq_2\Big)i\sigma_2\otimes q_1$$
$$=\Big(\frac{\phi_+}{2}\otimes q_1+\frac{\phi_-}{2}\otimes iq_2+\frac{\chi_+}{2}\otimes q_0+\frac{\chi_-}{2}\otimes q_3\Big)i\sigma_2\otimes 1$$
$$=\Big(\frac{\phi_+i\sigma_2}{2}\otimes q_1+\frac{\phi_-i\sigma_2}{2}\otimes iq_2+\frac{\chi_+i\sigma_2}{2}\otimes q_0+\frac{\chi_-i\sigma_2}{2}\otimes q_3\Big)$$
$$\rightarrow\Big(\frac{\phi_+}{2}\otimes q_1+\frac{\phi_-}{2}\otimes iq_2+\frac{\chi_+}{2}\otimes q_0+\frac{\chi_-}{2}\otimes q_3\Big). \tag{3.28}$$

where for convenience we have absorbed the factor $i\sigma_2$ into the wave functions as the wave function is arbitrary up to a constant multiplicative matrix and we have used the same symbol for each of the transformed wave functions to simplify notation. In our work below we drop the direct product symbol $\otimes$, it being understood to apply whenever σ and q space matrices multiply one another. The four component spinors $\psi_\pm$ and $\eta_\pm$ are similarly transformed into matrices which can be expanded in terms of σ_0 and σ,

$$\begin{aligned}\psi_\pm &\rightarrow \psi_\pm = (\psi_{\pm 0}\sigma_0 + \boldsymbol{\psi}_\pm \cdot \boldsymbol{\sigma}),\\ \eta_\pm &\rightarrow \eta_\pm = (\eta_{\pm 0}\sigma_0 + \boldsymbol{\eta}_\pm \cdot \boldsymbol{\sigma}).\end{aligned} \tag{3.29}$$

With $\mathbf{A}$ a generic matrix, Eq. (3.25) leads to

$$\begin{aligned}\boldsymbol{\sigma}_1 \cdot \mathbf{A}\psi_+ &\rightarrow \mathbf{A}\cdot\boldsymbol{\sigma}(\psi_{+0}\sigma_0 + \psi_+ \cdot \boldsymbol{\sigma})\\ &= \mathbf{A}\cdot\boldsymbol{\psi}_+\sigma_0 + \mathbf{A}\cdot\boldsymbol{\sigma}\psi_{+0} + i\mathbf{A}\times\boldsymbol{\psi}_+\cdot\boldsymbol{\sigma},\\ \boldsymbol{\sigma}_2\cdot\mathbf{A}\psi_+ &\rightarrow -\mathbf{A}\cdot(\psi_{+0}\sigma_0 + \boldsymbol{\psi}_+\cdot\boldsymbol{\sigma})\sigma\\ &= -\mathbf{A}\cdot\boldsymbol{\psi}_+\sigma_0 - \mathbf{A}\cdot\boldsymbol{\sigma}\psi_{+0} + i\mathbf{A}\times\boldsymbol{\psi}_+\cdot\boldsymbol{\sigma},\\ \boldsymbol{\sigma}_1\cdot\mathbf{A}\boldsymbol{\sigma}_2\cdot\mathbf{A}\psi_+ &\rightarrow -\boldsymbol{\sigma}\cdot\mathbf{A}(\psi_{+0}\sigma_0 + \boldsymbol{\psi}_+\cdot\boldsymbol{\sigma})\boldsymbol{\sigma}\cdot\mathbf{A}\\ &= -\mathbf{A}^2\phi_{+0}\sigma_0 + (\mathbf{A}^2\boldsymbol{\psi}_+ - 2\mathbf{A}\cdot\boldsymbol{\psi}_+\mathbf{A})\cdot\boldsymbol{\sigma}\end{aligned} \tag{3.30}$$

which are needed to convert Eqs. (3.16), (3.17), and (3.18) into their matrix counterparts. In terms of matrix wave functions $\phi_\pm$, $\chi_\pm$, $\psi_\pm$, and $\eta_\pm$, we find that Eq. (3.20) becomes

$$\begin{aligned}\phi_{\pm 0} &= \exp(F-K)\psi_{\pm 0};\\ \boldsymbol{\phi}_\pm &= \exp(F+K)(\mathbf{1} - (1-\exp(-2K))\hat{\mathbf{r}}\,\hat{\mathbf{r}})\cdot\boldsymbol{\psi}_\pm,\\ \chi_{\pm 0} &= \exp(F-K)\eta_{\pm 0};\\ \boldsymbol{\chi}_\pm &= \exp(F+K)(\mathbf{1} - (1-\exp(-2K))\hat{\mathbf{r}}\,\hat{\mathbf{r}})\cdot\boldsymbol{\eta}_\pm.\end{aligned} \tag{3.31}$$

The 4×4 matrix wave function form ψ of the 16 component ψ used in our decay amplitude is thus

$$\psi = \exp(F)[\cosh K\Psi(\mathbf{r}) - \sinh K\boldsymbol{\Sigma}\cdot\hat{\mathbf{r}}\Psi(\mathbf{r})\boldsymbol{\Sigma}\cdot\hat{\mathbf{r}}], \tag{3.32}$$

in which

$$\Psi(\mathbf{r}) = \frac{1}{2\sqrt{2}}(\psi_+ q_1 + \psi_- i q_2 + \eta_+ q_0 + \eta_- q_3), \tag{3.33}$$

where $\psi_+ = \psi_{+0}\sigma_0 + \boldsymbol{\psi}_+\cdot\boldsymbol{\sigma}$ is the 2×2 matrix form of the solution of the above Schrödinger-like Pauli Eq. (3.21).

Using these four components, the remaining 12 components ψ_{-0}, $\boldsymbol{\psi}_-$, $\eta_{\pm 0}$, and $\boldsymbol{\eta}_\pm$ are obtained from Eqs. (3.16), (3.17), (3.18), and (3.31). In all of our decays the particle-antiparticle pairs have the same mass: $m_1 = m_1 \equiv m$ and so $\varepsilon_1 = \varepsilon_2 \equiv \varepsilon = w/2$. Using the definition

$$\begin{aligned}M_1 &= M_2 \equiv M = m\exp(L),\\ E_1 &= E_2 \equiv E = \varepsilon\exp(-\mathcal{G}),\end{aligned} \tag{3.34}$$

we show in [36] that

$$\begin{aligned}\eta_{+0} &= \frac{\exp(\mathcal{G}+2K)}{E}\Big[\mathbf{p} - \frac{i}{2}\boldsymbol{\nabla}(L+2F+2K)\Big]\\ &\quad\cdot[\mathbf{1} + Q_m\hat{\mathbf{r}}\,\hat{\mathbf{r}}]\cdot\boldsymbol{\psi}_+,\\ \boldsymbol{\eta}_+ &= \frac{\exp(-L)}{E}\Big\{\Big(\mathbf{p} - \frac{i}{2}\boldsymbol{\nabla}L\Big) + Q_p\Big(\hat{\mathbf{r}}\,\hat{\mathbf{r}}\cdot\mathbf{p} - \frac{i}{2}\boldsymbol{\nabla}L\Big)\Big\}\psi_{+0},\end{aligned} \tag{3.35}$$

and

$$\begin{aligned}\eta_{-0} &= 0,\\ \boldsymbol{\eta}_- &= -\frac{\exp(\mathcal{G})}{M}[\mathbf{1} + Q_p\hat{\mathbf{r}}\,\hat{\mathbf{r}}]\cdot\Big[i\mathbf{p} + \frac{1}{2}\boldsymbol{\nabla}(L-2\mathcal{G})\Big]\\ &\quad\times[\mathbf{1} + Q_m\hat{\mathbf{r}}\,\hat{\mathbf{r}}]\cdot\boldsymbol{\psi}_+.\end{aligned} \tag{3.36}$$

The final four components of the four by four matrix wave function found in [36] are

$$\begin{aligned}\psi_{-0} &= \Big\{\frac{(E^2+M^2)}{2EM} - \frac{\exp(2\mathcal{G})}{2ME}\Big[\mathbf{p} + \frac{i}{2}\boldsymbol{\nabla}L\Big]\\ &\quad\cdot\Big[\mathbf{p} - \frac{i}{2}\boldsymbol{\nabla}L\Big]\Big\}\psi_{+0},\\ \boldsymbol{\psi}_- &= \frac{(E^2+M^2)}{2EM}\boldsymbol{\psi}_+ - \frac{\exp(2\mathcal{G})}{2EM}[\mathbf{1} + Q_p\hat{\mathbf{r}}\,\hat{\mathbf{r}}]\\ &\quad\cdot\Big(\Big[\mathbf{p} - \frac{i}{2}\boldsymbol{\nabla}(L+6\mathcal{G})\Big]\Big[\mathbf{p} - \frac{i}{2}\boldsymbol{\nabla}(3L-2\mathcal{G})\Big]\\ &\quad\times[\mathbf{1} + Q_m\hat{\mathbf{r}}\,\hat{\mathbf{r}}]\cdot\boldsymbol{\psi}_+ + \Big[\mathbf{p} - \frac{i}{2}\boldsymbol{\nabla}(L+2\mathcal{G})\Big]\\ &\quad\times\Big\{\Big[\mathbf{p} - \frac{i}{2}\boldsymbol{\nabla}(L-2\mathcal{G})\Big]\times[\mathbf{1} + Q_m\hat{\mathbf{r}}\,\hat{\mathbf{r}}]\cdot\boldsymbol{\psi}_+\Big\}\Big),\end{aligned} \tag{3.37}$$

where

$$Q_p \equiv \exp(2K) - 1, \qquad Q_m \equiv \exp(-2K) - 1. \tag{3.38}$$

We also show in [36] how for both singlet and triplet states these solutions together with Eq. (3.33) are related to the solutions governed by the free Dirac spinors in the absence of interactions [see also Eq. (3.4) and discussion below (3.6)].

3. Covariant normalization conditions for the matrix wave function

In this section we discuss how the norm of our matrix wave function will differ from the naive form of

$$\frac{1}{8}\int d^3x\,\mathrm{Tr}_{q\sigma}\Psi^\dagger\Psi = \frac{1}{4}\int d^3x\,\mathrm{Tr}_\sigma(\psi_+^\dagger\psi_+ + \psi_-^\dagger\psi_- + \eta_+^\dagger\eta_+ + \eta_-^\dagger\eta_-) = 1. \tag{3.39}$$

In a series of papers in the context of constraint dynamics, H. Sazdjian has shown [38] how this norm must be modified so that, like its nonrelativistic counterpart, its constancy is connected to a conserved, in this two-body case, tensor current. The norm he developed was not for the solution of a quasipotential equation like Eq. (3.21) but rather developed from a set of two-body Dirac equations similar to those we use here. It deviated from one like the above by terms that depend on the interaction as well as the way in which the interaction depends on the c.m. energy. Later work [39] simplified the norm to one that is interaction independent when the interaction is independent of the energy. In terms of the 16 component spinor solutions ψ of the two-body Dirac equations given in Eqs. (2.18) we found the norm condition of

$$\int d^3x\bigg[\psi^\dagger\bigg(1+4w^2\beta_1\beta_2\frac{\partial\Delta}{\partial w^2}\bigg)\psi\bigg]\equiv\int d^3x\psi^\dagger\mathcal{L}\psi=1. \tag{3.40}$$

If the matrix Δ is c.m. energy independent, then the norm is like that of the (one body) Dirac equation (with no energy dependence of the interactions). We call the norm of Eq. (3.39) the naive norm (NN) and that of Eq. (3.40) the two-body Dirac norm (TBDN). The connection between the matrix interaction function Δ and the core scalar and vector interactions appearing in Eqs. (2.18) were found in [39]. There we showed that (2.18) has the hyperbolic structure

$$\begin{aligned}\mathcal{S}_1\psi&=(\cosh(\Delta)\mathbf{S}_1+\sinh(\Delta)\mathbf{S}_2)\psi=0,\\ \mathcal{S}_2\psi&=(\cosh(\Delta)\mathbf{S}_2+\sinh(\Delta)\mathbf{S}_1)\psi=0,\end{aligned} \tag{3.41}$$

in which

$$\begin{aligned}\mathbf{S}_1\psi&\equiv(\mathcal{S}_{10}\cosh(\Delta)+\mathcal{S}_{20}\sinh(\Delta))\psi=0,\\ \mathbf{S}_2\psi&\equiv(\mathcal{S}_{20}\cosh(\Delta)+\mathcal{S}_{10}\sinh(\Delta))\psi=0,\end{aligned} \tag{3.42}$$

with

$$\begin{aligned}\mathcal{S}_{10}\psi&=(-\beta_1\Sigma_1\cdot p+\epsilon_1\beta_1\gamma_{51}+m_1\gamma_{51})\psi,\\ \mathcal{S}_{20}\psi&=(\beta_2\Sigma_2\cdot p+\epsilon_2\beta_2\gamma_{52}+m_2\gamma_{52})\psi,\end{aligned} \tag{3.43}$$

and

$$\Delta=\tfrac{1}{2}\gamma_{51}\gamma_{52}[L(x_\perp)-\gamma_1\cdot\gamma_2\mathcal{G}(x_\perp)], \tag{3.44}$$

with L and $\mathcal{G}$ given in Eq. (3.34) (see also [40]). In matrix form the connection given in Eqs. (3.31), (3.24), and (3.28) between the matrix form of the wave function ψ of (2.18) and (3.41) and Ψ is given in Eq. (3.32) which we write in symbolic form

$$\psi\equiv\mathcal{K}\Psi(\mathbf{r}). \tag{3.45}$$

In Appendix B we show that in terms of the matrix wave function Ψ solution (3.33) to Eq. (3.21) the nomalization condition can be written as

$$\int d^3x\,\mathrm{Tr}\psi^\dagger\mathcal{L}\psi=\int d^3x\,\mathrm{Tr}(\mathcal{K}\Psi(\mathbf{r}))^\dagger\mathcal{L}\mathcal{K}\Psi(\mathbf{r})=1. \tag{3.46}$$

There we also give the matrix form of the operator $\mathcal{L}$. The deviation of the matrices $\mathcal{K}$ and $\mathcal{L}$ from the unit matrix will affect the decay rates. The decay amplitude (3.8) in terms of the matrix wave function $\Psi(\mathbf{r})$ is

$$\begin{aligned}\mathcal{M}_{X\to2\gamma}=e^2\sqrt{\frac{\pi}{2}}\int d^3r\mathrm{Tr}_{\sigma q}\bigg\{\mathcal{K}\Psi(\mathbf{r})\bigg[&\exp(-i\mathbf{k}\cdot\mathbf{r})q_3q_1\\ &\times\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}^{(\alpha_1)}(m-iq_3q_1\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}\frac{\exp(-mr)}{r}q_3q_1\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}^{(\alpha_2)}\\ &+\exp(i\mathbf{k}\cdot\mathbf{r})q_3q_1\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}^{(\alpha_2)}(m-iq_3q_1\boldsymbol{\sigma}\cdot\boldsymbol{\nabla})\\ &\times\frac{\exp(-mr)}{r}q_3q_1\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}^{(\alpha_1)}\bigg]\bigg\}.\end{aligned} \tag{3.47}$$

4. Scalar and vector wave functions in vector spherical harmonics

Given the above wave functions we now write down the total 4×4 matrix wave function in terms of ψ_+. The spin-zero part of the total wave function is governed by ψ_{+0}, the spin-one portion by $\boldsymbol{\psi}_+$. These wave functions appear in the forms

$$\begin{aligned}&\psi_+=\psi_{+0}\sigma_0+\boldsymbol{\psi}_+\cdot\boldsymbol{\sigma},\qquad \psi_{+0}=\frac{u^+_{j0j}}{r}Y_{jm},\\ &\boldsymbol{\psi}_+=\frac{u^+_{(j+1)1j}}{r}\mathbf{Y}_{jm+}+\frac{u^+_{(j-1)1j}}{r}\mathbf{Y}_{jm-}+\frac{u^+_{j1j}}{r}\mathbf{X}_{jm},\end{aligned} \tag{3.48}$$

where the labels on the radial wave function u refer to the lsj quantum numbers of the solutions to Eq. (3.21) and

$$\begin{aligned}&\mathbf{Y}_{jm+}=(a_+\hat{\mathbf{r}}+rb_+\mathbf{p})Y_{jm},\\ &\mathbf{Y}_{jm-}=(a_-\hat{\mathbf{r}}+rb_-\mathbf{p})Y_{jm},\qquad \mathbf{X}_{jm}=\frac{\mathbf{L}Y_{jm}}{\sqrt{j(j+1)}},\end{aligned} \tag{3.49}$$

are vector spherical harmonic eigenfunctions of $\mathbf{L}^2$ with eigenvalue $l(l+1)$ where $l=j+1, j-1, j$ respectively. The coefficients are

$$\begin{aligned}a_+&=-\sqrt{\frac{j+1}{2j+1}};\qquad a_-=\sqrt{\frac{j}{2j+1}},\\ b_+&=\frac{i}{j+1}\sqrt{\frac{j+1}{2j+1}};\qquad b_-=\frac{i}{j}\sqrt{\frac{j}{2j+1}}.\end{aligned} \tag{3.50}$$

In our work below, there will be no spin mixing and the unnatural parity solutions $(u^+_{j1j}/r)\mathbf{X}_{jm}$ will not contribute.

For spin-singlet states ($\boldsymbol{\psi}_+=0$), Eqs. (3.33), (3.35), and (3.37) imply the following combination of scalar and vector wave functions

$$\Psi|_{s=0}=\frac{1}{2\sqrt{2}}(\psi_{+0}\sigma_0q_1+\psi_{-0}\sigma_0iq_2+\boldsymbol{\eta}_+\cdot\boldsymbol{\sigma}q_0). \tag{3.51}$$

In Appendix C of [41] we show that the TBDN for spin-singlet states is

$$1 = \frac{1}{2}\int d^3x \exp(2F)\Big([\exp(-2K)(\psi_{+0}^\dagger\psi_{+0} + \psi_{-0}^\dagger\psi_{-0}) + \exp(2K)\boldsymbol{\eta}_+^\dagger\cdot\boldsymbol{\eta}_+ - 2\sinh 2K\boldsymbol{\eta}_+^\dagger\cdot\hat{\mathbf{r}}\boldsymbol{\eta}_+\cdot\hat{\mathbf{r}}] + 2w^2\frac{\partial L}{\partial w^2}[\exp(-2K)(\psi_{+0}^\dagger\psi_{+0} - \psi_{-0}^\dagger\psi_{-0}) - \exp(2K)\boldsymbol{\eta}_+^\dagger\cdot\boldsymbol{\eta}_+ + 2\sinh 2K\boldsymbol{\eta}_+^\dagger\cdot\boldsymbol{\eta}_+] + 2w^2\frac{\partial \mathcal{G}}{\partial w^2} \times [2\exp(-2K)(2\psi_{+0}^\dagger\psi_{+0} + \psi_{-0}^\dagger\psi_{-0})]\Big). \tag{3.52}$$

The naive norm is given by $\mathcal{L}$, $\mathcal{K} \to 1$ or equivalently by $\exp(F)$, $\exp(K) \to 1$, $\partial L/\partial w^2$, and $\partial\mathcal{G}/\partial w^2 \to 0$.

Appendix D of [42] gives from Eqs. (3.35) and (3.37) the needed radial forms for the contributing wave functions in terms of the radial portions of the solution to Eq. (3.21). It requires the radial form of Eq. (3.21) which is simply

$$\left[-\frac{1}{r}\frac{d^2}{dr^2}r + \frac{j(j+1)}{r^2} + \frac{1}{2}\boldsymbol{\nabla}^2 L - \frac{1}{4}(\boldsymbol{\nabla}L)^2\right]\frac{u_{j0j}^+}{r} = \mathcal{B}^2\exp(-2\mathcal{G})\frac{u_{j0j}^+}{r}, \tag{3.53}$$

where [40]

$$\mathcal{B}^2 \equiv E^2 - M^2,$$
$$-\mathcal{B}^2\exp(-2\mathcal{G}) = 2m_w S + S^2 + 2\varepsilon_w A - A^2. \tag{3.54}$$

Reference [42] gives us the relations between the contributing wave functions ψ_{+0}, ψ_{-0}, and $\boldsymbol{\eta}_+$. They are

$$\psi_{-0} \equiv \frac{u_{j0j}^-}{r}Y_{jm} = \frac{M}{E}\psi_{+0} = \frac{M}{E}\frac{u_{j0j}^+}{r}Y_{jm}, \tag{3.55}$$

and

$$\boldsymbol{\eta}_+ \equiv i\left(\frac{v_{(j-1)1j}^+}{r}\mathbf{Y}_{jm-} + \frac{v_{(j+1)1j}^+}{r}\mathbf{Y}_{jm+}\right),$$
$$\frac{v_{(j-1)1j}^+}{r} = \frac{\exp(\mathcal{G}-2K)}{E}\left[\exp(2K)\left(-\frac{d}{dr} - \frac{L'}{2}\right) - \frac{(j+1)}{r}\right]\frac{u_{j0j}^+}{r}\sqrt{\frac{j}{2j+1}},$$
$$\frac{v_{(j+1)1j}^+}{r} = \frac{\exp(\mathcal{G}-2K)}{E}\left[\exp(2K)\left(\frac{d}{dr} + \frac{L'}{2}\right) - \frac{j}{r}\right]\frac{u_{j0j}^+}{r} \times\sqrt{\frac{j+1}{2j+1}}. \tag{3.56}$$

For spin-triplet states ($\psi_{+0} = 0$), Eqs. (3.33), (3.35), (3.36), and (3.37) imply the combination

$$\Psi|_{s=1} = \frac{1}{2\sqrt{2}}(\boldsymbol{\psi}_+\cdot\boldsymbol{\sigma}q_1 + \boldsymbol{\psi}_-\cdot\boldsymbol{\sigma}iq_2 + \eta_{+0}q_0 + \boldsymbol{\eta}_-\cdot\boldsymbol{\sigma}q_3), \tag{3.57}$$

and the contributing wave functions are $\boldsymbol{\psi}_+$, $\boldsymbol{\psi}_-$, η_{+0}, and $\boldsymbol{\eta}_-$. In [41] we show that the TBDN for spin-triplet states is

$$\frac{1}{2}\int d^3x\exp(2F)\Big([\exp(2K)(\boldsymbol{\psi}_+^\dagger\cdot\boldsymbol{\psi}_+ + \boldsymbol{\psi}_-^\dagger\cdot\boldsymbol{\psi}_- + \boldsymbol{\eta}_-^\dagger\cdot\boldsymbol{\eta}_-) + \exp(-2K)\eta_{+0}^\dagger\eta_{+0} - 2\sinh 2K(\boldsymbol{\psi}_+^\dagger\cdot\hat{\mathbf{r}}\boldsymbol{\psi}_+\boldsymbol{\psi}_-^\dagger\cdot\hat{\mathbf{r}}\boldsymbol{\psi}_-\cdot\hat{\mathbf{r}} + \boldsymbol{\eta}_-^\dagger\cdot\hat{\mathbf{r}}\boldsymbol{\eta}_-\cdot\hat{\mathbf{r}})] + \Big\{\Big[2w^2\frac{\partial L}{\partial w^2}[\exp(2K)(\boldsymbol{\psi}_+^\dagger\cdot\boldsymbol{\psi}_+ - \boldsymbol{\psi}_-^\dagger\cdot\boldsymbol{\psi}_- + \boldsymbol{\eta}_-^\dagger\cdot\boldsymbol{\eta}_-) - \exp(-2K)\eta_{+0}^\dagger\eta_{+0} - 2\sinh 2K(\boldsymbol{\psi}_+^\dagger\cdot\hat{\mathbf{r}}\boldsymbol{\psi}_+\cdot\hat{\mathbf{r}} - \boldsymbol{\psi}_-^\dagger\cdot\hat{\mathbf{r}}\boldsymbol{\psi}_-\cdot\hat{\mathbf{r}} + \boldsymbol{\eta}_-^\dagger\cdot\hat{\mathbf{r}}\boldsymbol{\eta}_-\cdot\hat{\mathbf{r}})\Big] + 4w^2\frac{\partial\mathcal{G}}{\partial w^2}([-\exp(2K)(\boldsymbol{\psi}_-^\dagger\cdot\boldsymbol{\psi}_- + \boldsymbol{\eta}_-^\dagger\cdot\boldsymbol{\eta}_-) + 2\exp(-2K)\eta_{+0}^\dagger\eta_{+0} + 2\sinh 2K[(\boldsymbol{\psi}_+^\dagger\cdot\hat{\mathbf{r}}\boldsymbol{\psi}_+\cdot\hat{\mathbf{r}} + \boldsymbol{\psi}_-^\dagger\cdot\hat{\mathbf{r}}\boldsymbol{\psi}_-\cdot\hat{\mathbf{r}} + \boldsymbol{\eta}_-^\dagger\cdot\hat{\mathbf{r}}\boldsymbol{\eta} - \cdot\hat{\mathbf{r}})]]\Big\}\Big) = 1. \tag{3.58}$$

The naive norm (NN) is obtained from Eq. (3.39) or from the above with $\mathcal{K}$, $\mathcal{L} \to 1$ (or $\exp(F)$, $\exp(K) \to 1$, and $\partial L/\partial w^2$, $\partial\mathcal{G}/\partial w^2 \to 0$). In [42] we show that Eq. (3.37) gives $\boldsymbol{\psi}_-$ from

$$\boldsymbol{\psi}_- = \frac{(E^2+M^2)}{2EM}\boldsymbol{\psi}_+ - \frac{\exp(2\mathcal{G})}{2EM}[\mathcal{B}^2\exp(-2\mathcal{G})\boldsymbol{\psi}_+ + \mathcal{J}] = \frac{M}{E}\boldsymbol{\psi}_+ - \frac{\exp(2\mathcal{G})}{2EM}\mathcal{J}, \tag{3.59}$$

in which

$$\mathcal{J} = \frac{1}{2j+1}\bigg(\bigg\{\Phi_{--} - 2(j+1)\mathcal{B}^2\exp(-2\mathcal{G}) + 2\sqrt{j(j+1)}\Phi_{+-} + \frac{A_{mm}}{r^2} + \frac{B_{mm}}{r} + C_{mm} + \frac{F_{mm}}{r}\frac{d}{dr} + G_{mm}\frac{d}{dr}\bigg\}\frac{u_-}{r}\mathbf{Y}_{jm-}$$
$$+ \bigg\{\Phi_{-+} + \sqrt{j(j+1)}\bigg[2\Phi_{++} - 2\mathcal{B}^2\exp(-2\mathcal{G}) + \frac{A_{mp}}{r^2} + \frac{B_{mp}}{r} + C_{mp} + \frac{F_{mp}}{r}\frac{d}{dr} + G_{mp}\frac{d}{dr}\bigg]\bigg\}\frac{u_+}{r}\mathbf{Y}_{jm-}$$
$$+ \bigg\{-\Phi_{++} - 2j\mathcal{B}^2\exp(-2\mathcal{G}) + 2\sqrt{j(j+1)}\Phi_{-+} + \frac{A_{pp}}{r^2} + \frac{B_{pp}}{r} + C_{pp} + \frac{F_{pp}}{r}\frac{d}{dr} + G_{pp}\frac{d}{dr}\bigg\}\frac{u_+}{r}\mathbf{Y}_{jm+}$$
$$+ \bigg\{-\Phi_{+-} + \sqrt{j(j+1)}\bigg[2\Phi_{--} - 2\mathcal{B}^2\exp(-2\mathcal{G}) + \frac{A_{pm}}{r^2} + \frac{B_{pm}}{r} + C_{pm} + \frac{F_{\rm pm}}{r}\frac{d}{dr} + G_{pm}\frac{d}{dr}\bigg]\bigg\}\frac{u_-}{r}\mathbf{Y}_{jm+}\bigg), \quad (3.60)$$

(see [42] for explicit forms of the functions $A_{mm}, \ldots, G_{pm},$). This equation requires the coupled radial wave equations for spin-triplet states that follow from Eq. (3.21). They have the form [8]

$$\bigg[-\frac{1}{r}\frac{d^2}{dr^2}r + \frac{(j+2)(j+1)}{r^2} + \Phi_{++}\bigg]\frac{u^+_{(j+1)1j}}{r} + \Phi_{+-}\frac{u^+_{(j-1)1j}}{r} = \mathcal{B}^2\exp(-2\mathcal{G})\frac{u^+_{(j+1)1j}}{r},$$
$$\bigg[-\frac{1}{r}\frac{d^2}{dr^2}r + \frac{j(j-1)}{r^2} + \Phi_{--}\bigg]\frac{u^+_{(j-1)1j}}{r} + \Phi_{-+}\frac{u^+_{(j+1)1j}}{r} = \mathcal{B}^2\exp(-2\mathcal{G})\frac{u^+_{(j-1)1j}}{r}. \quad (3.61)$$

(See [42] for the explicit forms of $\Phi_{\pm\pm}$). Thus with Eq. (3.48) we have

$$\frac{u^-_{(j-1)1j}}{r} = \frac{M}{E}\frac{u^+_{(j-1)1j}}{r} - \frac{\exp(2\mathcal{G})}{2EM(2j+1)}\bigg\{\bigg[\Phi_{--} - 2(j+1)\mathcal{B}^2\exp(-2\mathcal{G}) + 2\sqrt{j(j+1)}\Phi_{+-} + \frac{A_{mm}}{r^2} + \frac{B_{mm}}{r} + C_{mm}$$
$$+ \bigg(\frac{F_{mm}}{r} + G_{mm}\bigg)\frac{d}{dr}\bigg]\frac{u^+_{(j-1)1j}}{r} + \bigg[\Phi_{-+} + \sqrt{j(j+1)}(2\Phi_{++} - 2\mathcal{B}^2\exp(-2\mathcal{G}) + \frac{A_{mp}}{r^2} + \frac{B_{mp}}{r} + C_{mp}$$
$$+ \bigg(\frac{F_{mp}}{r} + G_{mp}\bigg)\frac{d}{dr}\bigg)\bigg]\frac{u^+_{(j+1)1j}}{r}\bigg\}, \quad (3.62)$$

and

$$\frac{u^-_{(j+1)1j}}{r} = \frac{M}{E}\frac{u^+_{(j+1)1j}}{r} - \frac{\exp(2\mathcal{G})}{2EM(2j+1)}\bigg\{\bigg[-\Phi_{++} - 2j\mathcal{B}^2\exp(-2\mathcal{G}) + 2\sqrt{j(j+1)}\Phi_{-+} + \frac{A_{pp}}{r^2} + \frac{B_{pp}}{r} + C_{pp}$$
$$+ \bigg(\frac{F_{pp}}{r} + G_{pp}\bigg)\frac{d}{dr}\bigg]\frac{u^+_{(j+1)1j}}{r} + \bigg[-\Phi_{+-} + \sqrt{j(j+1)}\bigg(2\Phi_{--} - 2\mathcal{B}^2\exp(-2\mathcal{G}) + \frac{A_{pm}}{r^2} + \frac{B_{pm}}{r} + C_{pm}$$
$$+ \bigg(\frac{F_{pm}}{r} + G_{pm}\bigg)\frac{d}{dr}\bigg)\bigg]\frac{u^+_{(j-1)1j}}{r}\bigg\}. \quad (3.63)$$

In [42] we also show that

$$\eta_{+0} = i\frac{v_{j0j}}{r}Y_{jm}, \quad (3.64)$$

and

$$\boldsymbol{\eta}_- = i\frac{v^-_{j1j}}{r}\mathbf{X}_{jm}, \quad (3.65)$$

in which

$$\frac{v_{j0j}}{r} = \frac{\exp(\mathcal{G}+2K)}{E}\bigg\{\bigg[\frac{(j-1)-2Q_m}{r} - (Q_m+1)\frac{d}{dr} + \frac{5}{2}L'(Q_m+1)\bigg]\sqrt{\frac{j}{2j+1}}\frac{u^+_{(j-1)1j}}{r}$$
$$+ \bigg[\frac{(j+2)+2Q_m}{r} + (Q_m+1)\frac{d}{dr} - \frac{5}{2}L'(Q_m+1)\bigg]\sqrt{\frac{j+1}{2j+1}}\frac{u^+_{(j+1)1j}}{r}\bigg\}, \quad (3.66)$$

and

$$\frac{v^-_{j1j}}{r} = -\frac{\exp(\mathcal{G})}{M}\Bigg\{\Bigg[\Bigg(\frac{d}{dr} - \frac{j(Q_m+1)-1}{r} - \Bigg(3\mathcal{G} + \frac{L}{2}\Bigg)'\Bigg)\Bigg]\sqrt{\frac{j+1}{2j+1}}\frac{u^+_{(j-1)1j}}{r} + \Bigg[\Bigg(\frac{d}{dr} + \frac{j(Q_m+1)+2+Q_m}{r} - \Bigg(3\mathcal{G} + \frac{L}{2}\Bigg)'\Bigg)\Bigg]\sqrt{\frac{j}{2j+1}}\frac{u^+_{(j+1)1j}}{r}. \quad (3.67)$$

We use these various wave functions to compute composite 2γ decay amplitude Eq. (3.8) which after performing the q space trace gives

$$\mathcal{M}_{X\to 2\gamma} = -\frac{e^2\sqrt{\pi}}{2}\int d^3r \exp(F)\,\mathrm{Tr}_\sigma\Bigg(\exp(-i\mathbf{k}\cdot\mathbf{r})\Bigg\{i\Bigg[\cosh K\psi_- - \sinh K\boldsymbol{\sigma}\cdot\hat{\mathbf{r}}\psi_-\boldsymbol{\sigma}\cdot\hat{\mathbf{r}}\Bigg]\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}^{(\alpha_1)}(\boldsymbol{\sigma}\cdot\boldsymbol{\nabla})\frac{\exp(-mr)}{r} + [\cosh K\eta_+ - \sinh K\boldsymbol{\sigma}\cdot\hat{\mathbf{r}}\eta_+\boldsymbol{\sigma}\cdot\hat{\mathbf{r}}]\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}^{(\alpha_1)}m\frac{\exp(-mr)}{r}\Bigg\}\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}^{(\alpha_2)} + \exp(i\mathbf{k}\cdot\mathbf{r})\Bigg\{i[\cosh K\psi_- - \sinh K\boldsymbol{\sigma}\cdot\hat{\mathbf{r}}\psi_-\boldsymbol{\sigma}\cdot\hat{\mathbf{r}}]\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}^{(\alpha_2)}(\boldsymbol{\sigma}\cdot\boldsymbol{\nabla})\frac{\exp(-mr)}{r} + [\cosh K\eta_+ - \sinh K\boldsymbol{\sigma}\cdot\hat{\mathbf{r}}\eta_+\boldsymbol{\sigma}\cdot\hat{\mathbf{r}}]\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}^{(\alpha_2)}m\frac{\exp(-mr)}{r}\Bigg\}\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}^{(\alpha_1)}\Bigg). \quad (3.68)$$

The trace eliminates the contribution of the portion $\psi_+(\mathbf{r})q_1$ of the wave function.

5. Decay amplitude for ^1L_l composites

Substituting Eq. (3.51) into (3.68),

$$\mathcal{M}_{1L_l\to 2\gamma} = -\frac{e^2\sqrt{\pi}}{2}\int d^3r \exp(F)\,\mathrm{Tr}_\sigma(\{i\exp(-K)\psi_{-0}[\exp(-i\mathbf{k}\cdot\mathbf{r})\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}^{(\alpha_1)}(\boldsymbol{\sigma}\cdot\boldsymbol{\nabla})\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}^{(\alpha_2)} + \exp(i\mathbf{k}\cdot\mathbf{r})\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}^{(\alpha_2)}(\boldsymbol{\sigma}\cdot\boldsymbol{\nabla})\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}^{(\alpha_1)}]\} + \{[\exp(K)\boldsymbol{\eta}_+\cdot\boldsymbol{\sigma} - \sinh(K)2\hat{\mathbf{r}}\cdot\boldsymbol{\eta}_+\boldsymbol{\sigma}\cdot\hat{\mathbf{r}}] \times [\exp(-i\mathbf{k}\cdot\mathbf{r})\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}^{(\alpha_1)}m\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon} + \exp(i\mathbf{k}\cdot\mathbf{r})\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}^{(\alpha_2)}m\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}^{(\alpha_1)}]\})\frac{\exp(-mr)}{r}, \quad (3.69)$$

performing the remaining trace and using (3.55) and (3.56) gives

$$\mathcal{M}_{1L_l\to 2\gamma} = -\sqrt{\pi}e^2\boldsymbol{\epsilon}^{(\alpha_1)}\times\boldsymbol{\epsilon}^{(\alpha_2)}\cdot\int d^3r\exp(F)[\exp(-i\mathbf{k}\cdot\mathbf{r}) - \exp(+i\mathbf{k}\cdot\mathbf{r})]\Bigg\{\exp(-K)\frac{u^-_{j0j}}{r}Y_{jm}\hat{\mathbf{r}}\Bigg(\frac{\exp(-mr)}{r}\Bigg)' + \Bigg\{-\exp(K)\Bigg(\frac{v^+_{(j-1)1j}}{r}\mathbf{Y}_{jm-} + \frac{v^+_{(j+1)1j}}{r}\mathbf{Y}_{jm+}\Bigg) + 2\sinh(K)Y_{jm}\hat{\mathbf{r}}\Bigg(-\frac{v^+_{(j+1)1j}}{r}\sqrt{\frac{j+1}{2j+1}} + \frac{v^+_{(j-1)1j}}{r}\sqrt{\frac{j}{2j+1}}\Bigg)\Bigg\}m\frac{\exp(-mr)}{r}\Bigg], \quad (3.70)$$

with unit vectors defined in terms of the photon decay momenta and transverse polarization vectors

$$\hat{\mathbf{z}} = \hat{\mathbf{k}}, \qquad \frac{(\hat{\mathbf{x}} \pm i\hat{\mathbf{y}})}{\sqrt{2}} = \boldsymbol{\epsilon}^{(\pm)}. \quad (3.71)$$

The integral forms appearing in Eq. (3.70) are treated in Appendix E of [43] in which we show (with $g(r)$ appropriately defined)

$$\int d^3r\exp(-i\mathbf{k}\cdot\mathbf{r})\hat{\mathbf{r}}g(r)Y_{jm}(\boldsymbol{\Omega}) = 4\pi\sum_{j'=|j-1|}^{j+1}(-i)^{j'}\sqrt{\frac{(2j+1)}{4\pi}}\langle j1;00|j'0\rangle\int_0^\infty drr^2j_{j'}(kr)g(r)[\hat{\mathbf{k}}\langle j1;00|j'0\rangle\delta_{m0} - \boldsymbol{\epsilon}^{(-)}\langle j1;-11|j'0\rangle\delta_{m-1} + \boldsymbol{\epsilon}^{(+)}\langle j1;1-11|j'0\rangle\delta_{m1}]. \quad (3.72)$$

We also show (with $f_\pm(r)$ appropriately defined)

$$\int d^3r \exp(-i\mathbf{k}\cdot\mathbf{r})f_\pm(r)\mathbf{Y}_{jm\pm}(\Omega) = 4\pi\hat{\mathbf{k}}\sqrt{\frac{(2j+1)}{4\pi}}\delta_{m0}\int_0^\infty drr^2 j_j(kr)krb_\pm f_\pm(r) + 4\pi\sqrt{\frac{(2j+1)}{4\pi}}\sum_{j'=|j-1|}^{j+1}(-i)^{j'} \times \langle j1;00|j'0\rangle\int_0^\infty drr^2 f_\pm(r)[(a_\pm - 2ib_\pm)j_{j'}(kr) - ib_\pm j'_{j'}(kr)kr][\hat{\mathbf{k}}\langle j1;00|j'0\rangle\delta_{m0} - \boldsymbol{\epsilon}^-\langle j1;-11|j'0\rangle\delta_{m-1} + \boldsymbol{\epsilon}^+\langle j1;1-11|j'0\rangle\delta_{m1}], \tag{3.73}$$

in which

$$j'_j(kr) = \frac{jj_{j-1}(kr) - (j+1)j_{j+1}(kr)}{2j+1}. \tag{3.74}$$

Thus

$$\mathcal{M}_{{}^1L_l\to 2\gamma} = -\sqrt{2j+1}\boldsymbol{\epsilon}^{(\alpha_1)}\times\boldsymbol{\epsilon}^{(\alpha_2)}\cdot\hat{\mathbf{k}}\bigg\{F_{j=l}(1+(-)^j)\delta_{m0} + \sum_{j'=|j-1|}^{j+1}G_{j=l}^{(j')}(1-(-)^{j'})\frac{(1-(-1)^{j+j'})}{2} \times\langle j1;00|j'0\rangle\langle j1;00|j'0\rangle\delta_{m0}\bigg\}, \tag{3.75}$$

in which

$$F_{j=l} = -2i\pi e^2(-i)^j\int_0^\infty drmr\exp(-mr)j_j(kr)kr\exp(F+K)\bigg(\frac{1}{j+1}\sqrt{\frac{j+1}{2j+1}}\frac{v^+_{(j+1)1j}(r)}{r} + \frac{1}{j}\sqrt{\frac{j}{2j+1}}\frac{v^+_{(j-1)1j}(r)}{r}\bigg)$$

$$G_{j=l}^{(j')} = -2\pi e^2(-i)^{j'}\int_0^\infty dr\exp(-mr)\exp(F)(j_{j'}(kr)\bigg\{(mr+1)\exp(-K)\frac{u^-_{j0j}}{r} - 2mr\sinh(K)\bigg(-\frac{v^+_{(j+1)1j}}{r}\sqrt{\frac{j+1}{2j+1}} + \frac{v^+_{(j-1)1j}}{r}\sqrt{\frac{j}{2j+1}}\bigg)\bigg\} + mr\bigg\{\bigg[\bigg(-1+\frac{2}{j+1}\bigg)j_{j'}(kr) + \frac{1}{j+1}j'_{j'}(kr)kr\bigg]\sqrt{\frac{j+1}{2j+1}}\frac{v^+_{(j+1)1j}(r)}{r} + \bigg[\bigg(1+\frac{2}{j}\bigg)j_{j'}(kr) + \frac{1}{j}j'_{j'}(kr)kr\bigg]\sqrt{\frac{j}{2j+1}}\frac{v^+_{(j-1)1j}(r)}{r}\bigg\}\exp(K)\bigg). \tag{3.76}$$

Notice that decay amplitude (3.75) is zero for j odd consistent with the Landau-Yang Theorem. We call this amplitude the two-body (Dirac) decay amplitude (TBDA). What we call the naive decay amplitudes (NDA) would correspond the use of the naive norm ($\mathcal{K} = \mathcal{L} = 1$) together with $\exp(F)$, $\exp(K) \to 1$ in Eq. (3.76).

6. Decay amplitude for ${}^3L_{l\pm 1}$ composites

Using Eq. (3.57) in (3.68) leaves us with

$$\mathcal{M}_{{}^3L_{j=l\pm1}\to 2\gamma} = -\frac{e^2}{2}\sqrt{\pi}\int d^3r\exp(F)\,\mathrm{Tr}_\sigma(\exp(-i\mathbf{k}\cdot\mathbf{r})\{[i\exp K\boldsymbol{\psi}_-\cdot\boldsymbol{\sigma} - 2\sinh K\boldsymbol{\psi}_-\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}\cdot\hat{\mathbf{r}}]\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}^{(\alpha_1)}\boldsymbol{\sigma}\cdot\boldsymbol{\nabla} + m\exp(-K)\eta_{+0}\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}^{(\alpha_1)}]\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}^{(\alpha_2)}\} + \exp(+i\mathbf{k}\cdot\mathbf{r})\{i[\exp K\boldsymbol{\psi}_-\cdot\boldsymbol{\sigma} - 2\sinh K\boldsymbol{\psi}_-\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}\cdot\hat{\mathbf{r}}]\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}^{(\alpha_2)}\boldsymbol{\sigma}\cdot\boldsymbol{\nabla} + m\exp(-K)\eta_{+0}\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}^{(\alpha_2)}\}\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}^{(\alpha_1)}\})\frac{\exp(-mr)}{r}. \tag{3.77}$$

Notice that only two of the four portions of the triplet wave function (3.57) survive that trace. Performing the σ space trace and using Eqs. (3.59) and (3.64) together with

$$\hat{\mathbf{r}}\cdot\boldsymbol{\psi}_- = -\sqrt{\frac{j+1}{2j+1}}\frac{u^-_{(j+1)1j}}{r}Y_{jm} + \frac{u^-_{(j-1)1j}}{r}\sqrt{\frac{j}{2j+1}}Y_{jm}, \tag{3.78}$$

we obtain

$$\mathcal{M}_{^3L_{j=l\pm1}\to2\gamma} = -i\sqrt{\pi}e^2\int d^3r[\exp(-i\mathbf{k}\cdot\mathbf{r})+\exp(i\mathbf{k}\cdot\mathbf{r})]\exp(F)\bigg(\bigg(\frac{\exp(-mr)}{r}\bigg)'\exp K\bigg\{\frac{u^-_{(j+1)1j}}{r}[\mathbf{Y}_{jm+}(\mathbf{\Omega})\cdot\boldsymbol{\epsilon}^{(\alpha_1)}\hat{\mathbf{r}}\cdot\boldsymbol{\epsilon}^{(\alpha_2)} + \mathbf{Y}_{jm+}(\mathbf{\Omega})\cdot\boldsymbol{\epsilon}^{(\alpha_2)}\hat{\mathbf{r}}\cdot\boldsymbol{\epsilon}^{(\alpha_1)} - \mathbf{Y}_{jm+}(\mathbf{\Omega})\cdot\hat{\mathbf{r}}\boldsymbol{\epsilon}^{(\alpha_1)}\cdot\boldsymbol{\epsilon}^{(\alpha_2)}] + \frac{u^-_{(j-1)1j}}{r}[\mathbf{Y}_{jm-}(\mathbf{\Omega})\cdot\boldsymbol{\epsilon}^{(\alpha_1)}\hat{\mathbf{r}}\cdot\boldsymbol{\epsilon}^{(\alpha_2)} + \mathbf{Y}_{jm-}(\mathbf{\Omega})\cdot\boldsymbol{\epsilon}^{(\alpha_2)}\hat{\mathbf{r}}\cdot\boldsymbol{\epsilon}^{(\alpha_1)} - \mathbf{Y}_{jm-}(\mathbf{\Omega})\cdot\hat{\mathbf{r}}\boldsymbol{\epsilon}^{(\alpha_1)}\cdot\boldsymbol{\epsilon}^{(\alpha_2)}]\bigg]\bigg\} - 4\sinh K\bigg(\frac{\exp(-mr)}{r}\bigg)'\bigg[-\sqrt{\frac{j+1}{2j+1}}\frac{u^-_{(j+1)1j}}{r} + \frac{u^-_{(j-1)1j}}{r}\sqrt{\frac{j}{2j+1}}\bigg]Y_{jm}\hat{\mathbf{r}}\cdot\boldsymbol{\epsilon}^{(\alpha_1)}\hat{\mathbf{r}}\cdot\boldsymbol{\epsilon}^{(\alpha_2)} + \bigg[m\exp(-K)\frac{v^+_{j0j}}{r} - 2(m+1/r)\sinh K\bigg(-\sqrt{\frac{j+1}{2j+1}}\frac{u^-_{(j+1)1j}}{r} + \frac{u^-_{(j-1)1j}}{r}\sqrt{\frac{j}{2j+1}}\bigg)Y_{jm}(\mathbf{\Omega})\boldsymbol{\epsilon}^{(\alpha_1)}\cdot\boldsymbol{\epsilon}^{(\alpha_2)}\frac{\exp(-mr)}{r}\bigg). \tag{3.79}$$

With Eq. (3.71), the simplest terms in the above expression include forms like

$$\int d^3r\exp(-i\mathbf{k}\cdot\mathbf{r})g(r)Y_{jm}(\mathbf{\Omega}) = 4\pi(-i)^jY_{jm}(\mathbf{\Omega}_k)\int_0^\infty drr^2j_j(kr)g(r) \to \sqrt{4\pi(2j+1)}(-i)^j\delta_{m0}\int_0^\infty drr^2j_j(kr)g(r). \tag{3.80}$$

Stepping up in complexity we have the transverse parts of the dyad form

$$\int d^3r\exp(-i\mathbf{k}\cdot\mathbf{r})G(r)\hat{\mathbf{r}}\,\hat{\mathbf{r}}. \tag{3.81}$$

In [43] we show that transverse portion is

$$(\mathbf{1}-\hat{\mathbf{k}}\,\hat{\mathbf{k}})\cdot\int d^3r\exp(-i\mathbf{k}\cdot r)G(r)\hat{\mathbf{r}}\,\hat{\mathbf{r}}\cdot(1-\hat{\mathbf{k}}\,\hat{\mathbf{k}}) = (\boldsymbol{\epsilon}^{(+)}\boldsymbol{\epsilon}^{(-)}+\boldsymbol{\epsilon}^{(-)}\boldsymbol{\epsilon}^{(+)})\frac{1}{3}\sqrt{4\pi(2j+1)}\bigg\{\delta_{m0}\int_0^\infty r^2dr(-i)^jj_j(kr)G(r) - \sum_{j'=|j-2|}^{j+2}\frac{(1+(-1)^{j+j'})}{2}\langle j2;00|j'0\rangle\langle j2;00|j'0\rangle\delta_{m0} \times\int_0^\infty r^2dr(-i)^{j'}j_{j'}(kr)G_\pm(r) + \bigg\{(\boldsymbol{\epsilon}^{(+)}\boldsymbol{\epsilon}^{(+)})\sqrt{\frac{8\pi(2j+1)}{3}} \times\sum_{j'=|j-2|}^{j+2}\frac{(1+(-1)^{j+j'})}{2}\langle j2;00|j'0\rangle\langle j2;-22|j'0\rangle\delta_{m-2} + (\boldsymbol{\epsilon}^{(-)}\boldsymbol{\epsilon}^{(-)})\sqrt{\frac{8\pi(2j+1)}{3}}\sum_{j'=|j-2|}^{j+2}\frac{(1+(-1)^{j+j'})}{2} \times\langle j2;00|j'0\rangle\langle j2;2-2|j'0\rangle\delta_{m2}\bigg\}\int_0^\infty r^2dr(-i)^{j'}j_{j'}(kr)G(r). \tag{3.82}$$

Finally, we need the trace as well as transverse parts of the dyad forms

$$\int d^3r\exp(-i\mathbf{k}\cdot\mathbf{r})F_\pm(r)\mathbf{Y}_{jm\pm}(\Omega)\hat{\mathbf{r}} = \int d^3r\exp(-i\mathbf{k}\cdot\mathbf{r})[a_\pm F_\pm(r)\hat{\mathbf{r}}Y_{jm} + b_\pm rF_\pm(r)\mathbf{p}Y_{jm})\hat{\mathbf{r}}. \tag{3.83}$$

In [43] we show that with Eq. (3.71) the trace portion of Eq. (3.83) is

$$(\boldsymbol{\epsilon}^{(+)}\boldsymbol{\epsilon}^{(-)}+\boldsymbol{\epsilon}^{(-)}\boldsymbol{\epsilon}^{(+)})\int d^3r\exp(-i\mathbf{k}\cdot\mathbf{r})F_\pm(r)\mathbf{Y}_{jm\pm}(\Omega)\cdot\hat{\mathbf{r}} = (\boldsymbol{\epsilon}^{(+)}\boldsymbol{\epsilon}^{(-)} + \boldsymbol{\epsilon}^{(-)}\boldsymbol{\epsilon}^{(+)})a_\pm\sqrt{4\pi(2j+1)}(-i)^j\delta_{m0}\int_0^\infty r^2drF_\pm(r)j_j(kr), \tag{3.84}$$

while the transverse part is

$$(\mathbf{1}-\hat{\mathbf{k}}\,\hat{\mathbf{k}})\cdot\int d^3r\exp(-i\mathbf{k}\cdot\mathbf{r})F_\pm(r)\mathbf{Y}_{jm\pm}(\Omega)\hat{\mathbf{r}}\cdot(1-\hat{\mathbf{k}}\,\hat{\mathbf{k}})=(\boldsymbol{\epsilon}^{(+)}\boldsymbol{\epsilon}^{(-)}+\boldsymbol{\epsilon}^{(-)}\boldsymbol{\epsilon}^{(+)})\frac{1}{3}\sqrt{4\pi(2j+1)}$$
$$\times\bigg\{\delta_{m0}\int_0^\infty r^2dr(-i)^j j_j(kr)[(a_\pm+3ib_\pm)F_\pm(r)+ib_\pm rF'_\pm(r)]$$
$$-\sum_{j'=|j-2|}^{j+2}\frac{(1+(-1)^{j+j'})}{2}\langle j2;00|j'0\rangle\langle j2;00|j'0\rangle\delta_{m0}$$
$$\times\int_0^\infty r^2dr(-i)^{j'}j_{j'}(kr)[a_\pm F_\pm(r)+ib_\pm rF'_\pm(r)]\bigg\}$$
$$+\bigg\{(\boldsymbol{\epsilon}^{(+)}\boldsymbol{\epsilon}^{(+)})\sqrt{\frac{8\pi(2j+1)}{3}}\sum_{j'=|j-2|}^{j+2}\frac{(1+(-1)^{j+j'})}{2}$$
$$\times\langle j2;00|j'0\rangle\langle j2;-22|j'0\rangle\delta_{m-2}+(\boldsymbol{\epsilon}^{(-)}\boldsymbol{\epsilon}^{(-)})\sqrt{\frac{8\pi(2j+1)}{3}}$$
$$\times\sum_{j'=|j-2|}^{j+2}\frac{(1+(-1)^{j+j'})}{2}\langle j2;00|j'0\rangle\langle j2;2-2|j'0\rangle\delta_{m2}\bigg\}$$
$$\times\int_0^\infty r^2dr(-i)^{j'}j_{j'}(kr)[a_\pm F_\pm(r)+ib_\pm rF'_\pm(r)]. \tag{3.85}$$

After integrations by parts, substitution of values of $a_\pm$, $ib_\pm$ and combining with the other portions, we obtain

$$\mathcal{M}_{{}^3L_{j=l\pm1}\to2\gamma}=A_{j=l\pm1}(1+(-)^j)\sqrt{(2j+1)}\boldsymbol{\epsilon}^{(\alpha_1)}\cdot\boldsymbol{\epsilon}^{(\alpha_2)}\delta_{m0}+\sum_{j'=|j-2|}^{j+2}\sqrt{(2j+1)}\langle j2;00|j'0\rangle\frac{(1+(-1)^{j+j'})}{2}$$
$$\times(1+(-)^{j'})2B^{(j')}_{j=l\pm1}(\boldsymbol{\epsilon}^{(\alpha_1)}\cdot\boldsymbol{\epsilon}^{(\alpha_2)}\langle j2;00|j'0\rangle\delta_{m0}-\sqrt{6}[\boldsymbol{\epsilon}^{(\alpha_1)}\cdot(\boldsymbol{\epsilon}^{(+)}\boldsymbol{\epsilon}^{(+)})\cdot\boldsymbol{\epsilon}^{(\alpha_2)}\langle j2;-22|j'0\rangle\delta_{m-2}$$
$$+\boldsymbol{\epsilon}^{(\alpha_1)}\cdot(\boldsymbol{\epsilon}^{(-)}\boldsymbol{\epsilon}^{(-)})\cdot\boldsymbol{\epsilon}^{(\alpha_2)}\langle j2;2-2|j'0\rangle\delta_{m2}]), \tag{3.86}$$

in which

$$A_{j=l\pm1}=\frac{i2\pi e^2}{3}i^j\int_0^\infty dr\exp(-mr)\exp(F)\bigg(\bigg\{-3j_j(kr)\bigg\{mr\exp(-K)\frac{v_{j0j}}{r}-2(mr+1)\sinh K\bigg[-\sqrt{\frac{j+1}{2j+1}}\frac{u^-_{(j+1)1j}}{r}$$
$$+\frac{u^-_{(j-1)1j}}{r}\sqrt{\frac{j}{2j+1}}\bigg]\bigg\}+(mr+1)\exp K\bigg\{\bigg[\Big(j_j(kr)+\frac{2}{j+1}j'_j(kr)kr\Big)\sqrt{\frac{j+1}{2j+1}}\frac{u^-_{(j+1)1j}}{r}+\Big(-j_j(kr)+\frac{2}{j}j'_j(kr)kr\Big)$$
$$\times\sqrt{\frac{j}{2j+1}}\frac{u^-_{(j-1)1j}}{r}\bigg]\bigg\}-4\sinh K(mr+1)\bigg[-\sqrt{\frac{j+1}{2j+1}}\frac{u^-_{(j+1)1j}}{r}+\frac{u^-_{(j-1)1j}}{r}\sqrt{\frac{j}{2j+1}}\bigg]j_j(kr)$$
$$B^{(j')}_{j=l\pm1}=-\frac{i2\pi e^2}{3}i^{j'}\int_0^\infty dr\exp(-mr)\exp(F)(mr+1)\bigg(\exp K\bigg\{\bigg[\Big(\frac{3}{j+1}-1\Big)j_{j'}(kr)+\frac{1}{j+1}j'_{j'}(kr)kr\bigg]\sqrt{\frac{j+1}{2j+1}}\frac{u^-_{(j+1)1j}}{r}$$
$$+\bigg[\Big(\frac{3}{j}+1\Big)j_{j'}(kr)+\frac{1}{j}j'_{j'}(kr)kr\bigg]\sqrt{\frac{j}{2j+1}}\frac{u^-_{(j-1)1j}}{r}\bigg\}-2\sinh K(mr+1)\bigg[-\sqrt{\frac{j+1}{2j+1}}\frac{u^-_{(j+1)1j}}{r}$$
$$+\frac{u^-_{(j-1)1j}}{r}\sqrt{\frac{j}{2j+1}}\bigg]j_j(kr)\bigg) \tag{3.87}$$

As in the singlet case we obtain zero amplitude (3.86) for odd j. We also call these amplitudes the two-body Dirac decay amplitudes. Again, the corresponding naive decay amplitudes would correspond the use of the naive norm ($\mathcal{K}=\mathcal{L}=1$) together with $\exp(F)$, $\exp(K)\to1$ in Eq. (3.87).

B. Decay rates

From the above two sets of amplitudes we construct the decay rates. In our present case, we have

$$\varepsilon_{\gamma1}=\varepsilon_{\gamma2}=\frac{w}{2},\qquad b=|\mathbf{p}_\gamma|=\frac{w}{2}. \tag{3.88}$$

Also, we are not interested in the decay of a state with a definite magnetic quantum number. Rather we are interested in the average over all m. The Lagrangian that leads to the Feynman amplitude for the decay process is Lorentz invariant. Consequently the amplitude and our bound state adaptation conserves total j, m. This implies that we can sum over final states in an unrestricted way that is most convenient, without picking only special helicities that one expects to contribute. The details of the decay amplitude should do this automatically. Using the general decay rate formula [44] we obtain

$$\Gamma(X\rightarrow 2\gamma)=\frac{1}{2!}\frac{1}{(2j+1)w^2(2\pi)^6}\int d\Omega_k\frac{2\pi\varepsilon_{\gamma1}\varepsilon_{\gamma2}b}{w}\times\sum_{m,\boldsymbol{\epsilon}^{(\alpha_1)},\boldsymbol{\epsilon}^{(\alpha_2)}}|\mathcal{M}_{X\rightarrow 2\gamma}|^2 =\frac{1}{(2j+1)16(2\pi)^5}\int d\Omega_k\sum_{m,\boldsymbol{\epsilon}^{(\alpha_1)},\boldsymbol{\epsilon}^{(\alpha_2)}}|\mathcal{M}_{X\rightarrow 2\gamma}|^2, \tag{3.89}$$

in which we carry out the initial state m average and final state polarization sum independently. For spin-singlet states with the decay amplitude (3.75) this becomes

$$\begin{aligned}\Gamma({}^1L_l\rightarrow 2\gamma)=&\frac{1}{(2j+1)16(2\pi)^5}\int d\Omega_k\sum_{m,\boldsymbol{\epsilon}^{(\alpha_1)},\boldsymbol{\epsilon}^{(\alpha_2)}}|\boldsymbol{\epsilon}^{(\alpha_1)}\\&\times\boldsymbol{\epsilon}^{(\alpha_2)}\cdot\hat{\mathbf{k}}|^2(2j+1)|F_{j=l}(1+(-)^j)\delta_{m0}\\&+\sum_{j'=|j-1|}^{j+1}G_{j=l}^{(j')}(1-(-)^{j'})\frac{(1-(-1)^{j+j'})}{2}\\&\times\langle j1;00|j'0\rangle\langle j1;00|j'0\rangle\delta_{m0}|^2\\=&\frac{1}{4(2\pi)^4}|F_{j=l}(1+(-)^j)\delta_{m0}\\&+\sum_{j'=|j-1|}^{j+1}G_{j=l}^{(j')}(1-(-)^{j'})\frac{(1-(-1)^{j+j'})}{2}\\&\times\langle j1;00|j'0\rangle\langle j1;00|j'0\rangle\delta_{m0}|^2.\end{aligned}\tag{3.90}$$

We have summed over the following four independent polarization combinations

$$\boldsymbol{\epsilon}^{(\alpha_1)},\boldsymbol{\epsilon}^{(\alpha_2)}=\boldsymbol{\epsilon}^{(\pm)},\boldsymbol{\epsilon}^{(\pm)},\qquad \boldsymbol{\epsilon}^{(\alpha_1)}\cdot\hat{\mathbf{k}}=\boldsymbol{\epsilon}^{(\alpha_2)}\cdot\hat{\mathbf{k}}=0, \tag{3.91}$$

with

$$\sum_{\boldsymbol{\epsilon}^{(\alpha_1)},\boldsymbol{\epsilon}^{(\alpha_2)}}|\boldsymbol{\epsilon}^{(\alpha_1)}\times\boldsymbol{\epsilon}^{(\alpha_2)}\cdot\hat{\mathbf{k}}|^2=\sum_{\boldsymbol{\epsilon}^{(\alpha_1)},\boldsymbol{\epsilon}^{(\alpha_2)}}[1-|\boldsymbol{\epsilon}^{(\alpha_1)}\cdot(\boldsymbol{\epsilon}^{(\alpha_2)})^*|^2]=2. \tag{3.92}$$

Note that only the zero helicity states (corresponding to both photons being either left or right handed polarized)

$$\begin{aligned}\boldsymbol{\epsilon}^{(\alpha_1)},\boldsymbol{\epsilon}^{(\alpha_2)}&=\frac{1}{\sqrt{2}}(\hat{\mathbf{x}}+i\hat{\mathbf{y}}),\qquad \frac{1}{\sqrt{2}}(\hat{\mathbf{x}}-i\hat{\mathbf{y}})\equiv\boldsymbol{\epsilon}^{(+)},\boldsymbol{\epsilon}^{(-)},\\ \boldsymbol{\epsilon}^{(\alpha_1)},\boldsymbol{\epsilon}^{(\alpha_2)}&=\frac{1}{\sqrt{2}}(\hat{\mathbf{x}}-i\hat{\mathbf{y}}),\qquad \frac{1}{\sqrt{2}}(\hat{\mathbf{x}}+i\hat{\mathbf{y}})\equiv\boldsymbol{\epsilon}^{(-)},\boldsymbol{\epsilon}^{(+)},\end{aligned}\tag{3.93}$$

give nonzero contributions to the rate factor $1-|\boldsymbol{\epsilon}^{(\alpha_1)}\cdot(\boldsymbol{\epsilon}^{(\alpha_2)})^*|^2$. The total helicity ± 2 states

$$\begin{aligned}\boldsymbol{\epsilon}^{(\alpha_1)},\boldsymbol{\epsilon}^{(\alpha_2)}&=\frac{1}{\sqrt{2}}(\hat{\mathbf{x}}+i\hat{\mathbf{y}}),\qquad \frac{1}{\sqrt{2}}(\hat{\mathbf{x}}+i\hat{\mathbf{y}})\equiv\boldsymbol{\epsilon}^{(+)},\boldsymbol{\epsilon}^{(+)},\\ \boldsymbol{\epsilon}^{(\alpha_1)},\boldsymbol{\epsilon}^{(\alpha_2)}&=\frac{1}{\sqrt{2}}(\hat{\mathbf{x}}-i\hat{\mathbf{y}}),\qquad \frac{1}{\sqrt{2}}(\hat{\mathbf{x}}+i\hat{\mathbf{y}})\equiv\boldsymbol{\epsilon}^{(+)},\boldsymbol{\epsilon}^{(+)},\end{aligned}\tag{3.94}$$

give zero contribution. Performing the angular integration gives [45]

$$\Gamma({}^1S_0\rightarrow 2\gamma)=\frac{1}{(2\pi)^4}|F_0+G_0^{(1)}\langle 01;00|10\rangle|^2=\frac{1}{(2\pi)^4}|F_0+G_0^{(1)}|^2, \tag{3.95}$$

and

$$\begin{aligned}\Gamma({}^1D_2\rightarrow 2\gamma)&=\frac{1}{4(2\pi)^4}\left|2F_2+\sum_{j'=1,3}^{3}2G_{j=l}^{(j')}(1-(-)^{j'})\times\langle j1;00|j'0\rangle\langle j1;00|j'0\rangle\right|^2\\&=\frac{1}{(2\pi)^4}|F_2+G_2^{(1)}\langle 21;00|10\rangle^2+G_2^{(3)}\langle 21;00|30\rangle^2|^2\\&=\frac{1}{(2\pi)^4}\left|F_2+\frac{2}{5}G_2^{(1)}+\frac{3}{5}G_2^{(3)}\right|^2.\end{aligned}\tag{3.96}$$

Using Eq. (3.86) for triplet states ${}^3L_{l\pm1}$ our rate formula is

$$\begin{aligned}\Gamma({}^3L_{j=l\pm1}\rightarrow 2\gamma)=&\frac{1}{8(2\pi)^4}\sum_{m,\boldsymbol{\epsilon}^{(\alpha_1)},\boldsymbol{\epsilon}^{(\alpha_2)}}\Bigg|A_{j=l\pm1}\boldsymbol{\epsilon}^{(\alpha_1)}\cdot\boldsymbol{\epsilon}^{(\alpha_2)}(1+(-)^j)\delta_{m0}+\sum_{j'=|j-2|}^{j+2}(1+(-)^{j'})\frac{(1+(-1)^{j+j'})}{2}\langle j2;00|j'0\rangle\\&\times(B_{j=l\pm1}^{(j')}2[\boldsymbol{\epsilon}^{(\alpha_1)}\cdot\boldsymbol{\epsilon}^{(\alpha_2)}\langle j2;00|j'0\rangle\delta_{m0}-\sqrt{6}[\boldsymbol{\epsilon}^{(\alpha_1)}\cdot(\boldsymbol{\epsilon}^{(+)}\boldsymbol{\epsilon}^{(+)})\\&\cdot\boldsymbol{\epsilon}^{(\alpha_2)}\langle j2;-22|j'0\rangle\delta_{m-2}+\boldsymbol{\epsilon}^{(\alpha_1)}\cdot(\boldsymbol{\epsilon}^{(-)}\boldsymbol{\epsilon}^{(-)})\cdot\boldsymbol{\epsilon}^{(\alpha_2)}\langle j2;2-2|j'0\rangle\delta_{m2}])\Bigg|^2.\end{aligned}\tag{3.97}$$

Notice from Eqs. (3.93) and (3.94) that this rate in general includes both helicity zero and helicity two contributions.

In the case of $^3P_0$ decay we have $j = m = 0$ and so performing the polarization sum gives [45]

$$\Gamma(^3P_0 \to 2\gamma) = \frac{1}{2(2\pi)^4} \sum_{\boldsymbol{\epsilon}^{(\alpha_1)},\boldsymbol{\epsilon}^{(\alpha_2)}} |A_0 + 2B_0^{(2)}\langle 02;00|20\rangle^2|^2 |\boldsymbol{\epsilon}^{(\alpha_1)} \cdot \boldsymbol{\epsilon}^{(\alpha_2)}|^2 = \frac{1}{(2\pi)^4} |A_0 + 2B_0^{(2)}|^2. \tag{3.98}$$

This rate includes only helicity zero contributions.

In the case of $^3P_2$ decay we have [45]

$$\begin{aligned}
\Gamma(^3P_2 \to 2\gamma) = {}& \frac{1}{2(2\pi)^4} \sum_{m,\boldsymbol{\epsilon}^{(\alpha_1)},\boldsymbol{\epsilon}^{(\alpha_2)}} \bigg| A_2 \boldsymbol{\epsilon}^{(\alpha_1)} \cdot \boldsymbol{\epsilon}^{(\alpha_2)} \delta_{m0} + \sum_{j'=0,2,4} \langle 22;00|j'0\rangle 2B_2^{(j')} (\boldsymbol{\epsilon}^{(\alpha_1)} \cdot \boldsymbol{\epsilon}^{(\alpha_2)} \langle 22;00|j'0\rangle \delta_{m0} \\
& - \sqrt{6}[\boldsymbol{\epsilon}^{(\alpha_1)} \cdot (\boldsymbol{\epsilon}^{(+)}\boldsymbol{\epsilon}^{(+)}) \cdot \boldsymbol{\epsilon}^{(\alpha_2)} \langle 22;-22|j'0\rangle \delta_{m-2} + \boldsymbol{\epsilon}^{(\alpha_1)} \cdot (\boldsymbol{\epsilon}^{(-)}\boldsymbol{\epsilon}^{(-)}) \cdot \boldsymbol{\epsilon}^{(\alpha_2)} \langle 22;2-2|j'0\rangle \delta_{m2}]) \bigg|^2 \\
= {}& \frac{1}{(2\pi)^4} [|A_2 - 2[B_2^{(0)}\langle 22;00|00\rangle^2 + B_2^{(2)}\langle 22;00|20\rangle^2 + B_2^{(4)}\langle 22;00|40\rangle^2]|^2 + 12|[B_2^{(0)}\langle 22;00|00\rangle\langle 22;-22|00\rangle \\
& + B_2^{(2)}\langle 22;00|20\rangle\langle 22;-22|20\rangle + B_2^{(4)}\langle 22;00|40\rangle\langle 22;-22|40\rangle]|^2 + 12|[B_2^{(0)}\langle 22;00|00\rangle\langle 22;2-2|00\rangle \\
& + B_2^{(2)}\langle 22;00|20\rangle\langle 22;2-2|20\rangle + B_2^{(4)}\langle 22;00|40\rangle\langle 22;2-2|40\rangle]|^2] \\
= {}& \frac{1}{(2\pi)^4} \bigg[\bigg| A_2 + \frac{2B_2^{(0)}}{5} + \frac{2B_2^{(2)}}{7} + \frac{36B_2^{(4)}}{35} \bigg|^2 + 24 \bigg| \frac{B_2^{(0)}}{5} - \frac{2B_2^{(2)}}{7} + \frac{3B_2^{(4)}}{35} \bigg|^2 \bigg].
\end{aligned} \tag{3.99}$$

1. Positronium decays

For these decays we ignore the effects of the potentials on the norms and decay amplitudes since they are relatively weak, ($\mathcal{K} = \mathcal{L} = 1$) or ($\exp(F)$, $\exp(K) \to 1$).

(a) $^1S_0$ Decay—The amplitude for $^1S_0$ positronium decay is from Eq. (3.75)

$$\mathcal{F}_{^1S_0} = (F_0 + G_0^{(1)}), \tag{3.100}$$

where for the weak potentials we expect in QED

$$\begin{aligned}
F_0 &= -2i\pi e^2 \int_0^\infty dr m r \exp(-mr) j_0(kr) kr \frac{v_{110}^+(r)}{r} \\
G_0^{(1)} &= i2\pi e^2 \int_0^\infty dr \exp(-mr)\bigg((mr+1) j_1(kr) \\
&\quad \times \frac{u_{000}^-(r)}{r} + mr[j_1(kr) - j_0'(kr)kr] \frac{v_{110}^+(r)}{r}\bigg),
\end{aligned} \tag{3.101}$$

with

$$\begin{aligned}
\frac{u_{000}^-}{r} &= \frac{M}{E} \frac{u_{000}^+}{r} = \frac{m}{E} \frac{u_{000}^+}{r} \\
&= \frac{2m}{w\sqrt{1 + 2\alpha/(wr)}} \frac{u_{000}^+}{r},
\end{aligned} \tag{3.102}$$

and

$$\begin{aligned}
\frac{v_{110}^+}{r} &= \frac{\exp(\mathcal{G})}{E}\left(\frac{d}{dr} + \frac{L'}{2}\right) \frac{u_{000}^+}{r} = \frac{\exp(\mathcal{G})}{E} \frac{d}{dr} \frac{u_{000}^+}{r} \\
&= \frac{2}{w(1 + 2\alpha/(wr))} \frac{d}{dr} \frac{u_{000}^+}{r}.
\end{aligned} \tag{3.103}$$

This wave function is one of the small component ones. For positronium, $w = 2m + O(\alpha^2)$ and so

$$\frac{M}{E} = \sqrt{\frac{mr}{mr + \alpha}}(1 + O(\alpha^2)), \tag{3.104}$$

and (with $k = m(1 + O(\alpha^2))$

$$\begin{aligned}
F_0 &= -2i\pi e^2 \int_0^\infty dr r^2 m \frac{\exp(-mr)}{r} j_1(mr) \\
&\quad \times mr\left(\frac{r}{mr + \alpha}\right) \frac{d}{dr} \psi_{000}\bigg) \\
G_0^{(0)} &= -2\pi e^2 \int_0^\infty dr \exp(-mr)\bigg((mr+1) j_1(mr) \\
&\quad \times \sqrt{\frac{mr}{mr + \alpha}} \psi_{000} + mr[j_1(kr) + j_0'(kr)kr] \\
&\quad \times \frac{1}{m} \frac{d}{dr} \psi_{000}\bigg),
\end{aligned} \tag{3.105}$$

with the nonrelativistic wave function given by

$$\psi_{000} = \frac{(m\alpha)^{3/2}}{\sqrt{8\pi}} \exp(-\alpha m r) = \frac{R(r)}{\sqrt{4\pi}}, \tag{3.106}$$

replacing the relativistic one u_{000}^+/r. In Appendix F of [46] we give the details of our formalism leading to the well known form for the decay rate,

$$\Gamma = \frac{|G_0^{(1)}|^2}{(2\pi)^4} = |R(0)|^2 \frac{\alpha^2}{m^2} = \frac{m\alpha^5}{2}. \tag{3.107}$$

showing that the small component portion F_0 does not contribute to the singlet decay rate at this order.

(b) $^3P_{0,2}$ Decay—The branching ratio for these decay have not been measured since the decays of those states is so largely dominated by the dipole transition to the $^3S_1$ state. Nevertheless, it will be of value to determine if our covariant formalism yields the standard results given in [47,48]. The relevant amplitudes given in Eq. (3.87) for weak potentials ($\mathcal{L}, \mathcal{K} = 1$) or ($\exp(F), \exp(K) \to 1$) are

$$A_{j=l\pm1} = i\frac{2\pi e^2}{3}(-i)^j \int_0^\infty dr \exp(-mr)\bigg((mr+1)\bigg\{\bigg[j_j(kr) + \frac{2}{(j+1)}j_j'(kr)kr\bigg]\sqrt{\frac{j+1}{2j+1}}\frac{u^-_{(j+1)1j}(r)}{r} + \bigg[-j_j(kr) + \frac{2}{j}j_j'(kr)kr\bigg]\sqrt{\frac{j}{2j+1}}\frac{u^-_{(j-1)1j}(r)}{r}\bigg\} - 3j_j(kr)mr\frac{v^+_{j0j}(r)}{r}\bigg),$$

$$B^{(j')}_{j=l\pm1} = i\frac{2\pi e^2}{3}(-i)^{j'} \int_0^\infty dr \exp(-mr)(mr+1)\bigg\{\bigg[\bigg(-1+\frac{3}{j+1}\bigg)j_{j'}(kr) + \frac{1}{j+1}j_{j'}'(kr)kr\bigg]\sqrt{\frac{j+1}{2j+1}}\frac{u^-_{(j+1)1j}(r)}{r} + \bigg[\bigg(1+\frac{3}{j}\bigg)j_{j'}(kr) + \frac{1}{j}j_{j'}'(kr)kr\bigg]\sqrt{\frac{j}{2j+1}}\frac{u^-_{(j-1)1j}(r)}{r}\bigg\}. \tag{3.108}$$

The connection between the wave functions $u^-_{(j\pm1)1j}$ and $u^+_{(j\pm1)1j}$ (see Eqs. (3.62) and (3.63) appears complicated, but specializing as in the singlet case, we find that the terms beyond the first include higher order α terms from the various potential.

For the nonrelativistic wave functions we have

$$\frac{u^+_{(j\pm1)1j}}{r} = R_{(j\pm1)1j}(r) = r^{j\pm1}\chi_{(j\pm1)1j}(r).$$

We also need the small component wave function

$$\frac{v_{j0j}}{r} = \frac{\exp(3\mathcal{G})}{m}\bigg\{\bigg[\frac{(j-1)-2Q_m}{r} - (Q_m+1)\frac{d}{dr}\bigg] \times \sqrt{\frac{j}{2j+1}}\frac{u^+_{(j-1)1j}}{r} + \bigg[\frac{(j+2)+2Q_m}{r} + (Q_m+1)\frac{d}{dr}\bigg]\sqrt{\frac{j+1}{2j+1}}\frac{u^+_{(j+1)1j}}{r}\bigg\}, \tag{3.109}$$

For the $^3P_0$ state we have

$$A_0 = \frac{i2\pi e^2}{3}\int_0^\infty dr \exp(-mr)\bigg\{-3mrj_0(kr)\frac{v^+_{000}}{r} + (mr+1)(j_0(kr) + 2j_0'(kr)kr)\frac{u^-_{110}}{r}\bigg\},$$

$$B_0^{(2)} = \frac{i2\pi e^2}{3}\int_0^\infty dr \exp(-mr)(mr+1)(2j_2(kr) + j_2'(kr)kr)\frac{u^-_{110}}{r}. \tag{3.110}$$

and the decay rate (3.98) involves the amplitude combination

$$\mathcal{F}_{^3P_0} = (A_0 + 2B_0^{(2)}) = \frac{2\pi i}{3}\int_0^\infty dr \exp(-mr)\bigg\{(mr+1)[j_0(kr) + 2j_0'(kr)kr + 4j_2(kr) + 2j_2'(kr)kr]\frac{u^-_{110}(r)}{r} - 3j_0(kr)mr\frac{v^+_{000}(r)}{r}\bigg\}, \tag{3.111}$$

in which (from [42] we find for this state that Φ_{++} cancels with the remaining portions of $\mathcal{J}$)

$$\frac{u^-_{110}}{r} = \frac{M}{E}\frac{u^+_{110}}{r}. \tag{3.112}$$

We also have

$$\frac{v^+_{000}}{r} = \frac{\exp(\mathcal{G})}{E}\bigg[\frac{2}{r} + \frac{d}{dr}\bigg]\frac{u^+_{110}}{r}, \tag{3.113}$$

and

$$\frac{d}{dr}R_{110}(r)|_{r=0} = \frac{d}{dr}r\chi_{110}(r)|_{r=0} = \chi_{110}(0). \tag{3.114}$$

Our multicomponent results uses these relations in Eq. (3.111). In [46] we present the details that allows us to obtain the result of

$$\Gamma(^3P_0 \to 2\gamma) = \frac{3m\alpha^7}{256}. \tag{3.115}$$

We point out there that in the limit in which the variation of the positronium wave function is neglected (the nonrelativistic approximation and single component result) we obtain vanishing amplitude in the $^3P_0$ case. As stressed in [47] the inclusion of the small components of the wave functions is essential for this decay.

For the $^3P_2$ amplitude $j = 2$, $l = j - 1 = 1$, $j' = 0$, 2, 4. The relevant decay amplitudes are (ignoring angular momentum coupling)

$$A_2 = -i\frac{2\pi e^2}{3}\int_0^\infty dr\exp(-mr)\bigg((mr+1)(-j_2(kr) + j_2'(kr)kr)\sqrt{\frac{2}{5}}\frac{u_{112}^-(r)}{r} - 3j_2(kr)mr\frac{v_{202}^+(r)}{r}\bigg), \tag{3.116}$$

and

$$\begin{aligned} B_2^{(0)} &= -i\frac{2\pi e^2}{3}\int_0^\infty dr\exp(-mr)(mr+1) \\ &\quad\times\bigg\{\bigg[\frac{5}{2}j_0(kr) + \frac{1}{2}j_0'(kr)kr\bigg]\sqrt{\frac{2}{5}}\frac{u_{112}^-(r)}{r}\bigg\}, \\ B_2^{(2)} &= +i\frac{2\pi e^2}{3}\int_0^\infty dr\exp(-mr)(mr+1) \\ &\quad\times\bigg\{\bigg[\frac{5}{2}j_2(kr) + \frac{1}{2}j_2'(kr)kr\bigg]\sqrt{\frac{2}{5}}\frac{u_{112}^-(r)}{r}\bigg\}, \\ B_2^{(4)} &= -i\frac{2\pi e^2}{3}\int_0^\infty dr\exp(-mr)(mr+1) \\ &\quad\times\bigg\{\bigg[\frac{5}{2}j_4(kr) + \frac{1}{2}j_4'(kr)kr\bigg]\sqrt{\frac{2}{5}}\frac{u_{112}^-(r)}{r}\bigg\}, \end{aligned} \tag{3.117}$$

with the neglect of orbital mixing where

$$\begin{aligned} \frac{u_{112}^-}{r} &= \sqrt{\frac{mr}{mr+\alpha}}\frac{u_{112}^+}{r} - \frac{12}{5m^2}\sqrt{\bigg(\frac{mr}{mr+\alpha}\bigg)^3} \\ &\quad\times\bigg\{\bigg[-\frac{1}{r^2}\bigg(\sqrt{\frac{mr+\alpha}{mr}}-1\bigg) \\ &\quad+\frac{1}{r}\bigg(\sqrt{\frac{mr}{mr+\alpha}} - \sqrt{\frac{mr+\alpha}{mr}}\bigg)\frac{d}{dr}\bigg]\frac{u_{112}^+}{r}\bigg\}, \\ \frac{v_{202}}{r} &= \frac{1}{m}\sqrt{\bigg(\frac{mr}{mr+\alpha}\bigg)^3}\bigg[\frac{3-2\sqrt{\frac{mr+\alpha}{mr}}}{r} \\ &\quad-\sqrt{\frac{mr+\alpha}{mr}}\frac{d}{dr}\bigg]\sqrt{\frac{2}{5}}\frac{u_{112}^+}{r}. \end{aligned} \tag{3.118}$$

Using the above expressions for v_{202}^+ and $u_{112}^-(r)$ with

$$\frac{u_{112}^+}{r} = R_{112}(r) = r\chi_{112}(r), \tag{3.119}$$

and Eq. (3.99) leads to [46]

$$\Gamma(^3P_2 \to 2\gamma) = \frac{m\alpha^7}{320}, \tag{3.120}$$

and the ratio $\Gamma(^3P_0 \to 2\gamma)/\Gamma(^3P_2 \to 2\gamma) = \frac{15}{4}$. Even though our approach leads to the earlier results of [47,48] it is of interest to see how our constraint formalism based approach differs from other approaches. We first note that in the constraint approach, the general frame form of the c.m. decay amplitude of Eqs. (3.5) and (3.6) is

$$\int d^4p\,\mathrm{Tr}\Gamma(p_-, p_+; k_1, k_2)\delta(p\cdot\hat{P})\Psi(\mathrm{p}), \tag{3.121}$$

in which

$$\begin{aligned}\Gamma(p_-, p_+; k_1, k_2) = e^2\bigg[&\boldsymbol{\gamma}\cdot\boldsymbol{\epsilon}^{(\alpha_1)}\frac{m-\gamma\cdot(p_- - k_1)}{(p_- - k_1)^2+m^2} \\ &\times\boldsymbol{\gamma}\cdot\boldsymbol{\epsilon}^{(\alpha_2)} + \boldsymbol{\gamma}\cdot\boldsymbol{\epsilon}^{(\alpha_2)} \\ &\times\frac{m-\gamma\cdot(p_- - k_2)}{(p_- - k_2)^2+m^2}\boldsymbol{\gamma}\cdot\boldsymbol{\epsilon}^{(\alpha_1)}\bigg].\end{aligned} \tag{3.122}$$

In the constraint approach, from Eqs. (2.6) and (2.13)

$$p_- = \frac{\hat{P}}{2} + p, \qquad p_+ = \frac{\hat{P}}{2} - p. \tag{3.123}$$

The c.m. form is seen to follow directly from this since there we have $p = (0, \mathbf{p})$ and

$$\begin{aligned} p_- &= \bigg(\frac{w}{2}, \mathbf{p}\bigg); \qquad p_+ = \bigg(\frac{w}{2}, -\mathbf{p}\bigg), \\ k_1 &= \bigg(\frac{w}{2}, \mathbf{k}\bigg); \qquad k_2 = \bigg(\frac{w}{2}, -\mathbf{k}\bigg). \end{aligned} \tag{3.124}$$

This interpretation of the amplitude follows directly from the constraint formalism and is distinct from that used in other approaches which assume an on shell form for the amplitude (see e.g. [15] which uses $p_i^0 = \sqrt{m^2 + \mathbf{p}_i^2}$). The decay amplitude we use incorporates an off-mass-shell assumption which is true for constituent particles of the bound state. The constraint modification of the off-mass-shell amplitude in addition places it on energy shell. This gives us the Yukawa modification seen in Eq. (3.8) not appearing in other approaches.

2. Meson decays

(a) *η_c, $\eta_{c'}$ Decays*—For the η_c the state vector is

$$|\eta_c\rangle = \frac{1}{\sqrt{3}}\sum_{r,g,b}|c\bar{c}\rangle, \tag{3.125}$$

with the charge of the charmed quark equal to $2e/3$. Since the interaction is color independent the resultant TBDA is

$$\begin{aligned}\mathcal{F}_{\eta_c} &= \frac{4\sqrt{3}}{9}(F_0 + G_0^{(1)}) \\ &= \frac{4\sqrt{3}e^2}{9}2\pi i\int_0^\infty dr\exp(-mr)\bigg(j_1(kr)\exp(F-K) \\ &\quad\times\frac{u_{000}^-}{r}(1+mr) + mr\frac{v_{110}^+}{r}\{j_1(kr)\exp(F) \\ &\quad\times[\exp(K) + 2\sinh(K)] + kr\exp(F+K) \\ &\quad\times[j_1'(kr) - j_0(kr)]\}\bigg)\end{aligned} \tag{3.126}$$

In (3.126) we take numerical wave functions from the work of [7]. The remaining parts of our multicomponent wave functions are

$$\frac{u_{000}^{-}}{r} = \frac{M}{E}\frac{u_{000}^{+}}{r} \tag{3.127}$$

and

$$\frac{v_{110}^{+}}{r} = \frac{\exp(\mathcal{G})}{E}\left(\frac{d}{dr} + \frac{L'}{2}\right)\frac{u_{000}^{+}}{r}, \tag{3.128}$$

which appear in that equation and satisfy the TBDN condition Eq. (3.52). Its radial form is [41]

$$\frac{1}{2}\int_0^\infty drr^2 \exp(2F)\left(\exp(-2K)\left[\left(\frac{u_{000}^{+}}{r}\right)^2 + \left(\frac{u_{000}^{-}}{r}\right)^2 + \left(\frac{v_{110}^{+}}{r}\right)^2\right] + 2w^2\frac{\partial L}{\partial w^2}\exp(-2K) \times\left[\left(\frac{u_{000}^{+}}{r}\right)^2 - \left(\frac{u_{000}^{-}}{r}\right)^2 - \left(\frac{v_{110}^{+}}{r}\right)^2\right] + 4w^2\frac{\partial \mathcal{G}}{\partial w^2}\exp(-2K)\left[2\left(\frac{u_{000}^{+}}{r}\right)^2 + \left(\frac{u_{000}^{-}}{r}\right)^2\right]\right) = 1. \tag{3.129}$$

Our NDA approximate to (3.126) becomes

$$\mathcal{F}_{\eta_c} = \frac{4\sqrt{3}e^2}{9}2\pi i\int_0^\infty dr\exp(-mr)\left(j_1(kr)\frac{u_{000}^{-}}{r} \times(1+mr) + mr\frac{v_{110}^{+}}{r}\{j_1(kr) + kr[j_1'(kr) - j_0(kr)]\}\right). \tag{3.130}$$

The multicomponent forms given by the TBDA and TBDN in (3.126) and (3.129) respectively give a decay rate of 9.18 keV, while that obtained from the corresponding multicomponent forms given by the NDA and NN is 9.15 keV. If we further ignore the small components in these latter forms by taking $u_{000}^{-} = u_{000}^{+}$ and $v_{110}^{+} = 0$, then the decay rate is 9.09 keV. Including first order QCD radiative corrections [49] damps these decay rates by a factor of $(1 + \alpha_s/\pi(\pi^2/3 - 20/3))$ [50][1] giving us 9.18 into 6.20 and 9.15 into 6.18 keV (and 9.09 to 6.14 keV when ignoring small components). These are to be compared with the observed rate of 7.44 ± 1.0 keV. For the η_c' our results are 4.81 and 2.79 keV (and 2.68 keV), respectively, compared with the observed rate of $1.3 \pm .6$ keV. The QCD radiative corrections reduce these from 4.81 to 3.36 and 2.79 to 1.95 keV (and 2.68 to 1.87 keV) to be compared with the observed rate of 1.3 ± 0.6 keV. The overall additional effects of using the TBDN and TBDA above that of the NN and NDA appear to be very small for the η_c but for the η_c' they are substantial (but in the wrong direction). It is of interest to trace the origin of these contrasting behaviors. The square root of the norm (starting with a normed u_{000}^{+}/r) for the η_c in the TBDN and TBDA case is 1.64, compared with 1.03 in the NN and NDA case. The respective raw decay amplitudes (with the norm effects taken out) are 0.252 and 0.160. These are both substantial differences. However, including the norm effect in the amplitude cancels out these differences giving us about a 0.155 amplitude in both cases. This cancellation hides the substantial effects of both the TBDA and TBDN. Things are different in the case of the η_c'. There the square root of the norm in the TBDN and TBDA case is 1.22, compared with 1.008 in the NN and NDA case. The respective raw decay amplitudes (with the norm effects taken out) are -0.138 and -0.087. Unlike the case of the η_c the effect of including the norm in the amplitude does not cancel out these differences giving us about a -0.113 amplitude in the first case and a -0.086 amplitude in the second. The ratio of the TBDA to the NDA are 1.58 in both cases. However, the square root norm ratios are quite different, being 1.59 in the case of the η_c but only 1.21 in the case of the η_c'. This difference for the η_c' may point to a limitation of the linear confining model used in working out the wave functions near threshold for the $\eta_{c'}$ decay.

(b) χ_0 *Decay*—The $^3P_0$ decay amplitudes are from Eqs. (3.87) and in the combination from (3.98)

$$\begin{aligned}\mathcal{F}_{\chi_0} &= \frac{4\sqrt{3}}{9}(A_0 + 2B_0^{(2)}) \\ &= \frac{4\sqrt{3}e^2}{9}\frac{2\pi i}{3}\int_0^\infty dr\exp(-mr) \\ &\quad\times\exp(F)\Big\{(mr+1)\{\exp K[j_0(kr) + 2j_0'(kr)kr \\ &\quad + 4j_2(kr) + 2j_2'(kr)kr] + 2j_0(kr)\sinh K\} \\ &\quad\times\frac{u_{110}^{-}(r)}{r} - 3\exp(-K)j_0(kr)mr\frac{v_{000}^{+}(r)}{r}\Big\},\end{aligned} \tag{3.131}$$

with the same color and flavor factors as before, in which (see [42])

$$\frac{u_{110}^{-}}{r} = \frac{M}{E}\frac{u_{110}^{+}}{r}. \tag{3.132}$$

We also have

$$\frac{v_{000}^{+}}{r} = \frac{\exp(\mathcal{G})}{E}\left[\frac{2}{r} - \frac{5}{2}L' + \frac{d}{dr}\right]\frac{u_{110}^{+}}{r}. \tag{3.133}$$

Our multicomponent TBDA results uses these relations in Eq. (3.131). The TBDN condition (3.58) becomes [41]

[1] For the running coupling constant α_s we use the value $\alpha_s = 12\pi/(27\log(10 + (w/0.31\ \text{GeV})^2))$ given in Ref. [1]

$$\frac{1}{2}\int drr^2\exp(2F)\bigg(\exp(-2K)\bigg\{\bigg(\frac{u^+_{110}}{r}\bigg)^2+\bigg(\frac{u^-_{110}}{r}\bigg)^2+\bigg(\frac{v^+_{000}}{r}\bigg)^2+\bigg\{2w^2\frac{\partial L}{\partial w^2}\bigg(\bigg(\frac{u^+_{110}}{r}\bigg)^2-\bigg(\frac{u^-_{110}}{r}\bigg)^2-\bigg(\frac{v^+_{000}}{r}\bigg)^2\bigg)+4w^2\frac{\partial\mathcal{G}}{\partial w^2}\bigg(-\bigg(\frac{u^-_{110}}{r}\bigg)^2+2\bigg(\frac{v^+_{000}}{r}\bigg)^2\bigg)\bigg\}+8w^2\frac{\partial\mathcal{G}}{\partial w^2}\sinh 2K\bigg(\frac{u^+_{110}}{r}\bigg)^2\bigg)=1 \tag{3.134}$$

The NDA becomes

$$\mathcal{F}_{\chi_0}=\frac{4\sqrt{3}e^2}{9}\frac{2\pi i}{3}\int_0^\infty dr\exp(-mr)\bigg\{(mr+1)\times[j_0(kr)+2j_0'(kr)kr+4j_2(kr)+2j_2'(kr)kr]\frac{u^-_{110}(r)}{r}-3\exp(-K)j_0(kr)mr\frac{v^+_{000}(r)}{r}\bigg\}. \tag{3.135}$$

Our multicomponent TBDA and TBDN result from (3.131) and (3.134) is 3.90 keV, while that obtained from the corresponding NDA form of (3.135) is 3.28 keV. If we further ignore the small components in these latter forms by taking $u^-_{110}=u^+_{110}$ and $v^+_{000}=0$, then the decay rate is 0.646 keV. The QCD radiative corrections [49] modify these by a factor of $(1+(_s/)(_2/3-28/9))$ from 3.90 to 3.96 and from 3.28 to 3.34 keV (and 0.646 to 0.656 keV). These are to be compared with the observed rate of 2.6 ± 0.65 keV. The multicomponent effects are substantial even if we do not include the effects of the TBDA and TBDN. Those additional effects are small compared with the effects of including the multicomponents by themselves. This parallels that which occurs in the $^3P_0$ positronium decay where the amplitude vanishes without the multicomponent (small) parts of the wave function.

(c) *χ_2 Decay*—The $^3P_2$ decay amplitudes (3.87) appear from Eq. (3.99) in the separate combination

$$\begin{aligned}\mathcal{F}(\mathcal{K})_{\chi_2}&=\frac{4\sqrt{3}}{9}\bigg[A_2+\frac{2B_2^{(0)}}{5}+\frac{2B_2^{(2)}}{7}+\frac{36B_2^{(4)}}{35}\bigg)\bigg]\\&=\frac{4\sqrt{3}e^2}{9}\frac{2\pi i}{3}\int_0^\infty dr\exp(-mr)\exp(F)\bigg[-(mr+1)\exp K\bigg\{\bigg[j_2(kr)+\frac{2}{3}j_2'(kr)kr\bigg]\sqrt{\frac{3}{5}}\frac{u^-_{312}(r)}{r}\\&\quad+\sqrt{\frac{2}{5}}\frac{u^-_{112}(r)}{r}[-2j_2(kr)+j_2'(kr)kr]\bigg\}+j_2(kr)\bigg\{3mr\exp(-K)\frac{v_{202}}{r}\\&\quad-2(mr+1)\sinh K\bigg[-\sqrt{\frac{3}{5}}\frac{u^-_{312}}{r}+\frac{u^-_{112}}{r}\sqrt{\frac{2}{5}}\bigg]\bigg\}-\frac{2}{5}(mr+1)\bigg(\exp K\bigg\{\frac{1}{3}j_0'(kr)kr\sqrt{\frac{3}{5}}\frac{u^-_{312}(r)}{r}\\&\quad+\bigg[\frac{5}{2}j_0(kr)+\frac{1}{2}j_0'(kr)kr\bigg]\sqrt{\frac{2}{5}}\frac{u^-_{112}(r)}{r}\bigg\}-2\sinh K(mr+1)\bigg[-\sqrt{\frac{3}{5}}\frac{u^-_{312}}{r}+\sqrt{\frac{2}{5}}\frac{u^-_{112}}{r}\bigg]j_2(kr)\bigg)\\&\quad+\frac{2}{7}(mr+1)\bigg(\exp K\bigg\{\frac{1}{3}j_2'(kr)kr\sqrt{\frac{3}{5}}\frac{u^-_{312}(r)}{r}+\bigg[\frac{5}{2}j_2(kr)+\frac{1}{2}j_2'(kr)kr\bigg]\sqrt{\frac{2}{5}}\frac{u^-_{112}(r)}{r}\bigg\}\\&\quad-2\sinh K(mr+1)\bigg[-\sqrt{\frac{3}{5}}\frac{u^-_{312}}{r}+\sqrt{\frac{2}{5}}\frac{u^-_{112}}{r}\bigg]j_2(kr)\bigg)-\frac{36}{35}(mr+1)\bigg(\exp K\bigg\{\frac{1}{3}j_4'(kr)kr\sqrt{\frac{3}{5}}\frac{u^-_{312}(r)}{r}\\&\quad+\bigg[\frac{5}{2}j_4(kr)+\frac{1}{2}j_4'(kr)kr\bigg]\sqrt{\frac{2}{5}}\frac{u^-_{112}(r)}{r}\bigg\}-2\sinh K(mr+1)\bigg[-\sqrt{\frac{3}{5}}\frac{u^-_{312}}{r}+\sqrt{\frac{2}{5}}\frac{u^-_{112}}{r}\bigg]j_2(kr)\bigg)\bigg]\end{aligned} \tag{3.136}$$

and

$$\mathcal{G}(K)_{\chi_2} = \frac{4\sqrt{3}}{9}\left[\frac{B_2^{(0)}}{5} - \frac{2B_2^{(2)}}{7} + \frac{3B_2^{(4)}}{35}\right]$$
$$= -\frac{4\sqrt{3}e^2}{9}\frac{2\pi i}{3}\int_0^\infty dr \exp(-mr)\exp(F)(mr+1)\left[\frac{1}{5}\left(\exp K\left\{\frac{1}{3}j_0'(kr)kr\sqrt{\frac{3}{5}}\frac{u_{312}^-(r)}{r} + \left[\frac{5}{2}j_0(kr) + \frac{1}{2}j_0'(kr)kr\right]\right.\right.\right.$$
$$\left.\times\sqrt{\frac{2}{5}}\frac{u_{112}^-(r)}{r}\right\} - 2\sinh K\left[-\sqrt{\frac{3}{5}}\frac{u_{312}^-}{r} + \sqrt{\frac{2}{5}}\frac{u_{112}^-}{r}\right]j_2(kr)\Bigg) + \frac{2}{7}\left(\exp K\left\{\left[\frac{1}{3}j_2'(kr)kr\right]\sqrt{\frac{3}{5}}\frac{u_{312}^-(r)}{r}\right.\right.$$
$$+\left[\frac{5}{2}j_2(kr) + \frac{1}{2}j_2'(kr)kr\right]\sqrt{\frac{2}{5}}\frac{u_{112}^-(r)}{r}\Bigg\} - 2\sinh K\left[-\sqrt{\frac{3}{5}}\frac{u_{312}^-}{r} + \sqrt{\frac{2}{5}}\frac{u_{112}^-}{r}\right]j_2(kr)\Bigg) + \frac{3}{35}\left(\exp K\left\{\left[\frac{1}{3}j_4'(kr)kr\right]\right.\right.$$
$$\left.\left.\left.\times\sqrt{\frac{3}{5}}\frac{u_{312}^-(r)}{r} + \left[\frac{5}{2}j_4(kr) + \frac{1}{2}j_4'(kr)kr\right]\sqrt{\frac{2}{5}}\frac{u_{112}^-(r)}{r}\right\} - 2\sinh K\left[-\sqrt{\frac{3}{5}}\frac{u_{312}^-}{r} + \sqrt{\frac{2}{5}}\frac{u_{112}^-}{r}\right]j_2(kr)\right)\right], \quad (3.137)$$

in which

$$\frac{u_{112}^-}{r} = \frac{E}{M}\frac{u_{112}^+}{r} - \frac{\exp(2\mathcal{G})}{10EM}\left\{\left[\Phi_{--} + 4\exp(2\mathcal{G})(E^2 - M^2) - 2\sqrt{6}\Phi_{+-} + \frac{A_{mm}}{r^2} + \frac{B_{mm}}{r} + C_{mm}\right.\right.$$
$$\left. + \left(\frac{F_{mm}}{r} + G_{mm}\right)\frac{d}{dr}\right]\frac{u_{112}^+}{r} + \sqrt{6}\left[\frac{6\Phi_{-+}}{\sqrt{6}} + 2\Phi_{++} - 2\exp(2\mathcal{G})(E^2 - M^2) + \frac{A_{mp}}{r^2} + \frac{B_{mp}}{r} + C_{mp}\right.$$
$$\left.\left. + \left(\frac{F_{mp}}{r} + G_{mp}\right)\frac{d}{dr}\right]\frac{u_{312}^+}{r}\right\}, \quad (3.138)$$

with $\Phi_{--}, \Phi_{-+}, A_{mm}, \ldots, G_{mp}$ given in [42] and

$$\frac{u_{312}^-}{r} = \frac{M}{E}\frac{u_{312}^+}{r} - \frac{\exp(2\mathcal{G})}{10EM}\left\{\left[-\Phi_{++} - 4\exp(2\mathcal{G})(E^2 - M^2) + 2\sqrt{6}\Phi_{-+} + \frac{A_{pp}}{r^2} + \frac{B_{pp}}{r} + C_{pp}\right.\right.$$
$$\left. + \left(\frac{F_{pp}}{r} + G_{pp}\right)\frac{d}{dr}\right]\frac{u_{312}^+}{r} + \sqrt{6}\left[-\frac{\Phi_{+-}}{\sqrt{6}} + 2\Phi_{--} - 2\exp(2\mathcal{G})(E^2 - M^2) + \frac{A_{pm}}{r^2} + \frac{B_{pm}}{r} + C_{pm}\right.$$
$$\left.\left. + \left(\frac{F_{pm}}{r} + G_{pm}\right)\frac{d}{dr}\right]\frac{u_{112}^+}{r}\right\} \quad (3.139)$$

with the expression for $\Phi_{++}, \Phi_{+-}, A_{pp}, \ldots, G_{pm}$ also in [42]. The other radial wave functions are

$$\frac{v_{202}}{r} = \frac{\exp(\mathcal{G} + 2K)}{E}\left\{\left[\frac{1 - 2Q_m}{r} - (Q_m + 1)\frac{d}{dr} - \frac{5L'}{2}(Q_m + 1)\right]\sqrt{\frac{2}{5}}\frac{u_{112}^+}{r} + \left[\frac{4 + 2Q_m}{r} + (Q_m + 1)\frac{d}{dr}\right.\right.$$
$$\left.\left. - \frac{5L'}{2}(Q_m + 1)\right]\sqrt{\frac{3}{5}}\frac{u_{312}^+}{r}\right\} \quad (3.140)$$

and

$$\frac{v_{212}^-}{r} = -\frac{\exp(\mathcal{G})}{M}\left\{\left[\left(\frac{d}{dr} - \frac{1}{r} - \frac{2Q_m}{r} + \frac{(L + 6\mathcal{G})'}{2}\right)\right]\sqrt{\frac{3}{5}}\frac{u_{112}^+}{r} + \left[\left(\frac{d}{dr} + \frac{4}{r} + \frac{3Q_m}{r} + \frac{(L + 6\mathcal{G})'}{2}\right)\right]\sqrt{\frac{2}{5}}\frac{u_{312}^+}{r}.\right. \quad (3.141)$$

Only the first of these latter two wave functions contributes to the decay amplitude. All wave functions contribute to the TBDN condition (3.58) which has the form [41]

$$1=\frac{1}{2}\int drr^2\exp(2F)\bigg(\bigg(\frac{u_{112}^{+}}{r}\bigg)^2\bigg\{\bigg[\exp(2K)-\frac{4}{5}\sinh 2K\bigg]\bigg[1+2w^2\frac{\partial L}{\partial w^2}\bigg]+\frac{4}{5}\sinh 2K\bigg(4w^2\frac{\partial\mathcal{G}}{\partial w^2}\bigg)\bigg\}+\bigg(\frac{u_{112}^{-}}{r}\bigg)^2 \times\bigg\{\bigg[\exp(2K)-\frac{4}{5}\sinh 2K\bigg]\bigg[1-2w^2\frac{\partial L}{\partial w^2}\bigg]+\frac{4}{5}\sinh 2K\bigg(4w^2\frac{\partial\mathcal{G}}{\partial w^2}\bigg)\bigg\}+\bigg(\frac{u_{312}^{+}}{r}\bigg)^2\bigg\{\bigg[\exp(2K)-\frac{6}{5}\sinh 2K\bigg] \times\bigg[1+2w^2\frac{\partial L}{\partial w^2}\bigg]+\frac{6}{5}\sinh 2K\bigg(8w^2\frac{\partial\mathcal{G}}{\partial w^2}\bigg)\bigg\}+\bigg(\frac{u_{312}^{-}}{r}\bigg)^2\bigg\{\bigg[\exp(2K)-\frac{6}{5}\sinh 2K\bigg]\bigg[1-2w^2\frac{\partial L}{\partial w^2}-4w^2\frac{\partial\mathcal{G}}{\partial w^2}\bigg]\bigg\} +\bigg(\frac{u_{312}^{+}}{r}\bigg)\bigg(\frac{u_{112}^{+}}{r}\bigg)\frac{4\sqrt{6}}{5}\sinh 2K\bigg[1+2w^2\frac{\partial L}{\partial w^2}-4w^2\frac{\partial\mathcal{G}}{\partial w^2}\bigg]+\bigg(\frac{u_{312}^{-}}{r}\bigg)\bigg(\frac{u_{112}^{-}}{r}\bigg)\frac{4\sqrt{6}}{5}\sinh 2K\bigg[1-2w^2\frac{\partial L}{\partial w^2}-4w^2\frac{\partial\mathcal{G}}{\partial w^2}\bigg] +\bigg(\frac{v_{202}^{+}}{r}\bigg)^2\exp(-2K)\bigg(1-2w^2\frac{\partial L}{\partial w^2}+8w^2\frac{\partial\mathcal{G}}{\partial w^2}\bigg)+\bigg(\frac{v_{212}^{-}}{r}\bigg)^2\exp(2K)\bigg(\bigg(1+2w^2\frac{\partial L}{\partial w^2}+4w^2\frac{\partial\mathcal{G}}{\partial w^2}\bigg)\bigg). \tag{3.142}$$

Our multicomponent results uses these relations in Eq. (3.136) and (3.137). Our corresponding NDAs are

$$\mathcal{F}(\mathcal{K})_{\chi_2}=\frac{4\sqrt{3}e^2}{9}\frac{2\pi i}{3}\int_0^\infty dr\exp(-mr)\bigg[-(mr+1)\bigg\{\bigg[j_2(kr)+\frac{2}{3}j_2'(kr)kr\bigg]\sqrt{\frac{3}{5}}\frac{u_{312}^{-}(r)}{r}+\sqrt{\frac{2}{5}}\frac{u_{112}^{-}(r)}{r}[-2j_2(kr) + j_2'(kr)kr]\bigg\}+j_2(kr)3mr\frac{v_{202}}{r}\bigg]\bigg\}-\frac{2}{5}(mr+1)\bigg(\bigg\{\bigg[\frac{1}{3}j_0'(kr)kr\sqrt{\frac{3}{5}}\frac{u_{312}^{-}(r)}{r}+\bigg[\frac{5}{2}j_0(kr)+\frac{1}{2}j_0'(kr)kr\bigg] \times\sqrt{\frac{2}{5}}\frac{u_{112}^{-}(r)}{r}\bigg\}\bigg)+\frac{2}{7}(mr+1)\bigg(\bigg\{\bigg[\frac{1}{3}j_2'(kr)kr\sqrt{\frac{3}{5}}\frac{u_{312}^{-}(r)}{r}+\bigg[\frac{5}{2}j_2(kr)+\frac{1}{2}j_2'(kr)kr\bigg]\sqrt{\frac{2}{5}}\frac{u_{112}^{-}(r)}{r}\bigg\}\bigg) -\frac{36}{35}(mr+1)\bigg(\bigg\{\bigg[\frac{1}{3}j_4'(kr)kr\sqrt{\frac{3}{5}}\frac{u_{312}^{-}(r)}{r}+\bigg[\frac{5}{2}j_4(kr)+\frac{1}{2}j_4'(kr)kr\bigg]\sqrt{\frac{2}{5}}\frac{u_{112}^{-}(r)}{r}\bigg\}\bigg)\bigg], \tag{3.143}$$

and

$$\mathcal{G}(\mathcal{K})_{\chi_2}=-\frac{4\sqrt{3}e^2}{9}\frac{2\pi i}{3}\int_0^\infty dr\exp(-mr)(mr+1)\bigg[\frac{1}{5}\bigg(\bigg\{\bigg[\frac{1}{3}j_0'(kr)kr\sqrt{\frac{3}{5}}\frac{u_{312}^{-}(r)}{r} +\bigg[\frac{5}{2}j_0(kr)+\frac{1}{2}j_0'(kr)kr\bigg]\sqrt{\frac{2}{5}}\frac{u_{112}^{-}(r)}{r}\bigg\}\bigg)+\frac{2}{7}\bigg(\bigg\{\bigg[\frac{1}{3}j_2'(kr)kr\bigg]\sqrt{\frac{3}{5}}\frac{u_{312}^{-}(r)}{r}+\bigg[\frac{5}{2}j_2(kr)+\frac{1}{2}j_2'(kr)kr\bigg]\sqrt{\frac{2}{5}}\frac{u_{112}^{-}(r)}{r}\bigg\}\bigg) +\frac{3}{35}\bigg(\bigg\{\bigg[\frac{1}{3}j_4'(kr)kr\bigg]\sqrt{\frac{3}{5}}\frac{u_{312}^{-}(r)}{r}+\bigg[\frac{5}{2}j_4(kr)+\frac{1}{2}j_4'(kr)kr\bigg]\sqrt{\frac{2}{5}}\frac{u_{112}^{-}(r)}{r}\bigg\}\bigg)\bigg]. \tag{3.144}$$

Our multicomponent TBDA and TBDN result from (3.136), (3.137), and (3.142) is 1.43 keV, while that obtained from the corresponding multicomponent NDA result of (3.143) and (3.144) is 0.836 keV. If we further ignore the small and tensor coupled components in these latter forms by taking $u_{112}^{-}=u_{112}^{+}$ and $u_{312}^{-}=u_{312}^{+}=v_{202}^{+}=v_{212}^{+}=0$, then the decay rate is 0.033 keV. THe QCD radiactive corrections [49] modify these by a factor of $(1-16\alpha_8/\pi)$ from 1.43 to 0.743 and 0.836 to 0.435 keV (0.033 to 0.017 keV). These are to be compared with the observed rate of $0.528\pm.09$ keV. Full tensor couplings are included in the first two results. As with the $^3P_0$ decay the NDA and NN multicomponent effects are substantial even if we do not include those of the TBDA and TBDN. Those effects are themselves significantly larger than the effects of the NDA and NN.

(d) π^0 Decay The π^0 state vector is

$$|\pi^0\rangle=\sum_{c=r,g,b}\frac{1}{\sqrt{2}}(|\bar{u}u\rangle-|\bar{d}d\rangle)_c\frac{1}{\sqrt{3}}, \tag{3.145}$$

where the charge of the u is $+2e/3$ that of the d is $-e/3$. Thus, the amplitude for its annihilation is modified by a factor of $\sqrt{3}[(2/3)^2-(-1/3)^2]/\sqrt{2}$. Otherwise the wave function discussion is the same as in the section on η_c decay. So we obtain

$$\mathcal{F}_{\pi_0}=\sqrt{\frac{3}{2}}\frac{e^2}{3}2\pi i\int_0^\infty dr\exp(-mr)\bigg(j_1(kr) \times\exp(F-K)\frac{u_{000}^{-}}{r}(1+mr)+mr\frac{v_{110}^{+}}{r} \times\{j_1(kr)\exp(F)[\exp(K)+2\sinh(K)] +kr\exp(F+K)[j_1'(kr)-j_0(kr)]\}\bigg). \tag{3.146}$$

Our multicomponent TBDA and TBDN result from this is 24.7 eV, while that obtained from the multicomponent NDA and NN result is 94.4 eV. If we further ignore the small components in the weak potential form by taking $u_{000}^{-}=u_{000}^{+}$ and $v_{110}^{+}=0$, then the decay rate is 89.5 eV. QCD radiative corrections modify these by a factor of $(1+(\alpha_s/\pi)\times(\pi^2/3-20/3))$ from 24.7 eV to 8.73 eV and 94.4 eV to 33.5 eV (89.5 eV to 31.5 eV). These are to be compared with the observed rate of 7.72 ± .04 eV, so that our primary result of 8.73 is off by only 13%. The influence of including the TBDA and TBDN multicomponent effects in the norm and the amplitude are substantial when compared to that of including just the NDA and NN effects and bring our pion decay rate reasonably close to the observed rate.

(e) π_2 Decay—For this spin-singlet decay the relevant amplitude is

$$\begin{aligned}\mathcal{F}_{\pi_2}&=\sqrt{\frac{3}{2}}\frac{1}{3}\left(F_2+\frac{2}{5}G_2^{(1)}+\frac{3}{5}G_2^{(3)}\right)\\&=i\sqrt{\frac{3}{2}}\frac{e^2}{3}2\pi\int_0^\infty drmr\exp(-mr)\left[\exp(F+K)mrj_2(kr)kr\left(\frac{1}{3}\sqrt{\frac{3}{5}}\frac{v_{312}^{+}(r)}{r}+\frac{1}{2}\sqrt{\frac{2}{5}}\frac{v_{112}^{+}(r)}{r}\right)\right.\\&\quad+\frac{2}{5}\left((mr+1)j_1(kr)\exp(F-K)\frac{u_{202}^{-}(r)}{r}+\exp(F+K)mr\left\{\left[-\frac{j_1(kr)}{3}+\frac{1}{3}j_1'(kr)kr\right]\sqrt{\frac{3}{5}}\frac{v_{312}^{+}(r)}{r}\right.\right.\\&\quad\left.+\left[2j_1(kr)+\frac{1}{2}j_1'(kr)kr\right]\sqrt{\frac{2}{5}}\frac{v_{112}^{+}(r)}{r}\right\}-2mr\exp(F)\sinh(K)j_1(kr)\left(-\frac{v_{312}^{+}}{r}\sqrt{\frac{3}{5}}+\frac{v_{112}^{+}}{r}\sqrt{\frac{2}{5}}\right)\Bigg)\\&\quad-\frac{3}{5}\left((mr+1)j_3(kr)\exp(F-K)\frac{u_{202}^{-}(r)}{r}+\exp(F+K)mr\left\{\left[-\frac{j_3(kr)}{3}+\frac{1}{3}j_3'(kr)kr\right]\sqrt{\frac{3}{5}}\frac{v_{312}^{+}(r)}{r}\right.\right.\\&\quad\left.\left.\left.+\left[2j_3(kr)+\frac{1}{2}j_3'(kr)kr\right]\sqrt{\frac{2}{5}}\frac{v_{112}^{+}(r)}{r}\right\}-2mr\exp(F)\sinh(K)j_3(kr)\left(-\frac{v_{312}^{+}}{r}\sqrt{\frac{3}{5}}+\frac{v_{112}^{+}}{r}\sqrt{\frac{2}{5}}\right)\right)\right],\end{aligned}\tag{3.147}$$

where

$$\frac{u_{202}^{-}}{r}=\frac{M}{E}\frac{u_{202}^{+}}{r},\tag{3.148}$$

and

$$\frac{v_{112}^{+}}{r}=\frac{\exp(\mathcal{G}-2K)}{E}\left[\exp(2K)\left(-\frac{d}{dr}-\frac{L'}{2}\right)-\frac{3}{r}\right]\frac{u_{202}^{+}}{r}\sqrt{\frac{2}{5}}\quad\frac{v_{312}^{+}}{r}=\frac{\exp(\mathcal{G}-2K)}{E}\left[\exp(2K)\left(\frac{d}{dr}+\frac{L'}{2}\right)-\frac{2}{r}\right]\frac{u_{202}^{+}}{r}\sqrt{\frac{3}{5}},\tag{3.149}$$

together with the normalization condition (3.52) (for details see [41])

$$\begin{aligned}&\frac{1}{2}\int_0^\infty drr^2\exp(2F)\left(\exp(-2K)\left[\left(\frac{u_{202}^{+}}{r}\right)^2+\left(\frac{u_{202}^{-}}{r}\right)^2\right]+\exp(2K)\left[\left(\frac{v_{112}^{+}}{r}\right)^2+\left(\frac{v_{312}^{+}}{r}\right)^2\right]\right.\\&-2\sinh 2K\left(-\frac{v_{312}^{+}}{r}\sqrt{\frac{3}{5}}+\frac{v_{112}^{+}}{r}\sqrt{\frac{2}{5}}\right)^2+2w^2\frac{\partial L}{\partial w^2}\left[\exp(-2K)\left[\left(\frac{u_{202}^{+}}{r}\right)^2-\left(\frac{u_{202}^{-}}{r}\right)^2\right]-\exp(2K)\left[\left(\frac{v_{112}^{+}}{r}\right)^2+\left(\frac{v_{312}^{+}}{r}\right)^2\right]\right.\\&\left.\left.+2\sinh 2K\left(-\frac{v_{312}^{+}}{r}\sqrt{\frac{3}{5}}+\frac{v_{112}^{+}}{r}\sqrt{\frac{2}{5}}\right)^2\right]+4w^2\frac{\partial\mathcal{G}}{\partial w^2}\exp(-2K)\left[2\left(\frac{u_{202}^{+}}{r}\right)^2+\left(\frac{u_{202}^{-}}{r}\right)^2\right]\right)=1.\end{aligned}\tag{3.150}$$

Our multicomponent TBDA and TBDN result from this is results is 180 eV while the NDA and NN result is 142 eV. If we further ignore the small components in the latter form by taking $u_{202}^{-}=u_{202}^{+}$ and $v_{112}^{+}=v_{112}^{+}=0$, then the decay rate is 4.66 eV. The experimental situation is unclear. Earlier results had very large widths on the order of 1 keV. In the latest compilation, one result is listed as <70 eV and one at <190 eV both at the 90% confidence level. In any event, the multicomponent effects here are substantial which ever result we use. The difference between the results are small compared with the effects of including the multicomponents by themselves.

(f) ${}^1D_2(3872)$ Decay—The quark-content of this state is unsure. If we assume this is a ${}^1D_2$ spin singlet, then the relevant decay amplitude has the same wave function structures as with the π_2 except for the flavor factor. Our multicomponent TBDA and TBDN result from this is 65.7 eV while the NDA and NN result is 73.2 eV. If we further ignore the small components in the latter form by taking $u^{-}_{202} = u^{+}_{202}$ and $v^{+}_{112} = v^{+}_{112} = 0$, then the decay rate is 168 keV. The experimental situation is unclear. Using ratios given in the latest table (for the ($c\bar{c}$, $l = 2$, $j = 2$ state) we take the observed value to be 435 eV. As with the π_2 decay, both multicomponent effects here are substantial. The difference between them are small compared with the effects of including the multicomponents by themselves. However, unlike the π_2 the effects are to reduce rather than enhance the rate.

(g) a_2 Decay—Except for the quark content (same as with π_0, π_2) this particle has a $^3P_2$ decay amplitude and wave functions given as with the χ_2. Except for the quark content (same as with π_0, π_2) this particle has a $^3P_2$ decay amplitude and wave functions given as with the χ_2. Our multicomponent TBDA and TBDN decay rate result is 31.5 keV reducing to 9.02 keV when QCD radiative effects are included. The corresponding NDA and NN rate result is 10.9 keV reducing to 3.12 keV when QCD radiative effects are included. The observed rate of $1.00 \pm .06$ keV. Including TBDN and TBDA effects in the norm and the amplitude are substantial and produce too large a decay rate.

(h) f_2' Decay—With an $s\bar{s}$ quark content, this particle has a $^3P_2$ decay amplitude otherwise similar to that of the above a_2. Our strong potential, multicomponent decay rate is 2.36 keV reducing to 760 eV when QCD effects are included. The corresponding weak component rates is 1.08 keV reducing to 348 eV when QCD radiative effects are included. The observed rate is 81 ± 9.6 eV. As with the a_2, including TBDN and TBDA effects in the norm and the amplitude are substantial and produce too large a decay rate.

IV. DISCUSSION AND EARLIER RESULTS

A. Charmonium

Table III (units are in keV) compares our results (both the ones that come from TBDA and NDA multicomponent results) with a variety of other quark models (ones that have not yet been subjected to the tests imposed on the TBDE and which for the most part do not include the light mesons in their spectroscopic calculations).

Ackleh and Barnes [15] independently developed a similar approach to the one we developed for positronium decay [14] and then applied it to spin-singlet quarkonium decay into two gammas. As in our approach, they include the effects of the bound state wave function on the initial decaying particle. Gupta, Johnson, and Repko [50] follow a similar approach. The two numbers displayed in their column correspond to two distinct approaches used in incorporating off shell effects. The first is similar to that used in [15] where the energy factors which arise from the Feynman amplitude are treated on mass shell ($E = \sqrt{\mathbf{p}^2 + m^2}$; energy conservation, which would have $E = w/2$, is not used) while the second set of numbers come from treating the particle on energy shell ($E = w/2$ but with $m^2 = E^2 - \mathbf{p}^2$). Our approach is different from both of these in that it is on energy shell, $E = w/2$, but with $m^2 \neq E^2 - \mathbf{p}^2$ it is off-mass-shell. The energy factors that appear in our equations are those required from the way in which the constraint formalism eliminates the c.m. relative energy—see Eq. (2.12) and (2.13). (See also our discussion in our section on $3P_{0,2}$ positronium.) To be more explicit, the portion of the Feynman propagator $(p_- - k_1)^2 + m^2 - i0$ in the approaches of [15] and the first of [50] is treated as $2\mathbf{p}\cdot\mathbf{k} + w\sqrt{\mathbf{p}^2 + m^2}$, in the second of [50] as $2\mathbf{p}\cdot\mathbf{k} + w^2/2$ and in the constraint approach as $(\mathbf{p} - \mathbf{k})^2 + m^2$. The treatment of the spin-dependent aspects of the wave function in [15,50,51] is similar to that appearing in our earlier paper on positronium decay in [14]. We point out, however, that in our paper here, the spin-dependent aspects of the wave function do not arise from the free spinor factors in the Feynman decay amplitude, but rather from the multicomponent structure of the interacting TBDE. The treatment appearing in [51] uses a quasipotential wave equation that gives a Schrödinger-like equation for the bound states. Their amplitude treatment is otherwise similar to that of [15] except that they include (as we do here) QCD radiative corrections. Another treatment is that of [52]. They list decay rates of 5.5 and 2.1 keV for the η_c and η_c' respectively, similar to the results of [51] (see also recent result of [53]). Their treatment of the spin-dependent aspects of the wave function appearing is more like ours except that they use the Salpeter truncation of the Bethe-Salpeter equation but with energy factors in the amplitude treated on shell as in [15]. The treatment in [54] is similar to that of [52] except that it involves the four-dimensional Bethe-Salpeter

TABLE III. Charmonium 2γ decay rates

	Expt	TBDE-TBDA.	TBDE-NDA	[15]	[50]	[51]	[54]
$\eta_c(1^1S_0 2976)$	7.4 ± 1.0	6.20	6.18	4.8	10.94,10.81	5.5	3.50
$\eta_c(2^1S_0 3263)$	1.3 ± 0.6	3.36	1.95	3.7	...	1.8	1.38
$\chi_0(1^3P_0 3415)$	$2.6 \pm .65$	3.96	3.34	...	6.38,8.13	2.9	1.39
$\chi_2(1^3P_2 3556)$	$0.53 \pm .09$	0.743	0.435	...	0.57,1.14	0.50	0.440

amplitude constructed from the Salpeter solution. As with [52] they use a combination of scalar and timelike confining potentials. Unlike [51,52], and like the treatment of [15] and the present one, [54] then goes on to treat the light quark pseudoscalar decays. For the $^1D_2(3872)$ particle, Ackleh and Barnes obtain a result of 20 eV compared to ours of 27 eV. None of the other authors include this particle.

B. Light quark mesons

Our formalism at this stage does not include the effects of flavor mixing and consequently we do not compute the rates for η, $\eta' \rightarrow 2\gamma$. This leaves us with the 2 γ decays of π^0, π_2, a_2, f_2'. We present the results of the decay width in Table IV, including those approaches above that give predictions for some of these decays. The units are in eV.

Our pion rate is comparable to the others. However, the assumptions of [15] are quite different from ours. First they use a nonrelativistic potential model for the wave function. As pointed out by [54–56] standard approaches to the pion decay fail miserably for such models, typically too large by 3 orders of magnitude (by comparison our result is only off by 20%). Ackleh and Barnes, however, included, as did Hayne and Isgur [57] in an earlier paper, a phenomenological resonance mass factor motivated by an effective field theory Lagrangian ($\sim \frac{1}{2} g\phi F_{\mu\nu}\tilde{F}_{\mu\nu}$) which greatly suppresses the ''bare'' rate. The approach we have taken above did not include such a factor. Of course, as they point out, the factor implied by that effective field theory is not contained within the positroniumlike model that they and we use. (The range of values in their column correspond to a range of assumed constituent masses). The approach taken by Münz [54] is much closer in spirit to the one we employ. He uses the framework of the Salpeter equation for the formulation of two photon decays and finds that including relativistic effects, and the negative energy, i.e. small components of the wave function, is important even for heavy quarkonia. In addition, unlike our approach, which in the c.m. frame would have momentum space wave function dependence only on the relative three momentum, he works out a decay matrix element which includes relative four-momentum dependence (including relative energy dependence in the c.m.). It is his claim that in this way, not only does the amplitude depend on off-mass-shell annihilating quark pairs (through the wave function) but also the exchanged quark within the diagram that are both off mass shell and off energy shell. In contrast to our Eq. (3.6) his amplitude involves an additional integral over the relative energy. However, he finds it necessary to introduce a cutoff factor in his spectral analysis for the one-gluon exchange. In addition he finds that he must assume not only a different confinement mechanism for the light and heavy quarks, but different confinement strengths. The spectral results we obtain do not treat the heavy and light quark bound states differently. Further, in the two models that he considers, he finds that it is not possible to formulate the one-gluon exchange gauge invariantly and so uses the Feynman gauge in one parameter set for a semirelativistic model and the Coulomb gauge in the other. By contrast, the constraint approach displays gauge invariance, and, for simplicity uses the Feynman gauge.

There are other approaches that develop formalisms with natural suppression of the $\pi^0 \rightarrow 2\gamma$ width in the quark model. Guisasu and Koniuk [56], using a multipair structure in the context of the Bethe-Salpeter formalism, show how the extremely bound highly relativistic nature of the pion suppress the decay rate. The authors of [55] also show how the assumption of a completely diagonalized QCD Hamiltonian (with meson eigenstates predominantly $q\bar{q}$), implies bound state effects can greatly suppress the width.

V. CONCLUDING REMARKS

The two-body Dirac equations are based on Dirac's constraint formalism and a minimal interaction structure for the effective particle of relative motion (first used by Todorov) confirmed by both classical [28] and quantum field theory [18]. This formalism displays spectral results with flavor independent interactions in very good experimental agreement for most of the meson spectra. At the same time, and we have stressed the importance of this in a recent publication [7], the formalism when treated in a nonperturbative manner naturally accounts for the perturbative results of QED bound states. So far this has not been fully replicated in any other approach. In a natural way it leads not only to good singlet-triplet ground state splittings for the light meson, but also a Goldstone behavior for the pion. This we showed is tied to the same relativistic structures that account for the nonperturbative positronium and muonium results. Based on this and the successful off shell treatment of positronium two photon decay we had reason to anticipate that the quarkonium decays to two photons would be reasonable. We have found this to be particularly true for the π_0 and η_c. There we found relativistic effects, including most importantly the full multi-

TABLE IV. Light quark meson 2γ decays.

	Expt.	TBDE-TBDA	TBDE-NDA	[15]	[54]
$\pi_0(^1S_0 0.135)$	$7.72 \pm .04$	8.73	33.5.	$3.4 \rightarrow 6.4$	3.81,5.07
$\pi_2(^1D_2)$	<70, 190	181	142	$110 \rightarrow 270$	73.2129
$a_2(^3P_2 1.318)$	1000 ± 60	9020	3120	...	766 900
$f_2'(^3P_2 1.525)$	81 ± 10	760	34	...	...

component wave function and the influence of the TBDN and TBDA, to be of crucial importance. The results compare favorably with models based on two-body formalisms not tested as extensively as that of the two-body Dirac equations. Our pion results are unlike some of the competing approaches in that no additional effective field theory assumptions were made that go beyond the relativistic potential model approach, and spectral results for all mesons are obtained in a flavor independent way. In light of this our results are not too unreasonable. Still one may speculate on assumptions made in the constraint approach which may be relaxed. For example, it may very well be that even though spectral results are independent of the method by which the relative time is controlled in the constraint formalism [26], the decay and other amplitudes may depend on this effect. The work in [26] (see Appendix A in that paper), allows one to show how the relative energy restriction Eq. (2.12) which in quantum form is $P\cdot p\psi = 0$ or $\psi(p) = \delta(p\cdot\hat{P})\psi(p_\perp)$, could be replaced in the amplitude Eq. (3.121) by a more general form, say

$$\psi(p) = \delta(p\cdot\hat{P})\psi(p_\perp) \to \Delta(p\cdot\hat{P})\psi(p_\perp) \tag{5.1}$$

in which Δ is a distribution with nonzero support [58]. In future work, having shown that the meson wave functions of [7] used in this paper give in most circumstances reasonable results, one will now consider applications of them to the meson-meson scattering process such as discussed in the beginning of this paper.

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APPENDIX A: DIRAC MATRICES FOR THE TWO-BODY DIRAC EQUATIONS

$$\beta_1 = \begin{bmatrix} 1_8 & 0 \\ 0 & -1_8 \end{bmatrix}, \quad \gamma_{51} = \begin{bmatrix} 0 & 1_8 \\ 1_8 & 0 \end{bmatrix}, \quad \beta_1\gamma_{51} \equiv \rho_1 = \begin{bmatrix} 0 & 1_8 \\ -1_8 & 0 \end{bmatrix}, \quad \beta_2 = \begin{bmatrix} \beta & 0 \\ 0 & \beta \end{bmatrix}, \quad \beta = \begin{bmatrix} 1_4 & 0 \\ 0 & -1_4 \end{bmatrix},$$
$$\gamma_{52} = \begin{bmatrix} \gamma_5 & 0 \\ 0 & \gamma_5 \end{bmatrix}, \quad \gamma_5 = \begin{bmatrix} 0 & 1_4 \\ 1_4 & 0 \end{bmatrix}, \quad \beta_2\gamma_{52} \equiv \rho_2 = \begin{bmatrix} \rho & 0 \\ 0 & \rho \end{bmatrix}, \quad \rho = \begin{bmatrix} 0 & 1_4 \\ -1_4 & 0 \end{bmatrix}, \quad \gamma_{51}\gamma_{52} = \begin{bmatrix} 0 & \gamma_5 \\ \gamma_5 & 0 \end{bmatrix},$$
$$\rho_1\rho_2 = \begin{bmatrix} 0 & \rho \\ -\rho & 0 \end{bmatrix} \quad \beta_1\gamma_{51}\gamma_{52} = \begin{bmatrix} 0 & \gamma_5 \\ -\gamma_5 & 0 \end{bmatrix}, \quad \beta_2\gamma_{52}\gamma_{51} = \begin{bmatrix} 0 & \rho \\ \rho & 0 \end{bmatrix}, \quad \beta_i = -\gamma_i\cdot\hat{P}, \quad \Sigma_i = \gamma_{5i}\beta_i\gamma_{\perp i}. \tag{A1}$$

APPENDIX B: INTERACTION DEPENDENT MODIFICATIONS OF THE NORM

In order to implement the interaction dependent modification

$$\int d^3x\,\mathrm{Tr}\left[\psi^\dagger\left(1 + 4w^2\beta_1\beta_2\frac{\partial\Delta}{\partial w^2}\right)\psi\right] = 1 \tag{B1}$$

of Eq. (3.40) we first need the matrix connection

$$\psi = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{bmatrix} = \frac{1}{2}(\beta_1 + \gamma_{51}\gamma_{52})\begin{bmatrix} \phi_+ \\ \chi_+ \\ \chi_- \\ \phi_- \end{bmatrix} = \frac{1}{2}(\beta_1 + \gamma_{51}\gamma_{52})\exp F(\cosh K + \sinh K\boldsymbol{\Sigma}_1\cdot\hat{\mathbf{r}}\boldsymbol{\Sigma}_2\cdot\hat{\mathbf{r}})\begin{bmatrix} \psi_+ \\ \eta_+ \\ \eta_- \\ \psi_- \end{bmatrix} \equiv \mathcal{L}_0\begin{bmatrix} \psi_+ \\ \eta_+ \\ \eta_- \\ \psi_- \end{bmatrix}, \tag{B2}$$

between the Dirac spinor solutions of Eqs. (2.18) and those of (3.21).

The transformation between the 16 component column vector form of the wave function that satisfies our quasipotential Eq. (3.21) and the one which satisfies the two-body Dirac equation in hyperbolic form is given in Eq. (B2). The corresponding 4×4 matrix form is

$$\psi = \frac{\exp(F)}{2\sqrt{2}}[\cosh K(\psi_+ q_1 + \psi_- i q_2 + \eta_+ q_0 + \eta_- q_3) - \sinh K \mathbf{\Sigma} \cdot \hat{\mathbf{r}}(\psi_+ q_1 + \psi_- i q_2 + \eta_+ q_0 + \eta_- q_3)\mathbf{\Sigma} \cdot \hat{\mathbf{r}}] \equiv \mathcal{K}\Psi(\mathbf{r}) \tag{B3}$$

where

$$\Psi(\mathbf{r}) = \frac{1}{2\sqrt{2}}(\psi_+ q_1 + \psi_- i q_2 + \eta_+ q_0 + \eta_- q_3). \tag{B4}$$

Whereas the normalization condition (3.40) in 16 component form is

$$\begin{aligned}\int d^3x\Big[\psi^\dagger\Big(1 + 4w^2\beta_1\beta_2\frac{\partial\Delta}{\partial w^2}\Big)\psi\Big] = \int d^3x\Big\{\Big[\psi_+^\dagger \quad \eta_+^\dagger \quad \eta_-^\dagger \quad \psi_-^\dagger\Big]\frac{1}{2}(\beta_1 + \gamma_{51}\gamma_{52})(A + B\mathbf{\Sigma}_1 \cdot \hat{\mathbf{r}}\boldsymbol{\psi}_2 \cdot \hat{\mathbf{r}})(A + B\mathbf{\Sigma}_1 \cdot \hat{\mathbf{r}}\mathbf{\Sigma}_2 \cdot \hat{\mathbf{r}})\\ \times\frac{1}{2}(\beta_1 + \gamma_{51}\gamma_{52})\begin{bmatrix}\psi_+\\ \eta_+\\ \eta_-\\ \psi_-\end{bmatrix}\Big\} + 4w^2\int d^3x\Big\{\Big[\psi_+^\dagger \quad \eta_+^\dagger \quad \eta_-^\dagger \quad \psi_-^\dagger\Big]\\ \times\frac{1}{2}(\beta_1 + \gamma_{51}\gamma_{52})(A + B\mathbf{\Sigma}_1 \cdot \hat{\mathbf{r}}\mathbf{\Sigma}_2 \cdot \hat{\mathbf{r}})\frac{1}{2}\Big[\rho_1\rho_2\frac{\partial L}{\partial w^2} + (\gamma_{51}\gamma_{52} - \mathbf{\Sigma}_1 \cdot \mathbf{\Sigma}_2)\frac{\partial \mathcal{G}}{\partial w^2}\Big]\\ \times(A + B\mathbf{\Sigma}_1 \cdot \hat{\mathbf{r}}\mathbf{\Sigma}_2 \cdot \hat{\mathbf{r}})\frac{1}{2}(\beta_1 + \gamma_{51}\gamma_{52})\begin{bmatrix}\psi_+\\ \eta_+\\ \eta_-\\ \psi_-\end{bmatrix}\Big\}\end{aligned} \tag{B5}$$

since the matrix form of

$$\begin{aligned}\Big(1 + 4w^2\beta_1\beta_2\frac{\partial\Delta}{\partial w^2}\Big)\psi &= \Big[1 + 2w^2\Big(\rho_1\rho_2\frac{\partial L}{\partial w^2} + (\gamma_{51}\gamma_{52} - \mathbf{\Sigma}_1 \cdot \mathbf{\Sigma}_2)\frac{\partial\mathcal{G}}{\partial w^2}\Big)\Big]\psi\\ &\to \mathcal{K}\Psi(\mathbf{r}) + \Big[-2w^2\frac{\partial L}{\partial w^2}\rho\mathcal{K}\Psi(\mathbf{r})\rho + 2w^2\frac{\partial\mathcal{G}}{\partial w^2}(\gamma_5\mathcal{K}\Psi(\mathbf{r})\gamma_5 + \mathbf{\Sigma}\mathcal{K}\Psi(\mathbf{r}) \cdot \mathbf{\Sigma})\Big]\\ &= \mathcal{K}\Psi(\mathbf{r}) + \Big[2w^2\frac{\partial L}{\partial w^2}iq_2\mathcal{K}\Psi(\mathbf{r})iq_2 + 2w^2\frac{\partial\mathcal{G}}{\partial w^2}(q_1\mathcal{K}\Psi(\mathbf{r})q_1 + \mathbf{\Sigma}\mathcal{K}\Psi(\mathbf{r}) \cdot \mathbf{\Sigma})\Big]\end{aligned} \tag{B6}$$

in terms of matrix wave functions the norm condition (3.40) is

$$\begin{aligned}1 &= \int d^3x\,\mathrm{Tr}\psi^\dagger\mathcal{L}\psi = \int d^3x\,\mathrm{Tr}(\mathcal{K}\Psi(\mathbf{r}))^\dagger\mathcal{L}\mathcal{K}\Psi(\mathbf{r})\\ &\equiv \int d^3x\,\mathrm{Tr}\Big\{(\mathcal{K}\Psi(\mathbf{r}))^\dagger\mathcal{K}\Psi(\mathbf{r}) + (\mathcal{K}\Psi(\mathbf{r}))^\dagger\\ &\quad\times\Big[2w^2\frac{\partial L}{\partial w^2}iq_2\mathcal{K}\Psi(\mathbf{r})iq_2\\ &\quad + 2w^2\frac{\partial\mathcal{G}}{\partial w^2}(q_1\mathcal{K}\Psi(\mathbf{r})q_1 + \mathcal{K}\Psi(\mathbf{r}) \cdot \mathbf{\Sigma})\Big]\Big\}.\end{aligned} \tag{B7}$$

Substituting Eq. (B3) and its conjugate, taking traces and using for the spin-singlet case (3.51)

$$\begin{aligned}\psi_+ &= \psi_{+0}\sigma_0; & \psi_- &= \psi_{-0}\sigma_0;\\ \eta_+ &= \boldsymbol{\eta}_+ \cdot \boldsymbol{\sigma}; & \eta_- &= 0,\end{aligned} \tag{B8}$$

together with matrix identities such as

$$\begin{aligned}\boldsymbol{\sigma}(\boldsymbol{\sigma} \cdot \hat{\mathbf{r}}\boldsymbol{\eta}_+ \cdot \boldsymbol{\sigma}\boldsymbol{\sigma} \cdot \hat{\mathbf{r}}) \cdot \boldsymbol{\sigma} &= \boldsymbol{\sigma}(2\boldsymbol{\eta}_+ \cdot \hat{\mathbf{r}}\boldsymbol{\sigma} \cdot \hat{\mathbf{r}} - \boldsymbol{\eta}_+ \cdot \boldsymbol{\sigma}) \cdot \boldsymbol{\sigma}\\ &= -(2\boldsymbol{\eta}_+ \cdot \hat{\mathbf{r}}\boldsymbol{\sigma} \cdot \hat{\mathbf{r}} - \boldsymbol{\eta}_+ \cdot \boldsymbol{\sigma})\\ &\quad - \boldsymbol{\eta}_+^\dagger \cdot \boldsymbol{\sigma}(2\boldsymbol{\eta}_+ \cdot \hat{\mathbf{r}}\boldsymbol{\sigma} \cdot \hat{\mathbf{r}} - \boldsymbol{\eta}_+ \cdot \boldsymbol{\sigma})\\ &= -2\boldsymbol{\eta}_+^\dagger \cdot \hat{\mathbf{r}}\boldsymbol{\eta}_+ \cdot \hat{\mathbf{r}} + \boldsymbol{\eta}_+^\dagger \cdot \boldsymbol{\eta}_+,\end{aligned} \tag{B9}$$

leads to Eq. (3.52) in the text. (For details see [41]) From that equation and Eqs. (3.55) and (3.56) we obtain the general radial forms of the norm [41] given in the text. Using for the spin-triplet case (3.57)

$$\psi_+ = \boldsymbol{\psi}_+ \cdot \boldsymbol{\sigma}; \qquad \psi_- = \boldsymbol{\psi}_- \cdot \boldsymbol{\sigma}; \qquad \eta_+ = \eta_{+0}\sigma_0; \qquad \eta_- = \boldsymbol{\eta}_- \cdot \boldsymbol{\sigma}. \tag{B10}$$

gives Eq. (3.58) in the text (For details see [41]). From that equation and Eqs. (3.48), (3.64), and (3.65) and

$$\boldsymbol{\psi}_- = \frac{u^-_{(j+1)1j}}{r}\mathbf{Y}_{jm+} + \frac{u^-_{(j-1)1j}}{r}\mathbf{Y}_{jm-} \tag{B.25}$$

we obtain the general radial forms [41] given in the text.

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