Operator product expansion in the production and decay of the $X(3872)$

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The $X(3872)$ seems to be a weakly bound hadronic molecule whose constituents are two charm mesons. Its binding energy is much smaller than all the other energy scales in QCD. This separation of scales can be exploited through factorization formulas for production and decay rates of the X. In a low-energy effective field theory for the constituents of the X, the factorization formulas can be derived using the operator product expansion. The derivations are carried out explicitly for the simplest effective theory in which the constituents interact through a contact interaction that produces a large scattering length. The long-distance factors in the operator product expansions for various observables are calculated nonperturbatively in the interaction strength of the contact interaction. After renormalization of the coupling constant, all remaining ultraviolet divergences can be absorbed into the short-distance factors in the operator product expansions.

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I. INTRODUCTION

The $X(3872)$ is a narrow resonance near 3872 MeV discovered by the Belle Collaboration in 2003 through its decay into $J/\psi\pi^+\pi^-$ [1]. Its existence was subsequently confirmed by the CDF, *BABAR*, and D0 Collaborations [2–4]. Its mass M_X is extremely close to the threshold for the charm mesons D^0 and $\bar{D}^{*0}$ [5]:

$$M_X - (M_{D^0} + M_{D^{*0}}) = +0.7 \pm 1.1 \text{ MeV}. \quad (1)$$

The $X(3872)$ is narrower than most of the known charmonium states [1]:

$$\Gamma_X < 2.3 \text{ MeV} \quad (90\% \text{ C.L.}). \quad (2)$$

The observation of the decay $X \to J/\psi\gamma$ implies that the X has charge conjugation quantum number $C = +$ [6]. Analyses of the discovery decay mode $X \to J/\psi\pi^+\pi^-$, including the angular correlations between the J/ψ and the pions and the $\pi^+\pi^-$ invariant mass distribution, strongly favor the spin and parity quantum numbers $J^P = 1^+$ [7]. These properties are compatible with the identification of X as a weakly bound molecule whose constituents are a superposition of charm meson pairs [8–24]:

$$X = \frac{1}{\sqrt{2}}(D^{*0}\bar{D}^0 + D^0\bar{D}^{*0}). \quad (3)$$

If this identification is confirmed, the $X(3872)$ would be the first unambiguously identified member of a new class of hadrons: *mesonic molecules* [25–29].

If the $X(3872)$ is a weakly bound mesonic molecule, it shares an important feature with the simplest baryonic molecule, the deuteron. Their binding energies are both small compared to the natural energy scale associated with the exchange of the lightest meson, the pion. That natural energy scale is $m_\pi^2/(2M_{12})$, where M_{12} is the reduced mass of the two constituents. The binding energy 2.2 MeV of the deuteron is small compared to the natural scale of about 20 MeV. The measurement of the mass of the X in Eq. (1) implies that its binding energy $(M_{D^0} + M_{D^{*0}}) - M_X$ is between -2.4 MeV and 1.2 MeV at the 90% confidence level. The small width in Eq. (2) further suggests that the mass of the X must be below the threshold for the charm mesons: $M_X < M_{D^0} + M_{D^{*0}}$. Thus the binding energy of the X is small compared to the natural scale of about 10 MeV. The deuteron has an S-wave coupling to its constituents, the proton and the neutron. The quantum numbers $J^{PC} = 1^{++}$ of the X implies that it also has an S-wave coupling to its constituents. The combination of the small binding energy compared to the natural energy scale and the S-wave coupling to the constituents implies that the deuteron and the $X(3872)$ have *universal* properties that are determined by the large scattering length a of their constituents [13]. The simplest example of a universal result is a simple formula for the binding energy of the molecule: $E_X = 1/(2M_{12}a^2)$. The universality of few-body systems with a large scattering length has many applications in atomic, nuclear, and particle physics [30]. The universal features of the $X(3872)$ were first exploited by Voloshin to describe its decays into $D^0\bar{D}^0\pi^0$ and $D^0\bar{D}^0\gamma$, which can proceed through decay of the constituent D^{*0} or $\bar{D}^{*0}$ [11]. Universality has also been applied to the production process $\Upsilon(4S) \to \pi^+\pi^- + X$ [15], to the production process $B \to K + X$ [17,20], to the line shape of the X [21], and to decays of X into J/ψ and pions [23].

The tiny binding energy of the $X(3872)$ provides a new energy scale that is much smaller than the other scales in QCD, including the pion mass m_π and the scale Λ_{QCD} associated with nonperturbative effects. In Ref. [21], this separation of scales was exploited by using *factorization formulas* to separate certain observables into long-distance factors that involve only energy scales comparable to the binding energy and short-distance factors that involve all the higher energy scales of QCD. The long-distance factors can be calculated using an effective field theory for the constituents of the X that describes the lowest energy scale.

In this paper, we show how the factorization formulas can be derived using the *operator product expansion* for the effective field theory that describes the constituents of the X. The effective field theory that describes the $X(3872)$ is complicated by the spin 1 of the constituent D^{*0} and by its charge conjugation quantum number $C = +$, which implies that it is the superposition of $D^{*0}\bar{D}^0$ and $D^0\bar{D}^{*0}$ given in Eq. (3). Another complication is that the $D^0\bar{D}^0\pi^0$ threshold is only about 8 MeV below the $D^{*0}\bar{D}^0$ threshold [24]. We will therefore illustrate the operator product expansion formalism using a simpler model in which these complications are absent. The simplest such model is a scalar meson model in which the constituents are spin-0 mesons with a contact interaction that gives a large positive scattering length a. The generalization to a realistic model with charm mesons is then straightforward.

In Sec. II, we define the minimal charm meson model that can describe the $X(3872)$ and the simpler scalar meson model. In Sec. III, we explain how the operator product expansion can be used to separate scales in short-distance production and decay rates. In Sec. IV, we give exact nonperturbative results for long-distance observables in the scalar meson model. In Sec. V, we apply the operator product expansion to short-distance production and decay rates in the scalar meson model and we calculate the long-distance factors in the operator product expansion. We show that, after renormalization of the coupling constant, all remaining dependence on the ultraviolet cutoff can be eliminated by renormalization of Wilson coefficients in the operator product expansion. In Sec. VI, we show how the factorization formulas can be simplified by expanding in inverse powers of the large scattering length. In Sec. VII, we extend the results of Secs. IV, V, and VI to the charm meson model. We summarize our results in Sec. VIII.

II. EFFECTIVE FIELD THEORIES

In this section, we define the scalar meson model and the minimal charm meson model. We also introduce an interpolating field for the S-wave bound state in each of these models. These models are straightforward generalizations to heavy mesons of the leading-order approximations to effective field theories for nucleons developed by Weinberg [31,32] and by Kaplan, Savage, and Wise [33,34]. At higher orders, these effective field theories can be used to calculate systematically corrections associated with the nonzero range of the interactions. All calculations in this paper will be carried out using leading-order effective field theories.

A. Scalar meson model

We will illustrate the derivation of factorization formulas using the operator product expansion in a simpler model that we call the *scalar meson model*. This model has an S-wave bound state which we will refer to as X. The constituents of the bound state are scalar mesons D_1 and D_2 with masses M_1 and M_2 satisfying $M_1 < M_2$. The scalar meson model is a nonrelativistic quantum field theory with two complex scalar fields, $D_1(\vec{r}, t)$ and $D_2(\vec{r}, t)$. The free terms in the Lagrangian are

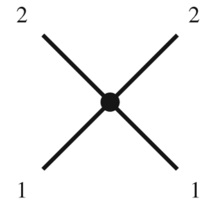

FIG. 1. Vertex for the D_1D_2 contact interaction.

$$\mathcal{L}_{\text{free}} = D_1^\dagger\left(i\frac{\partial}{\partial t} - M_1 + \frac{1}{2M_1}\nabla^2\right)D_1 + D_2^\dagger\left(i\frac{\partial}{\partial t} - M_2 + \frac{1}{2M_2}\nabla^2\right)D_2. \tag{4}$$

A superscript † on a field represents its complex conjugate. The interaction term in the Lagrangian for the scalar meson model is

$$\mathcal{L}_{\text{int}} = -\lambda_0 D_1^\dagger D_2^\dagger D_1 D_2. \tag{5}$$

The vertex for this interaction is illustrated in Fig. 1. The coupling constant λ_0 has mass dimension -2. The subscript on λ_0 emphasizes that it is a bare coupling constant that depends on the ultraviolet cutoff on the momenta of the particles in loop diagrams. If the $D_1^\dagger D_2^\dagger D_1 D_2$ interaction is treated nonperturbatively, there is an S-wave bound state.

It is convenient to introduce concise notations for the reduced mass of the D_1 and D_2 and for the sum of their masses:

$$M_{12} = \frac{M_1 M_2}{M_1 + M_2}, \tag{6a}$$

$$M_{1+2} = M_1 + M_2. \tag{6b}$$

If the scalar meson model is an effective field theory for a more fundamental theory in which the mesons D_1 and D_2 interact by the exchange of other mesons, the natural momentum scale for low-energy processes is the mass m of the lightest meson that can be exchanged. If $m \ll M_{12}$, the natural energy and momentum scales associated with the exchange of that meson are m^2/M_{12} and m, respectively. We assume that the binding energy of the molecule X is small compared to the natural energy scale:

$$M_{1+2} - M_X \ll m^2/M_{12}. \tag{7}$$

The scalar meson model describes the threshold region where the invariant mass M of D_1 and D_2 is very close to M_{1+2}:

$$|M - M_{1+2}| \ll m^2/M_{12}. \tag{8}$$

This constraint restricts the possible scattering states to $D_1 D_2$.

The bare coupling constant λ_0 of the scalar meson model must depend on the ultraviolet cutoff Λ in such a way that low-energy observables are independent of Λ. There are many alternative renormalization prescriptions that can be used to eliminate the explicit dependence on Λ and λ_0. One renormalization prescription is to eliminate λ_0 in favor of a renormalized coupling constant λ. Another renormalization prescription is to eliminate λ_0 in favor of the scattering length a of the two heavy mesons. The scattering length can be defined in terms of the T-matrix element for elastic $D_1 D_2$ scattering with zero relative momentum:

$$\mathcal{T}_{12\to 12}(p=0) \equiv -8\pi M_{1+2} a. \tag{9}$$

This is the T-matrix element for relativistically normalized particles in the initial and final states. If either D_1 or D_2 is an unstable particle, there is an inelastic scattering channel for $D_1 D_2$. This implies that the scattering length a has a negative imaginary part. The most convenient renormalization prescription for our purposes is to eliminate λ_0 in favor of the energy $E_{\rm pole}$ at which the Green's function for $D_1 D_2 \to D_1 D_2$ has a pole [21]. That energy can be expressed in the form

$$E_{\rm pole} = M_{1+2} - \gamma^2/(2M_{12}), \tag{10}$$

where γ is the complex binding momentum:

$$\gamma = \gamma_{\rm Re} + i\gamma_{\rm Im}. \tag{11}$$

Unitarity requires $\gamma_{\rm Im}$ to be positive. We assume that $\gamma_{\rm Re}$ is also positive, in which case the energy $E_{\rm pole}$ is the complex energy of the unstable bound state we denote by X. The real part of $E_{\rm pole}$ defines the pole mass of the molecule:

$$M_X = M_{1+2} - (\gamma_{\rm Re}^2 - \gamma_{\rm Im}^2)/(2M_{12}). \tag{12}$$

The imaginary part of $E_{\rm pole}$ multiplied by -2 can be interpreted as the width of the molecule:

$$\Gamma_X = 2\gamma_{\rm Re}\gamma_{\rm Im}/M_{12}. \tag{13}$$

The magnitude of the complex parameter γ is assumed to be small compared to the natural momentum scale: $|\gamma| \ll m$. This implies the condition on the binding energy in Eq. (7).

B. Minimal charm meson model

The charm mesons D^0 and D^{*0} with nonrelativistic energies and momenta can be described by a nonrelativistic quantum field theory with a complex spin-0 field $D(\vec{r}, t)$ and a 3-component complex spin-1 field $\vec{D}(\vec{r}, t)$. Their antiparticles $\bar{D}^0$ and $\bar{D}^{*0}$ can be described by corresponding fields $\bar{D}(\vec{r}, t)$ and $\vec{\bar{D}}(\vec{r}, t)$. The free terms in the Lagrangian density for these particles are

$$\begin{aligned}\mathcal{L}_{\rm free} = {}& D^\dagger\Big(i\frac{\partial}{\partial t} - M_{D^0} + \frac{1}{2M_{D^0}}\nabla^2\Big)D \\ &+ \bar{D}^\dagger\Big(i\frac{\partial}{\partial t} - M_{D^0} + \frac{1}{2M_{D^0}}\nabla^2\Big)\bar{D} \\ &+ \vec{D}^\dagger\cdot\Big(i\frac{\partial}{\partial t} - M_{D^{*0}} + \frac{1}{2M_{D^{*0}}}\nabla^2\Big)\vec{D} \\ &+ \vec{\bar{D}}^\dagger\cdot\Big(i\frac{\partial}{\partial t} - M_{D^{*0}} + \frac{1}{2M_{D^{*0}}}\nabla^2\Big)\vec{\bar{D}}.\end{aligned} \tag{14}$$

The simplest interaction term that can produce an S-wave bound state in the $C = +$ channel is

$$\mathcal{L}_{\rm int} = -\lambda_0(\bar{D}\vec{D} + D\vec{\bar{D}})^\dagger\cdot(\bar{D}\vec{D} + D\vec{\bar{D}}). \tag{15}$$

We will refer to the effective field theory with Lagrangian given by Eqs. (14) and (15) as the *minimal charm meson model*. If the interaction in Eq. (15) is treated nonperturbatively, there is an S-wave bound state with spin 1 that can be identified with the $X(3872)$. The effects of decays of the X can be taken into account by allowing the coupling constant λ_0 in Eq. (15) to have an imaginary part.

An ultraviolet cutoff Λ is required to regularize ultraviolet divergences generated by the interaction term in Eq. (15). The natural scale for the ultraviolet cutoff is the pion mass m_π. The bare coupling constant λ_0 must depend on Λ in such a way that low-energy observables are independent of the cutoff. There are many alternative renormalization prescriptions that can be used to eliminate the explicit dependence on Λ and λ_0. For example, the complex parameter λ_0 can be eliminated in favor of a renormalized coupling constant λ or in favor of the complex scattering length of the charm mesons. The most convenient renormalization prescription for our purposes is to eliminate λ_0 in favor of the mass and width of the $X(3872)$ or equivalently the complex binding momentum γ. An alternative statement of this renormalization prescription is that the Green's function for $D^{*0}\bar{D}^0 D^{*0}\bar{D}^0$ has a pole at the energy $E_{\rm pole}$ given by Eq. (10).

C. Interpolating fields for X

In the scalar meson model, the local composite operator $D_1^\dagger D_2^\dagger(x)$ has a nonzero amplitude to create X from the vacuum. Thus $D_1 D_2(x)$ can be used as an interpolating field for X. The resulting propagator for X is

$$i\Delta_X(E, P) = \int d^4x e^{iP\cdot x}\langle\varnothing|D_1 D_2(x) D_1^\dagger D_2^\dagger(0)|\varnothing\rangle, \tag{16}$$

where $P\cdot x = P^\mu x_\mu$ and $P^\mu = (E, \vec{P})$ is the 4-momentum of the X. The propagator is a function of E and $P = |\vec{P}|$. The Galilean invariance of the scalar meson model implies that it depends only on the combination $E - P^2/(2M_{12})$. Our renormalization prescription implies that this propagator at $\vec{P} = 0$ has a pole in E at the complex energy $E_{\rm pole}$ given in Eq. (10). The behavior of the propagator near the pole defines a wave-function normalization constant Z_X:

$$i\Delta_X(E, 0) \longrightarrow \frac{iZ_X}{E - E_{\rm pole} + i\varepsilon}. \qquad (17)$$

Because the composite operator D_1D_2 has mass dimension 3, the propagator $i\Delta_X(E, P)$ has mass dimension 2 and Z_X has mass dimension 3.

T-matrix elements involving X in the final state can be obtained from connected Green's functions involving the operator D_1D_2 by using the LSZ formalism [35]. The connected Green's function in momentum space with an external line associated with a D_1D_2 operator is amputated by multiplying by the inverse propagator for X, evaluated on the energy shell $E = E_{\rm pole}$, and then multiplied by $Z_X^{1/2}$ to obtain the T-matrix element. T-matrix elements involving X in the initial state can be obtained in a similar way from connected Green's functions involving the operator $D_1^\dagger D_2^\dagger$.

In the charm meson model, $X(3872)$ is identified as a bound state whose constituents are the $C = +$ superposition of charm mesons in Eq. (3). A convenient interpolating field for the X is the local composite operator $D^i\bar{D}(x) + D\bar{D}^i(x)$. The resulting propagator for the X is

$$i\Delta_X^{ij}(E, P) = \int d^4x e^{iP\cdot x}\langle\varnothing|(D^i\bar{D}(x) + D\bar{D}^i(x)) \times (D^j\bar{D}(0) + D\bar{D}^j(0))^\dagger|\varnothing\rangle. \qquad (18)$$

If $\vec{P} = 0$, the propagator has a pole at $E = E_{\rm pole}$. Its behavior near the pole defines a normalization factor Z_X:

$$i\Delta_X^{ij}(E, 0) \longrightarrow \frac{iZ_X}{E - E_{\rm pole} + i\varepsilon}\delta^{ij}. \qquad (19)$$

III. OPERATOR PRODUCT EXPANSION

The scalar meson model defined by the Lagrangian in Eqs. (4) and (5) can be a low-energy approximation to a more fundamental Lorentz-invariant quantum field theory. If the fundamental quantum field theory includes a high-energy process that can create D_1D_2 with invariant mass near M_{1+2}, that process will involve momenta ranging from the highest energy scale to momenta smaller than m. If *long-distance* effects involving momenta much smaller than m can be separated from *short-distance* effects involving momenta of order m and larger, we can use the scalar meson model to calculate the long-distance effects. Expressions for physical quantities in which short-distance effects and long-distance effects are separated into multiplicative factors are called *factorization formulas*. The tool required to separate long-distance effects from short-distance effects is the *operator product expansion*.

A. Short-distance production processes

The general production process for D_1D_2 has the form $A \rightarrow B + D_1D_2$, where A and B each represent one or more particles. There can be analogous production processes for X. We consider the production of D_1D_2 near their threshold, so that their invariant mass M satisfies the inequality in Eq. (8). We also assume that the relative momentum $\vec{p}$ of the D_1 and D_2 is small compared to m. We call $A \rightarrow B + D_1D_2$ a *short-distance production process* if all the particles in A and B have momenta in the D_1D_2 rest frame that are of order m or larger. This condition implies that the amplitude for $A \rightarrow B + D_1D_2$ can be expanded in powers of the small energy difference $M - M_{1+2}$ divided by m^2/M_{12} and larger energy scales, and in powers of the small relative momentum $\vec{p}$ divided by m and larger momentum scales. The operator product expansion can be used to express the T-matrix elements in the forms

$$\mathcal{T}[A \rightarrow B + D_1D_2] = \sqrt{4M_1M_2}\sum_n \mathcal{C}_A^{B,n}\langle D_1D_2|\mathcal{O}_n(x=0)|\varnothing\rangle, \qquad (20a)$$

$$\mathcal{T}[A \rightarrow B + X] = \sqrt{2M_X}\sum_n \mathcal{C}_A^{B,n}\langle X|\mathcal{O}_n(x=0)|\varnothing\rangle. \qquad (20b)$$

The sums are over local operators $\mathcal{O}_n$ in the effective field theory. They can be restricted to operators with a nonzero matrix element between $\langle D_1D_2|$ and the vacuum state $|\varnothing\rangle$. The arguments of the operator are the origin $x = 0$ of space and time. The operator matrix elements are evaluated in the rest frame of D_1D_2 or X. We use the standard nonrelativistic normalizations for the states $\langle D_1D_2|$ or $\langle X|$ in the operator matrix elements. We use the standard relativistic normalizations for the initial and final states in the T-matrix elements. The factors of $\sqrt{4M_1M_2}$ and $\sqrt{2M_X}$ account for the differences between the normalizations of the states in the operator matrix elements and the T-matrix elements. The Wilson coefficients $\mathcal{C}_A^{B,n}$ in Eq. (20) are functions of the 4-momenta and polarization 4-vectors of the particles in A and B and of the total 4-momentum P^μ of a D_1D_2 pair that is produced exactly at threshold with invariant mass M_{1+2}. They also depend on energy scales of order m^2/M_{12} and higher and on momentum scales of order m and higher. The only dependence on whether the final state includes D_1D_2 or X is in the operator matrix elements. The leading terms in the expansions of the T-matrix elements in powers of $M - M_{1+2}$ and $\vec{p}$ are the terms with the lowest dimension operator $D_1^\dagger D_2^\dagger(0)$. In Feynman diagrams, the local operator $D_1^\dagger D_2^\dagger$ is represented by an open dot from which a D_1 line and a D_2 line emerge, as illustrated in Fig. 2. The Feynman rule for this vertex is 1.

The operator product expansions in Eqs. (20) provide the desired separation of long-distance effects and short-distance effects only if the local operators $\mathcal{O}_n$ are chosen

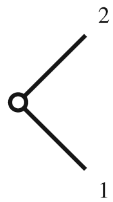

FIG. 2. Vertices for the D_1D_2 and $D_1^\dagger D_2^\dagger$ operators.

to be renormalized operators that have ultraviolet-finite matrix elements. The simplest local composite operators, such as D_1D_2, generally have matrix elements that are ultraviolet divergent. However, the multiplicative renormalizability of local composite operators implies that the relation between the simple local operators $\mathcal{O}_n$ and their renormalized counterparts $\mathcal{O}_n^{(R)}$ can be expressed as

$$\mathcal{O}_n(x) = \sum_m (Z^{-1})_{nm}\mathcal{O}_m^{(R)}(x), \tag{21}$$

where Z is an infinite-dimensional matrix of renormalization constants. The separation of short-distance effects and long-distance effects in the T-matrix elements in Eqs. (20) is accomplished by making the substitutions in Eq. (21) for the operators $\mathcal{O}_n$. The long-distance factors are matrix elements of the renormalized local operators $\mathcal{O}_m^{(R)}(x=0)$. The short-distance factors are the coefficients of these matrix elements, which are sums of products of Wilson coefficients $\mathcal{C}_A^{B,n}$ and renormalization constants $(Z^{-1})_{nm}$. We will find it more convenient to work with simple composite operators rather than renormalized operators. The combination of the operator product expansion and the multiplicative renormalizability of these operators will be used to separate short-distance effects from long-distance effects.

B. Short-distance decay processes

The fundamental theory may also allow transitions $D_1D_2 \to C$ from D_1D_2 with invariant mass M_C satisfying $|M_C - M_{1+2}| \ll m^2/M_{12}$ to a final state C that includes particles other than D_1 and D_2. If $M_C = M_X$, there can be analogous transitions $X \to C$. If the sum of the masses of the particles in C is substantially smaller than M_{1+2}, some of the particles in C must emerge with large momenta. We define a *short-distance transition* to be one for which all the particles in C have momenta in the center-of-momentum frame that are of order m or larger. The T-matrix element for such a process can be expanded in powers of the small energy difference $M_C - M_{1+2}$ divided by m^2/M_{12} and larger energy scales, and in powers of the small relative momentum $\vec{p}$ of the D_1 and D_2 divided by m and larger momentum scales. The operator product expansion can be used to express the T-matrix elements in the forms

$$\mathcal{T}[D_1D_2 \to C] = \sqrt{4M_1M_2}\sum_n \mathcal{C}_n^C \langle\emptyset|\mathcal{O}_n(x=0)|D_1D_2\rangle, \tag{22a}$$

$$\mathcal{T}[X \to C] = \sqrt{2M_X}\sum_n \mathcal{C}_n^C \langle\emptyset|\mathcal{O}_n(x=0)|X\rangle. \tag{22b}$$

The sums over local operators $\mathcal{O}_n$ of the D_1D_2 model can be restricted to those with a nonzero matrix element between the vacuum $\langle\emptyset|$ and $|D_1D_2\rangle$. The only dependence on the initial states is in the operator matrix elements. The leading terms in the expansions of the T-matrix elements in powers of $M_C - M_{1+2}$ and $\vec{p}$ are the terms with the lowest dimension operator $D_1D_2(0)$. The complete separation of short-distance and long-distance effects is accomplished by using Eq. (21) to eliminate the operators $\mathcal{O}_n$ in favor of renormalized operators.

C. Line shape

If the fundamental theory includes short-distance processes that allow the production of X via $A \to B + X$ and the decay of X via $X \to C$, it also allows the process $A \to B + C$, where C represents the same particles but with a variable invariant mass M_C instead of M_X. This process has a resonant enhancement when M_C is near the D_1D_2 threshold as specified by Eq. (8). If each of the particles in A and B is well separated in momentum space from each of the particles in C, the T-matrix element for this process can be described within the effective field theory by a double operator product expansion:

$$\mathcal{T}[A \to B + C] = \mathcal{C}_A^{B,C} + \sum_{m,n}\mathcal{C}_A^{B,n}\mathcal{C}_m^C \times \int d^4x e^{iP\cdot x}\langle\emptyset|\mathcal{O}_m(x)\mathcal{O}_n(0)|\emptyset\rangle. \tag{23}$$

The sum over operators $\mathcal{O}_m$ can be restricted to those with a nonzero matrix element between $\langle D_1D_2|$ and the vacuum state $|\emptyset\rangle$. The sum over operators $\mathcal{O}_n$ can be restricted to those with a nonzero matrix element between the vacuum $\langle\emptyset|$ and $|D_1D_2\rangle$. The Wilson coefficients $\mathcal{C}_A^{B,n}$ and $\mathcal{C}_m^C$ are the same ones that appear in the operator product expansions in Eqs. (20) and (22). In the Fourier transform of the vacuum-to-vacuum matrix element in Eq. (23), the 4-vector is $P^\mu = (M_C, \vec{0})$. The leading term in the expansion of the T-matrix element in powers of $M_C - M_{1+2}$ divided by m^2/M_{12} and larger energy scales is the term with the lowest dimension operators $D_1D_2(x)$ and $D_1^\dagger D_2^\dagger(0)$. The first term $\mathcal{C}_A^{B,C}$ on the right side of Eq. (23) takes into account the direct production of C at short distances. This term can be expanded in powers of the small energy difference $M_C - M_{1+2}$ divided by m^2/M_{12} and larger energy scales. The leading term in the expansion is a constant independent of M_C. The complete separation of short-distance and long-distance effects is accomplished by using Eq. (21) to eliminate the operators $\mathcal{O}_n$ in favor of renormalized operators.

IV. LONG-DISTANCE PROCESSES

In this section, we give the results for several quantities in the scalar meson model that depend only on long distances: the Green's function for $D_1D_2 \to D_1D_2$, the cross section for elastic D_1D_2 scattering, and the propagator for the bound state X. The results in this section are straightforward generalizations to heavy mesons of results for nucleons that were calculated using the leading-order approximations to the effective field theories developed in Refs. [31–34].

A. Green's function for $D_1D_2 \to D_1D_2$

In the scalar meson model, all the observables for processes near the D_1D_2 threshold are related in a simple way to the Green's function for $D_1D_2 \to D_1D_2$. We denote the amputated connected Green's function for $D_1D_2 \to D_1D_2$ by $i\mathcal{A}_0(E)$, because it depends only on the total energy E in the D_1D_2 rest frame. It can be calculated nonperturbatively by summing the geometric series represented by Fig. 3 to all orders in λ_0:

$$i\mathcal{A}_0(E) = \frac{-i}{1/\lambda_0 - L_0(E)}, \qquad (24)$$

where $iL_0(E)$ is the amplitude for the propagation of the D_1D_2 pair between successive contact interactions. The function $L_0(E)$ has an ultraviolet divergence that can be isolated into an additive term that is independent of E. It can be expressed as

$$L_0(E) = L_0(M_{1+2}) + \frac{M_{12}}{2\pi}\sqrt{-2M_{12}E_{12} - i\varepsilon}, \qquad (25)$$

where E_{12} is the energy relative to the D_1D_2 threshold:

$$E_{12} = E - M_{1+2}. \qquad (26)$$

The ultraviolet divergence is contained in the term $L_0(M_{1+2})$. If we use an ultraviolet momentum cutoff $\Lambda \gg |M_{12}E_{12}|^{1/2}$, this term has a linear ultraviolet divergence: $L_0(M_{1+2}) = -(M_{12}/\pi^2)\Lambda$. If we use dimensional regularization, this term is $L_0(M_{1+2}) = 0$, because dimensional regularization sets power ultraviolet divergences equal to 0.

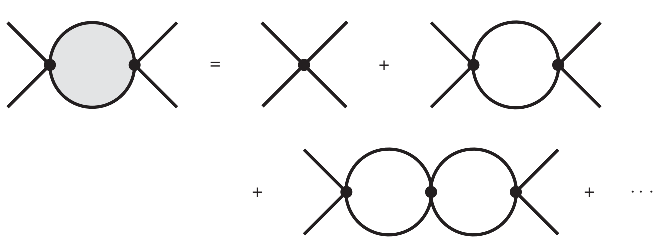

FIG. 3. The amputated connected Green's function $i\mathcal{A}_0(E)$ for $D_1D_2 \to D_1D_2$ at 0th order in g. It can be obtained by summing a geometric series of diagrams.

One can identify the combination $1/\lambda_0 - L_0(M_{1+2})$ on the right side of Eq. (24) as a renormalized inverse coupling constant $1/\lambda$. One possible renormalization prescription is to eliminate $1/\lambda_0$ in favor of $1/\lambda$. Another possible renormalization prescription is to eliminate λ_0 in favor of the scattering length a defined in Eq. (9). The most convenient renormalization prescription for our purposes is to demand that the pole in the amplitude $\mathcal{A}_0(E)$ be at the energy $E_{\rm pole}$ given in Eq. (10). Inserting the expression for $L_0(E)$ in Eq. (25) into the amplitude in Eq. (24) and eliminating λ_0 in favor of γ, the amplitude reduces to

$$\mathcal{A}_0(E) = \frac{2\pi/M_{12}}{-\gamma + \sqrt{-2M_{12}E_{12} - i\varepsilon}}. \qquad (27)$$

The renormalized expression for the amplitude $\mathcal{A}_0(E)$ in Eq. (27) follows from the renormalization prescription in Eq. (10) and is independent of the regularization scheme. The expression for the bare coupling constant is

$$\lambda_0 = \frac{1}{L_0(M_{1+2}) + M_{12}(\gamma_{\rm Re} + i\gamma_{\rm Im})/(2\pi)}. \qquad (28)$$

With dimensional regularization, $L_0(M_{1+2}) = 0$, so Eq. (28) gives a finite relation between λ_0 and the binding momentum γ: $\lambda_0 = 2\pi/(M_{12}\gamma)$. We will see later that the naive use of dimensional regularization can be misleading.

B. Elastic D_1D_2 scattering

We can use the amplitude in Eq. (27) to determine the T-matrix element for the elastic scattering of D_1 and D_2 with relative momentum p. In the D_1D_2 center-of-momentum frame, the total energy of the D_1 and D_2 is

$$E_{\rm cm}(p) = M_{1+2} + p^2/(2M_{12}). \qquad (29)$$

The energy variable E_{12} defined in Eq. (26) reduces to $E_{12} = p^2/(2M_{12})$. The T-matrix element is obtained by evaluating the amplitude $\mathcal{A}_0(E)$ in Eq. (27) at the energy $E_{\rm cm}(p)$ in Eq. (29) and multiplying by the factor $4M_1M_2$ to account for the relativistic normalization of states:

$$\mathcal{T}_{12\to 12}(p) = -\frac{8\pi M_{1+2}}{\gamma + ip}. \qquad (30)$$

The complex D_1D_2 scattering length defined by Eq. (9) is therefore simply

$$a = 1/(\gamma_{\rm Re} + i\gamma_{\rm Im}). \qquad (31)$$

We obtain the cross section for elastic D_1D_2 scattering by squaring the T-matrix element, integrating over the phase space of the D_1 and D_2 in the final state, and multiplying by a flux factor. The energy $E_{\rm cm}(p)$ in the D_1D_2 rest frame is assumed to be close to M_{1+2}, as specified by the condition in Eq. (8). The product of the phase space factor $\lambda^{1/2}(E_{\rm cm}(p), M_1, M_2)/(8\pi M_{1+2}^2)$ and the flux factor $1/(4M_{1+2}p)$ can therefore be approximated by $1/(16\pi M_{1+2}^2)$. The cross section is

$$\sigma[D_1D_2(\vec{p}) \to D_1D_2] = \frac{4\pi}{|\gamma_{\rm Re} + i(\gamma_{\rm Im} + p)|^2}. \quad (32)$$

The argument $(\vec{p})$ of D_1D_2 in the initial state implies that the D_1 and D_2 have momenta $-\vec{p}$ and $+\vec{p}$, respectively. The momenta of the D_1 and D_2 in the final state are not specified because they have been integrated over.

C. Propagator for X

If the local composite operator $D_1D_2(x)$ is used as an interpolating field for X, the propagator for X is given in Eq. (16). The diagrams for the propagator of X are shown in Fig. 5. In the rest frame $P = 0$, these diagrams form a geometric series whose sum is

$$i\Delta_X(E, 0) = \frac{iL_0(E)}{1 - \lambda_0 L_0(E)}. \quad (33)$$

This propagator has a pole in E at the same energy $E_{\rm pole}$ given in Eq. (10) as the amplitude $\mathcal{A}_0(E)$ in Eq. (24). Near the pole, the behavior of the propagator at $P = 0$ is given in Eq. (17). Using $L_0(E_{\rm pole}) = 1/\lambda_0$, we determine the wave-function normalization factor to be

$$Z_X = \frac{2\pi\gamma}{M_{12}^2\lambda_0^2}. \quad (34)$$

Using the expression for $\mathcal{A}_0(E)$ in Eq. (24), the propagator for X in Eq. (33) can be expressed as

$$\Delta_X(E, 0) = -\mathcal{A}_0(E)\frac{L_0(E)}{\lambda_0}. \quad (35)$$

The alternative expression for $\mathcal{A}_0(E)$ in Eq. (27) shows that, after renormalization of the coupling constant, it does not depend on the ultraviolet cutoff Λ. Thus the propagator in Eq. (35) depends on Λ only through the factor $L_0(E)/\lambda_0$. That there is some dependence on Λ is not a surprise, because we have used the simple composite operator $D_1D_2(x)$ as the interpolating field for X rather than a renormalized operator. As we shall see in Sec. V, $\lambda_0 D_1D_2(x)$ is a renormalized operator whose matrix elements between the vacuum $\langle\emptyset|$ and $|D_1D_2\rangle$ or $|X\rangle$ do not depend on the ultraviolet cutoff. The propagator for the renormalized operator $\lambda_0 D_1D_2(x)$ is obtained by multiplying the expression in Eq. (35) by λ_0^2. But this propagator depends on Λ through the factor $\lambda_0 L_0(E)$. To obtain a renormalized propagator that does not depend on Λ, one must add the Λ-dependent constant $i\lambda_0$ to the propagator for the renormalized operator $\lambda_0 D_1D_2(x)$:

$$i\lambda_0^2\Delta_X(E, 0) + i\lambda_0 = -i\mathcal{A}_0(E). \quad (36)$$

Thus the renormalized propagator is essentially just the amplitude $\mathcal{A}_0(E)$ in Eq. (27). The need for adding the constant term in Eq. (36) is related to the fact that, if an external source coupled to a composite operator is added to the Lagrangian, renormalization sometimes requires the addition of terms with higher powers of the source [36]. For example, the addition of the term $J^\dagger D_1D_2$ + H.c. creates new ultraviolet divergences that can only be canceled by a $J^\dagger J$ term. Such a term is required even in the absence of any interactions. To implement the LSZ prescription for T-matrix elements for processes with X in the initial or final state, it is not necessary to use a renormalized propagator. We will use the unrenormalized propagator for X in Eq. (35) for this purpose.

V. SHORT-DISTANCE PROCESSES

In this section, we consider processes in the scalar meson model that involve both short-distance and long-distance effects: short-distance production rates, short-distance decay rates, and the line shape of the bound state in a short-distance decay mode. We use the operator product expansion to derive factorization formulas in which those short-distance effects and long-distance effects are separated. After renormalization of the coupling constant, all remaining dependence on the ultraviolet cutoff can be removed by renormalization of the Wilson coefficients in the operator product expansion. To the best of our knowledge, these results have not been derived previously using effective field theories for particles with large scattering lengths.

A. Short-distance production of X and D_1D_2

We consider the short-distance production processes $A \to B + D_1D_2$ and $A \to B + X$, where A and B both represent one or more particles whose momenta in the D_1D_2 or X rest frame are all of order m or larger. The operator product expansions of the T-matrix element for such short-distance processes are given in Eqs. (20). The leading terms in the expansions are the ones with the local operator $D_1^\dagger D_2^\dagger(0)$. If we keep only these leading terms, the Lorentz-invariant T-matrix elements reduce to

$$\mathcal{T}[A \to B + D_1D_2] = \sqrt{4M_1M_2}\mathcal{C}_A^{B,12}\langle D_1D_2|D_1^\dagger D_2^\dagger(0)|\emptyset\rangle, \quad (37a)$$

$$\mathcal{T}[A \to B + X] = \sqrt{2M_X}\mathcal{C}_A^{B,12}\langle X|D_1^\dagger D_2^\dagger(0)|\emptyset\rangle. \quad (37b)$$

We first consider the T-matrix element for the production of D_2D_2 via the short-distance process $A \to B + D_1D_2$. We take the D_1 and D_2 in Eq. (37a) to have relative momentum $\vec{p}$ in the D_1D_2 rest frame and unspecified total momentum. Their invariant mass is $M = E_{\rm cm}(p)$, where $E_{\rm cm}(p)$ is given in Eq. (29). This invariant mass is assumed to be close to M_{1+2}, as specified by the inequality in Eq. (8). The Feynman diagrams for the vacuum-to-D_1D_2 matrix element, which are shown in Fig. 4, form a geometric series. The matrix element in Eq. (37a) is therefore

$$\langle D_1D_2(\vec{p})|D_1^\dagger D_2^\dagger(0)|\emptyset\rangle = \frac{1}{1 - \lambda_0 L_0(E_{\rm cm}(p))}. \quad (38)$$

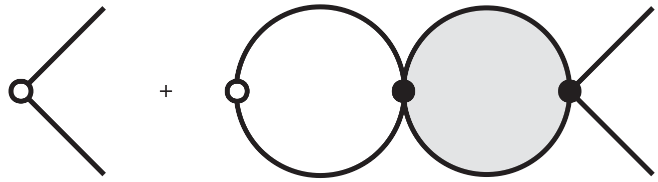

FIG. 4. Diagrams at 0th order in g for the vacuum-to-D_1D_2 matrix element of the operator $D_1^\dagger D_2^\dagger$.

The argument ($\vec{p}$) of the D_1D_2 state implies that the D_1 and D_2 have momenta $-\vec{p}$ and $+\vec{p}$, respectively. The sum of the diagrams in Fig. 4 differs from the geometric series of diagrams for $i\mathcal{A}_0(E)$ in Fig. 3 only by the multiplicative factor $1/(-i\lambda_0)$. Since $\mathcal{A}_0(E)$, which after renormalization of the coupling constant is given by Eq. (27), does not depend on the ultraviolet cutoff, the operator $\lambda_0 D_1^\dagger D_2^\dagger(x)$ is a renormalized operator whose matrix elements do not depend on the ultraviolet cutoff. The T-matrix element in Eq. (37a) can be separated into a short-distance factor and a long-distance factor by taking the long-distance factor to be the matrix element of the renormalized operator:

$$\mathcal{T}[A \to B + D_1D_2(\vec{p})] = -\sqrt{4M_1M_2}(\mathcal{C}_A^{B,12}/\lambda_0) \times \mathcal{A}_0(E_{\rm cm}(p)). \quad (39)$$

After renormalization of the coupling constant, $\mathcal{A}_0(E_{\rm cm}(p))$ is given by the expression in Eq. (27), which does not depend on the ultraviolet cutoff Λ. The T-matrix element in Eq. (39) will not depend on Λ if the short-distance factor $\mathcal{C}_A^{B,12}/\lambda_0$ does not depend on Λ. Equivalently, the dependence on Λ can be removed by a multiplicative renormalization of the Wilson coefficient $\mathcal{C}_A^{B,12}$.

We next consider the T-matrix element for the production of X via the short-distance process $A \to B + X$. The operator product expansion of the T-matrix element for this process is given in Eq. (37b). The vacuum-to-X matrix element can be obtained via the LSZ formalism from the connected Green's function for the operator $D_1^\dagger D_2^\dagger$ that acts on the vacuum and an operator D_1D_2 associated with the X in the final state. This connected Green's function is identical to the propagator $i\Delta_X(E, 0)$ given in Eq. (33). The matrix element in Eq. (37b) is the normalized on-shell amputated connected Green's function. The Green's function is amputated by multiplying by the inverse propagator $[i\Delta_X(E, 0)]^{-1}$, which simply gives 1. In the rest frame of X where its momentum is $\vec{P} = 0$, the Green's function is put on shell by setting the energy E equal to the energy $E_{\rm pole}$ in Eq. (10), although the absence of any dependence on E makes this condition moot. Finally, the Green's function is normalized by multiplying by the factor $Z_X^{1/2}$, where Z_X is given in Eq. (34). Thus the vacuum-to-X matrix element is simply

$$\langle X|D_1^\dagger D_2^\dagger(0)|\varnothing\rangle = \frac{(2\pi\gamma)^{1/2}}{M_{12}\lambda_0}. \quad (40)$$

The only factors in the T-matrix element in Eq. (37b) that are sensitive to short distances are the Wilson coefficient $\mathcal{C}_A^{B,12}$ and the factor of $1/\lambda_0$ from the matrix element. The T-matrix element in Eq. (37b) can be expressed as the product of a short-distance factor and a long-distance factor:

$$\mathcal{T}[A \to B + X] = \sqrt{2M_X}(\mathcal{C}_A^{B,12}/\lambda_0)\frac{(2\pi\gamma)^{1/2}}{M_{12}}. \quad (41)$$

The short-distance factor $\mathcal{C}_A^{B,12}/\lambda_0$ is the same one that appears in Eq. (39).

The factored expressions for the T-matrix elements in Eqs. (39) and (41) lead to factored expressions for the production rates. The rates for producing X and D_1D_2 are obtained by squaring the amplitudes and integrating over the appropriate phase space. If A consists of a single particle, the decay rate into $B + X$ can be expressed in the factored form

$$\Gamma[A \to B + X] = \Gamma_A^B \frac{2\pi}{M_{12}} |\gamma_{\rm Re}^2 + \gamma_{\rm Im}^2|^{1/2}, \quad (42)$$

where Γ_A^B is a short-distance factor with dimensions of mass:

$$\Gamma_A^B = \frac{M_{1+2}}{M_A M_{12}} \int \frac{d^3P_{1+2}}{(2\pi)^3 2E_{1+2}} \int \prod_{i\in B} \frac{d^3p_i}{(2\pi)^3 2E_i} \times |\mathcal{C}_A^{B,12}/\lambda_0|^2 (2\pi)^4 \delta^{(4)}\Big(P_A^\mu - P_{1+2}^\mu - \sum_{i\in B} p_i^\mu\Big). \quad (43)$$

The 4-momentum P_{1+2}^μ is that of a D_1D_2 pair exactly at threshold with invariant mass M_{1+2}. We have chosen the long-distance factor in Eq. (42) to be the square of the long-distance factor in the T-matrix element in Eq. (41) multiplied by M_{12} to make it dimensionless. All factors from integrating over the phase space of the particles in the final state are included in the short-distance factor. The differential rate for producing D_1D_2 with invariant mass $M = E_{\rm cm}(p)$ given by Eq. (29) can be expressed in the factored form

$$\frac{d\Gamma}{dM}[A \to B + D_1D_2(\vec{p})] = \Gamma_A^B \frac{2p}{|\gamma_{\rm Re} + i(\gamma_{\rm Im} + p)|^2}. \quad (44)$$

The short-distance factor Γ_A^B is the same as in Eq. (42). The long-distance factor is the product of $4M_1M_2|\mathcal{A}_0(E_{\rm cm}(p))|^2$ from the T-matrix element in Eq. (39), the phase space factor $\lambda^{1/2}(E_{\rm cm}(p), M_1, M_2)/(8\pi E_{\rm cm}(p)^2)$, the kinematic factor $E_{\rm cm}(p)/\pi$ associated with the differential dM, and a factor $M_{12}/(2M_{1+2})$ to compensate for the choice of prefactor in Eq. (43). Using the expression for the invariant mass

$E_{\rm cm}(p)$ in Eq. (29) and replacing $E_{\rm cm}(p)$ by M_{1+2} in every factor that is insensitive to p, the product of the phase space and kinematic factors can be reduced to $p/(4\pi^2)$.

B. Short-distance decay of X

The fundamental theory may have short-distance processes through which X can decay. We consider the short-distance decay process $X \to C$, where C represents two or more particles whose momenta in the X rest frame are of order m or larger. The operator product expansion for this decay process is given in Eq. (22b). The leading term in the expansion is the one with the operator $D_1 D_2(x)$. If we keep only this term, the T-matrix element reduces to

$$\mathcal{T}[X \to C] = \sqrt{2M_X}\mathcal{C}_{12}^{C}\langle \emptyset | D_1 D_2(0) | X\rangle. \tag{45}$$

The X-to-vacuum matrix element can be calculated in a similar way to the vacuum-to-X matrix element in Eq. (40)[1]:

$$\langle \emptyset | D_1 D_2(0) | X\rangle = \frac{(2\pi\gamma)^{1/2}}{M_{12}\lambda_0}. \tag{46}$$

This matrix element depends on the ultraviolet cutoff only through the factor $1/\lambda_0$. In the T-matrix element in Eq. (45), the separation of short-distance and long-distance effects can be accomplished by taking the long-distance factor to be the matrix element of the renormalized operator $\lambda_0 D_1 D_2(0)$:

$$\mathcal{T}[X \to C] = \sqrt{2M_X}(\mathcal{C}_{12}^{C}/\lambda_0)\frac{(2\pi\gamma)^{1/2}}{M_{12}}. \tag{47}$$

This T-matrix element will not depend on the ultraviolet cutoff Λ if the short-distance factor $\mathcal{C}_{12}^{C}/\lambda_0$ does not depend on Λ. Equivalently, the dependence on Λ can be removed by a multiplicative renormalization of the Wilson coefficient $\mathcal{C}_{12}^{C}$.

The decay rate of X into the particles represented by C is obtained by squaring the T-matrix element and integrating over the phase space of those particles. It can be expressed in the factored form

$$\Gamma[X \to C] = \Gamma^{C}\frac{2\pi}{M_{12}}|\gamma_{\rm Re}^2 + \gamma_{\rm Im}^2|^{1/2}, \tag{48}$$

where Γ^C is a short-distance factor with dimensions of mass:

$$\Gamma_C = \frac{1}{M_{12}}\int \prod_{j\in C}\frac{d^3p_j}{(2\pi)^3 2E_j}|\mathcal{C}_{12}^{C}/\lambda_0|^2(2\pi)^4\delta^{(4)}\times\left(P_{1+2}^{\mu} - \sum_{j\in C}p_j^{\mu}\right). \tag{49}$$

We have chosen the long-distance factor to be the square of the long-distance factor in the T-matrix element in Eq. (47) multiplied by M_{12} to make it dimensionless.

The separation of short-distance and long-distance effects for the transition $D_1(-\vec{p})D_2(\vec{p}) \to C$ can be accomplished in a similar way. The operator product expansion for the T-matrix element is given in Eq. (22a). The T-matrix element can be expressed as the product of the same short-distance factor as in Eq. (47) and a long-distance factor that includes a factor of $\mathcal{A}_0(E_{\rm cm}(p))$, where $E_{\rm cm}(p)$ is the energy in Eq. (29).

C. Line shape of X in a short-distance decay mode

If the fundamental theory includes short-distance processes that allow the production of X via $A \to B + X$ and the decay of X via $X \to C$, it also allows the process $A \to B + C$, where C represents the same particles but with a variable invariant mass M_C instead of M_X. We assume that M_C is near the D_1D_2 threshold as specified by Eq. (8). If every particle in A and B has momentum in the rest frame of C of order m or larger, the T-matrix element for this process can be expressed as the double operator product expansion in Eq. (23). The Wilson coefficient $C_A^{B,C}$ can be expanded in powers of $M_C - M_{1+2}$ divided by m^2/M_{12} and higher energy scales. The leading term in this expansion is simply a constant independent of M_C. The leading terms in the double sum come from the operators $\mathcal{O}_n(0) = D_1^\dagger D_2^\dagger(0)$ and $\mathcal{O}_m(x) = D_1D_2(x)$. According to Eq. (16), the Fourier transform of the matrix element of these operators in the C rest frame is just the X propagator, which is given in Eq. (33) or (35), evaluated at $E = M_C$. Thus the T-matrix element reduces to

$$\mathcal{T}[A \to B + C] = \mathcal{C}_A^{B,C} + \mathcal{C}_A^{B,12}\mathcal{C}_{12}^{C}\frac{iL_0(M_C)}{1-\lambda_0 L_0(M_C)}. \tag{50}$$

The Wilson coefficients and the factor $L_0(E)/\lambda_0$ in the X propagator depend on the ultraviolet cutoff Λ. All the dependence on the energy can be isolated in a term that does not depend on Λ by using the fact that the combination in Eq. (36) does not depend on Λ. By subtracting and adding $i\mathcal{C}_A^{B,12}\mathcal{C}_{12}^{C}/\lambda_0$ to the two terms on the right side of Eq. (50), the T-matrix element can be expressed as

$$\mathcal{T}[A \to B + C] = (\mathcal{C}_A^{B,C} - i\mathcal{C}_A^{B,12}\mathcal{C}_{12}^{C}/\lambda_0) - i(\mathcal{C}_A^{B,12}/\lambda_0)(\mathcal{C}_{12}^{C}/\lambda_0)\mathcal{A}_0(M_C). \tag{51}$$

After renormalization of the coupling constant, $\mathcal{A}_0(E)$ is given in Eq. (27). The short-distance factors $\mathcal{C}_A^{B,12}/\lambda_0$ and

[1]Our notation might suggest that the matrix element in Eq. (46) is the complex conjugate of the matrix element in Eq. (40). However, $|X\rangle$ in Eq. (46) is an in state, while $\langle X|$ in Eq. (40) is the Hermitian conjugate of an out state. These two states are related by the S matrix: $|X, {\rm out}\rangle = \mathcal{S}|X, {\rm in}\rangle$.

$\mathcal{C}_{12}^{C}/\lambda_0$ in Eq. (51) cannot depend on Λ, because otherwise the T-matrix elements in Eqs. (39), (41), and (47) would depend on Λ. Thus the T-matrix element in Eq. (51) will not depend on Λ if the constant term $\mathcal{C}_A^{B,C} - i\mathcal{C}_A^{B,12}\mathcal{C}_{12}^{C}/\lambda_0$ does not depend on Λ. Equivalently, the dependence on Λ can be removed by an additive renormalization of the Wilson coefficient $\mathcal{C}_A^{B,C}$. It is convenient to express the T-matrix element in Eq. (51) in the form

$$\mathcal{T}[A \to B + C] = -i(\mathcal{C}_A^{B,12}/\lambda_0)(\mathcal{C}_{12}^{C}/\lambda_0) \times [\mathcal{A}_0(M_C) - (2\pi/M_{12})c_A^{B,C}], \quad (52)$$

where $c_A^{B,C}$ is a complex constant with dimension of length that is completely determined by short-distance factors. The natural scale for $c_A^{B,C}$ is $1/m$, where m is the mass of the lightest meson that can be exchanged between D_1 and D_2.

The factored expression for the T-matrix element in Eq. (52) leads to a factored expression for the rate for $A \to B + C$. The invariant mass distribution of the particles in C is obtained by squaring the T-matrix element and integrating over the momenta of all the particles in the final state. It is convenient to express the phase space integral in an iterated form corresponding to the production of the particles in B and an effective particle of mass M_C followed by the decay of that effective particle into the particles in C. If A is a single particle, the decay rate is

$$\Gamma[A \to B + C] = \frac{1}{2M_A}\int \frac{dM_C^2}{2\pi}\int \frac{d^3P_C}{(2\pi)^3 2E_C}\int \prod_{i\in B}\frac{d^3p_i}{(2\pi)^3 2E_i}(2\pi)^4\delta^{(4)}\Big(P_A^\mu - P_C^\mu - \sum_{i\in B} p_i^\mu\Big) \times \int \prod_{j\in C}\frac{d^3p_j}{(2\pi)^3 2E_j}(2\pi)^4\delta^{(4)}\Big(P_C^\mu - \sum_{j\in C} p_j^\mu\Big)|\mathcal{T}[A\to B + C]|^2. \quad (53)$$

The invariant mass M_C of the particles in C can be replaced by M_{1+2} everywhere except in the long-distance factor of the T-matrix element. In that long-distance factor, it can be expressed as

$$M_C = M_{1+2} + p_C^2/(2M_{12}), \quad (54)$$

where p_C^2 can be positive or negative. The variable p_C is pure imaginary if $M_C < M_{1+2}$ and real and positive if $M_C > M_{1+2}$. The differential decay rate of the particle A for M_C near M_{1+2} reduces to

$$\frac{d\Gamma}{dM_C}[A \to B + C] = (\Gamma_A^B\Gamma^C)2\pi\left|\frac{1}{(\gamma_{\rm Re} - |p_C|) + i\gamma_{\rm Im}} + c_A^{B,C}\right|^2, \qquad M_C < M_{1+2}, \quad (55a)$$

$$= (\Gamma_A^B\Gamma^C)2\pi\left|\frac{1}{\gamma_{\rm Re} + i(\gamma_{\rm Im} + p_C)} + c_A^{B,C}\right|^2, \qquad M_C > M_{1+2}. \quad (55b)$$

The short-distance factors Γ_A^B and Γ^C are the same as in Eqs. (42), (44), and (48). All other short-distance effects are contained in the complex constant $c_A^{B,C}$. The invariant mass distribution in Eq. (55) is continuous at $M_C = M_{1+2}$.

The separation of scales represented by the renormalized operator product expansions for the T-matrix elements in Eqs. (39), (41), (47), and (51) can be obscured by using dimensional regularization. In a generic regularization scheme with ultraviolet cutoff Λ, the short-distance quantities λ_0, $\mathcal{C}_A^{B,12}$, $\mathcal{C}_{12}^{C}$, and $\mathcal{C}_A^{B,C}$ are insensitive to γ. They depend on Λ in such a way that the combinations $\mathcal{C}_A^{B,12}/\lambda_0$, $\mathcal{C}_{12}^{C}/\lambda_0$, and $\mathcal{C}_A^{B,C} - i\mathcal{C}_A^{B,12}\mathcal{C}_{12}^{C}/\lambda_0$ do not depend on Λ. Dimensional regularization sets power ultraviolet divergences to zero. In particular, it sets $L_0(M_{1+2}) = 0$, so the expression for the bare coupling constant in Eq. (28) reduces to $\lambda_0 = 2\pi/(M_{12}\gamma)$. The Wilson coefficients $\mathcal{C}_A^{B,12}$, $\mathcal{C}_{12}^{C}$, and $\mathcal{C}_A^{B,C}$ do not depend on the ultraviolet cutoff of dimensional regularization. Compatibility with other regularization schemes requires, however, that they depend on γ in such a way that the combinations $\mathcal{C}_A^{B,12}/\lambda_0$, $\mathcal{C}_{12}^{C}/\lambda_0$, and $\mathcal{C}_A^{B,C} - i\mathcal{C}_A^{B,12}\mathcal{C}_{12}^{C}/\lambda_0$ are insensitive to γ. This requires that $\mathcal{C}_A^{B,12}$ and $\mathcal{C}_{12}^{C}$ have multiplicative factors of γ^{-1} and that $\mathcal{C}_A^{B,C}$ have an additive term with a factor γ^{-1}. Thus the Wilson coefficients in dimensional regularization are not short-distance factors. Their dependence on γ is similar to the dependence of the Wilson coefficients on Λ in other regularization schemes. That dependence cancels in the combinations of Wilson coefficients and λ_0 that appear in the renormalized operator product expansions for the T-matrix elements.

VI. LARGE SCATTERING LENGTH EXPANSION

Effective field theories can exploit a large separation of momentum scales by providing a simpler description of the lowest momentum scale. Another important feature of effective field theories is that they provide a systematic framework for improving the accuracy of the description to any desired order in the ratio of the small momentum scale and higher momentum scales. For example, the effective field theories for nucleons developed in Refs. [31–34] can be used to systematically calculate corrections associated with the nonzero range of the strong force. In this section, we discuss how the accuracy of the results for the scalar meson model in Secs. IV and V can be systematically improved using an expansion in the large scattering length. This expansion is a straightforward generalization to heavy mesons of the effective range expansion in the effective field theory for nucleons in Refs. [33,34]. We also explain how the expansion in the large scattering length can be exploited to simplify some of the results in Sec. V.

In the scalar meson model, the smallest momentum scale is the scale $|\gamma|$ associated with the large scattering length. In a more fundamental theory, there may be many larger momentum scales, but the most important at low energies is the mass m of the lightest meson that can be exchanged between D_1 and D_2. The model defined by the Lagrangian in Eqs. (4) and (5) reproduces all effects that are not suppressed by powers of $|\gamma|/m$. The model can be systematically improved so that it reproduces all corrections to any desired order in $|\gamma|/m$. We will discuss only the improvements required to reproduce corrections through first order in $|\gamma|/m$.

The only improvement in the effective theory that is required to decrease the errors to second order in $|\gamma|/m$ is to take into account the effective range r_s for S-wave scattering. This parameter can be defined by the low-momentum expansion for the inverse of the T-matrix element:

$$\frac{1}{\mathcal{T}(p)} = -\frac{1}{8\pi M_{1+2}}\left(\frac{1}{a} + ip - \frac{1}{2}r_s p^2 + \cdots\right). \qquad (56)$$

If we impose the renormalization condition that the Green's function for $D_1D_2 \to D_1D_2$ has a pole at the energy $E_{\rm pole}$ given in Eq. (10), one expression that will give the correct effective range is

$$\mathcal{A}(E) = \frac{-2\pi/M_{12}}{(\gamma + ip)[1 - r_s(\gamma - ip)/2]}, \qquad (57)$$

where $p = i\sqrt{-2M_{12}E_{12} - i\varepsilon}$. The corresponding T-matrix element for elastic D_1D_2 scattering is then

$$\mathcal{T}_{12\to 12}(p) = \frac{-8\pi M_{1+2}}{(\gamma + ip)[1 - r_s(\gamma - ip)/2]}. \qquad (58)$$

In short-distance observables, there are additional terms in the expansions in $|\gamma|/m$ coming from higher dimension operators in the operator product expansion. For each additional gradient in the operator, the operator matrix element will have an additional factor of order γ. The dimensions from these additional factors of γ must be compensated by factors of $1/m$ in the short-distance coefficients. Only operators with a single gradient can give contributions that are suppressed by one power of $|\gamma|/m$. The matrix element of the operator $\nabla^i(D_1D_2)$ between $|D_1D_2(\vec{p})\rangle$ or $|X\rangle$ and the vacuum $\langle\varnothing|$ vanishes in the center-of-momentum frame. The other independent operator with a single gradient is $\nabla^i D_1 D_2 - D_1\nabla^i D_2$. The matrix element $\langle\varnothing|\nabla^i D_1 D_2 - D_1\nabla^i D_2|X\rangle$ must vanish because the operator is a vector and there are no vectors associated with the state $|X\rangle$ in its center-of-mass frame. The matrix element $\langle\varnothing|\nabla^i D_1 D_2 - D_1\nabla^i D_2|D_1D_2(\vec{p})\rangle$ is nonzero and proportional to p^i. This operator gives a term in the T-matrix element for $A \to B + D_1D_2(\vec{p})$ in Eq. (39) that is linear in the momentum $\vec{p}$ but has a suppression factor of $1/m$ in the short-distance coefficient. Higher dimension operators in the operator product expansion will contribute to the T-matrix elements for $A \to B + X$ in Eq. (41), for $X \to C$ in Eq. (47), and for $A \to B + C$ in Eq. (52) only at second and higher orders in $|\gamma|/m$.

The systematic expansion in powers of $|\gamma|/m$ can be used to simplify the leading-order results for short-distance observables. The T-matrix element for $A \to B + C$ in Eq. (52) has a resonant term $\mathcal{A}_0(M_C)$ and a nonresonant term $c_A^{B,C}$. The resonant term $\mathcal{A}_0(M_C)$ includes a factor that is of order $1/|\gamma|$ when p_C is of order $|\gamma|$. The nonresonant term $c_A^{B,C}$ is completely determined by short-distance effects, so the natural scale for $c_A^{B,C}$ is $1/m$. For p_C of order $|\gamma|$, this amplitude is suppressed by $|\gamma|/m$ compared to the resonant term in Eq. (52). One can therefore set $c_A^{B,C} = 0$ by truncating the expansion at leading order in $|\gamma|/m$. The invariant mass distribution in Eq. (55) then reduces to

$$\frac{d\Gamma}{dM_C}[A \to B + C] = (\Gamma_A^B\Gamma^C)\frac{2\pi}{(\gamma_{\rm Re} - |p_C|)^2 + \gamma_{\rm Im}^2}, \qquad M_C < M_{1+2}, \qquad (59a)$$

$$= (\Gamma_A^B\Gamma^C)\frac{2\pi}{\gamma_{\rm Re}^2 + (\gamma_{\rm Im} + p_C)^2}, \qquad M_C > M_{1+2}. \qquad (59b)$$

This simple factorization formula was first derived in Ref. [21]. If the nonresonant amplitude $c_A^{B,C}$ in Eq. (55) is included, the systematic expansion in powers of $|\gamma|/m$ requires that all other terms that are first order in $|\gamma|/m$ also be included. This requires that the effective field theory be improved so that it takes into account the effective range.

The above derivation of the simple factorization formula in Eq. (59) is much cleaner than the derivation in Ref. [21]. In Ref. [21], the authors used an ultraviolet momentum cutoff Λ. They obtained results that did not depend on Λ by taking the limit $\Lambda \to \infty$. The expression for the bare coupling constant λ_0 in Eq. (28) has a term $L_0(M_{1+2})$ in the denominator. Since $L_0(M_C)$ and $L_0(M_{1+2})$ are both linear in Λ, the product $\lambda_0 L_0(M_C)$ approaches 1 in the limit $\Lambda \to \infty$. In this limit, the last factor in the second term on the right side of Eq. (50) reduces to

$$\frac{iL_0(M_C)}{1-\lambda_0 L_0(M_C)} \to -\frac{i}{\lambda_0^2}\mathcal{A}_0(M_C). \tag{60}$$

The factors of $1/\lambda_0$ can be combined with the Wilson coefficients $\mathcal{C}_A^{B,12}$ and $\mathcal{C}_{12}^{C}$ to obtain short-distance factors with finite limits as $\Lambda \to \infty$. In Ref. [21], the authors omitted the $\mathcal{C}_A^{B,C}$ term in Eq. (50). This gave the simple factorization formula in Eq. (59). In retrospect, omitting the $\mathcal{C}_A^{B,C}$ term in Eq. (50) can be justified by the observation that the natural scale for the coefficient $c_A^{B,C}$ in Eq. (52) is $1/m_\pi$. If m_π is identified with the ultraviolet cutoff Λ, then $c_A^{B,C} \to 0$ in the limit $\Lambda \to \infty$. The derivation in Ref. [21] blurred the distinction between the arbitrary, unphysical, ultraviolet cutoff Λ, which can be taken to ∞, and the physical, short-distance scale m_π, which is fixed. By maintaining the distinction between Λ and m_π, we were able to separate the renormalization of the operator product expansion from the expansion in inverse powers of the large scattering length and give a much cleaner derivation of the factorization formula.

VII. MINIMAL CHARM MESON MODEL

In this section, we generalize the results of Secs. IV, V, and VI for the scalar meson model to the minimal charm meson model defined by the Lagrangian in Eqs. (14) and (15). The bound state in this model is identified as the $X(3872)$. For simplicity of notation, we denote the masses of D^0 and D^{*0} by $M_1 = M_{D^0}$ and $M_2 = M_{D^{*0}}$. Thus M_{12} is the reduced mass of D^0 and D^{*0} and M_{1+2} is the sum of their masses.

A. Long-distance processes

The amplitude $L_0(E)$ in the charm meson model for the propagation of $D^{*0}\bar{D}^0$ or $D^0\bar{D}^{*0}$ between contact interactions is given by the same expression in Eq. (25) as in the scalar meson model. The Green's function for $D^{*0}\bar{D}^0 \to D^{*0}\bar{D}^0$ is diagonal in the vector indices of the spin-1 mesons. The diagonal entries are

$$i\mathcal{A}(E) = \frac{-i}{1/\lambda_0 - 2L_0(E)}. \tag{61}$$

This differs from the expression for $i\mathcal{A}_0(E)$ in Eq. (24) only in the factor of 2 multiplying $L_0(E)$, which accounts for the fact that the particles in each of the loops in Fig. 3 can be either $D^{*0}\bar{D}^0$ or $D^0\bar{D}^{*0}$. The Green's functions for $D^{*0}\bar{D}^0 \to D^0\bar{D}^{*0}$, $D^0\bar{D}^{*0} \to D^{*0}\bar{D}^0$, and $D^0\bar{D}^{*0} \to D^0\bar{D}^{*0}$ are also given by this same expression. If we use the renormalization prescription that the Green's function in Eq. (61) has a pole in E at the energy $E_{\rm pole}$ given in Eq. (10), the expression for the diagonal entries of the Green's function can be reduced to

$$\mathcal{A}(E) = \frac{\pi/M_{12}}{-\gamma + \sqrt{-2M_{12}E_{12} - i\varepsilon}}. \tag{62}$$

The complex parameter γ determines the mass and width of a bound state with spin 1 that we identify as the $X(3872)$.

The Green's function $\mathcal{A}(E)$ in Eq. (62) differs from $\mathcal{A}_0(E)$ in Eq. (27) by a factor of $1/2$. The T-matrix element for $D^{*0}\bar{D}^0 \to D^{*0}\bar{D}^0$ therefore differs from the expression in Eq. (30) by a factor of $1/2$. The resulting expression for the cross section for elastic $D^{*0}\bar{D}^0$ scattering therefore differs by a factor of $1/4$ from the cross section in Eq. (32) for the charm meson model:

$$\sigma[D^{*0}\bar{D}^0(\vec{p}) \to D^{*0}\bar{D}^0] = \frac{\pi}{|\gamma_{\rm Re} + i(\gamma_{\rm Im} + p)|^2}. \tag{63}$$

The argument $(\vec{p})$ of $D^{*0}\bar{D}^0$ in the initial state implies that the D^{*0} and $\bar{D}^0$ have momenta $+\vec{p}$ and $-\vec{p}$, respectively. In the final state, the relative momentum of the D^{*0} and $\bar{D}^0$ have been integrated over. The cross section for elastic $D^0\bar{D}^{*0}$ scattering and the cross sections for $D^{*0}\bar{D}^0 \to D^0\bar{D}^{*0}$ and $D^0\bar{D}^{*0} \to D^{*0}\bar{D}^0$ are also given by the expression on the right side of Eq. (63).

If the local composite operator $D^i\bar{D}(x) + D\bar{D}^i(x)$ is used as the interpolating field for the X, the propagator for X is given in Eq. (18). The Feynman diagrams for the propagator $i\Delta_X^{ij}(E,0)$ in the charm meson model differ from those in Fig. 5 for the scalar meson model only in that each loop receives contributions from two pairs of particles, $D^{*0}\bar{D}^0$ and $D^0\bar{D}^{*0}$. Thus the diagonal entries of the X propagator can be obtained from the propagator in Eq. (33) by replacing $L_0(E)$ by $2L_0(E)$:

$$i\Delta_X^{ij}(E,0) = \frac{i2L_0(E)}{1-2\lambda_0 L_0(E)}\delta^{ij}. \tag{64}$$

The normalization factor defined by Eq. (19) is

$$Z_X = \frac{\pi\gamma}{M_{12}^2\lambda_0^2}. \tag{65}$$

This differs by a factor of 2 from the normalization factor Z_X in Eq. (34) for the scalar meson model.

The minimal charm meson model is an effective field theory that exploits the small ratio between the scale $|\gamma|$

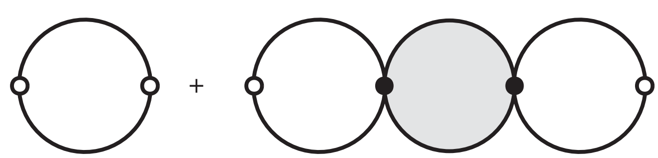

FIG. 5. Feynman diagrams for the X propagator at 0th order in g. The interpolating field for the X is $D_1D_2(x)$.

associated with a large scattering length and all the larger momentum scales of QCD. At very low energy, the most important of these larger momentum scales is the mass m_π of the pion. The minimal charm meson model takes into account all effects that are not suppressed by powers of $|\gamma|/m_\pi$. The model can be systematically improved so that it incorporates corrections to any desired order in $|\gamma|/m_\pi$. We will discuss only the improvements required to take into account corrections that are first order in $|\gamma|/m_\pi$.

At first order $|\gamma|/m_\pi$, it is necessary to take into account not only the large scattering length a_+ in the $C = +$ channel but also the effective range r_+. It is also necessary to take into account the scattering length a_- in the $C = -$ channel. These parameters can be defined by low-momentum expansions of the T-matrix elements analogous to Eq. (56). Since r_+ and a_- both have dimensions of length, the natural scales for these parameters are of order $1/m_\pi$. If we impose the renormalization condition that the $C = +$ channel amplitude has a pole in the energy at $E_{\rm pole}$ given in Eq. (10), the Green's functions in the two channels can be written as

$$\mathcal{A}_+(E) = \frac{-2\pi/M_{12}}{(\gamma + ip)[1 - r_+(\gamma - ip)/2]}, \tag{66a}$$

$$\mathcal{A}_-(E) = -(2\pi/M_{12})a_-, \tag{66b}$$

where $p = i\sqrt{-2M_{12}E_{12} - i\varepsilon}$ in Eq. (66a). The expression for the cross section for elastic $D^{*0}\bar{D}^0$ scattering that replaces Eq. (63) is

$$\sigma[D^{*0}\bar{D}^0(\vec{p}) \to D^{*0}\bar{D}^0] = \pi \left| \frac{1}{(\gamma + ip)[1 - r_+(\gamma - ip)/2]} + a_- \right|^2. \tag{67}$$

The cross section for elastic $D^0\bar{D}^{*0}$ scattering is given by the same expression. The cross sections for $D^{*0}\bar{D}^0 \to D^0\bar{D}^{*0}$ and $D^0\bar{D}^{*0} \to D^{*0}\bar{D}^0$ are given by the same expression except that a_- is replaced by $-a_-$.

B. Short-distance production of X, $D^{*0}\bar{D}^0$, and $D^0\bar{D}^{*0}$

The operator product expansion for the charm meson model can be used to separate the rates for short-distance production and decay processes into short-distance factors and long-distance factors. We first consider the short-distance production process $A \to B + X$ and the corresponding production processes for $D^{*0}\bar{D}^0$ and $D^0\bar{D}^{*0}$. A specific example of such a process is the discovery production process $B^+ \to K^+ + X$. The leading terms in the operator product expansions for these processes are those with the operators $D^{i\dagger}\bar{D}^\dagger(0)$ and $D^\dagger\bar{D}^{i\dagger}(0)$. The expressions for the T-matrix elements analogous to Eqs. (37) are

$$\mathcal{T}[A \to B + D^{*0}\bar{D}^0] = \sqrt{4M_1M_2}(\mathcal{C}_A^{B,i}\langle D^{*0}\bar{D}^0|D^{i\dagger}\bar{D}^\dagger(0)|\varnothing\rangle + \bar{\mathcal{C}}_A^{B,i}\langle D^{*0}\bar{D}^0|D^\dagger\bar{D}^{i\dagger}(0)|\varnothing\rangle), \tag{68a}$$

$$\mathcal{T}[A \to B + D^0\bar{D}^{*0}] = \sqrt{4M_1M_2}(\mathcal{C}_A^{B,i}\langle D^0\bar{D}^{*0}|D^{i\dagger}\bar{D}^\dagger(0)|\varnothing\rangle + \bar{\mathcal{C}}_A^{B,i}\langle D^0\bar{D}^{*0}|D^\dagger\bar{D}^{i\dagger}(0)|\varnothing\rangle), \tag{68b}$$

$$\mathcal{T}[A \to B + X] = \sqrt{2M_X}(\mathcal{C}_A^{B,i}\langle X|D^{i\dagger}\bar{D}^\dagger(0)|\varnothing\rangle + \bar{\mathcal{C}}_A^{B,i}\langle X|D^\dagger\bar{D}^{i\dagger}(0)|\varnothing\rangle). \tag{68c}$$

The matrix elements between the vacuum and the charm meson states are

$$\langle D^{*0}\bar{D}^0(\vec{p}, m)|D^{i\dagger}\bar{D}^\dagger(0)|\varnothing\rangle = \left(1 + \frac{\lambda_0 L_0(E_{\rm cm}(p))}{1 - 2\lambda_0 L_0(E_{\rm cm}(p))}\right)\varepsilon^{*i}(m), \tag{69a}$$

$$\langle D^{*0}\bar{D}^0(\vec{p}, m)|D^\dagger\bar{D}^{i\dagger}(0)|\varnothing\rangle = \frac{\lambda_0 L_0(E_{\rm cm}(p))}{1 - 2\lambda_0 L_0(E_{\rm cm}(p))}\varepsilon^{*i}(m), \tag{69b}$$

$$\langle D^0\bar{D}^{*0}(\vec{p}, m)|D^{i\dagger}\bar{D}^\dagger(0)|\varnothing\rangle = \frac{\lambda_0 L_0(E_{\rm cm}(p))}{1 - 2\lambda_0 L_0(E_{\rm cm}(p))}\varepsilon^{*i}(m), \tag{69c}$$

$$\langle D^0\bar{D}^{*0}(\vec{p}, m)|D^\dagger\bar{D}^{i\dagger}(0)|\varnothing\rangle = \left(1 + \frac{\lambda_0 L_0(E_{\rm cm}(p))}{1 - 2\lambda_0 L_0(E_{\rm cm}(p))}\right)\varepsilon^{*i}(m), \tag{69d}$$

where $\varepsilon^i(m)$ is the polarization vector for the spin-1 meson. The arguments $(\vec{p}, m)$ of the $D^{*0}\bar{D}^0$ and $D^0\bar{D}^{*0}$ states imply that the spin-1 and spin-0 mesons have momenta $+\vec{p}$ and $-\vec{p}$, respectively, and that the spin-1 meson has spin quantum number m. The matrix elements between the vacuum and the X are

$$\langle X(m)|D^{i\dagger}\bar{D}^\dagger(0)|\varnothing\rangle = \tfrac{1}{2}Z_X^{1/2}\varepsilon^{*i}(m), \tag{70a}$$

$$\langle X(m)|D^\dagger\bar{D}^{i\dagger}(0)|\varnothing\rangle = \tfrac{1}{2}Z_X^{1/2}\varepsilon^{*i}(m), \tag{70b}$$

where $\varepsilon^i(m)$ is the polarization vector for the X and the normalization constant Z_X is given in Eq. (65). The factors of $1/2$ in Eqs. (70) come from the fact that the Green's function for the operators $D^i\bar{D}(x) + D\bar{D}^i(x)$ and $D^{i\dagger}\bar{D}^\dagger(0)$ or $D^\dagger\bar{D}^{i\dagger}(0)$ are equal to the propagator for the X given in Eq. (64) multiplied by $1/2$. The T-matrix elements in Eqs. (69) can be expressed in a form in which short-distance and long-distance effects are separated:

$$\mathcal{T}[A \to B + D^{*0}\bar{D}^0(\vec{p}, m)] = \sqrt{4M_1M_2}\left[-\frac{\mathcal{C}_A^{B,i} + \bar{\mathcal{C}}_A^{B,i}}{2\lambda_0}\mathcal{A}(E_{\rm cm}(p)) + \frac{\mathcal{C}_A^{B,i} - \bar{\mathcal{C}}_A^{B,i}}{2}\right]\varepsilon^{*i}(m), \tag{71a}$$

$$\mathcal{T}[A \to B + D^0\bar{D}^{*0}(\vec{p}, m)] = \sqrt{4M_1M_2}\left[-\frac{\mathcal{C}_A^{B,i} + \bar{\mathcal{C}}_A^{B,i}}{2\lambda_0}\mathcal{A}(E_{\rm cm}(p)) - \frac{\mathcal{C}_A^{B,i} - \bar{\mathcal{C}}_A^{B,i}}{2}\right]\varepsilon^{*i}(m), \tag{71b}$$

$$\mathcal{T}[A \to B + X(m)] = \sqrt{2M_X}\left[\frac{\mathcal{C}_A^{B,i} + \bar{\mathcal{C}}_A^{B,i}}{2\lambda_0}\frac{(\pi\gamma)^{1/2}}{M_{12}}\right]\varepsilon^{*i}(m). \tag{71c}$$

These T-matrix elements do not depend on the ultraviolet cutoff if $(\mathcal{C}_A^{B,i} + \bar{\mathcal{C}}_A^{B,i})/\lambda_0$ and $\mathcal{C}_A^{B,i} - \bar{\mathcal{C}}_A^{B,i}$ do not depend on Λ. Equivalently, their dependence on Λ can be eliminated by renormalizations of the Wilson coefficients $\mathcal{C}_A^{B,i}$ and $\bar{\mathcal{C}}_A^{B,i}$.

Since the T-matrix element for $A \to B + X$ in Eq. (71c) is the product of a short-distance factor and a long-distance factor proportional to $\gamma^{1/2}$, the rate can be expressed as the product of a short-distance factor and a long-distance factor proportional to $|\gamma|$. If A consists of a single particle, the decay rate is

$$\Gamma[A \to B + X] = \Gamma_A^B \frac{2\pi}{M_{12}}|\gamma_{\rm Re}^2 + \gamma_{\rm Im}^2|^{1/2}. \tag{72}$$

We have chosen the long-distance factor to be the same as in Eq. (42).

The expressions for the invariant mass distributions for $D^{*0}\bar{D}^0$ and $D^0\bar{D}^{*0}$ that follow from the T-matrix elements in Eqs. (71a) and (71b) are much more complicated and depend on the types of particles in B. However, these expressions simplify if we keep only the leading terms in the expansions in $|\gamma|/m$. For p of order $|\gamma|$, the resonant amplitude $\mathcal{A}(E_{\rm cm}(p))$ in Eqs. (71a) and (71b) has a factor of order $1/|\gamma|$. The nonresonant terms in Eqs. (71a) and (71b) involve only short-distance factors and are insensitive to the scale $|\gamma|$. They are therefore suppressed relative to the resonant terms by $|\gamma|/m$. If we take p to be of order $|\gamma|$ and keep only the leading terms in $|\gamma|/m$, the invariant mass distributions reduce to

$$\frac{d\Gamma}{dM}[A \to B + D^{*0}\bar{D}^0(\vec{p})] = \Gamma_A^B \frac{p}{\gamma_{\rm Re}^2 + (\gamma_{\rm Im} + p)^2}, \tag{73a}$$

$$\frac{d\Gamma}{dM}[A \to B + D^0\bar{D}^{*0}(\vec{p})] = \Gamma_A^B \frac{p}{\gamma_{\rm Re}^2 + (\gamma_{\rm Im} + p)^2}. \tag{73b}$$

We have replaced $E_{\rm cm}(p)$ by M_{1+2} everywhere except in the long-distance factor. The short-distance factors Γ_A^B are the same as in Eq. (72). They cancel out of the ratio between Eq. (73a) or Eqs. (72) and (73b). This ratio differs from the ratio between Eq. (42) and Eq. (44) in the scalar meson model by the probability $\frac{1}{2}$ for the $D^{*0}\bar{D}^0$ and $D^0\bar{D}^{*0}$ to be in the $C = +$ channel.

The factorization formulas in Eqs. (72) and (73) were first derived in Refs. [17,20] for the case $\gamma_{\rm im} = 0$. They were applied to the decays of B mesons to $K + X$, $K + D^{*0}\bar{D}^0$, and $K + D^0\bar{D}^{*0}$. The factorization formulas were generalized to the case $\gamma_{\rm im} > 0$ in Ref. [21]. The short-distance coefficient Γ_A^B in Eq. (72) can be eliminated in favor of the decay rate $\Gamma[A \to B + X]$ using Eqs. (73):

$$\frac{d\Gamma}{dM}[A \to B + D^{*0}\bar{D}^0(\vec{p})] = \Gamma[A \to B + X]\frac{M_{12}p}{2\pi|\gamma||\gamma + ip|^2}. \tag{74}$$

The coefficient of $\Gamma[A \to B + X]$ agrees with Ref. [21]. The corresponding coefficient in Refs. [17,20] is larger by a factor of 2. The origin of this discrepancy is an error by a factor of $\sqrt{2}$ in the coalescence amplitude $\mathcal{A}[D^{*0}\bar{D}^0 \to X]$ in Refs. [17,20]. They used the universal prediction for this amplitude that was derived in Ref. [15]. The coalescence amplitude is determined by the residue of the pole in the energy for the amplitude for $D^{*0}\bar{D}^0 \to D^{*0}\bar{D}^0$:

$$\mathcal{A}[D^{*0}\bar{D}^0 \to D^{*0}\bar{D}^0] = \frac{-|\mathcal{A}[D^{*0}\bar{D}^0 \to X]|^2}{2M_X[E_{12} + 1/(2M_{12}a^2)]} \tag{75}$$
$$\text{as } E_{12} \to -1/(2M_{12}a^2).$$

The universal prediction for this amplitude was first derived in Ref. [15] and that result was used in Refs. [17,20]. The error in $\mathcal{A}[D^{*0}\bar{D}^0 \to X]$ in Ref. [15] came from an error in the amplitude $\mathcal{A}[D^{*0}\bar{D}^0 \to D^{*0}\bar{D}^0]$, which was larger by a factor of 2 than the correct expression in Eq. (62). The same error in the amplitude $\mathcal{A}[D^{*0}\bar{D}^0 \to D^{*0}\bar{D}^0]$ appears in Ref. [17].

We can exploit the fact that the minimal charm meson model is an effective field theory for a more fundamental Lorentz-invariant field theory, namely, the standard model. This implies that $\mathcal{C}_A^{B,i}\varepsilon^{*i}(m)$ and $\bar{\mathcal{C}}_A^{B,i}\varepsilon^{*i}(m)$ must have Lorentz-invariant expressions in terms of the 4-momenta and polarization 4-vectors of the particles in A and B, the 4-vector P_{1+2}^μ, and the polarization 4-vector ε^μ whose 3-vector part reduces to $\varepsilon^i(m)$ in the frame where $P_{1+2}^\mu = (M_{1+2}, \vec{0})$.

We proceed to deduce the constraints of Lorentz invariance on the discovery production process $B^+ \to K^+ + X$ and the corresponding production processes for $D^{*0}\bar{D}^0$ and $D^0\bar{D}^{*0}$. The Lorentz invariance of the fundamental theory is a particularly powerful constraint in this case. The only 4-vectors that the short-distance factors can depend on are the 4-momenta P_B^μ, P_K^μ, and P_{1+2}^μ, which satisfy $P_B^\mu = P_K^\mu + P_{1+2}^\mu$, and the polarization 4-vector ε^μ, which satisfies $P_{1+2} \cdot \varepsilon = 0$. The only independent Lorentz scalar that is linear in ε^* is $P_B \cdot \varepsilon^*$. Inner products of the 4-momenta can be expressed in terms of the masses M_B, M_K, and M_{1+2}. Thus Lorentz invariance implies that the T-matrix elements in (71) are determined by two complex constants C_+ and C_- defined by

$$\frac{\mathcal{C}_{B^+}^{K^+,i} + \bar{\mathcal{C}}_{B^+}^{K^+,i}}{2\lambda_0} \varepsilon^{*i}(m) = C_+ P_B \cdot \varepsilon^*(m), \tag{76a}$$

$$\frac{\mathcal{C}_{B^+}^{K^+,i} - \bar{\mathcal{C}}_{B^+}^{K^+,i}}{2} \varepsilon^{*i}(m) = C_- P_B \cdot \varepsilon^*(m). \tag{76b}$$

The T-matrix elements in Eqs. (71) reduce to

$$\mathcal{T}[B^+ \to K^+ + D^{*0}\bar{D}^0(\vec{p}, m)] = \sqrt{4M_1M_2}[-C_+ \mathcal{A}(E_{\rm cm}(p)) + C_-]P_B \cdot \varepsilon^*(m), \tag{77a}$$

$$\mathcal{T}[B^+ \to K^+ + D^0\bar{D}^{*0}(\vec{p}, m)] = \sqrt{4M_1M_2}[-C_+ \mathcal{A}(E_{\rm cm}(p)) - C_-]P_B \cdot \varepsilon^*(m), \tag{77b}$$

$$\mathcal{T}[B^+ \to K^+ + X(m)] = \sqrt{2M_X} C_+ \frac{(\pi\gamma)^{1/2}}{M_{12}} P_B \cdot \varepsilon^*(m). \tag{77c}$$

The decay rate for $B^+ \to K^+ + X$ can be expressed in the factored form in Eq. (72) with the short-distance factor

$$\Gamma_{B^+}^{K^+} = \frac{\lambda^{3/2}(M_B, M_K, M_{1+2})}{64\pi M_B^3 M_1 M_2} |C_+|^2. \tag{78}$$

The invariant mass distributions for the charm mesons in the decays of B^+ into $K^+ + D^{*0}\bar{D}^0$ and $K^+ + D^0\bar{D}^{*0}$ can be expressed in factored forms analogous to Eq. (44):

$$\frac{d\Gamma}{dM}[B^+ \to K^+ + D^{*0}\bar{D}^0(\vec{p})] = \Gamma_{B^+}^{K^+} p \left| \frac{1}{\gamma_{\rm Re} + i(\gamma_{\rm Im} + p)} + c_{B^+}^{K^+} \right|^2, \tag{79a}$$

$$\frac{d\Gamma}{dM}[B^+ \to K^+ + D^0\bar{D}^{*0}(\vec{p})] = \Gamma_{B^+}^{K^+} p \left| \frac{1}{\gamma_{\rm Re} + i(\gamma_{\rm Im} + p)} - c_{B^+}^{K^+} \right|^2, \tag{79b}$$

where $c_{B^+}^{K^+} = (M_{12}/2\pi)C_-/C_+$ is a complex constant. The nonresonant amplitude $c_{B^+}^{K^+}$ is required by the renormalization of the operator product expansion. It is completely determined by the short-distance coefficients C_- and C_+ which are insensitive to the small momentum scale $|\gamma|$. The smallest momentum scale to which they are sensitive is the pion mass m_π. Since $c_{B^+}^{K^+}$ has dimensions of length, the natural order of magnitude of $c_{B^+}^{K^+}$ is $1/m_\pi$. Thus, if p is of order $|\gamma|$, the nonresonant terms $\pm c_{B^+}^{K^+}$ in Eqs. (79) are suppressed by a factor of $|\gamma|/m_\pi$ compared to the resonant terms. If we keep only the leading terms in the expansion in $|\gamma|/m$, we can set $c_{B^+}^{K^+} = 0$. Thus, Eqs. (79) reduce to Eqs. (73). In a systematic expansion in powers of $|\gamma|/m_\pi$, the nonresonant amplitudes $\pm c_{B^+}^{K^+}$ in Eqs. (79) would be retained only if all other effects of the same order in $|\gamma|/m_\pi$ were also included. The effective field theory should be improved to take into account the effective range r_+ in the $C = +$ channel and the scattering length a_- in the $C = -$ channel. One should also include terms in the operator product expansion with the operators $\nabla^i D^j \bar{D} - D^j \nabla^i \bar{D}$ and $\nabla^i D \bar{D}^j - D \nabla^i \bar{D}^j$. These terms will have operator matrix elements with factors of $p^i \epsilon^j$ and Wilson coefficients suppressed by $1/m_\pi$.

We now apply the factorization formulas to the production process $B^+ \to K^{*+} + X$ and the corresponding production processes for $D^{*0}\bar{D}^0$ and $D^0\bar{D}^{*0}$. The only 4-vectors that the short-distance factors can depend on are the 4-momenta P_B^μ, $P_{K^*}^\mu$, and P_{1+2}^μ and the polarization 4-vector $\varepsilon_{K^*}^\mu$ of the K^*. Inner products of the 4-momenta can be expressed in terms of the masses M_B, M_{K^*}, and M_{1+2}. The independent Lorentz scalars that are linear in $\varepsilon_K^{*\mu} \varepsilon^\nu$ are $(P_B \cdot \varepsilon_K^*)(P_B \cdot \varepsilon^*)$, $M_B^2(\varepsilon_{K^*}^* \cdot \varepsilon^*)$, and $\epsilon_{\mu\nu\alpha\beta} P_B^\alpha P_{1+2}^\beta \varepsilon_K^{*\mu} \varepsilon^\nu$. Thus the constraint of Lorentz invariance reduces the T-matrix elements in Eq. (71) to six complex constants $D_\pm$, $E_\pm$, and $F_\pm$ defined by

$$\frac{\mathcal{C}_{B^+}^{K^{*+},i} + \bar{\mathcal{C}}_{B^+}^{K^{*+},i}}{2\lambda_0} \varepsilon^{*i}(m) = (D_+ M_B^2 g_{\mu\nu} + E_+ P_{B\mu} P_{B\nu} + F_+ \epsilon_{\mu\nu\alpha\beta} P_B^\alpha P_{1+2}^\beta) \varepsilon_K^{*\mu} \varepsilon^{*\nu}(m), \tag{80a}$$

$$\frac{\mathcal{C}_{B^+}^{K^{*+},i} - \bar{\mathcal{C}}_{B^+}^{K^{*+},i}}{2} \varepsilon^{*i}(m) = (D_- M_B^2 g_{\mu\nu} + E_- P_{B\mu} P_{B\nu} + F_- \epsilon_{\mu\nu\alpha\beta} P_B^\alpha P_{1+2}^\beta) \varepsilon_K^{*\mu} \varepsilon^{*\nu}(m). \tag{80b}$$

The decay rate for $B^+ \to K^{*+} + X$ can be expressed in the factored form in Eq. (72) with the short-distance factor $\Gamma_{B^+}^{K^{*+}}$. The invariant mass distributions for the charm mesons in the decays of B^+ into $K^{*+} + D^{*0}\bar{D}^0$ and $K^{*+} + D^0\bar{D}^{*0}$ can be expressed in factored forms analogous to Eqs. (79) but considerably more complicated. If we keep only the leading terms in the expansions in $|\gamma|/m$, these expressions reduce to Eqs. (73).

C. Short-distance decay of X

We now consider the decay of X into a short-distance decay mode C. Examples of such decay modes are the discovery mode $J/\psi\pi^+\pi^-$ and $J/\psi\gamma$. The expression for the T-matrix element analogous to Eq. (45) is

$$\mathcal{T}[X \to C] = \sqrt{2M_X}(\mathcal{C}^{C,i}\langle\emptyset|D^i\bar{D}(0)|X\rangle + \bar{\mathcal{C}}^{C,i}\langle\emptyset|D\bar{D}^i(0)|X\rangle). \tag{81}$$

The operator matrix elements are given by expressions analogous to those on the right sides of Eqs. (70) except that $\varepsilon^{*i}(m)$ is replaced by $\varepsilon^i(m)$. The factored expression for the T-matrix element is

$$\mathcal{T}[X(m) \to C] = \sqrt{2M_X}\frac{\mathcal{C}^{C,i} + \bar{\mathcal{C}}^{C,i}}{2\lambda_0}\frac{(\pi\gamma)^{1/2}}{M_{12}}\varepsilon^i(m). \tag{82}$$

The T-matrix element does not depend on the ultraviolet cutoff if $(\mathcal{C}^{C,i} + \bar{\mathcal{C}}^{C,i})/\lambda_0$ does not depend on Λ. If we were to consider the process $D^{*0}\bar{D}^0 \to C$, we would find that $\mathcal{C}^{C,i} - \bar{\mathcal{C}}^{C,i}$ must also be independent of Λ. Thus the ultraviolet divergences can be removed by renormalizations of the Wilson coefficients $\mathcal{C}^{C,i}$ and $\bar{\mathcal{C}}^{C,i}$. The decay rate for $X \to C$ can be expressed in a factored form analogous to Eq. (48):

$$\Gamma[X \to C] = \Gamma^C\frac{2\pi}{M_{12}}|\gamma_{\rm Re}^2 + \gamma_{\rm Im}^2|^{1/2}, \tag{83}$$

where Γ^C is a short-distance factor with dimension of mass. We have chosen the long-distance factor to be the same as in Eq. (48). The factorization formula in Eq. (83) was first derived in Ref. [21].

In Ref. [23], the decay rates of $X(3872)$ into $J/\psi\pi^+\pi^-$, $J/\psi\pi^+\pi^-\pi^0$, $J/\psi\gamma$, and $J/\psi\pi^0\gamma$ were calculated under the assumption that the decays are dominated by a direct coupling of X to J/ψ and the vector mesons ρ and ω followed by the decay of the virtual vector mesons into pions and photons. The decay rates were calculated in terms of coupling constants $G_{X\psi\rho}$ and $G_{X\psi\omega}$ and other parameters that were determined by vector-meson decays. The long-distance scale $|\gamma|$ enters the decay rates only through a factor of $\gamma^{1/2}$ in the coupling constants $G_{X\psi\rho}$ and $G_{X\psi\omega}$. Thus the decay rates in Ref. [23] satisfy the factorization formula in Eq. (83).

The Lorentz invariance of the more fundamental theory provides constraints on the T-matrix elements for specific short-distance decay processes. For the decay $X \to J/\psi\gamma$, the only 4-vectors the short-distance factors can depend on are the 4-momenta P_{1+2}^μ and P_ψ^μ or P_γ^μ and the polarization 4-vectors ε_ψ^μ and ε_γ^μ of the J/ψ and the photon. The coefficient of $\varepsilon_\psi^{*\mu}\varepsilon_\gamma^{*\nu}\varepsilon^\sigma$ must be a 3-index Lorentz tensor. There are six independent tensors that can be constructed from the 4-momenta, the metric, and the Levi-Civita tensor. Thus Lorentz invariance constrains the T-matrix element to be a linear combination of these six terms with constant coefficients. In the model of Ref. [23], the assumptions of the direct coupling of X to $J/\psi\rho$ and $J/\psi\omega$ and the vector-meson dominance of the coupling of the photon to hadrons were used to reduce the T-matrix to a single term proportional to $\epsilon_{\alpha\mu\nu\rho}P_\gamma^\alpha\epsilon_\psi^{*\mu}\epsilon_\gamma^{*\nu}\epsilon^\sigma$. For the process $X \to J/\psi\pi^+\pi^-$, the only 4-vectors the short-distance factors can depend on are the 4-momenta P_{1+2}^μ, P_ψ^μ, and $Q_\pi^\mu = P_{\pi^+}^\mu - P_{\pi^-}^\mu$ and the polarization 4-vector ε_ψ^μ of the J/ψ. The coefficient of $\varepsilon_\psi^{*\mu}\varepsilon^\nu$ must be a 2-index Lorentz tensor. There are eight independent tensors that can be constructed from the 4-momenta, the metric, and the Levi-Civita tensor. Thus Lorentz invariance constrains the T-matrix element to be a linear combination of these eight terms with coefficients that are functions of the two independent Lorentz scalars $Q_\pi \cdot P_\psi$ and $P_{1+2} \cdot P_\psi$. In the model of Ref. [23], the assumption of a direct coupling of the X to $J/\psi\rho$ was used to reduce the T-matrix element to a single term proportional to $\epsilon_{\alpha\beta\mu\nu}(P_{1+2} - P_\psi)^\alpha Q_\pi^\beta\varepsilon_\psi^{*\mu}\varepsilon^\nu$.

D. Line shape of X in a short-distance decay mode

We now consider the line shape of X in the process $A \to B + C$, where C is a short-distance decay mode of X. An example of such a process is the discovery process for the X: $B^+ \to K^+ + J/\psi\pi^+\pi^-$. The expression for the T-matrix element analogous to Eq. (50) is

$$\mathcal{T}[A \to B + C] = \mathcal{C}_A^{B,C} + (\mathcal{C}_A^{B,i}\mathcal{C}^{C,i} + \bar{\mathcal{C}}_A^{B,i}\bar{\mathcal{C}}^{C,i})iL_0(M_C) + (\mathcal{C}_A^{B,i} + \bar{\mathcal{C}}_A^{B,i})(\mathcal{C}^{C,i} + \bar{\mathcal{C}}^{C,i})\frac{i\lambda_0 L_0(M_C)^2}{1 - 2\lambda_0 L_0(M_C)}. \tag{84}$$

The factorized expression for the T-matrix element analogous to Eq. (51) is

$$\mathcal{T}[A \to B + C] = -i\frac{\mathcal{C}_A^{B,i} + \bar{\mathcal{C}}_A^{B,i}}{2\lambda_0}\frac{\mathcal{C}^{C,i} + \bar{\mathcal{C}}^{C,i}}{2\lambda_0}\mathcal{A}(M_C) + (\mathcal{C}_A^{B,C} + \Delta\mathcal{C}_A^{B,C}) + \frac{i}{2}(\mathcal{C}_A^{B,i} - \bar{\mathcal{C}}_A^{B,i})(\mathcal{C}^{C,i} - \bar{\mathcal{C}}^{C,i})[L_0(M_C) - L_0(M_{1+2})], \tag{85}$$

where $\Delta\mathcal{C}_A^{B,C}$ is

$$\Delta\mathcal{C}_A^{B,C} = -i\lambda_0 \frac{\mathcal{C}_A^{B,i} + \bar{\mathcal{C}}_A^{B,i}}{2\lambda_0} \frac{\mathcal{C}^{C,i} + \bar{\mathcal{C}}^{C,i}}{2\lambda_0} + \frac{i}{2}(\mathcal{C}_A^{B,i} - \bar{\mathcal{C}}_A^{B,i})(\mathcal{C}^{C,i} - \bar{\mathcal{C}}^{C,i})L_0(M_{1+2}). \tag{86}$$

After renormalization of the coupling constant, $\mathcal{A}(M_C)$ is given by the expression in (62), which does not depend on the ultraviolet cutoff Λ. The term $L_0(M_C) - L_0(M_{1+2})$, which is given in Eq. (25), also does not depend on Λ. The combinations $(\mathcal{C}_A^{B,i} + \bar{\mathcal{C}}_A^{B,i})\lambda_0$, $(\mathcal{C}^{C,i} + \bar{\mathcal{C}}^{C,i})/\lambda_0$, $\mathcal{C}_A^{B,i} - \bar{\mathcal{C}}_A^{B,i}$, and $\mathcal{C}^{C,i} - \bar{\mathcal{C}}^{C,i}$ cannot depend on Λ, because they appear as short-distance factors in other T-matrix elements such as those in Eqs. (71) and (82). Thus the T-matrix element in Eq. (85) will not depend on Λ if $\mathcal{C}_A^{B,C} + \Delta\mathcal{C}_A^{B,C}$ does not depend on Λ. Equivalently, the dependence on Λ can be removed by an additive renormalization of the Wilson coefficient $\mathcal{C}_A^{B,C}$.

The expression for the rate that follows from the T-matrix element in Eq. (85) is very complicated and depends on the types of particles in B. The expression simplifies if we keep only the leading term in the expansion in $|\gamma|/m_\pi$. If $|p_C|$ is of order $|\gamma|$, the resonant amplitude $\mathcal{A}(M_C)$ has a factor of order $1/|\gamma|$ while the term $L_0(M_C) - L_0(M_{1+2})$ has a factor of order $|\gamma|$. Additional factors of $|\gamma|$ must be accompanied by additional factors of m_π in the short-distance factors. Thus the second and third terms on the right side of Eq. (85) are suppressed by one and two powers of $|\gamma|/m_\pi$, respectively. If we keep only the leading term in $|\gamma|/m_\pi$, the invariant mass distribution for C in the decay of a single particle A into $B + C$ can be expressed in the simple factored form

$$\frac{d\Gamma}{dM_C}[A \to B + C] = \Gamma_A^{B,C} \frac{2\pi}{(\gamma_{\rm Re} - |p_C|)^2 + \gamma_{\rm Im}^2}, \qquad M_C < M_{1+2}, \tag{87a}$$

$$= \Gamma_A^{B,C} \frac{2\pi}{\gamma_{\rm Re}^2 + (\gamma_{\rm Im} + p_C)^2}, \qquad M_C > M_{1+2}. \tag{87b}$$

The invariant mass distribution is continuous at $M_C = M_{1+2} = M_{D^0} + M_{D^{*0}}$. The factorization formula in Eq. (87) was first derived in Ref. [21]. In contrast to the corresponding factorization formula in the scalar meson model which is given in Eq. (59), the short-distance factor $\Gamma_A^{B,C}$ is not simply the product of the short-distance factors Γ_A^B and Γ^C in Eqs. (73) and (83), respectively. The reason for this is that the short-distance factors associated with the initial and final states in the T-matrix element in (85) are connected by the vector index i.

VIII. SUMMARY

The $X(3872)$ seems to be a hadronic molecule consisting of a $C = +$ superposition of $D^{*0}\bar{D}^0$ and $D^0\bar{D}^{*0}$ that are weakly bound in the S-wave channel. The binding energy and the width of the X can be conveniently expressed in terms of the complex binding momentum γ defined in Eq. (10). The smallness of γ compared to the natural scale m_π together with the S-wave nature of the bound state imply that the X has universal properties that are completely determined by γ. The separation of scales between $|\gamma|$ and m_π can be exploited through factorization formulas for the production and short-distance decay rates of X. The factorization formulas express these rates as the sum of products of short-distance factors that are insensitive to γ and long-distance factors that are completely determined by γ.

We have shown how the factorization formulas can be derived using the operator product expansion for a low-energy effective field theory for the charm mesons, such as the minimal charm meson model. Using the operator product expansion, the rates are expressed as sums of products of Wilson coefficients and matrix elements of operators in the effective field theory. In the minimal charm meson model, the matrix elements can be calculated nonperturbatively and they depend on the ultraviolet cutoff Λ. Some of the dependence on Λ can be removed by the renormalization of the coupling constant. This can be accomplished conveniently by eliminating the bare coupling constant in favor of γ. The remaining dependence on Λ can be removed by renormalization of the Wilson coefficients in the operator product expansion. After eliminating all dependence on Λ, the rate can be expanded in powers of $|\gamma|/m_\pi$. The leading terms in the expansions are very simple. The leading terms in the rates for the production processes $A \to B + X$, $A \to B + D^{*0}\bar{D}^0$, and $A \to B + D^0\bar{D}^{*0}$ are given in Eqs. (72) and (73). The leading term in the rate for the short-distance decay process $X \to C$ is given in Eq. (83). The leading term for the line shape of X in the short-distance decay mode C is given in Eq. (87).

Our derivation of the factorization formulas using the operator product expansion makes it clear how these leading-order results can be extended systematically to higher orders in $|\gamma|/m_\pi$. This requires improving the effective field theory and including higher dimension operators in the operator product expansion. If accuracy to nth order in $|\gamma|/m_\pi$ is desired, the effective field theory must describe the scattering of charm mesons to that accuracy and the operator product expansion must include operators with up to n gradients. As the order in $|\gamma|/m_\pi$ increases, there are an increasing number of parameters in the effective field theory and an increasing number of Wilson coefficients in the relevant terms of the operator product expansion. The difficulty of determining all these parameters phenomenologically may limit the utility of the expansion to low orders in γ/m_π.

Our derivation of the simple factorization formulas in Eqs. (72), (73), (83), and (87) is conceptually cleaner than the previous derivations in Refs. [17,20,21]. Those previous derivations were awkward in that they required taking the limit $\Lambda \to \infty$ while also exploiting the fact that the natural scale of the ultraviolet cutoff is m_π. In the present derivation, all dependence on Λ is removed analytically through renormalization of the coupling constants and through renormalization of the Wilson coefficients in the operator product expansion without taking the limit $\Lambda \to \infty$. In a subsequent conceptually independent step, the rates are expanded in powers of $|\gamma|/m_\pi$. The leading terms in this expansion give the simple factorization formulas.

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