Gauge-gravity duality and thermalization of a boost-invariant perfect fluid

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We derive the equation for the quasinormal modes corresponding to the scalar excitation of a black hole moving away in the fifth dimension. This geometry is the AdS/CFT dual of a boost-invariant expanding perfect fluid in $\mathcal{N} = 4$ SUSY Yang-Mills theory at large proper-time. On the gauge-theory side, the dominant solution of the equation describes the decay back to equilibrium of a scalar excitation of the perfect fluid. Its characteristic proper-time can be interpreted as a thermalization time of the perfect fluid, which is a universal (and numerically small) constant in units of the unique scale of the problem. This may provide a new insight on the short thermalization-time puzzle encountered in heavy-ion collision phenomenology. A nontrivial scaling behavior in proper-time is obtained which can be interpreted in terms of a slowly varying adiabatic approximation.

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I. THERMALIZATION RESPONSE-TIME AND BLACK-HOLE QUASINORMAL MODES

In a recent paper [1], we have shown that the AdS/CFT dual of an expanding relativistic thermal and perfect fluid in $\mathcal{N} = 4$ SUSY Yang-Mills (SYM) theory can be identified as a black hole (BH) moving away in the fifth dimension [2]. In the above paper, a holographic renormalization procedure [3] using Fefferman-Graham coordinates [4] allowed to construct the gravity duals of a continuous 1-parametric set of 4-d stress-energy tensors of the gauge theory containing among others the duals of the *free streaming* and *perfect-fluid* cases. Interestingly, the corresponding family of geometries was shown to possess singularities except only for the latter case, which happens to be a BH moving away in the fifth dimension. This correspondence was shown to be valid at asymptotic proper-times, independently of initial conditions provided boost-invariance is preserved. This gives an AdS/CFT physical criterion for the emergence of perfect-fluid behavior at large proper times.

Starting from this correspondence, it is interesting to study the stability properties of this BH system, since it can bring a physical insight on the typical relaxation proper-times of the relativistic thermal and perfect fluid, which does not appear reachable from a direct strong coupling computation in the $\mathcal{N} = 4$ SYM field theory. As was done in the static BH case [5] in order to calculate the decay time of an excitation of the system, one computes the quasinormal modes (QNM) of the Einstein equations linearized around the background geometry. In particular the calculation of the QNM's corresponding to a scalar excitation canonically coupled to the metric in the gravitational dual configuration [5–8] may give an evaluation of the thermalization time of the dual gauge field-theoretic strongly-coupled system after a small deviation from equilibrium.

In the present paper, our aim is to extend the analysis from the static case to the black hole moving off in the fifth dimension, deriving the equation and making the evaluation of the corresponding QNM's in order to estimate the thermalization decay proper-time of the relativistically expanding perfect fluid in the $\mathcal{N} = 4$ SYM field theory.

The plan of our study is the following. In Sec. II, we shall (re)derive the QNM equation for the static BH using now the Fefferman-Graham coordinates. In Sec. III we derive the QNM equation for the gravitational dual of the expanding perfect fluid and give its solutions both in the minimally coupled scalar case and for transverse tensor perturbations. In Sec. IV, we discuss the features of our results and their possible physical relevance for the thermalization puzzle of the QCD quark-gluon plasma in heavy-ion collisions. We conclude and give an outlook.

II. QUASINORMAL MODES OF A STATIC BLACK HOLE IN FEFFERMAN-GRAHAM COORDINATES

Quasinormal modes (QNM's) formally define the response of a black-hole state to small perturbations, for instance due to a scalar-field excitation canonically coupled to the metric, with incoming (absorbing) boundary condition at the BH horizon. The main observation is that the resulting frequencies become complex and hence the perturbations are expected to die out exponentially [9].

Quasinormal frequencies have been calculated for numerous examples of static black holes in various numbers of dimensions, in particular, for the static planar AdS black hole which is the dual geometry to $\mathcal{N} = 4$ SYM theory at nonzero temperature. In this section we shall rederive the known results using the Fefferman-Graham coordinates

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which are suitable for the extension to the relevant nonstatic case. As already mentioned [1], the expanding geometry has a simple form when directly written in these coordinates.

In Fefferman-Graham coordinates the metric of a 5-d planar BH has the form:

$$ds^2 = \frac{1}{z^2}\left[-\frac{(1-\frac{z^4}{z_0^4})^2}{(1+\frac{z^4}{z_0^4})}dt^2 + \left(1+\frac{z^4}{z_0^4}\right)d\vec{x}^2 + dz^2\right], \quad (1)$$

where $(t, \vec{x})$ are the boundary coordinates, z is the fifth one, and z_0 is the location of the horizon in the bulk.

Quasinormal modes for a scalar perturbation of a black hole are obtained by solving the wave equation for a massless scalar field

$$\Delta\phi \equiv \frac{1}{\sqrt{-g}}\partial_i(\sqrt{-g}g^{ij}\partial_j\phi) = 0 \quad (2)$$

where g^{ij} is the metric tensor and g its determinant in the background geometry (1), assuming purely incoming boundary conditions at the horizon $z = z_0$ and Dirichlet conditions at the boundary (see e.g. [8]).

Inserting Eq. (1) into (2), the equation for a scalar field (with zero transverse momentum) has then the form:

$$-\frac{1}{z^3}\frac{(1-\frac{z^4}{z_0^4})^2}{(1+\frac{z^4}{z_0^4})}\partial_t^2\phi(t,z) + \partial_z\left(\frac{1}{z^3}\left(1-\frac{z^8}{z_0^8}\right)\partial_z\phi(t,z)\right) = 0 \quad (3)$$

A separation of variables

$$\phi(t,z) = e^{i\omega t}\phi(z) \quad (4)$$

leads to the ordinary differential equation

$$\partial_z\left(\frac{1}{z^3}\left(1-\frac{z^8}{z_0^8}\right)\partial_z\phi(z)\right) + \omega^2\frac{1}{z^3}\frac{(1-\frac{z^4}{z_0^4})^2}{(1+\frac{z^4}{z_0^4})}\phi(z) = 0. \quad (5)$$

Note that, by a change of variable

$$z/z_0 = \tanh^{1/4}(4z^*), \quad (6)$$

the equation takes the canonical form

$$\partial_{z^*}^2\phi(z^*) + \frac{\sqrt{8}e^{12z^*}}{\sinh^{3/2}(8z^*)}\omega^2\phi(z^*) = 0 \quad (7)$$

from which one can determine the quasinormal frequencies ω.

In fact, by a further change of variable

$$\tilde{z} = \frac{(1-(z/z_0)^2)^2}{1+(z/z_0)^4} = 1 - 2\frac{\tanh^{1/2}(4z^*)}{1+\tanh(4z^*)}, \quad (8)$$

one shows that this equation can be put in the form of the Heun equation

$$\phi'' + \frac{1-\tilde{z}^2}{\tilde{z}(1-\tilde{z})(2-\tilde{z})}\phi' + \frac{(\omega_{\text{static}}/\pi T)^2}{4\tilde{z}(1-\tilde{z})(2-\tilde{z})}\phi = 0 \quad (9)$$

which was obtained [6] starting with the conventional BH metric. The QNM solution of (9) dominant at large-time (i.e. with smallest imaginary part) is found to be $\omega_{\text{static}}/\pi T \sim 3.1194 - 2.74667i$ [6].

III. QUASINORMAL MODES FOR THE BOOST-INVARIANT BLACK-HOLE GEOMETRY

In this section we derive the equation for a scalar field in the background corresponding to the asymptotic expanding perfect-fluid geometry. The geometry has the form [1,10]:

$$ds^2 = \frac{1}{z^2}\left[-\frac{(1-v^4)^2}{(1+v^4)}d\tau^2 + (1+v^4)(\tau^2dy^2 + dx_i^2) + dz^2\right] \quad (10)$$

where $x_{i=1,2}$ are the transverse coordinates, while the proper-time τ and rapidity y are related to the longitudinal coordinates through $t = \tau\cosh y$ and $x_3 = \tau\sinh y$. v is a scaling variable

$$v = \frac{z}{\tau^{1/3}}. \quad (11)$$

To this background metric let us couple a scalar field which depends only on the proper-time τ, and on z. Equation (2) takes the form:

$$-\partial_\tau\left(\frac{\tau}{z^3}\frac{(1-v^4)^2}{1+v^4}\partial_\tau\phi\right) + \partial_z\left(\frac{\tau}{z^3}(1-v^8)\partial_z\phi\right) = 0. \quad (12)$$

Since the metric (10) has a simple dependence on the variable v let us rewrite the above equation in terms of τ and v. The matrix of differentials is

$$\partial_z \to \tau^{-(1/3)}\partial_v; \qquad \partial_\tau \to \partial_\tau - \frac{1}{3}\tau^{-(4/3)}\partial_v. \quad (13)$$

Since the perfect-fluid geometry is valid in our problem only at large proper-time, in the scaling limit $v = \text{const}$ and $\tau \to \infty$ we may consistently neglect the nondiagonal contribution in (13). Thus, the QNM calculation preserves the specific scaling property in $\tau/z^3 = 1/v^3$ of the perfect-fluid solution of the AdS/CFT correspondence. The resulting equation takes the form:

$$-\frac{1}{v^3}\frac{(1-v^4)^2}{1+v^4}\partial_\tau^2\phi(\tau,v) + \tau^{-(2/3)}\partial_v\left(\frac{1}{v^3}(1-v^8)\partial_v\phi(\tau,v)\right) = 0 \quad (14)$$

which has some similarity with the one for the static black hole, once substituting variables $(t,z) \to (\tau,v)$, see (3). The noticeable difference is the additional $\tau^{-2/3}$ factor, which leads to a nontrivial scaling in the (proper)time dependence.

Performing a separation of variables $\phi(\tau, v) = f(\tau)\phi(v)$ we get two decoupled equations. An important point is however to notice that the equation for the τ-dependence is no longer a plane wave as in (4) but is determined by the equation

$$\partial_\tau^2 f(\tau) = -\omega^2 \tau^{-(2/3)} f(\tau), \tag{15}$$

whose solutions are linear combinations of the Bessel functions

$$\sqrt{\tau} J_{\pm(3/4)}\left(\frac{3}{2}\omega\tau^{2/3}\right). \tag{16}$$

In the large τ behavior, the region of validity of the perfect-fluid geometry being asymptotic, the relevant behavior of the Bessel functions is

$$f(\tau) \sim \tau^{1/6} e^{(3/2)i\omega\tau^{2/3}}, \tag{17}$$

to be compared with the plane waves (4) of the static case. We will comment on the physical interpretation of (17) in the next section.

The resulting τ-independent factorized equation reads

$$\partial_v\left(\frac{1}{v^3}(1 - v^8)\partial_v \phi(v)\right) + \omega^2 \frac{1}{v^3} \frac{(1 - v^4)^2}{1 + v^4} \phi(v) = 0 \tag{18}$$

which, strikingly enough, is formally the same ordinary differential equation for the τ-independent part as Eq. (5) for the static black hole but with v playing the role of z. Hence we get the same profile functions ϕ of the quasinormal modes and their eigenfrequencies ω. However the variables in the perfect-fluid case v and τ are *different* from the z and t variables relevant for the static black hole. Indeed, restoring the z and τ dependence of the evolving solution $\phi(v)f(\tau)$ gives rise to a rather intricate spacetime dependence.

It is also interesting to derive the quasinormal modes for perturbations of the metric which on the gauge-theory side correspond to perturbations of the energy-momentum tensor. Of particular interest are perturbations of the component $T_{x_1x_2}$. Physically their exponential decay can be interpreted as transverse isotropization of the asymptotic hydrodynamic expansion of the plasma. On the gravity side they are interesting since, as shown in [11] for general static cases, their equation of motion is identical to the scalar one. In particular let us consider the metric component $g_{x_1x_2}$ and form the quantity

$$g_{x_2}^{x_1} \equiv z^2 e^{-c(v,z)} g_{x_1x_2}; \qquad e^{c(v,z)} \equiv 1 + v^4 \tag{19}$$

where $c(v, z)$ is the solution [1] found for the transverse component of the boost-invariant perfect-fluid metric (see Eq. (10)).

We have shown that in the case of the moving black hole, in the large τ limit, the quantity $g_{x_2}^{x_1}$ also satisfies the scalar equation of motion.

$$\partial_v\left(\frac{1}{v^3}(1 - v^8)\partial_v g_{x_2}^{x_1}(v)\right) + \omega^2 \frac{1}{v^3} \frac{(1 - v^4)^2}{1 + v^4} g_{x_2}^{x_1}(v) = 0 \tag{20}$$

where the proper-time dependence has been separated out in the same manner as for the scalar. Consequently this mode of the metric has the same set of quasinormal frequencies.

IV. CONCLUSION, DISCUSSION AND OUTLOOK

Let us summarize our results. We consider the problem of computing the response proper-time of an expanding relativistic and boost-invariant $\mathcal{N} = 4$ SYM perfect fluid, after a scalar excitation off thermodynamic equilibrium. In the dual gravitational geometry, which corresponds to a black-hole moving away in the fifth dimension, the characteristic proper-time is related to the frequency ω_c with the smallest imaginary part among quasinormal modes.

We derived the corresponding equations in the scalar and the transverse tensor channel. The resulting equations factorize into two decoupled equations defining, respectively, the explicit proper-time and scaling variable dependence of the modes. As in the static case, we find that the quasinormal frequencies have imaginary parts which correspond to exponential decay of perturbations towards the (expanding) equilibrium state. While formally the equation defining the frequencies has the same functional form as in the static case, they correspond to quite different variables, namely, the proper time τ and the scaling variable $v = z/\tau^{1/3}$.

The profile of the solution is given as a function of the scaling variable v, $\phi(v)$, which obeys the same equations as $\phi(z)$ in the static case. By contrast with the plane waves of the static case however, the frequencies are related to Bessel functions $\sqrt{\tau} J_{\pm(3/4)}(\frac{3}{2}\omega\tau^{2/3}) \rightarrow \tau^{1/6} e^{(3/2)i\omega\tau^{2/3}}$, hence leading to a nontrivial scaling in proper-time.

These results can be interpreted as follows. The temperature for the expanding perfect fluid is known to behave like $T \sim \tau^{-1/3}$ [12], which is consistent through the AdS/CFT correspondence with the evolving black-hole solution of [1]. In order to understand the nontrivial scaling in proper time (17), we note that the *static* quasinormal freqencies are proportional to the temperature i.e. $\omega_{\text{static}}(T) = \alpha \cdot T$, where α is a constant. In the expanding case we can consider an adiabatic approximation assuming that locally the temperature is fixed. Hence a plane-wave dependence

$$e^{i\omega_{\text{static}}(T)\cdot\tau} = e^{i\alpha T\tau} \sim e^{i\alpha\tau^{2/3}} \tag{21}$$

will give the scaling in $\tau^{2/3}$ seen in the overall solution. This adiabatic approximation deserves further study.

A similar discussion is valid for transverse perturbations of the metric. It would be interesting to extend the analysis to generic metric perturbations.

On a theoretical ground, a comment is in order about the well-known calculation of the viscosity to entropy ratio [13] in the *static* BH configuration. Quasinormal modes and viscosity calculation correspond both to poles of specific retarded Green functions for a scalar (for QNM's) and metric (for viscosity) deformation using Fourier transforms in the time variable t. This is the case for the static case, due to the plane-wave solution (4).

The difference between the scalar QNM's and viscosity (shear channel) is that the corresponding poles do not go to zero at small transverse momentum contrary to the viscosity case [8]. This is expected, since viscosity is a hydrodynamic excitation surviving at large-time scales, while normal QNM's lead to an exponential fall-off, at least within an AdS space. This is the origin of the finite ''thermalization response-time'' obtained for the scalar equation.

Now turning to the expanding geometry in proper-time, viscosity remains an interesting issue for future work, since it is not clear whether one can use the method of Fourier-transformed retarded Green functions to evaluate it, since the Fourier transforms seem not to be directly relevant for the proper-time dependence. An adiabatic approximation could help with this problem. We postpone this analysis for future work.

In a phenomenological perspective, it is instructive to make a parallel between our calculation in the framework of the $\mathcal{N} = 4$ SYM fluid, and the problems discussed about the thermalization proper-time of a QCD plasma formed in a heavy-ion collision at high energy (see [14] and references therein). Indeed, there are some indication that this typical proper-time is rather short, which is difficult to explain in terms of initial conditions dominated by a weak-coupling state, and thus with high viscosity and *a-priori* long thermalization time.

In [1], we have shown that the gauge/gravity correspondence was selecting the perfect-fluid solution at large proper-times. Since it seems that the model of a QCD perfect fluid could be favored by the phenomenological analyses, it is perhaps not completely unrealistic to look for some physical lessons of our present results. The main point is the rather strong stability of the asymptotic solution, since the response-time to a scalar excitation is short in terms of proper time. Combining Eq. (17) with the numerical value of the dominant QNM leads to a proper-time damping of the form

$$\exp\left(-\frac{3}{2}\cdot 2.7466\cdot \tau^{2/3}\right) \tag{22}$$

in the units where the horizon is fixed at $z_0 = \tau^{-1/3}$.

If these estimates would be also approximately valid for QCD, at least in the hypothesis of a deconfined phase for which the supersymmetric degrees of freedom would not play a major rôle, this can lead to a new point-of-view on the thermalization problem. Indeed, the stable, strongly interacting state represented by the perfect gauge fluid, could act as an ''attractor'' during the proper-time evolution of a collision with QCD plasma formation, such as for heavy-ion collisions. This would give a typical nonperturbative mechanism, *a priori* complementary to those related to the initial perturbative conditions. It would be interesting to see how one could merge the two ends of the evolution, the perturbative with the nonperturbative ones.

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