Dark energy cosmology from higher-order, string-inspired gravity, and its reconstruction

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In this paper we investigate the cosmological effects of modified gravity with string curvature corrections added to the Einstein-Hilbert action in the presence of a dynamically evolving scalar field coupled to Riemann invariants. The scenario exhibits several features of cosmological interest for the late universe. We show that higher-order stringy corrections can lead to a class of dark energy models consistent with recent observations. The models can give rise to quintessence without recourse to a scalar field potential. The detailed treatment of the reconstruction program for general scalar-Gauss-Bonnet gravity is presented for any given cosmology. The explicit examples of reconstructed scalar potentials are given for an accelerated (quintessence, cosmological constant, or phantom) universe. Finally, the relation with modified $F(G)$ gravity is established at the classical level and is extended to include third order terms on the curvature.

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I. INTRODUCTION

One of the most remarkable discoveries of our time is related to the late-time acceleration of our universe which is supported by observations of high redshift type Ia supernovae treated as standardized candles and, more indirectly, by observations of the cosmic microwave background (CMB) and galaxy clustering. The criticality of the universe supported by CMB observations fixes the total energy budget of the universe. The study of large scale structures reveals that dark matter contributes nearly 30% of the total cosmic budget. Then, there is a deficit of almost 70%; in the standard paradigm, the missing component is an exotic form of energy with large negative pressure dubbed *dark energy* [1–7]. The recent observations on baryon oscillations provide yet more independent support for the dark energy hypothesis [8].

The dynamics of our universe is described by the Einstein equations in which the contribution of the energy content of the universe is represented by the energy-momentum tensor appearing on the right-hand side (RHS) of these equations. The left-hand side (LHS) represents pure geometry given by the curvature of space-time. Gravitational equations in their original form with the energy-momentum tensor of normal matter cannot lead to acceleration. There are then two ways to obtain accelerated expansion, either by supplementing the energy-momentum tensor with the dark energy component or by modifying the geometry itself.

The dark energy problem is one of the most important problems of modern cosmology and, despite the number of efforts (for a recent review, see [3,4]), there is no consistent theory which may successfully describe the late-time acceleration of the universe. General relativity with the cosmological constant does not solve the problem because such theory is in conflict with radiation/matter domination eras. An alternative approach to dark energy is related to the modified theory of gravity (for a review, see [9]) in which dark energy emerges from the modification of the geometry of our universe. In this approach, there appears quite an interesting possibility to mimic dark energy cosmology from string theory. It was suggested in Refs. [10–12] that dark energy cosmology may result from string-inspired gravity. In fact, scalar-Gauss-Bonnet gravity from bosonic or type II strings was studied in the late universe [10,11] (for a review of the applications of such theory in the early universe, see [13]). It is also interesting that such theories may solve the initial singularity problem of the standard big-bang model (see [14] and references therein). Moreover, the easy account of next order (third order, Lovelock term) is also possible in this approach (for a recent discussion of such gravity, see [15]).

In this paper we examine string-inspired gravity with third order curvature corrections (the scalar-Gauss-Bonnet term and the scalar-Euler term) and explore the cosmological dynamics of the system attributing special attention

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to dark energy (nonphantom/phantom) solutions. We confront our result with the recent observations. We also outline the general program of reconstruction of scalar-Gauss-Bonnet gravity for any *a priori* given cosmology following the method [16] developed in the scalar-tensor theory.

The paper is organized as follows. In Sec. II, we consider the cosmological dynamics in the presence of string curvature corrections to the Einstein-Hilbert action. We analyze cosmological solutions in the Friedmann-Robertson-Walker (FRW) background; special attention is paid to dark energy which naturally arises in the model thanks to higher-order curvature terms induced by string corrections. A brief discussion on the comparison of theoretical results with recent observations is included. The stability of the dark energy solution is investigated in detail.

Section III is devoted to the study of late-time cosmology for scalar-Gauss-Bonnet gravity motivated by string theory but with the arbitrary scalar potentials. We explicitly show how such theory (actually, its potentials) may be reconstructed for any given cosmology. Several explicit examples of dark energy cosmology with a transition from deceleration to acceleration and (or) a cosmic speedup (quintessence, phantom, or de-Sitter) phase or with oscillating (currently accelerating) behavior of the scale factor are given. The corresponding scalar potentials are reconstructed. We show how such theory may be transformed to modified Gauss-Bonnet gravity which turns out to be just a specific parametrization of scalar-Gauss-Bonnet gravity at the classical level. Finally, we show how to include third order curvature terms in the above construction. A summary and outlook are given in the last section.

II. DARK ENERGY FROM HIGHER-ORDER STRING CURVATURE CORRECTIONS

In this section we shall consider higher-order curvature corrections to the Einstein-Hilbert action. To avoid technical complications we restrict the discussion up to third order Riemann invariants coupled to a dynamical field ϕ. The cosmological dynamics of the system will be developed in detail and general features of the solutions will be discussed. It is really interesting that the model can account for recent observations of dark energy.

A. General action

We begin with the following action,

$$\mathcal{S} = \int d^4x\sqrt{-g}\bigg[\frac{R}{2\kappa^2} - \frac{1}{2}\omega(\phi)g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi) + \mathcal{L}_c + \mathcal{L}_m\bigg], \tag{1}$$

where ϕ is a scalar field which, in a particular case, could be a dilaton. $\mathcal{L}_m$ is the Lagrangian of perfect fluid with energy density ρ_m and pressure p_m. Note that the scalar potential coupled to the curvature (nonminimal coupling) [17] does not appear in string-inspired gravity in the frame under consideration.

The quantum corrections are encoded in the term

$$\mathcal{L}_c = \xi_1(\phi)\mathcal{L}_c^{(1)} + \xi_2(\phi)\mathcal{L}_c^{(2)} \tag{2}$$

where $\xi_1(\phi)$ and $\xi_2(\phi)$ are the couplings of the field ϕ with higher curvature invariants. $\mathcal{L}_c^{(1)}$ and $\mathcal{L}_c^{(2)}$ are given by

$$\mathcal{L}_c^{(1)} = R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} - 4R_{\mu\nu}R^{\mu\nu} + R^2, \tag{3}$$

$$\mathcal{L}_c^{(2)} = E_3 + R^{\mu\nu}_{\alpha\beta}R^{\alpha\beta}_{\lambda\rho}R^{\lambda\rho}_{\mu\nu}. \tag{4}$$

The third order Euler density E_3 is proportional to

$$E_3 \propto \epsilon^{\mu\nu\rho\sigma\tau\eta}\epsilon_{\mu'\nu'\rho'\sigma'\tau'\eta'}R_{\mu\nu}{}^{\mu'\nu'}R_{\rho\sigma}{}^{\rho'\sigma'}R_{\tau\eta}{}^{\tau'\eta'}. \tag{5}$$

Since there does not exist $\epsilon^{\mu\nu\rho\sigma\tau\eta}$ if the space-time dimension D is less than 6, E_3 should vanish when $D < 6$, especially in four dimensions. By using

$$\epsilon^{\mu\nu\rho\sigma\tau\eta}\epsilon_{\mu'\nu'\rho'\sigma'\tau'\eta'} = \delta_\mu{}^{\mu'}\delta_\nu{}^{\nu'}\delta_\rho{}^{\rho'}\delta_\sigma{}^{\sigma'}\delta_\tau{}^{\tau'}\delta_\eta{}^{\eta'} \pm (\text{permutations}), \tag{6}$$

we can rewrite the expression (5) as

$$\begin{aligned} E_3 \propto 8(&R^3 - 12RR_{\mu\nu}R^{\mu\nu} + 3RR_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \\ &+ 16R_\mu{}^\nu R_\nu{}^\rho R_\rho{}^\mu + 24R_\mu{}^\nu R_\rho{}^\sigma R_{\nu\sigma}{}^{\mu\rho} \\ &- 24R_\mu{}^\nu R_{\nu\rho}{}^{\sigma\tau}R_{\sigma\tau}{}^{\mu\rho} + 2R_{\mu\nu}{}^{\rho\sigma}R_{\rho\sigma}{}^{\tau\eta}R_{\tau\eta}{}^{\mu\nu} \\ &- 8R_{\mu\nu}{}^{\rho\tau}R_{\rho\sigma}{}^{\mu\eta}R_{\tau\eta}{}^{\nu\sigma}). \end{aligned} \tag{7}$$

We should note in the RHS of (6), there are $6! = 720$ terms, which correspond to the sum of the absolute values of the coefficients in each term in the RHS of (7),

$$8(1 + 12 + 3 + 16 + 24 + 24 + 2 + 8) = 720. \tag{8}$$

In what follows we shall be interested in the cosmological applications of modified equations of motion and thus assume a flat FRW metric

$$ds^2 = -N^2(t)dt^2 + a^2(t)\sum_{i=1}^{d}(dx^i)^2, \tag{9}$$

where $N(t)$ is the lapse function. With the metric (9), the Riemann invariants read

$$\begin{aligned} \mathcal{L}_c^{(1)} &= 24H^2\bigg(\frac{\dot{H} + H^2}{N^4} - \frac{\dot{N}}{N^5}H\bigg), \\ \mathcal{L}_c^{(2)} &= \frac{24}{N^6}(H^6 + I^3) - \frac{72\dot{N}}{N^7}HI^2 \end{aligned} \tag{10}$$

where $I = \dot{H} + H^2$ and $H = \dot{a}/a$. It is straightforward though cumbersome to verify explicitly that third order Euler density E_3 is identically zero in the FRW background. The nonvanishing contribution in Eq. (10) comes from the second term in (4). To enforce the check in a particular case, we consider D dimensional de-Sitter space, where the Riemann curvature is given by

$$R_{\mu\nu}{}^{\rho\sigma} = H_0(\delta_\mu{}^\rho \delta_\nu{}^\sigma - \delta_\mu{}^\sigma \delta_\nu{}^\rho). \tag{11}$$

Here H_0 is a constant corresponding to the Hubble rate. In the de-Sitter background we have

$$E_3 \propto D(D-1)(D-2)(D-3)(D-4)(D-5), \tag{12}$$

which is obviously zero in the case of $D < 6$. For simplicity we shall limit the discussion to a homogeneous scalar field $\phi(t)$. Then the spatial volume can be integrated out from the measure in Eq. (1), which we rewrite as

$$\mathcal{S} = \int dt N a^3 \left[\frac{R}{2\kappa^2} + \mathcal{L}_c + \mathcal{L}_\phi + \mathcal{L}_m\right], \tag{13}$$

where $\mathcal{L}_\phi = -\frac{1}{2}\omega(\phi)(\nabla\phi)^2 - V(\phi)$. Varying the action (13) with respect to the lapse function N, we obtain [11]

$$\frac{3H^2}{\kappa^2} = \rho_m + \rho_\phi + \rho_c \tag{14}$$

where

$$\rho_\phi = \frac{1}{2}\omega\dot{\phi}^2 + V(\phi). \tag{15}$$

In Eq. (14), the energy density ρ_c is induced by quantum corrections and is given by the following expression:

$$\rho_c = \left(3H\frac{\partial \mathcal{L}_c}{\partial \dot{N}} + \frac{d}{dt}\frac{\partial \mathcal{L}_c}{\partial \dot{N}} - \frac{\partial \mathcal{L}_c}{\partial N} - \mathcal{L}_c\right)\Bigg|_{N=1}. \tag{16}$$

It would be convenient to rewrite ρ_c as

$$\rho_c = \xi_1(\phi)\rho_c^{(1)} + \xi_2(\phi)\rho_c^{(2)}. \tag{17}$$

Using Eqs. (10) we obtain the expressions of $\rho_c^{(1)}$ and $\rho_c^{(2)}$,

$$\rho_c^{(1)} = -24H^3\Xi_1, \tag{18}$$

$$\rho_c^{(2)} = -72HI^2\Xi_2 - 72(\dot{H}I^2 + 2IH\dot{I}) - 216H^2I^2 + 120(H^6 + I^3), \tag{19}$$

where $\Xi_1 = \dot{\xi}_1/\xi_1$ and $\Xi_2 = \dot{\xi}_2/\xi_2$. It is interesting to note that the contribution of the Gauss-Bonnet term [described by Eq. (10)] cancels in equations of motion for fixed ϕ, as it should be; it contributes for the dynamically evolving scalar field only. In the case of the third order curvature corrections, the Euler density is identically zero and hence it does not contribute to the equation of motion, in general. Second, $\mathcal{L}_c^{(2)}$ contributes for a fixed field as well as for dynamically evolving ϕ. It contains corrections of third order in curvature beyond the Euler density.

We should note that such higher-derivative terms in string-inspired gravity may lead to ghosts and related instabilities (for a recent discussion on scalar-Gauss-Bonnet gravity, see [18]). However, the ghost spectrum of such (quantum) gravity (for a review, see [19]) is more relevant in the early universe where the curvature is strong, but less relevant in the late universe. Moreover, in accordance with the modified gravity approach, the emerging theory is purely classical, effective theory which comes from some unknown gravity which has different faces at different epochs. (Actually, it could be that our universe currently enters the instable phase). For instance, in the near future the currently subleading terms may dominate in the modified gravity action which then has a totally different form. Hence, it is that (unknown) gravity, and not its classical limit given by Eq. (1) relevant during a specific epoch, whose spectrum should be studied. The point is best illustrated by the example of Fermi theory of weak interactions whose quantization runs into well-known problems. Finally, on phenomenological grounds, it is really interesting to include higher-order terms. At present the situation is remarkably tolerant in cosmology; many exotic constructions attract attention provided they can lead to a viable model of dark energy.

The equation of motion for the field ϕ reads from (13) as

$$\omega(\ddot{\phi} + 3H\dot{\phi}) + V' - \xi_1'\mathcal{L}_c^{(1)} - \xi_2'\mathcal{L}_c^{(2)} + \dot{\omega}\,\dot{\phi} - \omega'\frac{\dot{\phi}^2}{2} = 0. \tag{20}$$

In addition we have the standard continuity equation for the barotropic background fluid with energy density ρ_m and pressure p_m,

$$\dot{\rho}_m + 3H(\rho_m + p_m) = 0. \tag{21}$$

Equations (14), (20), and (21) are the basic equations for our system under consideration.

Let us note that in the string theory context with the dilaton field ϕ we have

$$V(\phi) = 0, \qquad \xi_1 = c_1\alpha' e^{2\phi/\phi_0}, \qquad \xi_2 = c_2\alpha'^2 e^{4\phi/\phi_0} \tag{22}$$

where $(c_1, c_2) = (0, 0, 1/8), (1/8, 0, 1/8)$ for type II, heterotic, and bosonic strings, respectively.

B. Fixed field case: General features of solutions

We now look for de-Sitter solutions in the case of $\phi =$ constant and $\rho_m = 0$. In this case the modified Hubble equations (14) gives rise to the de-Sitter solution

$$3 = 24\xi_2 H^4 \quad \text{or} \quad H = \left(\frac{1}{8\xi_2}\right)^{1/4} \tag{23}$$

where $\xi_2 = \frac{1}{8}\exp(-4\phi/\phi_0)$ for type II and bosonic strings. Normalizing ξ_2 to 1, we find that $H = 0.6$ (we have set $\kappa^2 = 1$ for convenience). Below we shall discuss the stability of the solution. There is no de-Sitter solution for the heterotic case. Actually, de-Sitter solutions were investigated in a similar background in Ref. [11] where higher-order curvature corrections up to 4th order were included. Since here we confine ourselves up to the third order and the fourth order terms are excluded from the expression of ρ_c, these terms come with different signs. Thus it becomes necessary to check whether or not the stability property of de-Sitter solutions is preserved order by order.

We further note that the modified Hubble equations (14) admit the following solution in the high curvature regime in the presence of the barotropic fluid with the equation of state (EoS) parameter w,

$$a(t) = a_0 t^{h_0}, \quad \text{or} \quad a(t) = a_0(t_s - t)^{h_0} \tag{24}$$

where

$$h_0 = \frac{2}{1+w}, \tag{25}$$

$$a_0 = \left[\frac{\xi_2}{\rho_0}(72(-h_0 I_0^2 + 2I_0 h_0 \dot{I}_0) + 216 h_0^2 I_0^2 - 120(h_0^6 + I_0^3))\right]^{-[1/(3(1+w))]}. \tag{26}$$

We have used $\rho_m = \rho_0 a^{-3(1+w)}$ for the background matter density and $I_0 = h_0(h_0 - 1)$, $\dot{I}_0 = -2h_0(h_0 - 1)$. For the effective equation of state dictated by the modified Hubble equations (14) we have

$$w_{\rm eff} = -1 - \frac{2}{3}\frac{\dot{H}}{H^2} = -1 + \frac{1+w}{3}. \tag{27}$$

It is interesting to note that the effective EoS parameter (27) may correspond to the inflationary solution in the presence of background fluid (radiation/matter). In the low curvature regime or at late times, $w_{\rm eff} = w$. In the presence of phantom matter, the effective EoS being less than -1 is typical for the big rip singularity. It is really not surprising that we have an inflationary solution at early epochs in the presence of a higher-order curvature correction to the Einstein-Hilbert action; an early example of this phenomenon is provided by R^2 *gravity*.

C. Autonomous form of equations of motion

Let us now cast the equations of motion in the autonomous form. Introducing the following notation ($\kappa^2 = 1$),

$$x = H, \quad y = \dot{H}, \quad u = \phi, \quad v = \dot{\phi}, \quad z = \rho_m, \tag{28}$$

we shall assume $\omega(\phi) = \nu = \text{const}$. We obtain the system of equations

$$\dot{x} = y, \quad \dot{y} = \frac{\frac{1}{2}\nu v^2 - 24\xi_1\Xi_1 x^3 + \xi_2[-72xI^2\Xi_2 - 72(yI^2 + 4Iyx^2) - 216x^2I^2 + 120(x^6 + I^3)] - 3x^2}{144I(x,y)\xi_2 x} + \frac{z}{144I\xi_2 x},$$
$$\dot{u} = v, \quad \dot{v} = \frac{-3\nu x v + \xi_1\mathcal{L}_c^{(1)} + \xi_2\mathcal{L}_c^{(2)}}{\nu}, \quad \dot{z} = -3x(1+w)z. \tag{29}$$

We shall first be interested in the case of a fixed field for which we have (assuming $\nu = 1$)

$$\dot{x} = y, \quad \dot{y} = \frac{[-72(yI(x,y)^2 + 4I(x,y)yx^2) + 216x^2I^2(x,y) + 120(x^6 + I^3(x,y)) - 3x^2 + z]}{144I(x,y)x}, \quad \dot{z} = -3x(1+w)z \tag{30}$$

where

$$I(x,y) = x^2 + y, \quad \mathcal{L}_c^{(1)} = 24x^2(y + x^2), \quad \mathcal{L}_c^{(2)} = 24(x^6 + I^3(x,y)). \tag{31}$$

In the presence of a perfect fluid, the de-Sitter fixed point is characterized by

$$x_c = 0.71, \quad y_c = 0, \quad z_c = 0. \tag{32}$$

Perturbing the system around the critical point and keeping the linear terms, we obtain

$$\dot{\delta x} = \delta y,$$
$$\dot{\delta y} = \left(\frac{21}{3}x_c^2 + \frac{10}{3x_c} + \frac{1}{48x_c^2}\right)\delta x + \left(\frac{2}{3}x_c + \frac{5}{6x_c^2} + \frac{1}{48x_c^3}\right)\delta y + \frac{1}{144x_c^3}\delta z,$$
$$\dot{\delta x} = -3x_c(1+w)\delta z. \tag{33}$$

The stability of the fixed points depends upon the nature of eigenvalues of the perturbation matrix

$$\lambda_{1,2} = \frac{1}{2}(a_{22} \pm \sqrt{4a_{21} + a_{22}^2}), \quad \lambda_3 = a_{33} = -3x_c(1+w). \tag{34}$$

For the fixed point given by (32), λ_1 is positive where as λ_2

is negative, making the de-Sitter solution an unstable node. In fact, λ_1 remains positive for any $x_c > 0$, thereby making the conclusion independent of the choice of $\xi_2^{(0)}$ (see Fig. 1).

D. The dynamically evolving field ϕ and dark energy solutions

In what follows we shall be interested in looking for an exact solution of the equations of motion (14) and (20) which are of interest to us from the point of view of dark energy in the absence of the background fluid. In this case let us look for the following solution:

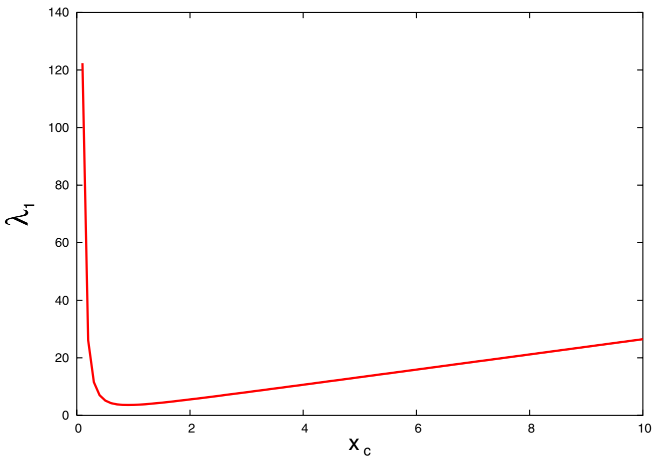

FIG. 1 (color online). Plot of the first eigenvalue λ_1 versus the critical point x_c. The eigenvalue remains positive if the critical point is varied from zero to larger values.

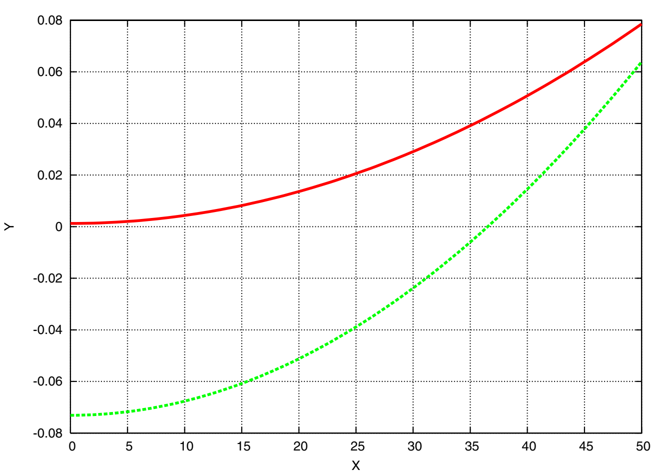

FIG. 2 (color online). Plot of $Y \equiv (48\xi_0^{(1)}/t_1^2) \times 10^5$ (black line) and $Y \equiv (96\xi_0^{(2)}/t_1^4)$ (gray line) versus $X \equiv \phi_0^2$, corresponding to $h_0 = 40$ or $w_{\rm DE} = -0.95$ (ν is assumed to be 1). The common region corresponding to positive values of the couplings gives possible models of dark energy induced by higher-order curvature corrections.

$$H = \frac{h_0}{t}, \qquad \phi = \phi_0 \ln\frac{t}{t_1} \ (\text{when } h_0 > 0),$$
$$H = \frac{h_0}{t_s - t}, \qquad \phi = \phi_0 \ln\frac{t_s - t}{t_1} \ (\text{when } h_0 < 0). \tag{35}$$

Substituting (35) in evolution equations (14) and (20) yields (we again set $\kappa^2 = 1$)

$$\nu(1 - 3h_0)\phi_0^2 + \frac{48\xi_1^{(0)}}{t_1^2} h_0^3(h_0 - 1) + \frac{96\xi_2^{(0)}}{t_1^2}(h_0^6 + I_0^3) = 0,$$
$$-3h_0^2 + \frac{\nu}{2}\phi_0^2 - \frac{48\xi_1^{(0)}}{t_1^2} h_0^3 + \frac{\xi_2^{(0)}}{t_1^4} J(h_0) = 0 \tag{36}$$

where

$$J = \tfrac{1}{96}(-288h_0 I_0^2 - 72(-h_0 I_0^2 + 2I_0 h_0 \dot{I}_0) - 216h_0 I_0^2 + 120(h_0^6 + I_0^3)),$$
$$I_0 = h_0(h_0 - 1), \qquad \dot{I}_0 = -2h_0(h_0 - 1). \tag{37}$$

Using Eqs. (36), we express the couplings through h_0 and ϕ_0,

$$\frac{48\xi_1^{(0)}}{t_1^2} = \left[\frac{3h_0^2 - \frac{\nu\phi_0^2}{2} + \nu(3h_0 - 1)\phi_0^2}{J(h_0)(h_0 - 1) + h_0^3(h_0^3 + (h_0 - 1)^3)}\right], \tag{38}$$

$$\frac{96\xi_2^{(0)}}{t_1^4} = \frac{1}{h_0^3}\left[-3h_0^2 + \frac{\nu\phi_0^2}{2} + \left[\frac{(3h_0^2 - \frac{\nu\phi_0^0}{2} + \nu(3h_0 - 1)\phi_0^2)J(h_0)}{J(h_0)(h_0 - 1) + h_0^3(h_0^3 + (h_0 - 1)^3)}\right]\right]. \tag{39}$$

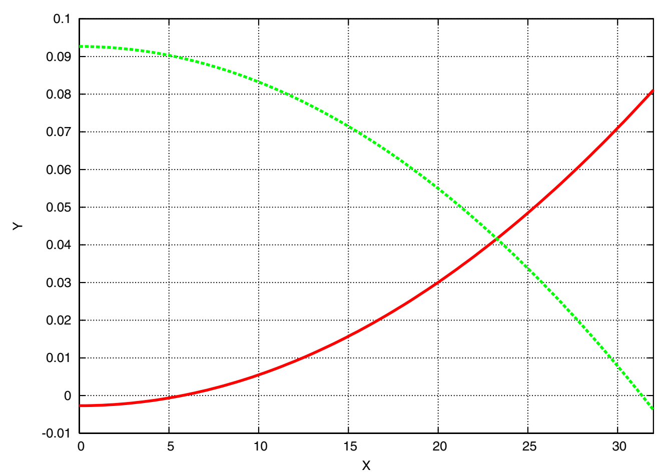

FIG. 3 (color online). Plot of $Y \equiv (48\xi_0^{(1)}/t_1^2) \times 10^5$ (gray line) and $Y \equiv (96\xi_0^{(2)}/t_1^4)$ (black line) versus $X \equiv \phi_0^2$, corresponding to $h_0 = -33.33$ or $w_{\rm DE} = -1.06$. The region bounded by $6 < X < 31.5$ corresponds to possible phantom dark energy models.

Let us note that the string couplings ($\xi_1(\phi) = \xi_1^{(0)} e^{n(\phi/\phi_0)}$, $\xi_2(\phi) = \xi_2^{(0)} e^{m(\phi/\phi_0)}$ with $m = n^2 = 4$) are generic to the solution described by (35); for other couplings, such a solution does not exist. We also note that Eqs. (38) and (39) reduce to the earlier results obtained in Ref. [10] (see Refs. [13,20,21] on the related theme) where similar investigations were carried out confining to only second order curvature invariants in the action (1).

There are several free parameters in the problem. In order to extract important information from Eqs. (38) and (39), we proceed in the following manner. We fix h_0 corresponding to the observed value of the dark energy equation of state parameter $w_{\rm DE}$ and impose the positivity condition on the couplings $\xi_1^{(0)}$ and $\xi_2^{(0)}$, leading to allowed values of the parameter ϕ_0^2. In the absence of the coupling $\xi_2(\phi)$, it was shown in Ref. [10] that, for a given value of h_0 from the allowed interval, the parameter ϕ_0 takes a fixed value. Our model incorporates higher-order curvature corrections allowing a one parameter flexibility in the values of ϕ_0. This gives rise to a comfortable choice of the equation of state consistent with observations.

The three-year WMAP data are analyzed in Ref. [22], which shows that the combined analysis of WMAP with the supernova Legacy survey (SNLS) constrains the dark energy equation of state $w_{\rm DE}$ pushing it towards the cosmological constant (see Table I). The marginalized best fit values of the equation of state parameter at 68% confidence level are given by $-1.14 \le w_{\rm DE} \le -0.93$. In case of a prior that universe is flat, the combined data give $-1.06 \le w_{\rm DE} \le -0.90$. Our model can easily accommodate these values of $w_{\rm DE}$. For instance, in the case of nonphantom dark energy (standard dark energy), we find a one parameter family of dark energy models with $h_0 \simeq 40$ ($w_{\rm DE} = -0.98$) corresponding to $\phi_0^2 > 41$. Likewise, in the case of phantom dark energy, we find that, for $h_0 \simeq -33$ ($w_{\rm DE} = -1.02$), the viable range of the parameter ϕ_0^2 is given by $6 < \phi_0^2 < 31.5$ (see Figs. 2 and 3). These values are consistent with the recent WMAP release and SNLS findings.

We should mention that the observations quoted above do not incorporate the dark energy perturbations which might severely constrain the phantom dark energy cosmologies. The combined data (CMB + LSS + SNLS) then force the dark energy equation of state to vary as $-1.001 < w_{\rm DE} < -0.875$ [22]. Our model can easily incorporate these numerical values of $w_{\rm DE}$ by constraining h_0 and ϕ_0^2 similar to the case of nonclustering dark energy. A word of caution: the evolution of dark energy perturbation across the phantom divide needs additional assumptions; a complete analysis should take into account the nonadiabatic perturbations which make dark energy gravitationally stable [4].

E. Stability of the dark energy solution

In what follows we shall examine the stability of the dark energy solution (35) induced by purely stringy corrections. In general, the analytical treatment becomes intractable; simplification, however, occurs in the limit of large h_0 corresponding to $w_{\rm eff} \simeq -1$.

Let us consider the following situation of interest to us:

$$\rho_m = 0, \qquad \omega = \nu = 1, \qquad \xi_1(\phi) = \xi_1^{(0)} e^{2\phi/\phi_0}, \qquad \xi_2(\phi) = \xi_2^{(0)} e^{4\phi/\phi_0}. \tag{40}$$

In order to investigate the stability around the dark energy solution defined by (35), we need a convenient set of variables to cast the evolution equations into an autonomous form. We now define the variables which are suited to our problem.

$$\mathcal{X} \equiv \frac{\dot\phi}{H}, \qquad \mathcal{Y} \equiv (\dot H + H^2)^2 \xi_2(\phi), \qquad \mathcal{Z} \equiv H^2 \xi_1(\phi), \qquad \frac{d}{dN} \equiv \frac{1}{H}\frac{d}{dt}. \tag{41}$$

With this choice, the evolution equations acquire the autonomous form

$$\begin{aligned}
\frac{d\mathcal{X}}{dN} &= -2\mathcal{X} + \xi_1^{(0)}\left(-\frac{\mathcal{X}}{\mathcal{Z}} + \frac{48}{\phi_0}\right)\left(\frac{\mathcal{Y}}{\xi_2^{(0)}}\right)^{1/2} + \frac{96}{\phi_0}\frac{\xi_2^{(0)}}{(\xi_1^{(0)})^2}\mathcal{Z}^2 + \frac{96\xi_1^{(0)}\mathcal{Y}}{\phi_0\mathcal{Z}}\left(\frac{\mathcal{Y}}{\xi_2^{(0)}}\right)^{1/2},\\
\frac{d\mathcal{Y}}{dN} &= -\frac{1}{24\kappa^2}\frac{\mathcal{X}^2}{144} - \frac{2}{3\phi_0}\mathcal{Z}\mathcal{X} - 2\mathcal{Y} + \frac{2\xi_1^{(0)}\mathcal{Y}}{3\mathcal{Z}}\left(\frac{\mathcal{Y}}{\xi_2^{(0)}}\right)^{1/2} + \frac{5\xi_2^{(0)}}{3(\xi_1^{(0)})^2}\mathcal{Z}^2,\\
\frac{d\mathcal{Z}}{dN} &= \left(-2 + \frac{2}{\phi_0}\mathcal{X}\right)\mathcal{Z} + 2\xi_1^{(0)}\left(\frac{\mathcal{Y}}{\xi_2^{(0)}}\right)^{1/2}.
\end{aligned} \tag{42}$$

We have used the field equation (20) and Eq. (29) for $\ddot H - \dot y$ in deriving the above autonomous form of equations. For

TABLE I. Observational constraints on the (nonclustering) dark energy equation of state $w_{\rm DE}$ dictated by the combined analysis of WMAP + SNLS data [22] and the numerical values of model parameters consistent with the observations.

Dark energy	h_0	ϕ_0^2	$w_{\rm DE}$	Observational constraint on $w_{\rm DE}$	Constraint on $w_{\rm DE}$ with flatness prior
Nonphantom	40	$\phi_0^2 > 41$	-0.98	$-1.06^{+0.13}_{-0.08}$	$-0.97^{+0.07}_{-0.09}$
Phantom	-33.33	$6 < \phi_0^2 < 31.5$	-1.02		

our solution given by (35), we have

$$\mathcal{X} = \mathcal{X}_0 \equiv \frac{\phi_0}{h_0}, \qquad \mathcal{Z} = \mathcal{Z}_0 \equiv \frac{h_0^2 \xi_1^{(0)}}{t_1^2}, \\ \mathcal{Y} = \mathcal{Y}_0 = \frac{(-h_0 + h_0^2)^2 \xi_2^{(0)}}{t_1^4}. \tag{43}$$

It can be checked that $(\mathcal{X}_0, \mathcal{Y}_0, \mathcal{Z}_0)$ is a fixed point of (42). We then consider small perturbations around (43) or equivalently around the original solution (35),

$$\mathcal{X} = \mathcal{X}_0 + \delta\mathcal{X}, \qquad \mathcal{Y} = \mathcal{Y}_0 + \delta\mathcal{Y}, \\ \mathcal{Z} = \mathcal{Z}_0 + \delta\mathcal{Z}. \tag{44}$$

Substituting (44) in (42) and retaining the linear terms in perturbations, we find

$$\frac{d}{dN}\begin{pmatrix} \delta\mathcal{X} \\ \delta\mathcal{Y} \\ \delta\mathcal{Z} \end{pmatrix} = M \begin{pmatrix} \delta\mathcal{X} \\ \delta\mathcal{Y} \\ \delta\mathcal{Z} \end{pmatrix}. \tag{45}$$

Here M is a 3×3 perturbation matrix whose components are given by

$$\begin{aligned}
M_{11} &= -2 + \frac{-1 + h_0}{h_0}, \\
M_{12} &= -\frac{\phi_0 t_1^2}{2h_0^3} + \frac{24\xi_1^{(0)} t_1^2}{\phi_0 \xi_2^{(0)}(-h_0 + h_0^2)} + \frac{144(-1 + h_0)}{\phi_0 h_0}, \\
M_{13} &= \frac{\phi_0 t_1^2(-1 + h_0)}{h_0^4 \xi_1^{(0)}} + \frac{192\xi_2^{(0)} h_0^2}{\xi_1^{(0)}\phi_0 t_1^2} - \frac{96(-1 + h_0)^3 \xi_2^{(0)}}{\phi_0 h_0 \xi_1^{(0)} t_1^2}, \\
M_{21} &= \frac{1}{72} - \frac{2h_0^2 \xi_1^{(0)}}{3\phi_0 t_1^2}, \\
M_{22} &= -2 - \frac{(-1 + h_0)}{h_0}, \\
M_{23} &= -\frac{2}{3h_0} - \frac{2(-1 + h_0)^3 \xi_2^{(0)}}{3h_0 \xi_1^{(0)} t_1^2} + \frac{10h_0^2 \xi_2^{(0)}}{3\xi_1^{(0)} t_1^2}, \\
M_{31} &= \frac{2h_0^2 \xi_1^{(0)}}{\phi_0 t_1^2}, \\
M_{32} &= \frac{\xi_1^{(0)} t_1^2}{\xi_2^{(0)}(-h_0 + h_0^2)}, \\
M_{33} &= -2 + \frac{2}{h_0}.
\end{aligned} \tag{46}$$

Stability of the fixed point(s) depends upon the nature of the eigenvalues of the perturbation matrix M. If there is an eigenvalue whose real part is positive, the system becomes unstable. Here, for simplicity, we only consider the case of $h_0 \to \pm\infty$, which corresponds to the limit of $w_{\rm eff} \sim -1$. In this case, we find

$$\frac{\xi_1^{(0)}}{t_1^2} \to \frac{1}{40h_0^5}, \qquad \frac{\xi_2^{(0)}}{t_1^4} \to -\frac{1}{32h_0}, \tag{47}$$

and the eigenvalue equation is given by

$$0 = F(\lambda) \equiv -\lambda^3 - 6\lambda^2 - \frac{h_0^3}{40\phi_0}\lambda - \frac{7h_0^3}{40\phi_0}. \tag{48}$$

The values of λ satisfying $F(\lambda) = 0$ give eigenvalues of M. The solutions of (48) are given by

$$\lambda = \lambda_\pm \equiv \pm\frac{|h_0|\sqrt{-h_0}}{\phi_0\sqrt{40}} + \mathcal{O}(|h_0|), \\ \lambda = \lambda_0 \equiv -7 + \mathcal{O}(|h_0|^{-1/2}). \tag{49}$$

When $h_0 < 0$, the mode corresponding to λ_+ (λ_-) becomes stable (unstable). Since $\lambda_\pm$ are pure imaginary when $h_0 > 0$, the corresponding modes become stable in this case. On the other hand, the mode corresponding to λ_0 is always stable. Thus, the nonphantom dark energy solution (35) induced by string corrections to Einstein gravity is stable. Such a solution exists in the presence of a dynamically evolving field ϕ with $V(\phi) = 0$ coupled to Riemann invariants with couplings dictated by string theory. Dark energy can be realized in a variety of scalar field models by appropriately choosing the field potential. It is really interesting that we can obtain dark energy solutions in the string model without recourse to a scalar field potential.

Let us compare the results with those obtained in [10], where $\xi_2 = 0$ but $V(\phi) \neq 0$. The dark energy solution studied in Ref. [10] was shown to be stable when $h_0 > 0$ but unstable for $h_0 > 0$. The present investigations include ξ_2 and $V = 0$ which makes our model different from Ref. [10]; it is therefore not surprising that our results differ from Ref. [10]. Since $h_0 > 0$ corresponds to the quintessence phase and $h < 0$ to the phantom, the solution in the model (with ξ_0 and $V = 0$) is stable in the quintessence phase but unstable in the phantom phase. We should notice that the approximation we used to check the stability works fine for any generic value of h_0. For instance, $5 < h_0 < -667$ which corresponds to the variation of $w_{\rm DE}$ in the case where the dark energy perturbations are taken into account. We also carried out numerical verification of our results.

III. THE LATE-TIME COSMOLOGY IN SCALAR-GAUSS-BONNET GRAVITY

A number of scalar field models have recently been investigated in connection with dark energy (see Ref. [4] for details). The cosmological viability of these constructs depends upon how well the Hubble parameter predicted by them compares with observations. One could also follow the reverse route and construct the Lagrangian using the observational input; such a scheme might help in the search of best fit models of dark energy [4]. In what follows we shall describe how the reconstruction program is implemented in the presence of higher-order string curvature corrections.

A. The reconstruction of scalar-Gauss-Bonnet gravity

In this section we will show how scalar-Gauss-Bonnet gravity may be reconstructed for any requested cosmology using the method [16] developed in the scalar-tensor theory. We now include the orders up to the Gauss-Bonnet term (by technical reasons) but there is no principal problem to include higher-order terms studied in the previous section. It is interesting that the principal possibility appears to reconstruct the scalar-Gauss-Bonnet gravity for any (quintessence, cosmological constant, or phantom) dark energy universe. The last possibility seems to be quite attractive due to the fact [10] that the phantom universe could be realized in scalar-Gauss-Bonnet gravity without introducing a ghost scalar field. In this section, we show that, in scalar-Gauss-Bonnet gravity, any cosmology, including phantom cosmology, could be realized by properly choosing the potential and the coupling to the Gauss-Bonnet invariant with the canonical scalar.

The starting action is

$$S = \int d^4x\sqrt{-g}\left[\frac{R}{2\kappa^2} - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi) - \xi_1(\phi)G\right]. \tag{50}$$

Here G is the Gauss-Bonnet invariant $G \equiv \mathcal{L}_c^{(1)}$ (3) and the scalar field ϕ is canonical in (50). As in the previous section, it is natural to assume the FRW universe (9) with $N(t) = 1$ and the scalar field ϕ only depending on t. The FRW equations look like

$$0 = -\frac{3}{\kappa^2}H^2 + \frac{1}{2}\dot{\phi}^2 + V(\phi) + 24H^3\frac{d\xi_1(\phi(t))}{dt}, \tag{51}$$

$$0 = \frac{1}{\kappa^2}(2\dot{H} + 3H^2) + \frac{1}{2}\dot{\phi}^2 - V(\phi) - 8H^2\frac{d^2\xi_1(\phi(t))}{dt^2} - 16H\dot{H}\frac{d\xi_1(\phi(t))}{dt} - 16H^3\frac{d\xi_1(\phi(t))}{dt}, \tag{52}$$

and the scalar field equation is

$$0 = \ddot{\phi} + 3H\dot{\phi} + V'(\phi) + \xi_1'(\phi)G. \tag{53}$$

Combining (51) and (52), one gets

$$\begin{aligned} 0 &= \frac{2}{\kappa^2}\dot{H} + \dot{\phi}^2 - 8H^2\frac{d^2\xi_1(\phi(t))}{dt^2} - 16H\dot{H}\frac{d\xi_1(\phi(t))}{dt} + 8H^3\frac{d\xi_1(\phi(t))}{dt} \\ &= \frac{2}{\kappa^2}\dot{H} + \dot{\phi}^2 - 8a\frac{d}{dt}\left(\frac{H^2}{a}\frac{d\xi_1(\phi(t))}{dt}\right). \end{aligned} \tag{54}$$

Equation (54) can be solved with respect to $\xi_1(\phi(t))$ as

$$\xi_1(\phi(t)) = \frac{1}{8}\int^t dt_1\frac{a(t_1)}{H(t_1)^2}\int^{t_1}\frac{dt_2}{a(t_2)}\left(\frac{2}{\kappa^2}\dot{H}(t_2) + \dot{\phi}(t_2)^2\right). \tag{55}$$

Combining (51) and (55), the scalar potential $V(\phi(t))$ is

$$V(\phi(t)) = \frac{3}{\kappa^2}H(t)^2 - \frac{1}{2}\dot{\phi}(t)^2 - 3a(t)H(t)\int^t\frac{dt_1}{a(t_1)} \times\left(\frac{2}{\kappa^2}\dot{H}(t_1) + \dot{\phi}(t_1)^2\right). \tag{56}$$

We now identify t with $f(\phi)$ and H with $g'(t)$ where f and g are some unknown functions in analogy with Ref. [16] since we know this leads to the solution of the FRW equations subject to the existence of such functions. Then we consider the model where $V(\phi)$ and $\xi_1(\phi)$ may be expressed in terms of two functions f and g as

$$\begin{aligned} V(\phi) &= \frac{3}{\kappa^2}g'(f(\phi))^2 - \frac{1}{2f'(\phi)^2} - 3g'(f(\phi))e^{g(f(\phi))} \times\int^\phi d\phi_1 f'(\phi_1)e^{-g(f(\phi_1))} \times\left(\frac{2}{\kappa^2}g''(f(\phi_1)) + \frac{1}{f'(\phi_1)^2}\right), \\ \xi_1(\phi) &= \frac{1}{8}\int^\phi d\phi_1\frac{f'(\phi_1)e^{g(f(\phi_1))}}{g'(\phi_1)^2} \times\int^{\phi_1} d\phi_2 f'(\phi_2)e^{-g(f(\phi_2))} \times\left(\frac{2}{\kappa^2}g''(f(\phi_2)) + \frac{1}{f'(\phi_2)^2}\right). \end{aligned} \tag{57}$$

By choosing $V(\phi)$ and $\xi_1(\phi)$ as (57), we can easily find the following solution for Eqs. (51) and (52):

$$\phi = f^{-1}(t) \quad (t = f(\phi)), \qquad a = a_0e^{g(t)} \quad (H = g'(t)). \tag{58}$$

We can straightforwardly check that the solution (58) satisfies the field equation (53).

Hence, any cosmology expressed as $H = g(\phi)$ in the model (50) with (57) can be realized, including the model exhibiting the transition from the nonphantom phase to the phantom phase without introducing the scalar field with the wrong sign kinetic term.

In Einstein gravity, the FRW equations are given by

$$0 = -\frac{3}{\kappa^2}H^2 + \rho, \qquad 0 = \frac{1}{\kappa^2}(2\dot{H} + 3H^2) + p. \tag{59}$$

Here ρ and p are total energy density and pressure in the universe. By comparing (59) with (58) we find that the effective energy density $\tilde{\rho}$ and the pressure $\tilde{p}$ are given by

$$\tilde{\rho} = \frac{3}{\kappa^2}g'(t)^2, \qquad \tilde{p} = -\frac{3}{\kappa^2}g'(t)^2 - \frac{2}{\kappa^2}g''(t). \tag{60}$$

Since $t = g'^{-1}((\kappa)\sqrt{\rho/3})$, we obtain the following effective EoS:

$$\tilde{p} = -\tilde{\rho} - \frac{2}{\kappa^2} g''\left(g'^{-1}\left(\kappa\sqrt{\frac{\rho}{3}}\right)\right), \tag{61}$$

which contains all the cases where the EoS is given by $p = w(\rho)\rho$. Furthermore, since g'^{-1} could *not* always be a single-valued function, Eq. (61) contains a more general EoS given by

$$0 = F(\tilde{\rho}, \tilde{p}). \tag{62}$$

This shows the equivalence between scalar-tensor and ideal fluid descriptions.

Let us come back now to scalar-Gauss-Bonnet gravity. It is not difficult to extend the above formulation to include matter with a constant EoS parameter $w_m \equiv p_m/\rho_m$. Here ρ_m and p_m are energy density and pressure of the matter. Then, instead of (51) and (52) the FRW equations are

$$0 = -\frac{3}{\kappa^2}H^2 + \frac{1}{2}\dot{\phi}^2 + V(\phi) + \rho_m + 24H^3\frac{d\xi_1(\phi(t))}{dt}, \tag{63}$$

$$0 = \frac{1}{\kappa^2}(2\dot{H} + 3H^2) + \frac{1}{2}\dot{\phi}^2 - V(\phi) + p_m - 8H^2\frac{d^2\xi_1(\phi(t))}{dt^2} - 16H\dot{H}\frac{d\xi_1(\phi(t))}{dt} - 16H^3\frac{d\xi_1(\phi(t))}{dt}. \tag{64}$$

The energy conservation law

$$\dot{\rho}_m + 3H(\rho_m + p_m) = 0 \tag{65}$$

gives

$$\rho_m = \rho_{m0}a^{-3(1+w_m)}, \tag{66}$$

with a constant ρ_{m0}. Instead of (57), if we consider the model with

$$\begin{aligned} V(\phi) &= \frac{3}{\kappa^2}g'(f(\phi))^2 - \frac{1}{2f'(\phi)^2} - 3g'(f(\phi))e^{g(f(\phi))}\int^{\phi} d\phi_1 f'(\phi_1)e^{-g(f(\phi_1))} \\ &\quad \times \left(\frac{2}{\kappa^2}g''(f(\phi_1)) + \frac{1}{2f'(\phi_1)^2} + (1+w_m)g_0e^{-3(1+w_m)g(f(\phi_1))}\right), \\ \xi_1(\phi) &= \frac{1}{8}\int^{\phi} d\phi_1\frac{f'(\phi_1)e^{g(f(\phi_1))}}{g'(\phi_1)^2}\int^{\phi_1} d\phi_2 f'(\phi_2)e^{-g(f(\phi_2))}\left(\frac{2}{\kappa^2}g''(f(\phi_2)) + \frac{1}{2f'(\phi_2)^2} + (1+w_m)g_0e^{-3(1+w_m)g(f(\phi_2))}\right), \end{aligned} \tag{67}$$

we reobtain the solution (58) even if the matter is included. However, a constant a_0 is given by

$$a_0 = \frac{g_0}{\rho_0}. \tag{68}$$

One can consider some explicit examples [16]:

$$t = f(\phi) = \frac{\phi}{\phi_0}, \qquad g(t) = h_0 \ln\frac{t}{t_s - t}, \tag{69}$$

which give

$$H = h_0\left(\frac{1}{t} + \frac{1}{t_s - t}\right), \qquad \dot{H} = \frac{h_0 t_s(2t - t_s)}{t^2(t_s - t)^2}. \tag{70}$$

Then the universe is in the nonphantom phase when $t < t_s/2$ and in the phantom phase when $t > t_s/2$. There is also a big rip singularity at $t = t_s$. Especially in the case $w_m = 0$ (that is, matter is dust) and $h_0 = 2$, we reconstruct the scalar-Gauss-Bonnet gravity with the following potentials:

$$\begin{aligned} V(\phi) &= \frac{6\phi_0\phi_s}{\kappa^2\phi(\phi_s-\phi)} - \frac{1}{2}\phi_0^2 - \frac{4\phi_0^2\phi_s\phi}{(\phi_s-\phi)^3}\left[\frac{4}{\kappa^2}\left(\frac{\phi_s^2}{3\phi^3}-\frac{\phi_s}{\phi^2}\right) - \frac{\phi_s^2}{\phi} - 2\phi_s\ln\frac{\phi}{\phi_s} + \phi \right. \\ &\quad \left. + \frac{g_0}{\phi_0^2}\left(-\frac{\phi_s^8}{7\phi^7} + \frac{4\phi_s^7}{3\phi^6} - \frac{28\phi_s^6}{5\phi^5} + \frac{14\phi_s^5}{\phi^4} - \frac{70\phi_s^4}{3\phi^3} + \frac{28\phi_s^3}{\phi^2} - \frac{28\phi_s^2}{\phi} - 8\phi_s\ln\frac{\phi}{\phi_s} + \phi\right) + c_1\right], \\ \xi_1(\phi) &= \frac{1}{32\phi_0^2\phi_s^2}\left[\frac{4}{\kappa^2}\left(\frac{\phi_s^2\phi^2}{6} - \frac{\phi_s\phi^3}{3}\right) - \frac{\phi_s^2\phi^4}{4} - 2\phi_s\phi^5\left(\frac{1}{5}\ln\frac{\phi}{\phi_s} - \frac{1}{25}\right) + \frac{\phi^6}{6} \right. \\ &\quad + \frac{g_0}{\phi_0^2}\left(\frac{\phi_s^8}{14\phi^2} - \frac{4\phi_s^7}{3\phi} - \frac{28\phi_s^6}{5}\ln\frac{\phi}{\phi_s} + 14\phi_s^5\phi - \frac{35\phi_s^4\phi^2}{3} + \frac{28\phi_s^3\phi^3}{3} - 7\phi_s^2\phi^4 \right. \\ &\quad \left.\left. - 8\phi_s\phi^5\left(\frac{1}{5}\ln\frac{\phi}{\phi_s} - \frac{1}{25}\right) + \frac{\phi^6}{6}\right) + \frac{c_1\phi^5}{5} + c_2\right]. \end{aligned} \tag{71}$$

Here $\phi_s \equiv \phi_0 t_s$ and c_1, c_2 are constants of the integration.

Another example, without matter ($g_0 = 0$), is [23]

$$g(t) = h_0\left(t + \frac{\cos\theta_0}{\omega}\sin\omega t\right), \qquad f^{-1}(t) = \phi_0 \sin\frac{\omega t}{2}. \tag{72}$$

Here h_0, θ_0, ω, and ϕ_0 are constants. This leads to reconstruction of scalar-Gauss-Bonnet gravity with

$$\begin{aligned} V(\phi) &= \frac{3h_0}{\kappa^2}\left(1 + \cos\theta_0 - \frac{2\cos\theta_0}{\phi_0^2}\phi^2\right) \\ &\quad - \frac{\phi_0^2\omega^2}{8}\left(1 - \frac{\phi^2}{\phi_0^2}\right)^{1/2}, \\ \xi_1(\phi) &= -\frac{\omega\phi_0}{32h_0^3}\int^\phi d\phi_1\left(1 - \frac{\phi_1^2}{\phi_0^2}\right)^{-1/2} \\ &\quad \times\left(1 + \cos\theta_0 - \frac{2\cos\theta_0}{\phi_0^2}\phi_1^2\right)^{-2}. \end{aligned} \tag{73}$$

Then, from Eq. (72) we find

$$\begin{aligned} H &= h_0(1 + \cos\theta_0\cos\omega t) \geq 0, \\ \dot{H} &= -h_0\omega\cos\theta_0\sin\omega t. \end{aligned} \tag{74}$$

Then the Hubble rate H is oscillating but, since H is positive, the universe continues to expand and, if $h_0\omega\cos\theta_0 > 0$, the universe is in the nonphantom (phantom) phase when $2n\pi < \omega t < (2n+1)\pi$ [$(2n-1)\pi < \omega t < 2n\pi$] with integer n. Thus, the oscillating late-time cosmology in string-inspired gravity may be easily constructed.

One more example is [23]

$$g(t) = H_0 t - \frac{H_1}{H_0}\ln\cosh H_0 t. \tag{75}$$

Here we assume $H_0 > H_1 > 0$. Since

$$\begin{aligned} H &= g'(t) = H_0 - H_1\tanh H_0 t, \\ \dot{H} &= g''(t) = -\frac{H_0H_1}{\cosh^2 H_0 t} < 0, \end{aligned} \tag{76}$$

when $t \to \pm\infty$, the universe becomes asymptotically de-Sitter space, where H becomes a constant $H \to H_0 \mp H_1$ and therefore the universe is accelerating. When $t = 0$, we find

$$\frac{\ddot{a}}{a} = \dot{H} + H^2 = -H_1H_0 + H_0^2 < 0; \tag{77}$$

therefore the universe is decelerating. Then the universe is accelerating at first, becomes decelerating, and after that becomes accelerating again. As $\dot{H}$ is always negative, the universe is in the nonphantom phase. Furthermore, with the choice

$$w_m = 0, \qquad t = f(\phi) = \frac{1}{H_0}\tan\left(\frac{\kappa H_0}{2\sqrt{2}H_1}\phi\right), \tag{78}$$

we find the corresponding scalar-Gauss-Bonnet gravity:

$$\begin{aligned} V(\phi) &= \frac{3}{\kappa^2}(H_0 - H_1\tanh\varphi)^2 - \frac{H_1}{\sqrt{2}\kappa^2\cosh^2\varphi} - \frac{12g_0}{H_0}(H_0 - H_1\tanh\varphi)(1 + e^{2\varphi}) \\ &\quad \times\left[2\varphi - \ln(1 + e^{2\varphi}) + \frac{5}{6(1 + e^{2\varphi})} + \frac{5}{6(1 + e^{2\varphi})^2} + \frac{2}{6(1 + e^{2\varphi})^3}\right], \\ \xi_1(\phi) &= \frac{g_0}{2H_0}\int^\varphi d\varphi'\frac{1 + e^{2\varphi'}}{(H_0 - H_1\tanh\varphi')^2}\left[2\varphi' - \ln(1 + e^{2\varphi'}) + \frac{5}{6(1 + e^{2\varphi'})} + \frac{5}{6(1 + e^{2\varphi'})^2} + \frac{2}{6(1 + e^{2\varphi'})^3}\right]. \end{aligned} \tag{79}$$

Here

$$\varphi \equiv \tan\left(\frac{\kappa H_0}{2\sqrt{2}H_1}\phi\right). \tag{80}$$

Although it is difficult to give the explicit forms of $V(\phi)$ and $\xi_1(\phi)$, we may also consider the following example [16]:

$$g(t) = h_0\left(\frac{t^4}{12} - \frac{t_1 + t_2}{6}t^3 + \frac{t_1 + t_2}{2}t^2\right), \quad (3t_1 > t_2 > t_1 > 0, h_0 > 0). \tag{81}$$

Here h_0, t_1, t_2 are constants. Hence, the Hubble rate is

$$H(t) = h_0\left(\frac{t^3}{3} - \frac{t_1 + t_2}{2}t^2 + t_1 t_2 t\right), \quad \dot{H}(t) = h_0(t - t_1)(t - t_2). \tag{82}$$

Since $H > 0$ when $t > 0$ and $H < 0$ when $t < 0$, the radius of the universe $a = a_0 e^{g(t)}$ has a minimum when $t = 0$. From the expression of $\dot{H}$ in (82), the universe is in the phantom phase ($\dot{H} > 0$) when $t < t_1$ or $t > t_2$, and in the nonphantom phase ($\dot{H} < 0$) when $t_1 < t < t_2$ (for other string-inspired models with similar cosmology, see for instance [24]). Then, we may identify that the period $0 < t < t_1$ could correspond to the inflation and the period $t > t_2$ to the present acceleration of the universe (this is similar in spirit to unification of the inflation with the acceleration suggested in the other class of modified gravities in Ref. [25]). If we define the effective EoS parameter $w_{\rm eff}$ as

$$w_{\rm eff} = \frac{p}{\rho} = -1 - \frac{2\dot{H}}{3H^2}, \tag{83}$$

we find $w_{\rm eff} \to -1$ in the limit $t \to +\infty$. Although it is difficult to find the explicit forms of $V(\phi)$ and $\xi_1(\phi)$, one might give the rough forms by using the numerical calculations. From the expression of $V(\phi)$ and $\xi_1(\phi)$ in (57), if $f(\phi)$ is properly given, say as $t = f(\phi) = \phi/\phi_0$ with constant ϕ_0, there cannot be any singularity in $V(\phi)$ and $\xi_1(\phi)$ even if $t = t_1$ or $t = t_1$, which corresponds to the transition between the phantom and nonphantom phases. Then, the model (50) could exhibit the smooth transition between the phantom and nonphantom phases.

The next example is

$$g(t) = h_0 \ln\frac{t}{t_0}, \qquad t = f(\phi) = t_0 e^{\phi/\phi_0}. \tag{84}$$

Since

$$H = \frac{h_0}{t}, \tag{85}$$

we have a constant effective EoS parameter:

$$w_{\rm eff} = -1 + \frac{2}{3h_0}. \tag{86}$$

Equations (84) give

$$\begin{aligned} V(\phi) &= -\frac{1}{(h_0+1)t_0^2}\left(\frac{3h_0^2(1-h_0)}{\kappa^2} + \frac{\phi_0^2}{2}(1-5h_0)\right)e^{-(2\phi/\phi_0)} + \frac{3h_0(1+w_m)g_0}{(4+3w_m)h_0 - 1}e^{-((3(1+w_m)h_0\phi)/\phi_0)}, \\ \xi_1(\phi) &= -\frac{t_0^2}{16h_0^2(h_0+1)}\left(-\frac{2h_0}{\kappa^2} + \phi_0^2\right)e^{(2\phi/\phi_0)} + \frac{1}{8}\{3(1+w_m)h_0 - 4\}^{-1}\{(4+3w_m)h_0 - 1\}^{-1}(1+w_m) \\ &\quad \times g_0 t_0^4 e^{-((\{3(1+w_m)h_0-4\}\phi)/\phi_0)}. \end{aligned} \tag{87}$$

Thus, exponential functions appear, which are typical in string-inspired gravity.

As is clear from (85), if $h_0 > 1$, the universe is in the quintessence phase, which corresponds to $-1/3 < w_{\rm eff} < -1$ in (84). If $h_0 < 0$, the universe is in the phantom phase with $w_{\rm eff} < -1$. In the phantom phase, we choose t_0 to be negative and our universe corresponds to negative t, or if we shift the time coordinate t as $t \to t - t_s$, with a constant t_s, t should be less than t_s.

The model [10] corresponds to $g_0 = 0$ in (87). In the notations of Ref. [10], $t_0 = t_1$, $V(\phi) = V_0 e^{-(2\phi/\phi_0)}$, and $f(\phi) = f_0 e^{(2\phi/\phi_0)} = -\xi_1(\phi)$. Then, from the expression (87), one gets

$$V_0 = -\frac{1}{(h_0+1)t_0^2}\left(\frac{3h_0^2(1-h_0)}{\kappa^2} + \frac{\phi_0^2}{2}(1-5h_0)\right), \quad f_0 = \frac{t_0^2}{16h_0^2(h_0+1)}\left(-\frac{2h_0}{\kappa^2} + \phi_0^2\right), \tag{88}$$

which is identical (after replacing t_0 with t_1) with (16) in [10].

Thus, we demonstrated that arbitrary late-time cosmology (from specific quintessence or phantom to oscillating cosmology) may be produced by scalar-Gauss-Bonnet gravity with scalar potentials defined by such cosmology. The reconstruction of string-inspired gravity may always be done. Moreover, one can extend this formulation to include the higher-order terms in low-energy string effective action.

B. The relation with modified Gauss-Bonnet gravity

In this section we show that scalar-Gauss-Bonnet gravity may be transformed to another form of modified Gauss-Bonnet gravity where no scalars are present. In addition, the formulation may be extended to include higher-order terms too. Starting from (50), one may redefine the scalar field ϕ by $\phi = \epsilon\varphi$. The action takes the following form:

$$S = \int d^4x\sqrt{-g}\left[\frac{R}{2\kappa^2} - \frac{\epsilon^2}{2}\partial_\mu\phi\partial^\mu\phi - \tilde{V}(\varphi) - \tilde{\xi}_1(\varphi)G\right]. \tag{89}$$

Here

$$\tilde{V}(\varphi) \equiv V(\epsilon\varphi), \qquad \tilde{\xi}_1(\varphi) \equiv \xi_1(\epsilon\varphi). \tag{90}$$

If a proper limit of $\epsilon \to 0$ exists, the action (89) reduces to

$$S = \int d^4x\sqrt{-g}\left[\frac{R}{2\kappa^2} - \tilde{V}(\varphi) - \tilde{\xi}_1(\varphi)G\right]. \tag{91}$$

Then φ is an auxiliary field. By the variation of φ, we find

$$0 = \tilde{V}'(\varphi) - \tilde{\xi}_1'(\varphi)G, \tag{92}$$

which may be solved with respect to φ as

$$\varphi = \Phi(G). \tag{93}$$

Substituting (94) into the action (91), the $F(G)$ gravity follows [26]:

$$S = \int d^4x\sqrt{-g}\left[\frac{R}{2\kappa^2} - F(G)\right], \qquad F(G) \equiv \tilde{V}(\Phi(G)) - \tilde{\xi}_1(\Phi(G))G. \tag{94}$$

For example, in the case of (84), in the $\epsilon \to 0$ limit after redefining $\phi = \epsilon\varphi$ and $\phi_0 = \epsilon\varphi_0$, $V(\phi)$ and $\xi_1(\phi)$ reduce to

$$\begin{aligned}\tilde{V}(\varphi) &= \frac{3h_0^2(h_0-1)}{(h_0+1)t_0^2\kappa^2}e^{-(2\varphi/\varphi_0)} + \frac{3h_0(1+w_m)g_0}{(4+3w_m)h_0-1}e^{-((3(1+w_m)h_0\phi)/\phi_0)},\\ \tilde{\xi}_1(\varphi) &= \frac{t_0^2}{8h_0(h_0+1)\kappa^2}e^{2\phi/\phi_0} + \frac{1}{8}\{3(1+w_m)h_0-4\}^{-1}\\ &\quad\times\{(4+3w_m)h_0-1\}^{-1}(1+w_m)\\ &\quad\times g_0t_0^4e^{-((\{3(1+w_m)h_0-4\}\phi)/\phi_0)}.\end{aligned} \tag{95}$$

The solution corresponding to (84) is

$$g(t) = h_0\ln\frac{t}{t_0}, \qquad \varphi = \varphi_0\ln\frac{t}{t_0}. \tag{96}$$

If we further consider the case $g_0 = 0$, Eq. (92) gives

$$e^{-(4\varphi/\varphi_0)} = \frac{t_0^4}{24h_0^3(h_0-1)}G. \tag{97}$$

Equation (97) could have meaning only when $h_0 > 1$ or $h_0 < 0$ if G is positive. In this situation

$$F(G) = A_0G^{1/2}, \qquad A_0 \equiv \frac{1}{2(1+h_0)\kappa^2}\sqrt{\frac{3(h_0-1)h_0}{2}}. \tag{98}$$

The above model has been discussed in [26]. Actually, in [26] the following type of action has been considered:

$$S = \int d^4x\sqrt{-g}\left(\frac{1}{2\kappa^2}R + F(G)\right). \tag{99}$$

In the case that $F(G)$ is given by (98), in terms of [26], $A_0 = f_0$. Hence, A_0 (98) coincides with Eq. (26) of [26].

As a further generalization, we may also consider the string-inspired theory of the second section where the next order term is coupled with the scalar field:

$$S = \int d^4x\sqrt{-g}\left[\frac{R}{2\kappa^2} - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi) - \xi_1(\phi)G + \xi_2(\phi)\mathcal{L}_c^{(2)}\right]. \tag{100}$$

As in (89), we may redefine the scalar field ϕ by $\phi = \epsilon\varphi$. If a proper limit of $\epsilon \to 0$ exists, the action (100) reduces to

$$S = \int d^4x\sqrt{-g}\left[\frac{R}{2\kappa^2} - \tilde{V}(\varphi) - \tilde{\xi}_1(\varphi)G + \tilde{\xi}_2(\varphi)\mathcal{L}_c^{(2)}\right]. \tag{101}$$

Here

$$\tilde{\xi}_2 = \lim_{\epsilon\to 0}\xi_2(\epsilon\varphi). \tag{102}$$

Then φ could be regarded as an auxiliary field and one gets

$$0 = \tilde{V}'(\varphi) - \tilde{\xi}_1'(\varphi)G + \tilde{\xi}_2'\mathcal{L}_c^{(2)}, \tag{103}$$

which may be solved with respect to φ as

$$\varphi = \Psi(G, \mathcal{L}_c^{(2)}). \tag{104}$$

Substituting (104) into the action (101), we obtain $F(G, \mathcal{L}_c^{(2)})$-gravity theory:

$$\begin{aligned}S &= \int d^4x\sqrt{-g}\left[\frac{R}{2\kappa^2} - F(G, \mathcal{L}_c^{(2)})\right],\\ F(G) &\equiv \tilde{V}(\Phi(G, \mathcal{L}_c^{(2)})) - \tilde{\xi}_1(\Phi(G, \mathcal{L}_c^{(2)}))G\\ &\quad + \tilde{\xi}_2(\Phi(G, \mathcal{L}_c^{(2)}))\mathcal{L}_c^{(2)}.\end{aligned} \tag{105}$$

In the case of the string-inspired gravity,

$$V = V_0e^{-(2\phi/\phi_0)}, \qquad \xi_1 = \xi_0e^{2\phi/\phi_0}, \qquad \xi_2 = \eta_0e^{4\phi/\phi_0}. \tag{106}$$

Here ϕ_0, V_0, ξ_0, and η_0 are constants. We may consider the limit of $\epsilon \to 0$ after redefining $\phi = \epsilon\varphi$ and $\phi_0 = \epsilon\varphi_0$. Thus, Eq. (103) gives

$$e^{2\varphi/\varphi_0} = \Theta(G, \mathcal{L}_c^{(2)}) \equiv \frac{\xi_0 G}{2\eta_0 \mathcal{L}_c^{(2)}} + Y(G, \mathcal{L}_c^{(2)}). \tag{107}$$

Here

$$Y(G, \mathcal{L}_c^{(2)}) = y_+ + y_-, \qquad y_+ e^{(2/3)\pi i} + y_- e^{(4/3)\pi i}, \qquad y_+ e^{(4/3)\pi i} + y_- e^{(2/3)\pi i} \tag{108}$$

and

$$y_\pm \equiv \left\{\frac{V_0}{4\eta_0 \mathcal{L}_c^{(2)}} \pm \sqrt{\left(\frac{V_0}{4\eta_0 \mathcal{L}_c^{(2)}}\right)^2 - \left(\frac{\xi_0 G}{6\eta_0 \mathcal{L}_c^{(2)}}\right)^6}\right\}^{1/3}. \tag{109}$$

Hence, the action of the corresponding $F(G, \mathcal{L}_c^{(2)})$ theory is

$$S = \int d^4x\sqrt{-g}\left[\frac{R}{2\kappa^2} - F(G, \mathcal{L}_c^{(2)})\right],$$
$$F(G, \mathcal{L}_c^{(2)}) = \frac{V_0}{\Theta(G, \mathcal{L}_c^{(2)})} - \xi_0 \Theta(G, \mathcal{L}_c^{(2)}) G + \eta_0 \Theta(G, \mathcal{L}_c^{(2)})^2 \mathcal{L}_c^{(2)}. \tag{110}$$

Instead of (50), one may consider the model with one more scalar field χ coupled with the Gauss-Bonnet invariant:

$$S = \int d^4x\sqrt{-g}\left[\frac{R}{2\kappa^2} - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{\epsilon}{2}\partial_\mu\chi\partial^\mu\chi - V(\phi) - U(\chi) - (\xi_1(\phi) + \theta(\chi))G\right]. \tag{111}$$

This kind of action often appears in the models inspired by string theory [14]. In such models, one scalar ϕ may correspond to the dilaton and another scalar χ to the modulus. We now consider the case in which the derivative of χ, $\partial_\mu\chi$, is small or ϵ is very small. Then we may neglect the kinetic term of χ, and χ could be regarded as an auxiliary field. Repeating the process (92)–(94), we obtain the $F(G)$ gravity coupled with the scalar field ϕ:

$$S = \int d^4x\sqrt{-g}\left[\frac{R}{2\kappa^2} - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi) - \xi_1(\phi)G + F(G)\right]. \tag{112}$$

The relation between scalar-Gauss-Bonnet gravity and modified Gauss-Bonnet gravity (or two parametrizations of the same theory) is discussed in this section. We show that cosmological solutions obtained in one of such theories may be used (with different physical interpretation, compare with [27]) in another theory. It often turns out that it is easier to work with a specific parametrization of the same theory. Of course, only comparison with observational data may select the truly physical theory in the correct parametrization.

IV. CONCLUSION

In this paper we have studied several aspects of (dilaton) gravity in the presence of string corrections up to third order in curvature. The second order term is Euler density of 2nd order, called the Gauss-Bonnet term. The next-to-leading term contains higher-order Euler density (E_3) plus a term of 3rd order in curvature. The expression of E_3 is identically zero in space-time of dimension less than 6; the term beyond the Euler density contributes to the equation of motion even for a fixed field ϕ. We have verified that the de-Sitter solution which exists in the case of type II and bosonic strings is an unstable node. We show that in the presence of a barotropic fluid (radiation/matter), an inflationary solution exists in the high curvature regime for a constant field.

For a dynamically evolving field ϕ canonical in nature, there exists an interesting dark energy solution (35) characterized by $H = h_0/t$, $\phi = \phi_0 \ln t/t_1$ for $h_0 > 0$ [$H = h_0/(t_s - t)$, $\phi = \phi_0 \ln((t_s - t)/t_1)$ when $h_0 < 0$]. The three-year WMAP data taken with the SNL survey [22] suggest that $w_{\rm DE} = -1.06^{+0.13}_{-0.08}$. We have shown that choosing a range of the parameter ϕ_0^2 (which is amplified thanks to the third order curvature term contribution) we can easily obtain the observed values of $w_{\rm DE}$ for phantom as well as for nonphantom dark energy. We have demonstrated, in detail, the stability of the dark energy solution. For nonphantom energy, in the large h_0 limit, we presented an analytical solution which shows that one of the eigenvalues of the 3×3 perturbation matrix is real and negative where as the other two are purely imaginary, thereby establishing the stability of the solution (35). We have verified numerically that stability holds for all smaller and generic values of h_0 in this case. The phantom dark energy solution corresponding to $h_0 < 0$ turns out to be unstable. It is remarkable that string curvature corrections can account for late-time acceleration and dark energy can be realized without the introduction of a field potential.

We show how scalar-Gauss-Bonnet gravity may be reconstructed for any given cosmology. The corresponding scalar potentials for several dark energy cosmologies, including quintessence, phantom, cosmological constant, and oscillatory regimes, are explicitly found. This shows that, having the realistic scale factor evolution, the principal possibility appears to present string-inspired gravity where such evolution is realized. We explain how to transform scalar-Gauss-Bonnet gravity (even with account of the third order curvature term) to modified Gauss-Bonnet gravity [26], which seems to pass the solar system tests.

Different forms of modified gravity are attempted recently (for a review, see [9]) to describe the dark energy universe; these models provide a qualitatively simple resolution of dark energy/coincidence problems and deserve further consideration. It is quite likely that the time has come to reconsider the basics of general relativity in the late universe in the search for realistic modified gravity/dark energy theory.

We should also mention that, in the present study, we have tested the background model against observations. The study of perturbations in the scenario discussed here is quite complicated and challenging and in our opinion it deserves attention; we defer this investigation to our future work.

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