Reflection and transmission at dimensional boundaries

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An inhomogeneous Kaluza-Klein compactification of a higher dimensional spacetime may give rise to an effective 4D spacetime with distinct domains having different sizes of the extra dimensions. The domains are separated by domain walls generated by the extra dimensional scale factor. The scattering of electromagnetic and massive particle waves at such boundaries is examined here for models without warping or branes. We consider the limits corresponding to thin (thick) domain walls, i.e., limits where wavelengths are large (small) in comparison to wall thickness. We also obtain numerical solutions for a wall of arbitrary thickness and extract the reflection and transmission coefficients as functions of frequency. Results are obtained which qualitatively resemble those for electroweak domain walls and other ordinary domain walls for 4D theories.

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I. INTRODUCTION

The possible existence of unseen extra dimensions could have important implications for the effective four dimensional physics that we observe [1–9]. Different types of extra dimensional models include nonorbifolded Kaluza-Klein (KK) compactifications without branes [1,2], brane world models with large [3] extra dimensions or warped [4,5] spacetimes, infinite extra dimensions [5,6], and universal extra dimension models [7,8]. Here, we consider KK-type models where a higher dimensional spacetime without branes or warping is compactified to an effective 4D spacetime in an *inhomogeneous* way, so that a scale factor $b(x^\mu)$ associated with the extra dimension(s) can vary with 4D position x^μ. From a 4D perspective, the scale factor b then appears as a 4D scalar field with an effective potential $U(b)$ [10,11]. If this effective potential has different minima separated by barriers, the field b can settle into these different minima at different positions x^μ. The boundaries between the different domains then appear as domain walls in the 4D theory, where the field $b(x^\mu)$ varies across the wall, generally possessing both gradient and potential energy densities [10,12]. The "gravitational bags" of Ref. [13] are described by exact analytical solutions to the field equations of a 6d theory where the extra two dimensions are compact outside the bag, but become completely decompactified at the center of the bag ($b \to \infty$ as $r \to 0$). Dimension bubbles [12,14,15] are similar nontopological solitons, but filled with particles and radiation helping to stabilize the bubble, and having slightly different boundary conditions ($b \to$ finite as $r \to 0$). Domain wall networks may give rise to such types of bubbles where the value of b inside a bubble is different from that outside the bubble [10,12,14,15]. It is also possible that an evaporating black hole may spawn a "modulus bubble" surrounding the black hole [16]. The values of b on different sides of such a domain wall may both be microscopic, or not. For example, if b takes values b_1 and b_2 on different sides of a wall, we may have a ratio $b_2/b_1 \sim 10^{\pm 16}$ if $(b_2/b_1)^{\pm 1} \sim l_P/l_{\rm TeV}$, i.e., one of the values is characteristic of the Planck scale, $l_P \sim M_P^{-1}$, and the other is characteristic of a TeV scale, $l_{\rm TeV} \sim {\rm TeV}^{-1}$ [15]. On the other hand, it is possible that the value of b becomes macroscopically large on one side of the wall [10,13].

In the 4D theory that follows from a (brane-free) compactified higher dimensional theory, the field $b(x^\mu)$ couples to electromagnetic fields in the form of a dielectric function ε. Massive particle fields in the 4D theory have masses which depend on b; for example, a particle with a mass m_5 in a 5D theory gives rise to a (Kaluza-Klein zero mode) particle with mass $m = m_5/\sqrt{b}$ in the 4D theory [15]. Therefore, there will be a difference in the way that both electromagnetic waves and massive particle waves will propagate in the two different domains, and we might anticipate a dependence of the amount of transmission across a dimensional domain boundary which depends upon the change in the field b and the spatial rate at which b varies. If, for instance, a massive particle has an energy $E = (p_1^2 + m_1^2)^{1/2}$ in a domain region where $b = b_1$, but across the wall where $b = b_2$ the particle mass is $m_2 > m_1$, then for energies $E < m_2$ the particle will undergo a total reflection, since it is energetically forbidden in the b_2 region. For higher energies, $E > m_2$, we expect a partial reflection, with reflection and transmission coefficients depending upon particle energy [17]. However, we may also anticipate a dependence upon the domain wall width (as compared to wavelength), as discovered by Everett [18] in his study of wave transmission across electroweak domain walls. Ayala, Jalilian-Marian, McLerran, and Vischer [19], and Farrar and McIntosh [20] have also examined wave propagation across electroweak domain walls, obtaining energy-dependent reflection and transmission coefficients. We obtain results for wave transmission across dimensional boundaries which qualitatively resemble some of the basic results obtained by Refs. [18–20] for

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wave transmission across electroweak domain walls. More specifically, we find that the reflection coefficient depends upon both b_1 and b_2, with *reflection* being enhanced at *low* energies, where wavelengths are long in comparison to wall width (thin wall limit). On the other hand, *transmission* is enhanced and reflection becomes negligible at *high* energies, where wavelengths are short compared to wall width (thick wall limit). For the thin wall limit we approximate the wall as a discontinuous boundary, and in the thick wall limit we consider a slowly varying field $b(x^\mu)$. Analytical results can be obtained for these limiting cases. A numerical study is also made for a smooth, continuous transition region for a range of energies and wavelengths.

The amount of reflection and transmission of various modes at a dimensional boundary will generally depend upon the compactification details. Later, as an example, we show how the qualitative behavior for the reflection of electromagnetic fields and massive particles differs for the RS1 model [4]. Thus, if dimensional boundaries could be probed experimentally, information about the extra dimensions could be gathered from the boundary's reflectivity.

In Sec. II we present the effective 4D theory that emerges from the KK reduction of a higher dimensional theory without warping or branes, and illustrate how the extra dimensional scale factor b appears as a scalar field in the 4d theory. The reflection and transmission of electromagnetic and massive particle waves from a thin domain wall is considered in Sec. III, and thick walls are treated in Sec. IV. Our numerical study for a domain wall of arbitrary width is presented in Sec. V. Section VI contains a summary and discussion of results, and expectations concerning scattering from dimensional boundaries in the visible brane of the RS1 model are mentioned.

II. THE EFFECTIVE 4D THEORY

We consider a $D = (4 + n)$ dimensional spacetime having n compact extra spatial dimensions. The metric of the D dimensional spacetime is assumed to take a form

$$\begin{aligned} ds_D^2 &= \tilde{g}_{MN} dx^M dx^N \\ &= \tilde{g}_{\mu\nu}(x) dx^\mu dx^\nu + b^2(x^\mu)\gamma_{mn}(y) dy^m dy^n, \end{aligned} \tag{1}$$

where $x^M = (x^\mu, y^m)$ and $M, N = 0, 1, 2, 3, \cdots, D-1$ label all the spacetime coordinates, $\mu, \nu = 0, 1, 2, 3$, label the 4D coordinates, and m, n label those of the compact extra space dimensions. The extra dimensional scale factor is $b(x^\mu)$, which is assumed to be independent of the y coordinates and the extra dimensional metric $\gamma_{mn}(y)$ depends upon the geometry of the extra dimensional space and is related to $\tilde{g}_{mn}(x, y)$ by $\tilde{g}_{mn} = b^2\gamma_{mn}$.

The action for the D dimensional theory is

$$S_D = \int d^D x \sqrt{|\tilde{g}_D|}\left\{\frac{1}{2\kappa_D^2}[\tilde{R}_D[\tilde{g}_{MN}] - 2\Lambda] + \tilde{\mathcal{L}}_D\right\}, \tag{2}$$

where $\tilde{g}_D = \det\tilde{g}_{MN}$, $\tilde{R}_D$ is the Ricci scalar built from $\tilde{g}_{MN}$, Λ is a cosmological constant for the D dimensional spacetime, $\tilde{\mathcal{L}}_D$ is a Lagrangian for the fields in the D dimensions, $\kappa_D^2 = 8\pi G_D = V_y\kappa^2 = 8\pi G$, where G is the 4D gravitational constant, G_D is the D dimensional one, and $V_y = \int d^n y\sqrt{|\gamma|}$ is the coordinate "volume" of the extra dimensional space. We use a mostly negative metric signature, $\mathrm{diag}(\tilde{g}_{MN}) = (+, -, -, \cdots, -)$.

To express the action as an effective 4D action, we borrow the relations used in Ref. [11]:

$$\sqrt{|\tilde{g}_D|} = b^n\sqrt{-\tilde{g}}\sqrt{|\gamma|}, \tag{3}$$

$$\begin{aligned} \tilde{R}[\tilde{g}_{MN}] = \tilde{R}[\tilde{g}_{\mu\nu}] + b^{-2}\tilde{R}[\gamma_{mn}] - 2nb^{-1}\tilde{g}^{\mu\nu}\tilde{\nabla}_\mu\tilde{\nabla}_\nu b \\ - n(n-1)b^{-2}\tilde{g}^{\mu\nu}(\tilde{\nabla}_\mu b)(\tilde{\nabla}_\nu b), \end{aligned} \tag{4}$$

where the number of extra dimensions n is not to be confused with the tensor index n and $\tilde{R}[\tilde{g}_{\mu\nu}]$ is the Ricci scalar built from $\tilde{g}_{\mu\nu}$, etc. The metric $\tilde{g}_{\mu\nu}$ then acts as a 4D Jordan frame metric. We define D dimensional and 4D gravitation constants by $2\kappa_D^2 = 16\pi G_D$ and $2\kappa^2 = 16\pi G$, respectively, which are related by

$$\frac{1}{2\kappa^2} = \frac{1}{16\pi G} = \frac{V_y}{16\pi G_D} = \frac{V_y}{2\kappa_D^2}. \tag{5}$$

Following Ref. [11] we consider compact spaces of extra dimensions with constant curvature and a curvature parameter defined by

$$k = \frac{\tilde{R}[\gamma_{mn}]}{n(n-1)}. \tag{6}$$

Integrating over y in the action of (2), we have

$$\begin{aligned} S = \int d^4x\sqrt{-\tilde{g}}\bigg\{&\frac{1}{2\kappa^2}[b^n\tilde{R}[\tilde{g}_{\mu\nu}] - 2nb^{n-1}\tilde{g}^{\mu\nu}\tilde{\nabla}_\mu\tilde{\nabla}_\nu b \\ &- n(n-1)b^{n-2}\tilde{g}^{\mu\nu}(\tilde{\nabla}_\mu b)(\tilde{\nabla}_\nu b) \\ &+ n(n-1)kb^{n-2}] + b^n\left[\mathcal{L}_D - \frac{\Lambda}{\kappa^2}\right]\bigg\}, \end{aligned} \tag{7}$$

where $\mathcal{L}_D = V_y\tilde{\mathcal{L}}_D$. We define a 4D Einstein frame metric $g_{\mu\nu}$ by

$$\begin{aligned} \tilde{g}_{\mu\nu} = b^{-n}g_{\mu\nu}, \qquad \tilde{g}^{\mu\nu} = b^n g^{\mu\nu}, \\ \sqrt{-\tilde{g}} = b^{-2n}\sqrt{-g}. \end{aligned} \tag{8}$$

In terms of the 4D Einstein frame metric the action S in (7) becomes

$$S = \int d^4x\sqrt{-g}\left\{\frac{1}{2\kappa^2}\left[R[g_{\mu\nu}] + \frac{n(n+2)}{2}b^{-2}g^{\mu\nu}(\nabla_\mu b) \times (\nabla_\nu b) + n(n-1)kb^{-(n+2)}\right] + b^{-n}\left[\mathcal{L}_D - \frac{\Lambda}{\kappa^2}\right]\right\}, \tag{9}$$

where total derivative terms have been dropped. From the D dimensional source Lagrangian $\mathcal{L}_D = V_y\tilde{\mathcal{L}}_D$ we can define an effective 4D source Lagrangian $\mathcal{L}_4$,

$$\mathcal{L}_4 = b^{-n}\mathcal{L}_D, \tag{10}$$

where, again, n is the number of extra dimensions. Notice that the extra dimensional scale factor $b(x)$ plays the role of a scalar field in the 4D theory. It will have a corresponding effective potential $U(b)$ that is constructed from the curvature and cosmological constant terms in (9) along with terms from $\mathcal{L}_4$. When the potential U possesses two or more minima separated by barriers, domain walls associated with the scalar field $b(x)$ can appear in the 4D theory. A domain wall interpolates between two different values of b_1 and b_2 on the two sides, and the energy density and width of the wall are expected to depend on b_1 and b_2, the potential $U(b)$, and how rapidly the field b varies. One can envision a thin wall where there is a sudden jump between b_1 and b_2, a thick wall where $b(x)$ slowly varies, or intermediate cases where the wall may be considered as neither thin nor thick.

A. Electromagnetic and scalar boson fields in 4D

We are interested in the propagation of massive particles and electromagnetic fields through regions where the size of the extra dimensions, characterized by the scale factor $b(x)$, changes. We will focus on the simple case of a free scalar boson as a prototype of a massive particle, and thereby neglect particle spin. In this case it is easy to see how the 4D particle mass m depends upon the field b for Kaluza-Klein (KK) zero mode bosons. For the electromagnetic field, it seems natural to adopt a dielectric approach [14], where the field $b(x)$ gives rise to an effective permittivity $\varepsilon(x)$. (Attention is restricted to KK zero modes.)

Electromagnetic field—We write the $D = (4+n)$ dimensional Lagrangian for the electromagnetic (EM) fields as

$$\begin{aligned}\mathcal{L}_{\mathrm{EM}} &= -\tfrac{1}{4}\tilde{F}'^{MN}\tilde{F}'_{MN} = -\tfrac{1}{4}\tilde{g}^{MA}\tilde{g}^{NB}F'_{AB}F'_{MN},\\ F'_{MN} &= \partial_M A'_N - \partial_N A'_M.\end{aligned} \tag{11}$$

We assume the field b to take a value b_0 in the ambient 4D spacetime, and then define a rescaled gauge field $A_M = b_0^{n/2}A'_M$. For KK zero modes we assume A_M to be independent of y^m and set $A_m = 0$. In the 4D Einstein frame $\mathcal{L}_{\mathrm{EM}}$ becomes $\mathcal{L}_{\mathrm{EM}} = -\frac{1}{4}b^{2n}F'^{\mu\nu}F'_{\mu\nu} = -\frac{1}{4}\times(b^{2n}/b_0^n)F^{\mu\nu}F_{\mu\nu}$, with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. By (10) the effective 4D EM Lagrangian is

$$\mathcal{L}_{4,\mathrm{EM}} = -\frac{1}{4}\left(\frac{b}{b_0}\right)^n F^{\mu\nu}F_{\mu\nu} = -\frac{1}{4}\varepsilon F^{\mu\nu}F_{\mu\nu}, \tag{12}$$

where the effective dielectric function is

$$\varepsilon(x) = \left(\frac{b(x)}{b_0}\right)^n. \tag{13}$$

In ordinary 4D vacuum regions where $b = b_0$, we have $\varepsilon = 1$.

Scalar bosons—For the case of a free scalar boson, we start with a D dimensional Lagrangian

$$\mathcal{L}_S = \tilde{\partial}^M\phi^*\tilde{\partial}_M\phi - V(\phi) = \tilde{g}^{MN}\partial_M\phi^*\partial_N\phi - V(\phi). \tag{14}$$

For KK zero modes ϕ is independent of y^m and $\mathcal{L}_S = \tilde{g}^{\mu\nu}\partial_\mu\phi^*\partial_\nu\phi - V$. In terms of the 4D Einstein frame metric from (8) we then have $\mathcal{L}_S = b^n g^{\mu\nu}\partial_\mu\phi^*\partial_\nu\phi - V$. From (10) it follows that the effective 4D Lagrangian is

$$\mathcal{L}_{4,S} = g^{\mu\nu}\partial_\mu\phi^*\partial_\nu\phi - b^{-n}V. \tag{15}$$

Therefore a bosonic particle with a mass μ_0 in the $4+n$ dimensional theory appears as a KK zero mode bosonic particle with a 4D mass m given by $m = b^{-n/2}\mu_0$. In terms of the dielectric function ε and the mass $m_0 = b_0^{-n/2}\mu_0$ we can write

$$m = \varepsilon^{-1/2}m_0 \tag{16}$$

with $m \to m_0$ as $\varepsilon \to 1$. The 4D mass decreases in regions of larger b and ε.

III. WAVE PROPAGATION THROUGH THIN WALLS

We consider here the thin wall limit for EM and particle waves, i.e., the limit in which the effective wavelengths are large in comparison to the wall width δ, or frequencies are sufficiently small, $\omega \ll 1/\delta$. The cases of EM waves and massive particle waves are considered separately, but in a similar manner. The transition region where $b(x)$ varies is idealized as a sharp boundary, i.e., as a planar interface perpendicular to the x axis.

A. Electromagnetic waves

The effects of a rapidly varying b upon EM wave propagation was investigated in Ref. [14]. An EM contribution in the form of Eq. (12) to the effective 4d theory can be treated with a dielectric approach where the dielectric function, or permittivity ε, in a region of space is given by Eq. (13), where b_0 is the value of the extra dimensional scale factor in a normal region of 4D spacetime. (The coordinates y^m could be rescaled to set $b_0 = 1$, but we leave its value arbitrary.) The permeability of a region of space is seen from the Maxwell equations to be $\mu = 1/\varepsilon$ so that the index of refraction in a region of space is $n =$

$\sqrt{\varepsilon\mu} = 1$ and the "impedance" is $Z = \sqrt{\mu/\varepsilon} = 1/\varepsilon \propto b^{-n}$, where n is the number of extra dimensions. In Ref. [14] it was found that at the boundary between two different constant values of b given by b_1 and b_2, the reflection ratio for a plane wave of frequency ω is given by

$$A_{\rm ref} = \frac{E_R}{E_I} = \frac{1-(Z_T/Z_I)}{1+(Z_T/Z_I)} = \frac{1-(\varepsilon_I/\varepsilon_T)}{1+(\varepsilon_I/\varepsilon_T)} = \frac{\varepsilon_T - \varepsilon_I}{\varepsilon_T + \varepsilon_I}, \tag{17}$$

where $Z_{I(T)}$ denotes the value of Z in the incident (transmitting) region, etc. For a reflection coefficient $\mathcal{R} = A_{\rm ref}^2$ we have

$$\mathcal{R} = \left(\frac{\varepsilon_T - \varepsilon_I}{\varepsilon_T + \varepsilon_I}\right)^2 = \left(\frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_2 + \varepsilon_1}\right)^2 = \left(\frac{\varepsilon_2/\varepsilon_1 - 1}{\varepsilon_2/\varepsilon_1 + 1}\right)^2, \tag{18}$$

with an invariance under the interchange $\varepsilon_1 \longleftrightarrow \varepsilon_2$ indicating the same amount of reflection from either side of the wall. There is a small amount of reflection for $(\Delta\varepsilon/\varepsilon)^2 \ll 1$, or $(\varepsilon_2/\varepsilon_1) \approx 1$, but a large amount of reflection when either $(\varepsilon_2/\varepsilon_1) \gg 1$ or $(\varepsilon_2/\varepsilon_1) \ll 1$.

B. Massive particles

We will neglect possible effects due to particle spin and polarization and therefore examine the reflection and transmission associated with a free scalar boson field described by Eq. (15) with a potential $V = \mu_0^2\phi^*\phi$. The 4D boson mass m is then given by Eq. (16). We consider the sharp boundary to be located at $x = 0$ with $\varepsilon = \varepsilon_1$ for $x < 0$ and $\varepsilon = \varepsilon_2$ for $x > 0$. The field ϕ satisfies the Klein-Gordon equation $\Box\phi + m^2\phi = 0$ with a jump in m^2 at $x = 0$. An incident plane wave of energy $E = \omega$ described by ϕ_0 propagating toward the right is assumed to be incident from the left ($\varepsilon = \varepsilon_1$) on the interface, and a reflected plane wave ϕ_1 is assumed to propagate back toward the left in this region. A transmitted wave of energy ω is transmitted in the region where $\varepsilon = \varepsilon_2$. The plane waveforms are

$$\phi = \left\{ \begin{array}{ll} \phi_0 + \phi_1 = A_0 e^{i(p_1 x - \omega t)} + A_1 e^{i(-p_1 x - \omega t)}, & (x<0) \\ \phi_2 = A_2 e^{i(p_2 x - \omega t)}, & (x>0) \end{array} \right\}, \tag{19}$$

where $\omega^2 = p_1^2 + m_1^2 = p_2^2 + m_2^2$ with $p_{1,2}^2$ assumed to be nonnegative, and

$$m^2 = \left\{ \begin{array}{ll} m_1^2 = \varepsilon_1^{-1} m_0^2, & (x<0) \\ m_2^2 = \varepsilon_2^{-1} m_0^2, & (x>0) \end{array} \right\}. \tag{20}$$

Note that if $\omega < m_2$, then p_2^2 is negative and the wave is exponentially attenuated in the region $x > 0$. In other words, free particles are not kinematically allowed into the transmitting region when $\omega < m_2$, resulting in an effective total reflection.

Requiring continuity of ϕ and $\phi' = \partial_x\phi$ at the boundary $x = 0$ yields

$$\frac{A_1}{A_0} = -\frac{(p_2 - p_1)}{(p_2 + p_1)}, \qquad \frac{A_2}{A_0} = \frac{2p_1}{(p_2 + p_1)}, \qquad p_{1,2} = (\omega^2 - m_{1,2}^2)^{1/2} \geq 0. \tag{21}$$

The x-component of the current density is given by $j^x = -i\phi^*\overleftrightarrow{\partial}_x\phi$ so that we can define reflection and transmission coefficients

$$\mathcal{R} = -\frac{j_1}{j_0} = \left(\frac{A_1}{A_0}\right)^2 = \left(\frac{p_2 - p_1}{p_2 + p_1}\right)^2 = \left(\frac{1-(p_1/p_2)}{1+(p_1/p_2)}\right)^2, \tag{22}$$

$$\mathcal{T} = \frac{j_2}{j_0} = \frac{p_2}{p_1}\left(\frac{A_2}{A_0}\right)^2 = \frac{4p_1p_2}{(p_1+p_2)^2} = \frac{4p_1}{(1+(p_1/p_2))^2}, \tag{23}$$

with $\mathcal{R} + \mathcal{T} = 1$. Again, since $\mathcal{R}$ and $\mathcal{T}$ are symmetric under the interchange of indices, there is the same amount of reflection and transmission regardless of which side is the incident side. There is a small amount of reflection when $(\Delta p/p)^2 \ll 1$ or $p_2/p_1 \approx 1$, i.e., when $\varepsilon_2/\varepsilon_1 \approx 1$. On the other hand, there is a large amount of reflection when either $p_2/p_1 \ll 1$ or $p_2/p_1 \gg 1$, i.e., when there is a large difference between ε_1 and ε_2 with $(\varepsilon_2/\varepsilon_1) \gg 1$ or $(\varepsilon_2/\varepsilon_1) \ll 1$. We therefore get the same qualitative results for reflection/transmission for both EM waves and particles at a sharp boundary, or for small frequencies or low energies, $\omega \ll 1/\delta$ and $\omega > m_{1,2}$. For the case where the particle energy ω is less than the particle mass in the transmitting region, a free particle is kinematically forbidden in that region, with ϕ assumed to be rapidly damped, so that there is effectively total reflection back into the incident region.

IV. WAVE PROPAGATION THROUGH THICK WALLS

In this section we consider the opposite of the thin wall limit, i.e., the thick wall limit for EM and particle waves where the variation in $b(x)$ in the transition region is very gradual. This is equivalent to a limit where wavelengths are small in comparison to the wall thickness, or frequencies and energies are high, $E = \omega \gg 1/\delta$.

A. Electromagnetic waves

For sourceless EM fields the effective Lagrangian is given by Eqs. (12) and (13), $\mathcal{L}_{4,\rm EM} = -\frac{1}{4}\varepsilon F^{\mu\nu}F_{\mu\nu}$ where the effective permittivity is $\varepsilon = (b/b_0)^n$ with the property that $\varepsilon \to 1$ in the normal vacuum. Also, from the Maxwell field equations we have a permeability $\mu = 1/\varepsilon$. These give rise to an index of refraction $n = \sqrt{\varepsilon\mu} = 1$ and impedance $Z = \sqrt{\mu/\varepsilon} = 1/\varepsilon$. Summarizing,

$$\varepsilon = \frac{1}{\mu} = \left(\frac{b}{b_0}\right)^n, \qquad n = \sqrt{\varepsilon\mu} = 1, \qquad Z = \sqrt{\frac{\mu}{\varepsilon}} = \frac{1}{\varepsilon}. \tag{24}$$

We have the inhomogeneous (from $\mathcal{L}_{4,\mathrm{EM}}$) and homogeneous (Bianchi identity) Maxwell equations (source-free case)

$$\nabla_\mu(\varepsilon F^{\mu\nu}) = 0, \qquad \nabla_\mu {}^*F^{\mu\nu} = 0, \tag{25}$$

where the dual tensor is ${}^*F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$. With our metric $g_{\mu\nu} = \mathrm{diag}(+,-,-,-)$ we have the field tensors

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}, \qquad F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}, \tag{26}$$

with $E_x = E_1$, $B_x = B_1$, etc. being the physical fields and $F_{ij} = F^{ij}$ and we assume a flat 4D spacetime with $g_{\mu\nu} = \eta_{\mu\nu}$.

For a simple ansatz we consider waves propagating in the $+x$ direction and we take $\varepsilon = \varepsilon(x)$ to be a smooth, slowly varying, although somewhat arbitrary, function. We take the field components $E_y(x,t)$ and $B_z(x,t)$ to be nonvanishing and complex, in general, with $\vec{H} = \vec{B}/\mu = \varepsilon\vec{B}$. There is then a nonvanishing Poynting vector

$$\vec{S} = \mathrm{Re}(\vec{E} \times \vec{H}^*) = \varepsilon\,\mathrm{Re}(\vec{E} \times \vec{B}^*), \tag{27}$$

with a nonvanishing component S_x. The field Eqs. (25) can then be written in the form

$$\dot{E} + B' + \gamma' B = 0, \qquad E' = -\dot{B}, \tag{28}$$

where $E = E_y$, $B = B_z$, an overdot represents differentiation with respect to t, a prime stands for differentiation with respect to x, and we have defined $\gamma = \ln\varepsilon$. These equations can be used to obtain the wave equation

$$B'' - \ddot{B} + \gamma'' B + \gamma' B' = 0. \tag{29}$$

Approximate solutions—We consider the special case where the scale factor $b(x)$, $\varepsilon(x)$, and $\gamma(x)$ are very slowly varying functions of x in a planar domain wall oriented perpendicular to the x axis. The overall change in ε can be arbitrarily large, but the rate of change must be sufficiently small for the frequencies under consideration. The magnetic field is assumed to be of the form

$$B(x,t) = Ae^{i\phi(x)}e^{-i\omega t}, \tag{30}$$

where the amplitude A is a real constant. The wave Eq. (29) then gives an equation for the phase function ϕ,

$$i\phi'' - \phi'^2 + \omega^2 + \gamma'' + i\phi'\gamma' = 0. \tag{31}$$

Now, for the ordinary case where $\varepsilon = \mathrm{const}$, or $\gamma' = 0$, the wave eq. for $B(x,t)$ reduces to the ordinary wave equation, $B'' - \ddot{B} = 0$, giving $\phi(x) = \pm\omega x$, $\phi' = \pm\omega$, $\phi'' = 0$. Therefore, for the case of a very slowly changing γ and ϕ', we, as a first approximation, drop the terms involving γ'' and ϕ'' and approximate $\phi'\gamma' \approx \pm\omega\gamma'$ in (31), so that it reduces to the approximate equation

$$(\phi'_\pm)^2 = \omega^2 \pm i\omega\gamma', \qquad \phi'_\pm = \pm\omega\sqrt{1 \pm i\frac{\gamma'}{\omega}}, \tag{32}$$

i.e., we use $\phi'\gamma' \approx +\omega\gamma'$ for the ϕ_+ solution and use $\phi'\gamma' \approx -\omega\gamma'$ for the ϕ_- solution, giving $\phi'_+ = \omega\sqrt{1 + i\frac{\gamma'}{\omega}}$ and $\phi'_- = -\omega\sqrt{1 - i\frac{\gamma'}{\omega}}$ for the two solutions. We consider frequencies for which $|\gamma'/\omega| \ll 1$ so that the approximate solution is given by

$$\phi'_\pm = \pm\omega + i\frac{\gamma'}{2}, \qquad (|\gamma'| \ll \omega). \tag{33}$$

Integrating this gives

$$\phi_\pm(x) - \phi_\pm(x_0) = \pm\omega(x - x_0) + \frac{i}{2}[\gamma(x) - \gamma(x_0)]. \tag{34}$$

We choose to set the constants $\phi_\pm(x_0) = \pm\omega x_0$ so that the solution simplifies to

$$\phi_\pm(x) = \pm\omega x + \frac{i}{2}\ln\left(\frac{\varepsilon(x)}{\varepsilon(x_0)}\right) = \pm\omega x + i\ln\left(\frac{\varepsilon}{\varepsilon_0}\right)^{1/2}, \tag{35}$$

where $\varepsilon = \varepsilon(x)$ and $\varepsilon_0 = \varepsilon(x_0)$. We then have $e^{i\phi_\pm} = (\frac{\varepsilon}{\varepsilon_0})^{-1/2}e^{\pm i\omega x}$ and

$$B_\pm(x,t) = A\left(\frac{\varepsilon}{\varepsilon_0}\right)^{-1/2}e^{\pm i\omega x}e^{-i\omega t}. \tag{36}$$

This describes waves propagating in the $\pm x$ directions traveling at the speed of light in vacuum ($\omega/k = 1$) with an effective amplitude $A(\frac{\varepsilon}{\varepsilon_0})^{-1/2}$ which varies with x.

From the field Eqs. (28) we have $\dot{E} + B' + \gamma' B = 0$ so that, using $\dot{E} = -i\omega E$ and $B'_\pm = i\phi'_\pm B_\pm$ we obtain the electric field

$$E_\pm = -\frac{i}{\omega}(B'_\pm + \gamma' B_\pm) = \left(\pm 1 - i\frac{\gamma'}{2\omega}\right)B_\pm. \tag{37}$$

The approximate solutions for the EM fields are then

$$B_\pm(x,t) = B_{z,\pm} = A\left(\frac{\varepsilon}{\varepsilon_0}\right)^{-1/2}e^{\pm i\omega x}e^{-i\omega t},$$
$$E_\pm(x,t) = E_{y,\pm} = \left(\pm 1 - i\frac{\gamma'}{2\omega}\right)A\left(\frac{\varepsilon}{\varepsilon_0}\right)^{-1/2}e^{\pm i\omega x}e^{-i\omega t}. \tag{38}$$

Reflection and transmission—The EM energy flow is indicated by the Poynting vector $\vec{S} = \mathrm{Re}(\vec{E} \times \vec{H}^*)$, with $\vec{H} = \vec{B}/\mu = \varepsilon\vec{B}$, so that

$$S_x = \varepsilon\,\mathrm{Re}(\vec{E} \times \vec{B}^*)_x = \frac{\varepsilon}{2}(E_y B_z^* + E_y^* B_z), \tag{39}$$

with $E_\pm = (\pm 1 - i\frac{\gamma'}{2\omega})B_\pm$. We have $E_+B_+^* = (1 - i\frac{\gamma'}{2\omega})B_+B_+^*$, $E_-B_-^* = (-1 - i\frac{\gamma'}{2\omega})B_-B_-^*$, etc., so that

$$\begin{aligned} E_+B_+^* + E_+^*B_+ &= 2B_+^*B_+, \\ E_-B_-^* + E_-^*B_- &= -2B_-^*B_-, \end{aligned} \tag{40}$$

giving

$$\begin{aligned} (S_x)_\pm &= \varepsilon\,\mathrm{Re}(\vec{E}_\pm \times \vec{B}_\pm^*) = \pm\varepsilon|\vec{B}_\pm|^2 = \pm\varepsilon\left[A^2\frac{\varepsilon_0}{\varepsilon}\right] \\ &= \pm\varepsilon_0 A^2. \end{aligned} \tag{41}$$

The Poynting vector $S_x \propto A^2$ is therefore x independent, indicating that no energy is lost by reflection or absorption by the traveling wave, i.e., within the approximation we have used, the transmission amplitude is unity and the reflection amplitude is zero, giving transmission and reflection coefficients $\mathcal{T} \approx 1$, $\mathcal{R} \approx 0$ for high frequencies $\omega \gg |\gamma'|$. For a linear approximation of $\gamma(x)$ in a domain wall of width δ, we have $\gamma' \approx (\gamma - \gamma_0)/(x - x_0) = \ln(\varepsilon/\varepsilon_0)/\delta$ and the condition $\omega \gg |\gamma'|$ translates into $\omega \gg |\frac{\ln(\varepsilon/\varepsilon_0)}{\delta}|$. So, for sufficiently high frequencies the wall is transparent,

$$\mathcal{T} \approx 1, \qquad \mathcal{R} \approx 0, \qquad \omega \gg |\gamma'| \sim \left|\frac{\ln(\varepsilon/\varepsilon_0)}{\delta}\right|. \tag{42}$$

This resembles the situation found for "ordinary" electroweak domain walls separating different electroweak phases [18–20].

B. Massive particles

We now focus again on "free" scalar bosons obeying the Klein-Gordon equation (KGE) $\Box\phi + m^2(x)\phi = 0$, or

$$\ddot{\phi} - \phi'' + m^2(x)\phi = 0, \tag{43}$$

where $m^2(x) = b^{-n}(x)\mu_0^2 = \varepsilon^{-1}(x)m_0^2$ is a very slowly varying function of x. We write the scalar field as

$$\phi(x, t) = Ae^{i\psi(x)}e^{-i\omega t}, \tag{44}$$

where the amplitude A is a real constant, $\psi(x)$ is a phase function to be determined, and $\omega = E = \sqrt{p^2 + m^2}$ is a fixed, constant energy (although the momentum p and mass m vary with x, in general). For the case that $m =$ const, the solutions are $\phi = Ae^{\pm ipx}e^{-i\omega t}$. For the more general case, using Eq. (43), the KGE gives an equation for the phase function

$$i\psi'' - \psi'^2 + (\omega^2 - m^2) = 0. \tag{45}$$

Again, for the case that $m =$ const, we have $\psi = \pm px$ with $\psi'' = 0$.

Approximate solutions—For slowly varying $m^2(x)$ let us assume $|\psi''| \ll \psi'^2$ so that we can drop the ψ'' term in Eq. (45). This leaves us with $\psi' = \pm\omega(1 - m^2/\omega^2)^{1/2}$ and we find that for a sufficiently mildly varying function $m(x) = m_0/\sqrt{\varepsilon(x)} \propto b^{-n/2}(x)$ or at a sufficiently high energy ω our assumption $|\psi''| \ll \psi'^2$ is valid, i.e.,

$$\left|\frac{\psi''}{\psi'^2}\right| = \left|\frac{mm'}{\omega^3}\right| \frac{1}{(1 - \frac{m^2}{\omega^2})^{3/2}} \ll 1. \tag{46}$$

Since $\omega \geq m$, this condition will be satisfied when the first factor on the right hand side is small, i.e., $|mm'/\omega^3| = |\frac{nb'}{2b}\frac{m^2}{\omega^3}| \lesssim |\frac{b'/b}{\omega}| \ll 1$. This is satisfied for an arbitrary function $m(x)$ at a sufficiently high energy ω, where $|m'/m| \ll \omega$ or $|b'/b| \ll \omega$. We have $\psi'^2 \approx (\omega^2 - m^2)$ and can then write, approximately,

$$\psi_\pm(x) = \pm\left\{\omega x - \frac{1}{2}\int_{x_0}^{x}\frac{m^2}{\omega}dx\right\} \tag{47}$$

and

$$\phi_\pm(x, t) = A_\pm e^{i\psi_\pm(x)}e^{-i\omega t}. \tag{48}$$

Reflection and transmission—The current density $j^\mu = i\phi^*\overleftrightarrow{\partial^\mu}\phi$ then gives, approximately,

$$\begin{aligned} j_\pm^x &= 2A^2\psi'_\pm = \pm 2A_\pm^2(\omega^2 - m^2)^{1/2} \\ &= \pm 2A_\pm^2\omega\left(1 - \frac{m^2}{\omega^2}\right)^{1/2}. \end{aligned} \tag{49}$$

Denoting $m(-\infty) = m_1$ and $m(+\infty) = m_2$, we write the transmission coefficient as

$$\begin{aligned} \mathcal{T} &= \frac{j_{2+}^x}{j_{1+}^x} = \frac{j_+^x(\infty)}{j_+^x(-\infty)} = \left(\frac{\omega^2 - m_2^2}{\omega^2 - m_1^2}\right)^{1/2} \\ &\approx \left[1 - \frac{(m_2^2 - m_1^2)}{\omega^2}\right], \qquad m^2/\omega^2 \ll 1. \end{aligned} \tag{50}$$

Therefore, up to small corrections of $O(m^2/\omega^2)$, we have a transmission coefficient of unity;

$$\mathcal{R} \approx 0, \qquad \mathcal{T} \approx 1, \qquad \omega \gg m, \qquad \omega \gg |m'/m|. \tag{51}$$

From (42) and (51) we conclude that thick walls are essentially transparent to particle and EM radiation at very high energies.

V. NUMERICAL RESULTS

We have made a numerical study of the reflection and transmission of electromagnetic and matter waves at a dimensional boundary characterized by a smooth, continuous transition region with a dielectric function $\varepsilon(x)$ which tends to a value $\varepsilon \rightarrow \varepsilon_1$ as $x \rightarrow -\infty$ and a value $\varepsilon \rightarrow \varepsilon_2$ as $x \rightarrow +\infty$. We examined monochromatic waves that solve the field equations and have used various ε_1 and ε_2 values. From the currents for the waves, the reflection coefficient $\mathcal{R}$ and transmission coefficient $\mathcal{T}$ were computed. These coefficients can then be displayed as functions of ω for various values of ε_1 and/or ε_2. The numerical procedure that the code is based upon is described in the appendix.

For a simple representation of a smooth transition region interpolating between ε_1 at $x = -\infty$ and ε_2 at $x = +\infty$, we take $\varepsilon(x)$ to be given by the function

$$\varepsilon(x) = C + D\tanh\left(\frac{x}{\delta}\right) = \frac{1}{2}\left\{(\varepsilon_1 + \varepsilon_2) + (\varepsilon_2 - \varepsilon_1)\tanh\left(\frac{x}{\delta}\right)\right\}, \tag{52}$$

where $C = (\varepsilon_1 + \varepsilon_2)/2$ and $D = (\varepsilon_2 - \varepsilon_1)/2$. We also define $\gamma(x) = \ln\varepsilon(x)$, as before. The parameter δ characterizes the width of the domain wall function $\varepsilon(x)$. Dimensioned quantities are rescaled by factors of δ to give dimensionless ones. For example, we have dimensionless quantities (denoted with an overbar)

$$\bar{x} = \frac{x}{\delta}, \qquad \bar{\omega} = \omega\delta, \qquad \bar{m} = m\delta, \qquad \bar{k} = k\delta, \qquad \bar{\lambda} = \frac{\lambda}{\delta} = \frac{2\pi}{\bar{k}}, \tag{53}$$

where $k = \sqrt{\omega^2 - m^2(x)}$ is the particle momentum and $\omega = E$ is the particle energy. (The equations with dimensionless parameters can be obtained from those with dimensioned parameters by simply setting $\delta = 1$ and regarding the dimensioned parameters as dimensionless ones.)

A. Electromagnetic waves

We write the electric and magnetic fields, respectively, as $E(x, t) = E(x)e^{-i\omega t}$ and $B(x, t) = B(x)e^{-i\omega t}$, with $E(x) = E_y(x)$ and $B(x) = B_z(x)$ being complex-valued, in general. The field equations are given by (28) and lead to the wave Eq. (29) for $B(x, t)$. The monochromatic ansatz leads to the wave equation for $B(x)$:

$$B'' + \omega^2 B + \gamma'' B + \gamma' B' = 0, \tag{54}$$

where again the prime stands for differentiation with respect to x. Using Eq. (28) $E(x)$ is given by $E = -\frac{i}{\omega} \times (B' + \gamma' B)$. From Eq. (39) the x component of the Poynting vector is

$$S = \varepsilon \operatorname{Re}(EB^*) = \frac{\varepsilon}{2}(EB^* + E^*B). \tag{55}$$

The wave Eq. (54) is to be solved numerically, subject to boundary conditions, which are posted in the form of solutions to the wave equation in the asymptotic regions. That is, as $x \rightarrow \pm\infty$, we have $\gamma' \rightarrow 0$ and solutions for B and E are left-moving and right-moving plane wave solutions:

$$B(x) = \begin{cases} e^{i\omega x} + A_1 e^{i\delta_1} e^{-i\omega x}, & x \rightarrow -\infty \\ A_2 e^{i\delta_2} e^{i\omega x}, & x \rightarrow +\infty \end{cases} \tag{56}$$

and

$$E(x) = \begin{cases} e^{i\omega x} - A_1 e^{i\delta_1} e^{-i\omega x}, & x \rightarrow -\infty \\ A_2 e^{i\delta_2} e^{i\omega x}, & x \rightarrow +\infty \end{cases}, \tag{57}$$

where A_1 and δ_1 are the (real) amplitude and phase constant, respectively, for the reflected wave, and A_2 and δ_2 are those for the transmitted wave and the incident fields are $E_{\rm inc} = B_{\rm inc} = e^{i\omega x}$. The EM energy momentum is conserved with $S' = 0$. From Eqs. (55)–(57) we have

$$S = \begin{cases} S_{\rm inc} + S_-(-\infty) = \varepsilon_1(1 - A_1^2), & x \rightarrow -\infty \\ S_+(+\infty) = \varepsilon_2 A_2^2, & x \rightarrow +\infty \end{cases}, \tag{58}$$

where $S_{\rm inc} = \varepsilon_1$ is the energy-momentum of the incident beam, $S_- = S - S_{\rm inc}$ and S_+ represent the reflected and transmitted energy-momentum flow, respectively. The solutions should respect the condition $S' = 0$, or

$$\varepsilon_1(1 - A_1^2) = \varepsilon_2 A_2^2. \tag{59}$$

The reflection and transmission coefficients are then defined by

$$\mathcal{R} = -\frac{S_-(-\infty)}{S_{\rm inc}} = A_1^2, \qquad \mathcal{T} = \frac{S_+(+\infty)}{S_{\rm inc}} = \frac{\varepsilon_2}{\varepsilon_1} A_2^2. \tag{60}$$

The condition given by Eq. (59) then implies that $\mathcal{R} + \mathcal{T} = 1$.

Results obtained for the reflection and transmission coefficients, as functions of dimensionless frequency $\bar{\omega} = \omega\delta$, are illustrated in Fig. 1. These coefficients are monotonic functions and have limiting values that are in agreement with the analytical results obtained earlier for thin walls ($\omega \rightarrow 0$, $\lambda \gg \delta$), given by Eq. (18), and for thick walls ($\omega \gg \delta^{-1}$, $\lambda \ll \delta$), given by Eq. (42). From Fig. 1 the reflection is seen to be substantially reduced below its maximum value around a frequency $\omega \sim \delta^{-1}$, i.e., $\bar{\omega} \sim 1$. The numerical study indicates that the functions $\mathcal{R}$ and $\mathcal{T}$ are invariant under the interchange $\varepsilon_1 \leftrightarrow \varepsilon_2$, so that the amount of reflection does not depend on whether the beam is incident from the left or the right.

B. Massive particles

The boson field obeys the Klein-Gordon equation $\Box\phi + m^2(x)\phi = 0$ and we take $\phi(x, t) = \phi(x)e^{-i\omega t}$. From Eq. (16) we have $m^2(x) = m_0^2/\varepsilon(x)$, where m_0 is a constant. We define a dimensionless mass parameter $\bar{m}_0 = m_0\delta$ and set $\bar{m}_0 = 1$ for convenience, letting δ^{-1} set a mass scale. The Klein-Gordon equation for $\phi(x)$ can then be written in terms of dimensionless parameters, and takes

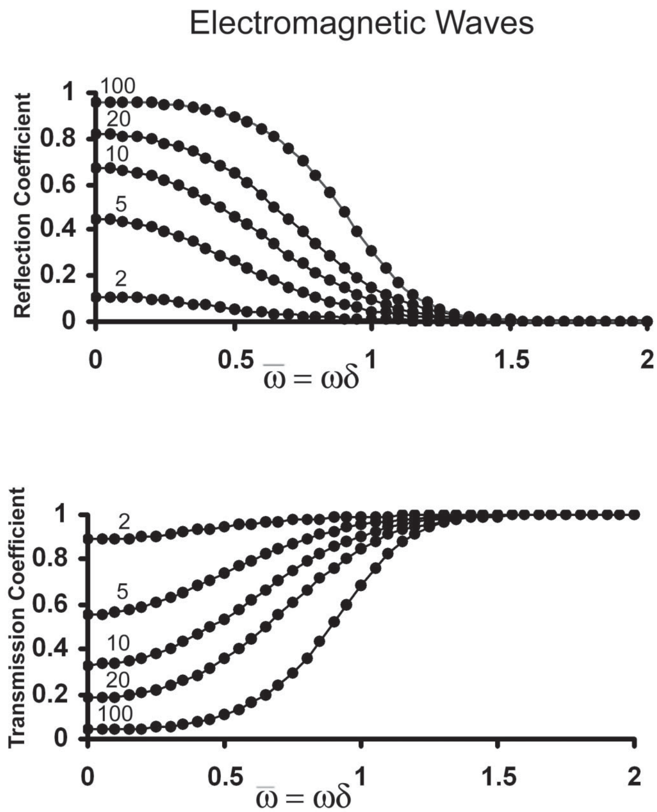

FIG. 1. Reflection and transmission coefficients for electromagnetic waves. The values for ε_2 (2, 5, 10, 20, 100) are given in the figure. In all cases ε_1 was set equal to 1. Top: Reflection coefficient $\mathcal{R}$ as a function of dimensionless frequency $\bar{\omega} = \omega\delta$. The coefficient agrees with the theoretical value of $\mathcal{R}(\bar{\omega} = 0) = (\frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_1 + \varepsilon_2})^2$. Bottom: Transmission coefficient as a function of $\bar{\omega}$.

the form

$$\phi'' - \left(\bar{\omega}^2 - \frac{1}{\varepsilon(x)}\right)\phi = 0 \tag{61}$$

or $\phi''(x) - \bar{k}^2(x)\phi(x) = 0$, where $\bar{m}_0 = 1$ and

$$\bar{k} = \sqrt{\bar{\omega}^2 - \bar{m}^2(x)} = \sqrt{\bar{\omega}^2 - \frac{1}{\varepsilon(x)}}. \tag{62}$$

In the asymptotic regions we have $\varepsilon \to \varepsilon_1$ as $x \to -\infty$ and $\varepsilon \to \varepsilon_2$ as $x \to +\infty$, so that

$$\bar{k} = \left\{ \begin{array}{ll} \bar{k}_1 = \sqrt{\bar{\omega}^2 - \frac{1}{\varepsilon_1}}, & x \to -\infty \\ \bar{k}_2 = \sqrt{\bar{\omega}^2 - \frac{1}{\varepsilon_2}}, & x \to +\infty \end{array} \right\} \tag{63}$$

and

$$\phi(x) = \left\{ \begin{array}{ll} \phi_{\rm inc} + \phi_1 = e^{ik_1x} + A_1 e^{i\delta_1} e^{-ik_1x}, & x \to -\infty \\ \phi_2 = A_2 e^{i\delta_2} e^{ik_2x}, & x \to +\infty \end{array} \right\}. \tag{64}$$

The bosonic current density $j^\mu(x,t) = i\phi^* \overleftrightarrow{\partial^\mu} \phi$ is conserved, so that $j'(x) = 0$, where the x component of the current is

$$j(x) = -i(\phi^*\phi' - \phi^{*\prime}\phi) = 2\,\mathrm{Im}(\phi^*\phi'). \tag{65}$$

From Eqs. (64) and (65) the asymptotic currents are

$$j = \left\{ \begin{array}{ll} j(-\infty) = j_{\rm inc} + j_-(-\infty) = 2k_1(1 - A_1^2), & x \to -\infty \\ j(+\infty) = j_+(+\infty) = 2k_2A_2^2, & x \to +\infty \end{array} \right\}, \tag{66}$$

where $j_- = -2k_1A_1^2$ is the current of the reflected wave, j_+ is the current of the transmitted right-moving wave, and $j_{\rm inc} = 2k_1$ is the current of the incident beam. We define the reflection and transmission coefficients as

$$\mathcal{R} = \frac{-j_-(-\infty)}{j_{\rm inc}} = A_1^2, \qquad \mathcal{T} = \frac{j_+(+\infty)}{j_{\rm inc}} = \frac{k_2}{k_1}A_2^2. \tag{67}$$

From $j' = 0$ it follows that the solutions must satisfy the condition $\mathcal{R} + \mathcal{T} = 1$.

The results for the reflection and transmission coefficients as functions of frequency $\bar{\omega} = \omega\delta$ obtained from the numerical solutions are shown in Fig. 2. Again, these are monotonic functions, as expected, and $\mathcal{T} \to 1$ in the high energy limit where the thick wall approximation becomes valid [see Eq. (51)] for $\lambda \ll \delta$, or $\bar{\omega} \gg 1/\varepsilon$. We have chosen $\varepsilon_1 = 1$ in Fig. 2, so that $k \geq 0$ implies that $\bar{\omega} \geq 1$. For $\bar{\omega}$ only a few percent larger than unity, there is an apparent deviation from the analytical results obtained in Eqs. (22) and (23) for the thin wall approximation where we used a sharp boundary. This is understood by writing the thin wall approximation $\lambda \gg \delta$ in terms of $\bar{\omega}$, i.e., $\frac{1}{\sqrt{\varepsilon}} \leq \bar{\omega} \ll 2\pi\sqrt{1 + \frac{1}{4\pi^2\varepsilon}}$. This inequality is hard to satisfy because of the presence of the lower bound, $\bar{\omega} \geq 1$ in the regions where a propagating wave is kinematically allowed, making the range of $\bar{\omega}$ rather restricted for the case where $\varepsilon \geq 1$. However, in the limit that $\bar{\omega} \to 1$, ($k_1 \to 0$) we numerically determine that $\mathcal{R} \to 1$, $\mathcal{T} \to 0$ as implied by Eqs. (22) and (23). Therefore, the analytical thin wall results are approached in the limit $\bar{\omega} \to 1$, but quickly deviate somewhat from the thin wall approximation in the plots of Fig. 2. As with the case of electromagnetic waves, the numerical study indicates that the functions $\mathcal{R}$ and $\mathcal{T}$ are invariant under the interchange $\varepsilon_1 \leftrightarrow \varepsilon_2$, so that the amount of reflection is independent of whether the beam is incident from the left or the right.

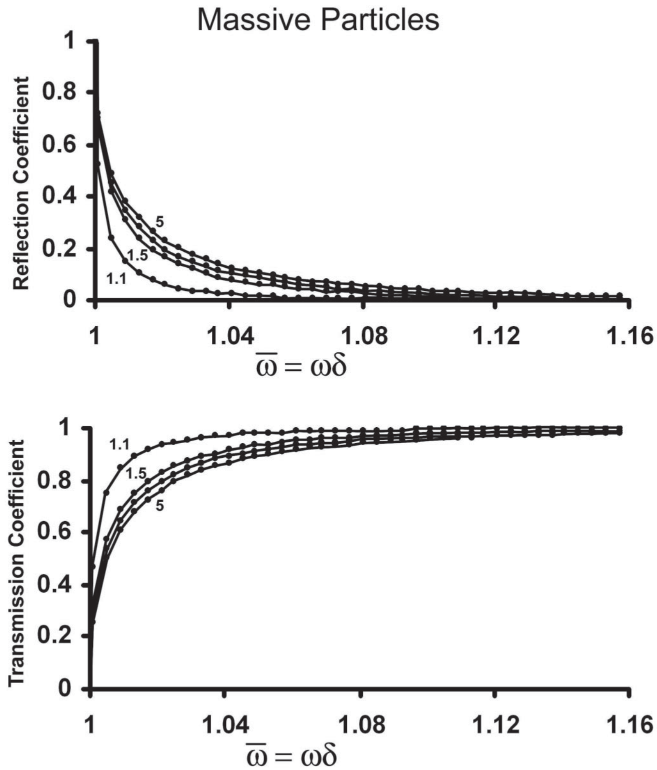

FIG. 2. Reflection and transmission coefficients for massive spinless bosons. The values of ε_2 (1.1, 1.5, 2.0, 5) are given in the figure except for $\varepsilon_2 = 2$ which is omitted for visual clarity. In all cases ε_1 was set equal to 1. Top: Reflection coefficient $\mathcal{R}$ as a function of dimensionless frequency $\bar{\omega} = \omega\delta$. Numerically we found $\mathcal{R} \to 1$ as $\bar{\omega} \to 1$. Bottom: Transmission coefficient $\mathcal{T}$ as a function of $\bar{\omega}$.

VI. SUMMARY AND DISCUSSION

We have considered a situation wherein an *inhomogeneous* compactification of a $4 + n$ dimensional spacetime, without warping or branes, with n compact extra space dimensions, gives rise to an effective 4D spacetime with distinct domains having different sizes of the extra dimensions. From a 4D point of view these domains are separated by domain walls arising from a 4D scalar field $b(x^\mu)$, which is also the scale factor for the extra dimensions in the higher dimensional spacetime. The field b giving rise to a domain wall takes an asymptotic value of b_1 on one side of the wall and an asymptotic value of b_2 on the other side, so that the domain wall serves as a dimensional boundary. We have focused on the reflection and transmission of both electromagnetic waves and massive bosonic particle waves across such dimensional boundaries. This has been done by examining the limiting cases of thin (thick) walls, i.e., wall thicknesses that are small (large) in comparison to the wavelengths of the propagating waves. A convenient parameter for describing the sizes of the extra dimensions is the "dielectric function" $\varepsilon(x) = (b(x)/b_0)^n$ where b takes an asymptotic, constant value of b_0 in a region of ordinary 4D vacuum.

The results we obtain for the reflection and transmission across dimensional boundaries is qualitatively similar to those obtained for ordinary domain walls in a 4D theory. Specifically, we find that at very high energies the boundaries are essentially transparent to EM and particle radiation, while at low energies, the degree of reflectivity can be quite high if either $\varepsilon_2/\varepsilon_1 \gg 1$ or $\varepsilon_2/\varepsilon_1 \ll 1$, that is, if there is a dramatic change in the size of the extra dimensions across the boundary. This could be realized, for example, when the extra dimensions remain microscopically small in both regions while $(b_2/b_1)^{\pm 1} \sim l_P/l_{\mathrm{TeV}}$, where $l_p \sim M_P^{-1}$ is the Planck length and $l_{\mathrm{TeV}} \sim \mathrm{TeV}^{-1}$. For particles with nonzero masses there is a threshold energy ($\omega \geq \omega_{\min}$, for which k_1 or k_2 becomes zero) for propagating waves, with $\mathcal{R} \to 1$ as $\omega \to \omega_{\min}$. Our numerical study substantiates these results for the case of a domain wall of arbitrary width δ, with $\mathcal{R}$ and $\mathcal{T}$ being monotonic functions smoothly connecting the thin wall and thick wall approximations for various wavelengths.

We note, however, that the results obtained here are valid only for extra dimensional models having compact extra dimensions without warping or branes. Braneworld models can exhibit different qualitative behaviors for the reflection and transmission of massless or massive modes. As an example, consider the RS1 model [4] consisting of one extra dimension compactified on an S^1/Z_2 orbifold. The background spacetime metric is

$$ds^2 = e^{-2kr|\phi|} g_{\mu\nu} dx^\mu dx^\nu - r^2 d\phi^2, \qquad (68)$$

where $-\pi \leq \phi \leq \pi$, with (x^μ, ϕ) and $(x^\mu, -\phi)$ identified. The two 3-branes are located at $\phi = 0$ (hidden brane) and $\phi = \pi$ (visible brane). The parameter k is a constant, $e^{-2kr|\phi|}$ is the warp factor, and r is the radius of the compactified extra dimension. Let us consider a situation wherein the radius r becomes a function of x^μ, corresponding to a case of inhomogeneous compactification. If the spatial variation of r is mild, we would expect the basic results obtained from the RS1 model with constant r to be approximately valid, at least qualitatively. In particular, a physical particle mass m on the visible brane is related to the mass parameter m_0 appearing in the 5D theory by

$$m = e^{-kr\pi} m_0. \qquad (69)$$

Therefore, if r varies with position x, the mass m on the visible brane is smaller for larger r, i.e., for a larger extra dimension. This is the same type of basic behavior found above for our unwarped, brane-free models, and we therefore expect the same qualitative type of reflection behavior for massive particles at a dimensional boundary. The story is different, however, for electromagnetic fields. To see this, we write the contribution to the EM fields on the visible brane as

$$S_v = \int d^4x \sqrt{-g_v} \Big(-\frac{1}{4} g_v^{\mu\alpha} g_v^{\nu\beta} F_{\alpha\beta} F_{\mu\nu} \Big), \qquad (70)$$

where the induced metric on the visible brane is $g_{v,\mu\nu} = e^{-2kr\pi}g_{\mu\nu}$, with $g_{\mu\nu}$ the 4d Einstein frame metric. The visible brane EM action can then be rewritten as

$$S_v = \int d^4x\sqrt{-g}\left(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}\right) \tag{71}$$

from which we see an effective dielectric function of unity, $\varepsilon = 1$. We therefore expect no reflection of EM fields from a dimensional boundary in the visible brane. This qualitative difference in the reflectivity of EM fields at a dimensional boundary could thus serve to distinguish the RS1 braneworld model from one with a brane-free, warp-free compactified extra dimension. Similarly, the qualitative and quantitative differences between various extra dimensional models regarding transmission and reflection of massless and massive modes could help to differentiate among various models.

ACKNOWLEDGMENTS

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APPENDIX A: NUMERICAL CONSIDERATIONS

Here we discuss some details of the numerical procedure used to extract solutions for the Klein-Gordon and electromagnetic wave equations (see previous section), which can be written in terms of a dimensionless frequency $\bar{\omega} = \omega\delta$ as

$$\phi'' + \bar{\omega}^2\phi - \varepsilon^{-1}\phi = 0, \tag{A1}$$

$$B'' + \gamma' B' + \gamma'' B + \bar{\omega}^2 B = 0 \tag{A2}$$

respectively, where $\varepsilon = \frac{1}{2}[(\varepsilon_1 + \varepsilon_2) + (\varepsilon_2 - \varepsilon_1)\tanh\bar{x}]$ and $\gamma = \ln\varepsilon$. The numerical procedure is essentially the same for either case. We discuss here our procedure for the Klein-Gordon equation. We assume complex solutions and write $\phi = \phi_R + i\phi_I$. The real and imaginary parts of ϕ both independently satisfy Eq. (A1). The function ε is constant far away from the transition region centered at $x = 0$. Therefore we assume pure oscillatory solutions in asymptotic boundary regions far from $x = 0$. Let ϕ^- and ϕ^+ be the boundary solutions at the positive and negative x boundary regions. The boundary solutions must then take the form

$$\phi^- = e^{ik_1x} + A_1e^{i\delta_1}e^{-ik_1x}, \tag{A3}$$

$$\phi^+ = A_2e^{i\delta_2}e^{ik_2x} + A_3e^{i\delta_3}e^{-ik_2x}, \tag{A4}$$

where $k = \sqrt{\bar{\omega}^2 - \varepsilon^{-1}}$. The boundary condition ϕ^- represents our ''initial'' conditions for the numerical integration of the differential equations associated with ϕ_R and ϕ_I. The second term in Eq. (A3) represents the reflected wave. Since we are interested in only right-moving waves at $x \to +\infty$, we seek solutions where $A_3 = 0$. Thus we search for solutions that connect ϕ^- to ϕ^+ with $A_3 = 0$. A two parameter search is needed to meet these conditions. The parameter space $(0 < A_1 < 1,\ -\pi < \delta_1 < \pi)$ is searched for the outgoing wave which has $A_3 = 0$. Numerically, this condition is met in the following manner: Eq. (A4) is examined at $x = \frac{2n\pi}{k_2}$ and $x = \frac{(4n+1)\pi}{2k_2}$, where n is a positive integer large enough so that a pure oscillatory solution is guaranteed. We then get

$$\phi_R^+(2n\pi/k_2) = \alpha_1 = A_2\cos\delta_2 + A_3\cos\delta_3, \tag{A5}$$

$$\phi_I^+(2n\pi/k_2) = \alpha_2 = A_2\sin\delta_2 + A_3\sin\delta_3, \tag{A6}$$

$$\phi_R^+((4n\pi + 1)/2k_2) = \beta_1 = -A_2\sin\delta_2 + A_3\sin\delta_3, \tag{A7}$$

$$\phi_I^+((4n\pi + 1)/2k_2) = \beta_2 = A_2\cos\delta_2 - A_3\cos\delta_3. \tag{A8}$$

After a little rearrangement we arrive at the equations we use for the numerical search:

$$A_2 = \frac{1}{2}\sqrt{(\alpha_1 + \beta_2)^2 + (\alpha_2 - \beta_1)^2}, \tag{A9}$$

$$A_3 = \frac{1}{2}\sqrt{(\alpha_1 - \beta_2)^2 + (\alpha_2 + \beta_1)^2}. \tag{A10}$$

Technically, we found that setting $n = 5$ and a linear interpolation to the solutions (A5) and (A6) at the specified x boundary values were of sufficient numerical accuracy for our purposes. From Eq. (A10) we see that the numerical solution must meet the condition that $\alpha_1 = \beta_2$ and $\alpha_2 = -\beta_1$. The reflection and transmission coefficients are then given by Eq. (67): $\mathcal{R} = A_1^2$ and $\mathcal{T} = \frac{k_2}{k_1}A_2^2$. Each two parameter search was conducted at a specific value of $\bar{\omega}$. Numerical integration of the differential equations for ϕ_R and ϕ_I was accomplished using an Adams Pece integrator in the numerical integration code entitled *DE* by Shampine and Gordon [21]. A code was written in FORTRAN and executed on IUN's AVIDD-N computer cluster.

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