

Large N reductions and holography

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The large N reductions in gauge theories are identified with dimensional reductions with homogeneous distribution of the eigenvalues of the gauge fields, and it is used to identify the corresponding closed string descriptions in the Maldacena duality. When one does not take the zero-radii limit, the large N reductions are naturally extended to the equivalences between the gauge theories and the “generalized” reduced models, which naturally contain the notion of T -dual equivalence. In the dual gravitational description, T -duality relates two type IIB supergravity solutions, the near horizon geometry of $D3$ -branes, and the near horizon geometry of D -instantons distributed densely and homogeneously on the dual torus. This is the holographic description of the generalized large N reductions.

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I. INTRODUCTION

The large N limit of gauge theories leads to a drastic reduction of dynamical degrees of freedom [1]. The quantities in a gauge theory in D dimension can be calculated from a much simpler reduced model, which is obtained by dropping off the spacetime dependence of the original gauge theory. The crucial condition for this large N reduction to take place, in the case of $SU(N)$ gauge group, is a homogeneous distribution of the eigenvalues of gauge fields, which preserves the $(Z_N)^D$ symmetry. This is essentially because the homogeneous distribution of the eigenvalues generates spacetime momentum from the gauge group [2–5].

On the otherhand, the celebrated Maldacena’s duality conjecture [6] states that the large N gauge theories have dual descriptions in terms of closed strings in higher dimensions, concretely realizing the ideas of the large N gauge theory—closed string duality [7] and holography [8] at the same time. It is interesting to ask how the large N reductions are realized in the dual closed string theory via the Maldacena duality. Recently in the Gopakumar’s program towards a precise formulation of the large N gauge theory—closed string duality [9], I studied ’t Hooft-Feynman diagrams of correlation functions in gauge theories compactified on a thermal circle, to read off the corresponding dual geometries [10]. It was mentioned that the technique used for calculating the thermal correlation functions was reminiscent to that appeared in the large N reductions. However, this aspect was not investigated in depth there. In the present article, I clarify its relation to the large N reductions, and its relevance for finding the dual holographic realization in the Maldacena duality. Some aspects of the large N reductions will be shown to have simple explanations in the dual closed string description.

One of the motivations for this study is that the reduced models are convenient for putting on computers, and there-

fore clarifying the holographic dual description of the large N reductions will lead to the test of the Maldacena duality by computer simulations.

Another main motivation is that this gives a concrete way to obtain a closed string theory from the matrix model of M -theory or the type IIB matrix model, via the well studied Maldacena duality.

II. LARGE N REDUCTIONS IN MALDACENA DUALITY

In this section, I first review and extend the argument of [10] for how to probe the dual geometry of the $(Z_N)^D$ symmetric phase by the correlation functions in gauge theories compactified on a D dimensional torus. Then, the large N reductions of the gauge theories are obtained as a limit where the size of the torus is taken to zero. The issue of stability of the $(Z_N)^D$ symmetric distribution of the eigenvalues of the gauge fields will be discussed with the comparison with the stability of the corresponding dual geometries.

As an example I take $D = 4$ case, where the boundary description is naturally identified with some $SU(N)$ gauge theory. Throughout this article I will work in the $N \rightarrow \infty$ planar limit [7]. I study the case where all fields are in adjoint representation of the gauge group.¹ When there are fermions, I put periodic boundary conditions on them in all the compactified directions.² This is necessary for obtaining the reduced model which reproduces the original gauge

¹One can also introduce fields in fundamental representation and repeat the arguments similar to the one below, but baryons may be missed from such arguments based on Feynman diagrams [11]. One may still expect from the dual holographic descriptions similar to what is discussed in this article that the large N reductions still take place.

²In the thermal case, fermions obey the antiperiodic boundary condition in the Euclidean time direction. As long as the phase is in the $(Z_N)^D$ symmetric phase the following argument apply, but which phase is realized depends also on these boundary conditions.

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theory results. The crucial condition for the large N reduction to take place is that the gauge field takes the configuration

$$A_\mu = \frac{1}{R_\mu} \text{diag}(\theta_\mu^1, \dots, \theta_\mu^N) \quad (1)$$

in an appropriate gauge, where θ_μ^a are distributed homogeneously between $[-\frac{1}{2}, \frac{1}{2}]$. The square expectation value of a Wilson loop winding around a cycle of T^4 is an order parameter of the $(Z_N)^4$ symmetry. It vanishes in the $(Z_N)^D$ symmetric phase: $\langle |W|^2 \rangle = 0$ [12], where

$$W = \frac{1}{N} \text{Tr} P \exp i \oint_0^{2\pi R_\mu} A_\mu dx^\mu, \quad \mu = 0, \dots, 3. \quad (2)$$

Here R_μ is the compactification radius in μ -th direction and P denotes the path ordering. Whether the above configuration is realized or not depends on the theory, here I am interested in a class of theories where this is the case. However, see the discussions on the Eguchi-Kawai reduction below.

Suppose one calculates some field theory correlator $\langle \mathcal{O}_1(K_{\mu 1}) \cdots \mathcal{O}_n(K_{\mu n}) \rangle$ of gauge invariant local single trace operators $\mathcal{O}(K_{\mu j})$, $\text{Tr} \Phi^{I_{j1}} \cdots \Phi^{I_{mj}}(K_{\mu j})$ for example. Here, $K_{\mu j}$ is an external momentum of the j -th operator which takes integer values in the unit of $\frac{1}{R_\mu}$, and Φ^I 's are adjoint scalars. I take the background gauge $D_\mu \mathcal{A}^\mu = \partial_\mu \mathcal{A}^\mu + i[A_\mu, \mathcal{A}^\mu] = 0$, with \mathcal{A}^μ being the fluctuating quantum part of the gauge field and the background configuration A_μ being (1). I quantize the theory through the BRS formalism. Then, the momenta $\frac{n_\mu^i}{R_\mu}$ always appear in the combination $\frac{1}{R_\mu} (n_\mu^i \delta_{ab} + \theta_\mu^a - \theta_\mu^b)$. Furthermore, in the planar limit one can always associate a loop momentum $\frac{n_\mu^i}{R_\mu}$ ($i = 1, \dots, \ell$ labels the loop momentum) with an index loop a_i , and they appear in a specific combination $\frac{1}{R_\mu} (n_\mu^i + \theta_\mu^{a_i})$ [3–5, 10].³ In the large N limit one can replace the index sums with the integrations:

$$\sum_{a_1 \cdots a_\ell} G\left(\frac{\theta_\mu^{a_i}}{R_\mu}\right) \rightarrow \left(N \prod_{\mu=0}^3 R_\mu \int_{-(1/2R_\mu)}^{1/2R_\mu} dP_{\mu i} \right) G(P_{\mu i}) \quad (3)$$

where $\frac{\theta_\mu^{a_i}}{R_\mu}$ was replaced with the continuous integration variable $P_{\mu i}$ in the $N \rightarrow \infty$ limit. As one sums over the gauge indices, the sums run over the homogenous distribution of the eigenvalues of the background gauge field. Thus the sum over the gauge indices can be replaced by the integration over the dual torus. This is the essential mechanism for the large N reductions. The integrand for the correlator is a function of $P_{\mu i} + \frac{n_{\mu i}}{R_\mu}$. Hence the correlator

has a form

$$\begin{aligned} & \left(\prod_{i=1}^{\ell} \sum_{n_{\mu i}=-\infty}^{\infty} \int_{-(1/2R_\mu)}^{1/2R_\mu} dP_{\mu i} \right) G\left(P_{\mu i} + \frac{n_{\mu i}}{R_\mu}, K_{\mu j}\right) \\ & = \left(\prod_{i=1}^{\ell} \int_{-\infty}^{\infty} dP_{\mu i} \right) G(P_{\mu i}, K_{\mu j}). \end{aligned} \quad (4)$$

Thus the full internal loop momentum integrations of the *uncompactified* theory has been recovered. In other words, in the large N limit the functional forms of the field theory correlators on T^4 with the background (1) coincide with those of the uncompactified theory (with a trivial gauge field configuration) to all orders in perturbation theory. However, notice that the external momenta $K_{\mu j}$ still take discrete values. Therefore when one performs Fourier transformation to the position space, one obtains the sum over images of the correlation functions of the uncompactified theory:

$$G(x_j^\mu)|_{T^4} = \sum_{m_j=-\infty}^{\infty} G(x_j^\mu + 2\pi m_j R_\mu)|_{R^4}. \quad (5)$$

The result (5) was recently obtained in [10] in the context of the Maldacena duality. It may be worth noting that the main ingredients in the derivation of (5) had appeared in the old study of the large N reductions [3]. The new viewpoint brought by [10] was its bulk interpretation: (5) has a simple interpretation in the corresponding dual geometry. In the Maldacena duality, the geometry of the bulk can be probed by the gauge theory correlators. Then, (5) means that the dual geometry probed by the gauge theory Feynman diagrams of the compactified theory with the background (1) is the same as that of the uncompactified theory, except for the periodic identifications in T^4 directions [10]. Recall that the result for correlation functions of composite operators (5) is not a trivial consequence of a simple compactification in the gauge theory side, but the configuration (1) was crucial: If one sums over images of each field's Feynman propagator on R^4 to obtain the propagator on T^4 (say $\langle \Phi^I(x_1) \Phi^J(x_2) \rangle$), which is appropriate for probing a geometry corresponding to $A_\mu = 0$ background but not the homogeneous configuration (1), one does not obtain the sum over images of the correlation functions of the composite operators.

Now I identify the large N reduction with the zero-radius limit $R_\mu \rightarrow 0$, so that in the first line of (4) the momentum summation can be truncated to $n_{\mu i} = 0$ part only. Then, the original T^4 momentum summations drop out, but the gauge index summations reproduce the R^4 momentum integrations. This is the essence of the perturbative ‘‘derivation’’ of the holographic dual description of the large N reduction.⁴

³The origin of this combination is the covariant derivative for adjoint fields. For a planar diagram with all adjoint field propagators, the number of index loop is one more than the number of momentum loops, but one index sum factors out and just gives an overall factor N [10].

⁴Derivation assuming that the Maldacena duality is correct for uncompactified theory. The derivation may be extended to the nonperturbative one by using the Schwinger-Dyson equation [1].

There are two main options for taking the zero-radius limit, corresponding to two types of the reduced models. The “quenched” reduced models [2] are essentially the models where the condition (1) is put by hand. This is actually sufficient for a purpose of calculating quantities of the *uncompactified* original gauge theories. By construction the dual geometry in this case is the same as that of the uncompactified theory, up to the periodic identifications in T^4 directions. To calculate quantities which is translationally invariant along the T^4 directions from the closed string side, one just needs to study translationally invariant solutions of classical equation of motions. The periodic identifications in T^4 directions, in particular $R_\mu \rightarrow 0$ limit,⁵ do not matter in this case. This is the closed string dual description of the quenched large N reduction. The fact that in the classical bulk theories one can truncate the equation of motions to the holographic radial direction,⁶ and the classical limit of the closed string theory corresponds to the planar limit, shows a beautiful correspondence between the two descriptions. The large N gauge theories may be said to be “classical” in this sense.

So far, I have been describing how the translationally invariant quantities can be obtained from the reduced model, but the reduced model can also be used to calculate the gauge theory quantities which depend on spacetime coordinates. This will be explained in the next subsection.

On the otherhand, in the spirit of the original reduced model of Eguchi and Kawai [1], the configuration (1) is not put by hand, but it must be realized as a dominant saddle point. Thus, whether the large N reduction takes place or not becomes a dynamical issue. This translates via the Maldacena duality into the issue of stability of the geometry dual to the uncompactified theory upon the zero-radius limit of the T^4 compactification.

The dynamical stability of the homogeneous distribution (1) against the small volume limit $R_\mu \rightarrow 0$ in gauge theories is a model dependent problem. Here I just make a few remarks on some aspects of it.

In the supersymmetric case, the results of [13] for S^1 compactification may seem to suggest the stability of the configuration (1). But since here all spacetime directions are compactified, the quantum fluctuations can be suppressed only by the large N effect. Therefore a separate study is actually in order. Below, I will discuss a role of fermions with the periodic boundary conditions, for the stability of the configuration (1).

⁵The limit is, however, slightly subtle for conformal field theories where the small volume limit can be undone by conformal transformation (or isometry in dual closed string description). It will be more appropriate to keep R_μ finite in such cases. This is discussed in the next section.

⁶In quantum theories, even if one is interested in translationally invariant quantities, spacetime dimensionality comes in through the loop integrals.

If the gauge theory contains a massless elementary fermionic field, the periodic boundary conditions on it may restrict the topology of the dual geometry to be $R_{\geq 0} \times T^4$ [14]. This is because if some circle of T^4 shrinks to zero at some distance in the holographic radial direction in the bulk, the bulk fermion which couples to the gauge theory operator containing the massless fermion cannot have the periodic boundary conditions.⁷ As argued above, the stability of the $R_{\geq 0} \times T^4$ topology in the bulk is necessary for the stability of the configuration (1) in the limit $R_\mu \rightarrow 0$ [10].⁸ This expectation from the closed string side may heuristically be explained in the reduced models if one recalls the procedure taken here for taking the large N limit. To see this, I first analyze a reduced model with $SU(2)$ gauge group and with one massless adjoint fermion, to estimate effective potential between two eigenvalues of the gauge field. In this case it is possible to integrate out the fermion [17], and it is easy to see that the presence of the massless adjoint fermion introduces a repulsive potential $\sim -\log L$ for $L \sim 0$, where L is a difference between the two eigenvalues. One may expect that there is a similar repulsive force between eigenvalues also in the $SU(N)$ reduced model. Then, recall that to obtain the reduced models from the gauge theories, I took $N \rightarrow \infty$ before taking the $R_\mu \rightarrow 0$ limit. To implement this condition starting from the reduced models, one should restrict eigenvalues of the reduced gauge fields between $-\frac{1}{2R_\mu}$ and $\frac{1}{2R_\mu}$.⁹ It is like putting particles with short distant repulsive force dense enough in a finite volume so that the resulting distribution becomes uniform, i.e. the configuration (1) is realized. On the otherhand, adding a mass term to the fermion weakens the repulsive force and it disappears if the mass is sufficiently large. It is natural because if the mass is taken large enough, the fermion will eventually decouple from the system. It suggests that in the corresponding classical solution of the dual closed string descriptions, fermionic fields which couple to a gauge invariant fermionic operator with that massive fermion are excluded from a region in the spacetime corresponding to the scale lower than the fermion mass scale, and they do not restrict the topology there. The complete exclusion of fermionic field from some region may require a singular

⁷However, this restriction may not be so strong if one takes into account other spacetime directions in the dual theory. See [15] for a recent interesting example where the circle in the asymptotic boundary is mixed with another circle corresponding to an internal symmetry in field theory side.

⁸The bulk topology may also be probed by using the classical closed string worldsheet as a dual description for the Wilson loop expectation values [16]. Precisely speaking, what is calculated in [16] is a generalization of the Wilson loop including adjoint scalars.

⁹This is a gauge invariant condition for the reduced models. The mutually commuting configuration (1) should emerge dynamically.

geometry in the supergravity description. The arguments given here are heuristic and deserves further study.

If one introduces a bosonic adjoint field Φ to a gauge theory instead of the fermion, it means introducing another space dimension in the dual closed string side. Here I study the simple situation where $\Phi = 0$ vacuum is realized in the gauge theory. To construct a corresponding reduced model, one should take the diagonal components of Φ to be zero by hand, much in the same spirit as in the quenched reduced models, but for the opposite type of configuration.¹⁰ Then, a calculation similar to the above shows, in $SU(2)$ case, that the bosonic adjoint field does not change the leading repulsive potential between eigenvalues produced by the fermionic field.

For purely bosonic theories, in the closed string side the AdS soliton [18] which is a possible vacuum state at finite R_μ corresponds to a partial breaking of the configuration (1), and one must also take into account the possibilities of various phase transitions, like to black holes, black strings [19,20] and so on, which can trigger instability of the configuration (1) upon taking the $R_\mu \rightarrow 0$ limit. As I mentioned earlier, the configuration (1) is crucial for the large N reduction. The instability of the geometries dual to the uncompactified theories upon compactification means that the Eguchi-Kawai reduction does not take place in those cases.

Correlators of local gauge invariant single trace operators from reduced models

As has been described in the previous subsections, the large N reduction is not merely a simple dimensional reduction in the gauge theory side, but the configuration (1) was crucial. The loop momentum integrations in the original gauge theory are recovered from the gauge index sums in the reduced model. One can also calculate correlation functions in gauge theories which depend on external momenta from reduced models, provided that the configuration (1) is realized. This is essentially because the gauge indices in the reduced models play the role of spacetime momenta in the original gauge theory. In the Maldacena duality, the correlation functions of local gauge invariant single trace operators are important because they correspond to closed string amplitudes in the dual theory. However, since trace operators do not have uncontracted gauge indices, additional techniques are required to calculate correlation functions which depend on external momenta.¹¹ In this subsection, I explain how to calculate such correlation functions from the reduced models.

¹⁰One may also try to show that this configuration is dynamically preferred in the reduced model [17].

¹¹If there are uncontracted gauge indices, these can be straightforwardly regarded as the external momenta in the large N reductions [3–5].

I take an operator made of adjoint scalars Φ^I (I labels the species of the scalars) as an example, generalizations to include fermions or dynamical gauge fields are straightforward. For an operator made of q scalars $\text{Tr}\Phi^{I_1} \cdots \Phi^{I_q}(k)$ in the gauge theory, the corresponding object in the reduced model is given by

$$\text{Tr}^{\Delta k} \Phi^{I_1} \cdots \Phi^{I_q} \tag{6}$$

where I have defined the ‘‘shifted trace’’ $\text{Tr}^{\Delta k}$ as

$$\text{Tr}^{\Delta k} AB = \sum_{P_1, P_2} A_{P_1, P_2} B_{P_2, P_1+k}. \tag{7}$$

I have reparametrized the matrix indices in terms of the continuous parameter P_μ already taking into account the $N \rightarrow \infty$ limit. See Eq. (3). k_μ is regarded as a shift in the matrix indices. The point is that the shift inserts the external momentum k_μ to the index line connected to the latter index of Φ^{I_q} . The shifted trace does not satisfy the cyclic property $\text{Tr}AB = \text{Tr}BA$, so the above mapping from the gauge theory to the reduced model is not unique. However this is not a problem, since for a planar diagram these cyclically shifted operators all give the same results. This is because the index loops and the incoming momenta always appear in the combination $P_{\mu i_1} - P_{\mu i_2} + k_\mu$, where i_1, i_2 are loop indices. k_μ can be assigned to either i_1 or i_2 , and the difference can be absorbed by a shift of the loop momentum, which is an integration variable. See Figs. 1–3.

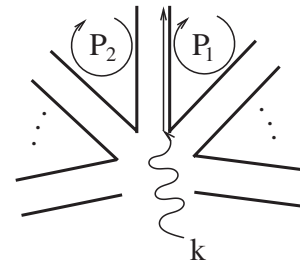


FIG. 1. The reduced model Feynman diagram for a correlator with incoming $\text{Tr}^{\Delta k} \Phi^{I_1} \cdots \Phi^{I_q}$ in the ’t Hooft’s double line notations. The figure expresses the assignment of the incoming momenta k to the P_1 index loop: $(P_1 + k) - P_2$.

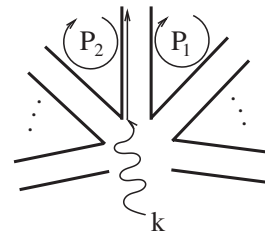


FIG. 2. $P_1 - (P_2 - k)$.

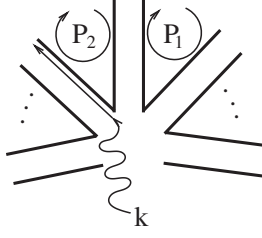


FIG. 3. Figure 2 \rightarrow Shift of the integration variable: $P_2 \rightarrow P_2 + k$. This figure corresponds to using $\text{Tr}^{\Delta k} \Phi^{I_2} \dots \Phi^{I_q} \Phi^{I_1}$.

III. FINITE RADII AND T -DUALITY

A. T -duality in gauge theories

In the previous section I identified the large N reductions with the dimensional reductions with the nontrivial gauge configuration (1). However, since the Maldacena duality is supposed to hold for any radii, it is natural to generalize the notion of the large N reductions to that case. In this subsection, I will explain that the equivalence between the original gauge theory and the reduced model still holds for finite radii, by appropriately generalizing the notion of the reduced model as a matrix model on a compact space, along the line of [21]. As found in [21], this naturally leads to the notion of T -dual equivalence in the matrix model. The corresponding dual closed string description of this T -dual equivalence [22] will be presented in the subsequent subsection.

For finite radii R_μ , the gauge theory calculation has equivalent T -dual descriptions in terms of the matrix model [21]. Each eigenvalues are interpreted as positions of D -instantons (in the string units) in the T -dual language. The radii of the dual torus \tilde{T}^4 are $\frac{\ell_s^2}{R_\mu}$, where ℓ_s is the string length. The summation over n_μ in (4) corresponds to the summation over images of D -instantons on the dual torus \tilde{T}^4 . To incorporate the images in the reduced models, one embeds the M^4 $SU(N)$ gauge groups into the diagonal blocks in $SU(N \times M^4)$ gauge group, where M is a positive integer which will be taken to infinity. The matrix components of the reduced fields are subject to an identification corresponding to the \tilde{T}^4 compactification. The background gauge field configuration (1) is generalized to

$$A_{\mu \vec{m} \vec{m}} = \frac{1}{R_\mu} \text{diag}(\theta_\mu^1, \dots, \theta_\mu^N) + \frac{m_\mu}{R_\mu} \quad (8)$$

where m_μ is a component of a four-vector \vec{m} which is an index for $SU(M^4)$. The off-diagonal components (in terms of $SU(M^4)$) are zero. In the matrix model on the torus, the fields in adjoint representation satisfy [21]

$$\Phi_{\vec{\theta}_1(\vec{m}+\vec{v}), \vec{\theta}_2(\vec{n}+\vec{v})} = \Phi_{\vec{\theta}_1 \vec{m}, \vec{\theta}_2 \vec{n}} \quad (9)$$

where I have labeled the $SU(N)$ gauge group indices in terms of $\vec{\theta}$, and \vec{m}, \vec{n} are $SU(M^4)$ indices. \vec{v} is an arbitrary four-vector with integer entries, which expresses a parallel

shift to an image. For simplicity, I study massless scalar fields Φ^I . Generalization to other fields is straightforward. The quadratic term of the reduced model is given by

$$\begin{aligned} & \frac{1}{M^4} \text{Tr}_{SU(N \times M^4)} [A_\mu, \Phi^I] [A_\mu, \Phi^I] \\ & \sim \sum_{\vec{m}, \vec{\theta}_1, \vec{\theta}_2} \Phi_{\vec{\theta}_1 \vec{m}, \vec{\theta}_2 \vec{0}}^I \left(\frac{1}{R_\mu} (\theta_{\mu 1} - \theta_{\mu 2} + m_\mu) \right)^2 \Phi_{\vec{\theta}_2 \vec{0}, \vec{\theta}_1 \vec{m}}^I \end{aligned} \quad (10)$$

where use has been made for (9). In the Maldacena duality, one studies the coupling of gauge invariant operators to their sources. For example, in the gauge theory the trace of q scalar fields have the coupling of the form

$$\int d^4 K \mathcal{J}_{I_1 \dots I_q} (-K) \text{Tr}_{SU(N)} \Phi^{I_1} \dots \Phi^{I_q}(K). \quad (11)$$

The source $\mathcal{J}_{I_1 \dots I_q}$ is identified with the boundary value of the corresponding field in closed string side. In the reduced models, the corresponding coupling is given by

$$\left(\prod_\mu \int_{-(1/2R_\mu)}^{1/2R_\mu} dk_\mu \sum_{m_\mu=-\infty}^{\infty} \right) \mathcal{J}_{I_1 \dots I_q} (-K) \text{Tr}_{SU(N \times M^4)}^{\Delta k, \Delta \vec{m}} \Phi^{I_1} \dots \Phi^{I_q} \quad (12)$$

where $K_\mu = k_\mu + \frac{m_\mu}{R_\mu}$, and I have introduced the “ \vec{m} -shifted trace” $\text{Tr}_{SU(M^4)}^{\Delta \vec{m}}$ for $SU(M^4)$ indices defined by

$$\begin{aligned} \text{Tr}_{SU(M^4)}^{\Delta \vec{m}} AB &= \sum_{\vec{m}_1, \vec{m}_2} A_{\vec{m}_1, \vec{m}_2} B_{\vec{m}_2, (\vec{m}_1 + \vec{m})} \\ &= \sum_{\vec{m}_1, \vec{m}_2} B_{\vec{m}_2, \vec{m}_1} A_{\vec{m}_1, (\vec{m}_2 + \vec{m})}. \end{aligned} \quad (13)$$

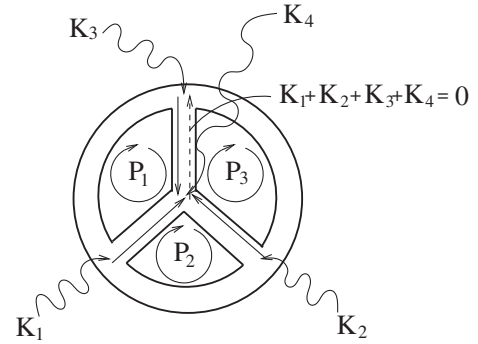


FIG. 4. An example: A four point function of the operators of the form $\text{Tr}_{SU(N \times M^4)}^{\Delta k, \Delta \vec{m}_s} \Phi^3$ ($s = 1, 2, 3, 4$). Assigning loop momenta to matrix index loops automatically take care of momentum conservation at each vertex. The $\text{Tr}_{SU(N \times M^4)}^{\Delta k, \Delta \vec{m}_s} \Phi^3$ adds incoming momentum $K_s = k_s + \frac{m_s}{R}$ between two index lines. Note that $K_1 + K_2 + K_3 + K_4 = 0$ by the total momentum conservation for incoming momenta. There are different but equivalent assignments for between which two index lines one puts an incoming momentum, see Fig. 5.

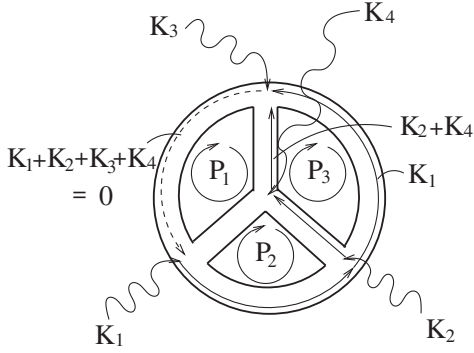


FIG. 5. Another way to calculate the same diagram.

The $SU(M^4)$ matrices A and B satisfy the same condition as in (9), and the “cyclic” property, i.e. the second line of (13) follows from that condition. The shifted trace for the $SU(N)$ indices P is defined as in (6). One can check that the reduced model with the above source term gives the same result of the original gauge theory in the diagrammatic perturbative calculations. See the example Fig. 4 and 5. In the dual D -instanton descriptions, the T -dual momenta are winding modes of closed strings, and the shift k and \tilde{m} in the trace $\text{Tr}_{SU(N \times M^4)}^{\Delta k, \Delta \tilde{m}}$ corresponds to a string stretched between a D -instanton and another D -instanton shifted by $\ell_s^2(k_\mu + \frac{m_\mu}{R_\mu})$.

Note that usually the sum over images is not taken in the reduced models. In this sense this is a generalization of the large N reduction.

B. The dual geometry

Finally, I explain in this subsection that the generalized large N reduction has a simple description in the dual geometry.¹² As a concrete example, I take $\mathcal{N} = 4$ super Yang-Mills theory on T^4 , which is identified as a worldvolume theory of $D3$ -branes, at strong coupling. At strong coupling supergravity approximation is valid and the dual geometry is $\text{AdS}_5 \times S^5$ with the periodic identifications in the T^4 directions, and the dilaton is constant.¹³ As will be shown below, this geometry can be obtained from a multi D -instanton solution in type IIB supergravity via T -duality, where D -instantons are densely and homogeneously dis-

¹²I thank R. Gopakumar and K. P. Yogendran for stimulating my thought on T -dual geometries at the early stage collaboration in [10].

¹³Since the discussions in the previous section were based on perturbation theory, the discussion in this section at strong coupling may should be regarded as complementary description. However, from the high symmetry of AdS_5 one may expect that the form of the geometry does not receive α' corrections [23]. If this is the case the following arguments should be the direct holographic dual description of the perturbative explanation of the generalized large N reduction discussed in the previous section.

tributed on the dual \tilde{T}^4 .¹⁴ The dense homogeneous distribution of the D -instantons is identified as a holographic dual of the dense and homogeneous distribution of the eigenvalues (1). Thus this is a holographic description of the equivalence between the gauge theory and the generalized reduced model.

The (Euclideanized) metric for the D -instantons in Einstein frame is flat: $\tilde{g}_{\mu\nu} = \delta_{\mu\nu}$, $\mu, \nu = 0, \dots, 9$ [25]. The solution can be obtained by solving the following equation for dilaton $\tilde{\phi}$:

$$\partial_\mu \partial^\mu e^{\tilde{\phi}} = 0. \quad (14)$$

When D -instantons are densely and homogeneously distributed in the \tilde{T}^4 directions, and overlapping on a point in the transverse six dimensions, the solution is given by

$$e^{\tilde{\phi}_\infty + \tilde{\phi}} = g_s \left(1 + \frac{c_0 g_s N \ell_s^4}{r^4} \right) \quad (15)$$

where r is the radial coordinate transverse to \tilde{T}^4 , N is a number of D -instantons on \tilde{T}^4 and $g_s = e^{\tilde{\phi}_\infty}$ is the string coupling constant. c_0 is a numerical constant related to the volume of the unit five-sphere, I suppress such numerical factors hereafter. In the near horizon limit $r \rightarrow 0$, the dilaton configuration becomes

$$e^{\tilde{\phi}} = \frac{g_s N \ell_s^4}{r^4} \quad (16)$$

and I obtain the $\text{AdS}_5 \times S^5$ metric in the string frame $\tilde{d}s_{st}^2 = e^{\tilde{\phi}/2} \tilde{d}s_E^2$:

$$\tilde{d}s_{st}^2 = \frac{\sqrt{g_s N \ell_s^4}}{r^2} [dr^2 + r^2 d\Omega_5 + d\tilde{x}_{//}^2] \quad (17)$$

where $\tilde{x}_{//}^\mu$ is a coordinate on \tilde{T}^4 with period $2\pi \tilde{R}_\mu$ ¹⁵ and $d\Omega_5$ is the volume form of the unit five-sphere. Now I perform T -dual transformation on \tilde{T}^4 . The T -dual metric is again $\text{AdS}_5 \times S^5$:

$$ds_{st}^2 = \frac{\sqrt{g_s N \ell_s^4}}{r^2} [dr^2 + r^2 d\Omega_5] + \frac{r^2}{\sqrt{g_s N \ell_s^4}} dx_{//}^2 \quad (18)$$

where $x_{//}^\mu$ is a coordinate on T^4 with period $2\pi R_\mu = 2\pi \frac{\ell_s^2}{\tilde{R}_\mu}$. Under the T -duality the dilaton transforms as [26]

$$\phi = \tilde{\phi} - \frac{1}{2} \log \det_{\tilde{T}^4} \tilde{g}_{\mu\nu st} = 0. \quad (19)$$

Thus one arrives at the $\text{AdS}_5 \times S^5$ geometry with the constant dilaton ($e^{\phi_\infty} = g_s$), as I have claimed. This is

¹⁴The T -dual relation of these geometries has appeared in [24]. The point of this subsection is to exhibit the parallel between the dual descriptions.

¹⁵Before taking the near horizon limit $\prod_{\mu=0}^3 \tilde{R}_\mu = \ell_s^4$ should hold in this solution so that N coincides with the number of D -instantons. After the near horizon limit this restriction can be removed by the isometry of AdS_5 .

the holographic description of the generalized large N reduction in the previous subsection. Notice the key role of the dense and homogeneous distribution of the D -instantons, which is dual to the dense and homogeneous distribution of the eigenvalues of the gauge field: It gives the geometry which is T -dual to the geometry just obtained by a simple T^4 identification of the *uncompactified* $D3$ -brane near horizon geometry.

Recall that the T -dual relation in the supergravity classical solutions can be derived from closed string world-sheet sigma model [22,26], whereas the matrix model T -duality was motivated and “explained” by the open string sigma model but was shown purely within the gauge theoretical language in [21]. The validity of these two descriptions may have an overlap in the Maldacena’s large N and the near horizon limit, as long as the conjecture is correct.¹⁶ Then, the T -dual equivalence of two geometries can be interpreted as a holographic dual description of the matrix model T -dual equivalence between the gauge theory and the generalized reduced model studied in the previous subsection.

IV. SUMMARY AND DISCUSSIONS

In this article, I have presented the holographic dual descriptions of the large N reductions in the Maldacena duality. This will be useful for deepening the understanding of both sides. The equivalence between the reduced model and the original gauge theory can be interpreted as a limit of the compactification with the homogeneous distribution of the eigenvalues of the gauge field. It was shown how this equivalence is reflected in the dual bulk geometry through the correlation functions of the local gauge invariant single trace operators. Since the Maldacena duality holds even for finite radii, it is natural to generalize the equivalence relation to that case. This was achieved by using the description of the matrix model on a compact space introduced in [21]. This description naturally contains the notion of T -duality. I pointed out that for finite radii the T -dual equivalence of two supergravity solutions are the holographic dual description of the T -dual equivalence between the gauge theory and the generalized reduced model.

The crucial condition for the large N reduction is the homogeneous distribution of the eigenvalues of the gauge

¹⁶Practically, one needs to be able to handle either the stringy corrections or the strongly coupled gauge theory. Note that although one obtains a smeared solution from $D3$ -brane solution even for finite N , when the number of the $D3$ -brane is small the gauge theory description is not rigorously related to this geometry. The ’t Hooft-Maldacena limit provides the correspondence between the gauge theory and the closed string theory, and the large N limit of the homogeneous distribution of the eigenvalues of the gauge field, which is dual to the dense and homogeneous distribution of the D -instantons, provides the effective smearing of the multi- D -instanton solution.

field (1). In the quenched reduced models this condition is forced by hand, whereas in the Eguchi-Kawai reduction the stability is a dynamical issue. The stability of the homogeneous distribution should reflect the stability of the supergravity solution dual to the uncompactified gauge theory upon compactification on T^4 . I pointed out an interesting possible role of fermions obeying the periodic boundary conditions in the T^4 directions.

I also presented a new technique for calculating position dependent correlation functions of local gauge invariant single trace operators in gauge theories from the reduced models.

Despite the evidences from the past studies, the Maldacena duality still remains as a conjecture. The holographic dual of the large N reductions established in this article will be useful for the quantitative tests of the Maldacena duality. Reduced models are suitable for studying the nonperturbative effects. In the Maldacena duality, it is also expected that the classical closed string descriptions capture the nonperturbative effects of the dual gauge theories. It will be interesting to study further how nonperturbative effects in the reduced models reflect themselves in the dual closed string descriptions. The large N reductions also provide an advantage for computer simulations [27]. As shown in this article, the large N reductions have more direct correspondence with the Maldacena duality compared with the lattice gauge theory, at least at present. If a computer simulation of a reduced model supports the dynamical stability of the configuration (1), it suggests that there is a dual closed string solution which is stable against the limit $R_\mu \rightarrow 0$. Then one can further calculate quantities in closed string theory using the reduced model. I hope that the holographic dual descriptions of the large N reductions described in this article will lead to the investigation of the Maldacena duality by computer simulations of reduced models.

I think relating the Matrix model of M -theory [28] or the IIB matrix model [29] to the Maldacena duality¹⁷ via the large N reductions discussed in this article is the most concrete way to study how closed strings emerge from these models, especially taking into account the recent developments in the understanding of the Maldacena duality [9,10,30].

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¹⁷I am aware that many researchers have resorted the idea of relating the matrix models with Maldacena duality from different directions.

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