PHYSICAL REVIEW D **74,** 045010 (2006)

Quantum properties of the four-dimensional generic chiral superfield model

A. T. Banin* and I. L. Buchbinder†
Department of Theoretical Physics, Tomsk State Pedagogical University, Tomsk 634041, Russia

N. G. Pletnev‡
Department of Theoretical Physics, Institute of Mathematics, Novosibirsk, 630090, Russia
(Received 11 July 2006; published 15 August 2006)

We study a problem of systematical evaluation of the quantum corrections for general 4D supersymmetric Kähler sigma models with chiral and antichiral superpotentials. Using manifestly reparametrization covariant techniques (the background-quantum splitting and proper-time representation) in the $\mathcal{N}=1$ superspace we show how to define unambiguously the one-loop effective action. We introduce the reparametrization covariant derivatives acting on superfields and prove that their algebra is analogous to algebra in super Yang-Mills (SYM) theory. This analogy allows us to use for evaluation of the effective action in the theory under consideration methods developed for SYM theory. The divergencies for the model are obtained. It is shown that on general Kähler manifold the one-loop counterterms have the structure of supersymmetric WZNW term in the form proposed in Ref. [16]. Leading finite contribution in covariant derivative expansion of the one-loop effective action (superfield a_3 coefficient) is calculated.

DOI: 10.1103/PhysRevD.74.045010 PACS numbers: 11.10.−z, 11.30.Pb, 12.60.Jv

I. INTRODUCTION

Nonlinear sigma models play a significant role in many areas of the field theory. An important class of such models is presented by 2D conformal field theories, which are exactly solvable (in a sense that S-matrix, the correlation functions of the different fields and the anomalous dimensions are completely determined on the base of conformal invariance) [1] and consistent at the quantum level [2]. Two-dimensional supersymmetric nonlinear sigma-models possess many remarkable properties. They are renormalized field theories. Moreover for a field theory on the world sheet the correlations between conformal invariance, (extended) supersymmetry and geometry of the complex manifolds of the full quantum theory yield to powerful restrictions on the background fields geometry in each order of the perturbation theory [3].

The remarkable properties of supersymmetric 2D nonlinear σ-models inspired interest to constructing the geometrical nonpolynomial theories of the supersymmetric matter in 4D space-time and studying their properties [4,5]. Supersymmetric nonlinear sigma-models have been formulated both for simple and for extended supersymmetries (see e.g. [6–8] for review). It was proved that in 4D the target space of rigid supersymmetric nonlinear σ-models must be the Kähler manifold [9] for $\mathcal{N}=1$ supersymmetry and hyper-Kähler manifold [2,3] for $\mathcal{N}=2$ supersymmetry. A number of supersymmetric sigma-models has been constructed within superstring theory [10] where the extra dimensions are wrapped up into a coset space. Unlike 2D models, the 4D nonlinear supersymmetric sigma-models are nonrenormalizable in power-counting as well as their 4D nonsupersymmetric predecessors. This is a main reason why the quantum aspects of such models are not well studied (see however some attempts in Refs. [11]).

Another large class of nonlinear sigma-models is formed by the 4D low-energy effective phenomenological theories. Dynamics of these models is invariant under a global group $\mathcal{G}$, whereas a vacuum is invariant only under some subgroup $\mathcal{H}$ [12]. It is well known that such models are nonrenormalizable under a power-counting analysis and requires the introduction of new couplings in each order of the loop expansion. However, the higher order loop terms involve higher powers of the momentum, and thus, the low-energy behavior is controlled only by the lower order terms which are unambiguous and do not undergo further renormalization (see e.g. [13]). These models possess the global anomalies which can be reproduced at the low-energy scale by the four-derivative Wess-Zumino-Novikov-Witten (WZNW) action [14].

Manifest $N=1$, 4D superfield form of the WZNW term has been considered in [15] but the proposed construction requires an infinite number of unspecified constants that appear in an undetermined function $\beta_{ij\bar{k}}$. In addition the auxiliary fields become propagating fields. The alternative form of the 4D, $\mathcal{N}=1$ ungauged supersymmetric WZNW model has been given in [16]. The interesting superfield sigma-models using the CNM (chiral/nonminimal) formulation have been constructed in [17] where chiral superfields exist in a tandem with complex linear superfields. The physical superfields Φ, $\bar{\Phi}$ and Γ, $\bar{\Gamma}$ in these models should be regarded, respectively, as a coordinates of the Kähler manifold and a tangent vector at a point Φ, $\bar{\Phi}$ of the same manifold [18]. The models are closely related

*Electronic address: atb@math.nsc.ru;
Department of Theoretical Physics Institute of Mathematics, Novosibirsk, 630090, Russia.
†Electronic address: joseph@tspu.edu.ru
‡Electronic address: pletnev@math.nsc.ru

to phenomenological models of pion physics and admit interpretations as low-energy stringlike models associated with QCD. Supergraph technique for these models was given in Ref. [19]. Next natural step here should be a development of a background field expansion in terms of normal coordinates and constructing of a computational procedure for finding counterterms and finite contributions to the effective action (see the references for early literature in [20,21]). This is one of our motivations to study quantum properties of the Kähler sigma model, which is a part of CNM sigma models.

One-loop divergencies for the standard $\mathcal{N} = 1$ and $\mathcal{N} = 2$ supersymmetric nonlinear sigma models on the Kähler and hyper-Kähler manifolds, respectively, demonstrate an existence of divergent terms with four order derivatives of background fields. The origin of such divergences is the nonrenormalizability of the model, and, therefore, to make quantum theory multiplicatively renormalizable we have to write terms with four order derivatives in the action from the very beginning. Such a situation is analogous e.g. to the relation between nonrenormalized Einstein quantum gravity and asymptotically free renormalized R^2 quantum gravity (see e.g. [22]). Unfortunately a detailed analysis of quantum properties of generic $4D$ $\mathcal{N} = 1, 2$ supersymmetrical models with some set of chiral supermultiplets has not been carried out so far.

The goal of this paper is to describe quantum aspects of the $4D$ generic chiral superfield model including supersymmetric sigma-model as a particular case. To be more precise, we formulate the heat kernel approach for the covariant computations of the one-loop effective action in the $4D$ generic chiral superfield model. Modern interest to this problem was inspired by recent development of generic chiral superfield models on nonanticommutative (NAC) superspaces (see e.g. [23–25]). Classical structure of such models has been thoroughly studied while their quantum properties requires a further analysis. In addition to the problems inherent with nonanticommutative models there is a known general difficulty in superfield sigma-models: the absence of chiral and simultaneously holomorphic normal coordinates on the generic Kähler manifolds does not allow to develop a loop expansion preserving all symmetries of the theory. This fact has been already mentioned in the pioneering papers [2,3]. Some papers were directly addressed to treatment of the above difficulty [26,27]. Unfortunately this difficulty can not be overcome in general since neither chiral metric nor chiral Levi-Civita connection do not exist on the Kähler manifold (even having isometries). Therefore the geodesics also do not exist in a subspace of chiral superfields and we can not utilize an expansion on a nontrivial superfield background in a way which preserves the chirality of the model. However as it has been pointed out some time ago (see e.g. [5]) this problem is unessential for one-loop calculations because the deviation from chirality $\bar{D}\sigma^i \sim \mathcal{O}(\sigma^2)$ is quadratic over quantum fluctuations and therefore the above difficulty arises only in the higher loops.

The paper is organized as follows. In Sec. II we present some mathematical grounds related to the model under consideration and discuss classical properties of the model. In Sec. III we consider the normal coordinate expansion for the Kähler potential and superpotentials. Then we introduce the specific covariant derivatives, formulate an algebra of these derivatives, study their basic properties and observe the analogies with SYM theories. As a result, we prove that the $4D$ superfield sigma-models are characterized by the objects which are analogues to superfield strengths in SYM theory. These objects have well-definite transformation properties and naturally arise as the building blocks for constructing the effective action. In Sec. IV we fulfil the one-loop calculations and find the divergencies and some finite corrections. Discussion and conclusions are given in Sec. V.

II. GENERIC CHIRAL SUPERFIELD MODEL

In this section we briefly discuss the basic notions of the generic chiral multiplet model in superspace (see e.g. [6]) which will be used further.

The model under consideration is a map from $\mathcal{N} = 1$ superspace into the Kähler space and is described by chiral Φ^i and antichiral $\bar{\Phi}^{\bar{i}}$ superfields whose components, the complex scalars ϕ^i, $\bar{\phi}^{\bar{i}}$, play the roles of coordinates on the Kähler manifold, whereas the fermions ψ^i, $\bar{\psi}^{\bar{i}}$ transform as vectors on a target manifold. The corresponding superspace action is written in terms of the Kähler potential $K(\Phi, \bar{\Phi})$ and chiral and antichiral superpotentials $P(\Phi)$, $\bar{P}(\bar{\Phi})$ and has the form

$$S = \int d^8z K(\Phi, \bar{\Phi}) + \int d^6z P(\Phi) + \int d^6\bar{z}\, \bar{P}(\bar{\Phi}). \tag{1}$$

This action is invariant under holomorphic reparametrization of superfields

$$\Phi^i \rightarrow \Phi^{i'} = f^i(\Phi), \qquad \bar{\Phi}^{\bar{i}} \rightarrow \bar{\Phi}^{\bar{i}'} = \bar{f}^{\bar{i}}(\bar{\Phi}). \tag{2}$$

and the Kähler transformations

$$K(\Phi, \bar{\Phi}) \rightarrow K(\Phi, \bar{\Phi}) + F(\Phi) + \bar{F}(\bar{\Phi}), \tag{3}$$

with some F, $\bar{F}$. The Kähler potential $K(\Phi, \bar{\Phi})$ defines the metric of the target manifold

$$g_{i\bar{j}} = \frac{\partial^2 K(\Phi, \bar{\Phi})}{\partial \Phi^i \partial \bar{\Phi}^{\bar{j}}} \equiv K_{,i\bar{j}}(\Phi, \bar{\Phi}). \tag{4}$$

This metric allows to find the components of the Levi-Civita connection

$$\Gamma^i_{kl} = g^{i\bar{m}} \partial_l g_{\bar{m}k}, \qquad \Gamma^{\bar{i}}_{\bar{k}\,\bar{l}} = g^{\bar{i}m} \partial_{\bar{l}} g_{m\bar{k}}, \qquad g_{i\bar{j};k} = g_{i\bar{j};\bar{k}} = 0. \tag{5}$$

Using the above connection we introduce the standard covariant derivatives of the geometrical objects with target space indices

$$A^i_{;l} = \partial_l A^i + \Gamma^i_{kl} A^k, \qquad A^{\bar{i}}_{;l} = \partial_l A^{\bar{i}},$$
$$A^i_{;\bar{l}} = \partial_{\bar{l}} A^i, \qquad A^{\bar{i}}_{;\bar{l}} = \partial_{\bar{l}} A^{\bar{i}} + \Gamma^{\bar{i}}_{\bar{k}\bar{l}} A^{\bar{k}}. \tag{6}$$

The Lagrangian for the component fields is obtained as the lowest component of the superfield $\mathcal{L} = D^\alpha \bar{D}^2 D_\alpha K$. The component form of the Kähler D-term is written from the expression

$$\begin{aligned} K|_D = {} & -ig_{i\bar{j}}\partial^{\alpha\dot{\alpha}}\Phi^i\partial_{\alpha\dot{\alpha}}\bar{\Phi}^{\bar{j}} - g_{i\bar{j}}(\mathcal{D}_{\alpha\dot{\alpha}}D^\alpha\Phi^i)\bar{D}^{\dot{\alpha}}\bar{\Phi}^{\bar{j}} \\ & + g_{i\bar{j}}D_\alpha\Phi^i(\mathcal{D}^{\alpha\dot{\alpha}}\bar{D}_{\dot{\alpha}}\bar{\Phi}^{\bar{j}}) + 2ig_{i\bar{j}}\bar{\mathcal{F}}^{\bar{j}}\mathcal{F}^i \\ & + \frac{i}{2}R_{i\bar{l}\bar{k}m}D^\alpha\Phi^m D_\alpha\Phi^i D^{\dot{\alpha}}\bar{\Phi}^{\bar{k}}D_{\dot{\alpha}}\bar{\Phi}^{\bar{l}}. \end{aligned} \tag{7}$$

Here we have introduced the reparametrization covariant derivatives

$$\mathcal{D}^\alpha A^i = D^\alpha A^i + \Gamma^i_{lk} D^\alpha \Phi^k A^l,$$
$$\mathcal{D}^{\dot{\alpha}} \bar{A}^{\bar{j}} = D^{\dot{\alpha}} \bar{A}^{\bar{j}} + \Gamma^{\bar{j}}_{\bar{l}\bar{k}} D^{\dot{\alpha}} \bar{\Phi}^{\bar{k}} \bar{A}^{\bar{l}}, \tag{8}$$

which will be important ingredients for constructing the effective action. The curvature tensor of the Kähler manifold has the form $R_{\bar{i}kl\bar{m}} = -g_{\bar{i}j}\partial_{\bar{m}}\Gamma^j_{kl}$ and we use following definitions for combinations of the auxiliary and spinoral fields that transform as tangent vectors

$$\mathcal{F}^i = \frac{1}{2}\mathcal{D}^\alpha D_\alpha \Phi^i, \qquad \bar{\mathcal{F}}^{\bar{i}} = \frac{1}{2}\mathcal{D}^{\dot{\alpha}} D_{\dot{\alpha}} \bar{\Phi}^{\bar{i}}. \tag{9}$$

Projection of (7) to the lowest superfield components results the most general $4D$ $\mathcal{N} = 1$ second-order Lagrangian [6,7] in the component form. The F-term superpotential defines of the scalar potential and Yukawa couplings.

The equations of motion for the model (1) in superfield form are given by

$$\bar{\mathcal{D}}^2 K_{,i} = P_{,i}, \qquad \mathcal{D}^2 K_{,\bar{i}} = \bar{P}_{,\bar{i}}. \tag{10}$$

In Sec. IV we study the one-loop effective action for the model (1).

III. BACKGROUND-QUANTUM SPLITTING

Covariant calculation of the quantum corrections for the model (1) is based on the loop expansion with help of background-quantum splitting which preserves the symmetry of the model. For construction of such a splitting it is useful to compare the superfield σ-model with conventional σ-model. The background-quantum splitting for conventional σ-model is realized in terms of Riemann normal coordinates what allows to retain the general coordinate invariance (for references in early literature, see e.g. [2,20]). Therefore the Riemann normal coordinates are the basic ingredient in constructing covariant loop expansion for σ-model effective action due to the following property: the geodesics passing through the origin have the same form $d^2y^i/d\lambda^2 = 0$ (here λ is a affine parameter, "time" along the geodesic) as the equations of straight lines passing through the origin of a Cartesian system of coordinates in a flat geometry.

The superfield σ-models are associated with Kähler geometry which possess a complex structure. The Riemann normal coordinates mix the holomorphic and antiholomorphic coordinates and hence violate the reparametrization symmetry since the set of holomorphic coordinate transformations (2) is only subset of a full set of general coordinate transformations on manifold parametrized by the coordinates Φ^i and $\bar{\Phi}^{\bar{i}}$. As a result, a general covariant loop expansion of the effective action preserving the complex structure is impossible in principle.

The problem of building a manifestly covariant background-quantum splitting for supersymmetrical σ-model was discussed by a number of authors (see e.g. [26,27]). In this section we compare two various approaches to this problem and demonstrate that for one-loop calculations such a problem does not exist really.

A. Chiral coordinate expansion

First of all we shortly discuss a well-known decomposition (see e.g. [26,27]) of the complex (anti)chiral superfields into background superfields ϕ^i, $(\bar{\phi}^{\bar{i}})$ and quantum superfields π^i, $(\bar{\pi}^{\bar{i}})$ fluctuating around them:

$$\Phi^i(z) = \phi^i(z) + \pi^i(z), \qquad \bar{\Phi}^{\bar{i}}(z) = \bar{\phi}^{\bar{i}}(z) + \bar{\pi}^{\bar{i}}(z). \tag{11}$$

In (11) the quantum fields $\pi^i(\bar{\pi}^{\bar{i}})$ are differences between coordinates on the manifold and therefore do not transform as the vectors under general coordinate transformations. Hence the expansion of the geometric objects on the manifold in a power series in $\pi(\bar{\pi})$ will not be covariant at each order. Instead we consider the velocity vectors $\sigma^i(\pi)$, $\bar{\sigma}^{\bar{i}}(\bar{\pi})$ which are chosen to play the role of the quantum fields. According to Ref. [26] we consider an affine parameter λ $(0 \leq \lambda \leq 1)$ along an arbitrary path from $\Phi(0) = \phi$ to $\Phi(1) = \phi + \pi$ and expand the field in the chiral coordinates

$$\Phi^i(\lambda) = \phi^i + \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} \phi^i_{(n)}. \tag{12}$$

Coordinate transformations on the curve are described by $\delta\Phi^i(\lambda) = f^i(\Phi(\lambda))$. Because of the transformation rule $\delta\frac{\partial\Phi^i(\lambda)}{\partial\lambda} = f^i_{,j}(\Phi(\lambda))\frac{\partial\Phi^j(\lambda)}{\partial\lambda}$ the quantity like $\frac{\partial\Phi^i(\lambda)}{\partial\lambda}$ is a contravariant chiral tangent vector at each value of λ, including the quantity $\sigma^i = \frac{\partial\Phi^i(\lambda)}{\partial\lambda}|_{\lambda=0}$. The coefficients $\phi^i_{(n)}$, $(\bar{\phi}^{\bar{i}}_{(n)})$ can be obtained directly from analysis of the geodesics equation

$$\frac{d^2\Phi^i}{d\lambda^2} + \Gamma^i_{jk}(\Phi, \bar{\Phi})\frac{d\Phi^j}{d\lambda}\frac{d\Phi^k}{d\lambda} = 0, \tag{13}$$

which, of course, is incompatible with the chirality condition on Φ^i, since the Γ^i_{jk} depends on both Φ^i and $\bar{\Phi}^{\bar{i}}$. Substituting the expansion in affine parameter for the curves (11) to above equation, we find the recursive formulae for the coefficients $\phi^i_{(n)}$, $(\bar{\phi}^{\bar{i}}_{(n)})$. For example

$$\begin{aligned}\phi^i_{(2)} &= -\Gamma^i_{jk}\sigma^j\sigma^k,\\ \phi^i_{(3)} &= -(\Gamma^i_{jk,l} - 2\Gamma^i_{jm}\Gamma^m_{kl})\sigma^j\sigma^k\sigma^l.\end{aligned} \tag{14}$$

In the this paper we concentrate only on one-loop analysis where we need only in the second order in an expansion of the classical action in quantum superfields. It is well-known (see e.g. [26]) that Taylor expansion of the Kähler potential in normal coordinates has the form

$$\begin{aligned}K(\phi + \pi, \bar{\phi} + \bar{\pi}) = K(\phi, \bar{\phi}) &+ K_{,i}\sigma^i + K_{,\bar{i}}\bar{\sigma}^{\bar{i}}\\ &+ \frac{1}{2}K_{;ij}\sigma^i\sigma^j + K_{,i\bar{j}}\sigma^i\bar{\sigma}^{\bar{j}}\\ &+ \frac{1}{2}K_{;\bar{i}\bar{j}}\bar{\sigma}^{\bar{i}}\bar{\sigma}^{\bar{j}} + \ldots,\end{aligned} \tag{15}$$

where subscript semicolon means covariant derivatives (6). The expansion of (anti) holomorphic superpotentials $P(\Phi)$, $\bar{P}(\bar{\Phi})$ can be also written down:

$$\begin{aligned}P(\phi + \pi) &= P(\phi) + P_{,i}(\phi)\sigma^i + \frac{1}{2}P_{,i;j}(\phi)\sigma^i\sigma^j + \ldots,\\ \bar{P}(\bar{\phi} + \bar{\pi}) &= \bar{P}(\bar{\phi}) + \bar{P}_{,\bar{i}}(\bar{\phi})\bar{\sigma}^{\bar{i}} + \frac{1}{2}\bar{P}_{,\bar{i};\bar{j}}(\bar{\phi})\bar{\sigma}^{\bar{i}}\bar{\sigma}^{\bar{j}} + \ldots\end{aligned} \tag{16}$$

It should be noted that the Eq. (13) for (11) defines a transformation to holomorphic normal coordinates. However the superfields σ^i lose the chirality properties $\bar{D}_{\dot{\alpha}}\pi^i = \bar{D}_{\dot{\alpha}}\sigma^i - \Gamma^i_{jk}\bar{D}_{\dot{\alpha}}\sigma^j\sigma^k + \frac{1}{2}R^i_{jk\bar{l}}\bar{D}_{\dot{\alpha}}\bar{\phi}^{\bar{l}}\sigma^j\sigma^k + \ldots = 0$. That means, the deviation from chirality is quadratically in quantum superfields σ. It leads to higher powers of σ in expansion of action and hence gives contribution beyond one loop. Therefore for one-loop calculations we can consider the σ as chiral superfiled.

However, that presented above background-quantum splitting is not a single one because except of widely used scalar multiplet representation by means of chiral scalar superfield there are the representations by means of chiral spinor superfield and by means of unconstrained complex scalar superfield prepotential. All mentioned representations are classically equivalent. Their quantum equivalence was studied in [28] (see also [6]).

B. Unconstrained field expansion

We already pointed out that manifestly supersymmetric expansion of the action (1), preserving the chirality properties on the base of Riemann or Kähler normal coordinates, is impossible in principle. Now we provide expansion of the Kähler potential using unconstrained complex scalar superfield prepotential $U^i(\lambda)$ (see Ref. [29]) and compare the result with described above normal coordinate expansion. Let the prepotential $U^i(\lambda)$ is such that $\Phi^i = \bar{D}^2U^i$ and defined by the equation of parallel transport for $\frac{dU^i}{d\lambda}$ along arbitrary nongeodesic curves $\Phi^i(\lambda)$:

$$\frac{d^2U^i}{d\lambda^2} + \Gamma^i_{jk}(\Phi, \bar{\Phi})\frac{d\Phi^j}{d\lambda}\frac{dU^k}{d\lambda} = 0. \tag{17}$$

This equation can be solved subject to the initial condition $U^i(\lambda = 0) = U^i_{\rm backgr}$, $\frac{dU^i}{d\lambda}(\lambda = 0) = \xi^i$ where $U^i_{\rm backgr}$ is a background prepotential and ξ^i is a quantum field which again is an unconstrained superfield. Explicitly:

$$U^i = U^i_{\rm backgr} + \lambda\xi^i - \frac{\lambda^2}{2}\Gamma^i_{jk}\bar{D}^2\xi^j\xi^k + \ldots \tag{18}$$

We observe that all higher order terms in the expansion (18) involve the background field via ϕ (not $U_{\rm backgr}$) so that the substitution of (18) into the Kähler potential will yield a Lagrangian which depends on ϕ^i and the quantum field ξ^i. Though the chiral normal coordinates do not exist, it is easy to show that this expansion is reparametrization covariant. The expansion coefficients at all orders in the quantum field ξ^i are constructed from geometrical objects, which are functions of the background field ϕ^i. The leading terms in the expansion Φ^i are given by

$$\begin{aligned}\Phi^i(\lambda) &= \phi^i + \pi^i\\ &= \phi^i + \lambda\bar{D}^2\xi^i - \frac{\lambda^2}{2}\bar{D}^2(\Gamma^i_{jk}\bar{D}^2\xi^j\xi^k) + \ldots\end{aligned}$$

and for Kähler potential we obtain

$$\begin{aligned}K(\phi + \pi, \bar{\phi} + \bar{\pi}) = K &+ K_{,i}\bar{D}^2\xi^i - K_{,i}\frac{1}{2}\bar{D}^2(\Gamma^i_{jk}\bar{D}^2\xi^j\xi^k)\\ &+ K_{,\bar{i}}D^2\bar{\xi}^{\bar{i}} - K_{,\bar{i}}\frac{1}{2}D^2(\Gamma^{\bar{i}}_{\bar{j}\bar{k}}D^2\bar{\xi}^{\bar{j}}\bar{\xi}^{\bar{k}})\\ &+ \frac{1}{2}K_{,i;j}\bar{D}^2\xi^i\bar{D}^2\xi^j + \frac{1}{2}K_{,\bar{i};\bar{j}}D^2\bar{\xi}^{\bar{i}}D^2\bar{\xi}^{\bar{j}}\\ &+ K_{,i;\bar{j}}\bar{D}^2\xi^iD^2\bar{\xi}^{\bar{j}}.\end{aligned} \tag{19}$$

One can see that the two expansions (15) and (19) are coordinated at the one-loop level and therefore it is no need to worry about nonchirality of the quantum superfields σ^i. Further we will use the chiral superfields approach [6,7,30].

To conclude this subsection we point out that though the above procedure is appropriate to construct a covariant background field expansion, use of the unconstrained prepotential means that the action (19) is invariant under quantum field gauge transformation $\delta(\xi^i - \frac{1}{2}\Gamma^i_{jk}\bar{D}^2\xi^j\xi^k + \ldots) = \bar{D}^{\dot{\alpha}}\omega^i_{\dot{\alpha}}$, $\delta\phi^i = 0$. That means we have to impose the gauges and introduce the corresponding ghosts following a quantization scheme for gauge theories with linearly dependent generators [31]. The treatment of an infinite tower of ghosts for a nonlinear sigma-model defined in terms of the nonminimal scalar multiplet has been carried out in [32] and it was found that the classical duality of the formulations in terms of chiral scalar superfields and in terms of complex general scalar superfileds takes place at least at the one-loop level.

C. The algebra of the covariant derivatives

In this section we consider algebra of covariant derivatives (8) related to general coordinate transformations (2). Covariant derivatives $\mathcal{D}_A$ transform every tensor superfield again into tensor superfield. As in global supersymmetry, spinor covariant derivatives $\mathcal{D}_\alpha$ and $\bar{\mathcal{D}}_{\dot\alpha}$ generate the full superalgebra of covariant derivatives $\mathcal{D}_A$. We demonstrate that this algebra is equivalent to the algebra of covariant derivatives for SYM theory. This fact allows us to use the general methods developed for finding the one-loop effective action in SYM theory.

Components of curvature tensor in Kählear geometry satisfies the relation $R^i_{jkl} \equiv 0$. It leads to the following property for anticommutators of covariant derivatives

$$\{\mathcal{D}^\alpha, \mathcal{D}^\beta\} = \{\bar{\mathcal{D}}^{\dot\alpha}, \bar{\mathcal{D}}^{\dot\beta}\} = 0. \qquad (20)$$

This relation can be treated as a representation-preserving constraint that make possible the existence of chiral scalar superfield $\bar{\mathcal{D}}_{\dot\alpha} A = 0$. The other anticommutation relations as conventional constraints means a definition for the vector component of the superconnection

$$\begin{aligned}\{\mathcal{D}^\alpha, \bar{\mathcal{D}}^{\dot\alpha}\}A^i &= i\mathcal{D}^{\alpha\dot\alpha i}{}_k A^k,\\ \mathcal{D}^{\alpha\dot\alpha i}{}_k &= \partial^{\alpha\dot\alpha}\delta^i_k - i\bar{\mathcal{D}}^{\dot\alpha}(\Gamma^i_{jk}D^\alpha\phi^j),\\ \{\mathcal{D}^\alpha, \bar{\mathcal{D}}^{\dot\alpha}\}A^{\bar i} &= i\mathcal{D}^{\alpha\dot\alpha \bar i}{}_{\bar k} A^{\bar k},\\ \mathcal{D}^{\alpha\dot\alpha \bar i}{}_{\bar k} &= \partial^{\alpha\dot\alpha}\delta^{\bar i}_{\bar k} - i\mathcal{D}^{\alpha}(\Gamma^{\bar i}_{\bar j\bar k}\bar D^{\dot\alpha}\bar\phi^{\bar j}).\end{aligned} \qquad (21)$$

The commutators of the covariant derivatives define the spinor superfield strengths

$$\begin{aligned}[\mathcal{D}_\beta, \mathcal{D}^{\alpha\dot\alpha}]A^{\bar i} &= i\delta^\alpha_\beta\mathcal{D}^2(\Gamma^{\bar i}_{\bar j\bar k}\bar D^{\dot\alpha}\bar\phi^{\bar j})A^{\bar k} = -\delta^\alpha_\beta \bar{\mathcal{W}}^{\dot\alpha\bar i}{}_{\bar k}A^{\bar k},\\ [\bar{\mathcal{D}}_{\dot\beta}, \mathcal{D}^{\alpha\dot\alpha}]A^{i} &= i\delta^{\dot\alpha}_{\dot\beta}\bar{\mathcal{D}}^2(\Gamma^{i}_{jk}D^{\alpha}\phi^{j})A^{k} = -\delta^{\dot\alpha}_{\dot\beta} \mathcal{W}^{\alpha i}{}_{k}A^{k}.\end{aligned} \qquad (22)$$

and the conjugation properties:

$$\begin{aligned}[\mathcal{D}_\beta, \mathcal{D}^{\alpha\dot\alpha}]A^{i} &= g^{i\bar i}g_{k\bar k}\mathcal{W}^{\dot\alpha\bar k}{}_{\bar i}A^{k},\\ [\bar{\mathcal{D}}_{\dot\beta}, \mathcal{D}^{\alpha\dot\alpha}]A^{\bar i} &= g_{k\bar k}g^{i\bar i}\delta^{\dot\alpha}_{\dot\beta}\mathcal{W}^{\alpha k}{}_{i}A^{\bar k}.\end{aligned}$$

Finally, a commutation relations of vector derivatives gives the definition for the vector component of the strength superfield:

$$\begin{aligned}i[\mathcal{D}_{\beta\dot\beta}, \mathcal{D}^{\alpha\dot\alpha}] &= -\frac{1}{2}\delta^{\dot\alpha}_{\dot\beta}\mathcal{D}_{(\beta}\mathcal{W}^{\alpha)} - \frac{1}{2}\delta^{\alpha}_{\beta}\bar{\mathcal{D}}_{(\dot\beta}\bar{\mathcal{W}}^{\dot\alpha)}\\ &= -\frac{1}{2}G^{\alpha\dot\alpha}_{\beta\dot\beta}.\end{aligned} \qquad (23)$$

Thus all the superfield strengths for the theory are expressed in terms of spinor superfields

$$\begin{aligned}\mathcal{W}^{\alpha i}{}_k &= -i\bar{\mathcal{D}}^2(g^{i\bar m}D^\alpha g_{\bar m k})\\ &= R^i_{jk\bar l}\partial^{\alpha\dot\alpha}\phi^j\bar D_{\dot\alpha}\bar\phi^{\bar l} + iR^i_{jk\bar l}D^\alpha\phi^j\bar{\mathcal{D}}^2\bar\phi^{\bar l}\\ &\quad + \frac{i}{2}\bar{\mathcal{D}}_{\dot\alpha}R^i_{jk\bar l}D^\alpha\phi^j\bar D^{\dot\alpha}\bar\phi^{\bar l},\\ \bar{\mathcal{W}}^{\dot\alpha\bar i}{}_{\bar k} &= -i\mathcal{D}^2(g^{\bar i m}\bar D^{\dot\alpha}g_{m\bar k})\\ &= R^{\bar i}_{\bar j\bar k l}\partial^{\alpha\dot\alpha}\bar\phi^{\bar j}D_\alpha\phi^l + iR^{\bar i}_{\bar j\bar k l}\bar D^{\dot\alpha}\bar\phi^{\bar j}\mathcal{D}^2\phi^l\\ &\quad + \frac{i}{2}\mathcal{D}_\alpha R^{\bar i}_{\bar j\bar k l}\bar D^{\dot\alpha}\bar\phi^{\bar j}D^\alpha\phi^l.\end{aligned} \qquad (24)$$

The spinor superfield strengths $\mathcal{W}^\alpha$ and $\bar{\mathcal{W}}^{\dot\alpha}$ evidently obey the Bianchi identities

$$\begin{aligned}\bar{\mathcal{D}}_{\dot\alpha}\mathcal{W}_\alpha = 0, \qquad \mathcal{D}_\alpha\bar{\mathcal{W}}_{\dot\alpha} = 0,\\ \mathcal{D}^\alpha\mathcal{W}_\alpha + \bar{\mathcal{D}}^{\dot\alpha}\bar{\mathcal{W}}_{\dot\alpha} = 0.\end{aligned} \qquad (25)$$

Using the above covariant derivatives we introduce two basic covariant differential operators acting on covariantly (anti)chiral superfields. These operators are obtained by covariantization of the identity $\bar D^2 D^2\Phi = \Box\Phi$ (where $\Box$ denotes the free d'Alembertian)

$$\begin{aligned}\Box_+ A^i &= \bar{\mathcal{D}}^2\mathcal{D}^2A^i\\ &= \Box_{\rm cov}A^i + i\mathcal{W}^{\alpha i}{}_k\mathcal{D}_\alpha A^k - \frac{i}{2}(\mathcal{D}_\alpha\mathcal{W}^{\alpha i}{}_k)A^k,\end{aligned} \qquad (26)$$

where $\Box_{\rm cov} = \frac{1}{2}\mathcal{D}^{\alpha\dot\alpha}\mathcal{D}_{\alpha\dot\alpha}$ and A^i is covariantly chiral superfield. Analogously for covariantly antichiral superfield $A^{\bar i}$ ones get

$$\begin{aligned}\Box_- A^{\bar i} &= \mathcal{D}^2\bar{\mathcal{D}}^2A^{\bar i}\\ &= \Box_{\rm cov}A^{\bar i} + i\bar{\mathcal{W}}^{\dot\alpha \bar i}{}_{\bar k}\bar{\mathcal{D}}_{\dot\alpha} A^{\bar k} - \frac{i}{2}(\bar{\mathcal{D}}_{\dot\alpha}\bar{\mathcal{W}}^{\dot\alpha \bar i}{}_{\bar k})A^{\bar k}.\end{aligned} \qquad (27)$$

The operators $\Box_+$ and $\Box_-$ obey the useful properties

$$\mathcal{D}^2\Box_+ = \Box_-\mathcal{D}^2, \qquad \bar{\mathcal{D}}^2\Box_- = \Box_+\bar{\mathcal{D}}^2.$$

As we will see further the operators (chiral and antichiral d'Alambertians) $\Box_+$ and $\Box_-$ play the crucial role for calculating the effective action.

We see that the strength superfields $\mathcal{W}$, $\bar{\mathcal{W}}$ demonstrate the properties similar to the SYM superfield strength properties. It is useful to compare spinor strengths definition for generic chiral superfiled model with the definition of the superstrengths for the supersymmetry gauge model $W_\alpha = i\bar{D}^2(e^{-V}D_\alpha e^V)$. One can note that the metric of the sigma model plays the same role as a prepotential for a gauge model. Then taking into account $U(n)$ gauge transformations of metric $g \to e^{\bar{\Lambda}(\bar{\phi})} g e^{-\Lambda(\phi)}$, we see that a corresponding connection should be defined as $g^{-1}D_\alpha g$. Such a connection will have the gauge transformation $e^{\Lambda}\mathcal{D}_\alpha e^{-\Lambda}$. Therefore one can impose "Wess-Zumino" gauge and in particular for the metrics this means

$$g_{i\bar{j}}(\phi, \bar{\phi}) = g_{i\bar{j}}(0) + R_{i\bar{j}k\bar{l}}(0)\phi^k\bar{\phi}^{\bar{l}} + \dots$$

Such a choice of gauge fixing is nothing but the Kähler normal coordinate expansion [27].

Thus, the obtained algebra of covariant derivatives is analogous to the algebra of covariant derivatives for $\mathcal{N} = 1$ SYM theory and then we can use the powerful results developed for quantum SYM theory [6,7]. But it should be kept in mind that many important results for SYM one-loop effective action were found in the constant background field approximation (constant strength approximation). However this approximation is not very appropriate and interesting for the model under consideration since it effectively means that ether all background fields Φ, $\bar{\Phi}$ are constant or $R_{i\bar{j}k\bar{l}} = 0$ and, therefore, a geometrical character of the model disappears.

IV. ONE-LOOP CALCULATIONS

We define the effective action $\Gamma[\phi, \bar{\phi}]$ on the base of background field generating functional $Z[\phi, \bar{\phi}] = e^{(i/\hbar)\Gamma[\phi,\bar{\phi}]}$ by integrating over the quantum fluctuations σ^I, $I = \{i, \bar{i}\}$. This definition leads at one loop to

$$Z[\phi] = e^{i\Gamma[\phi]} = e^{iS_0}\int D\sigma \det{}^{1/2}(g_{i\bar{j}}(\phi, \bar{\phi}))e^{i\int \sigma^I \mathcal{H}_{IJ}\sigma^J} = e^{iS_0}\det{}^{-(1/2)}[\mathcal{H}_I^J], \quad (28)$$

where $\mathcal{H}_I^J = \mathcal{H}_{IK}g^{KJ}$, while $\mathcal{H}_{IK}$ is the second functional derivatives of the action (1) over quantum fields

$$\mathcal{H}_{IJ} = \frac{\delta^2 S(\Phi, \bar{\Phi})}{\delta\sigma^I\delta\sigma^J}. \quad (29)$$

In the previous sections we obtained background-quantum splitting (15) and (16) for the classical action (1). It allows us to calculate the above functional derivatives and find the operator $\mathcal{H}_{IK}$ in an explicit form.

A. One-loop reparametrization invariant counterterms

In this subsection we find the divergent part of one-loop effective action $\Gamma^{(1)}$. This functional is expressed in terms of functional determinant of the operator (29)

$$\mathcal{H} = \begin{pmatrix} \frac{\delta^2 S}{\delta\sigma^i(z)\delta\sigma^j(z')} & \frac{\delta^2 S}{\delta\sigma^i(z)\delta\bar{\sigma}^{\bar{j}}(z')} \\ \frac{\delta^2 S}{\delta\bar{\sigma}^{\bar{i}}(z)\delta\sigma^j(z')} & \frac{\delta^2 S}{\delta\bar{\sigma}^{\bar{i}}(z)\delta\bar{\sigma}^{\bar{j}}(z')} \end{pmatrix} = \begin{pmatrix} \mathcal{H}_{++}(z,z') & \mathcal{H}_{+-}(z,z') \\ \mathcal{H}_{-+}(z,z') & \mathcal{H}_{--}(z,z') \end{pmatrix}.$$

The two-point functions $\mathcal{H}_{\pm\pm}(z, z')$ are covariantly chiral $(+)$ or covariantly antichiral $(-)$ with respect to the corresponding superspace argument. The functional derivatives for covariantly chiral (antichiral) superfields have following forms

$$\frac{\delta\sigma^i(z)}{\delta\sigma^j(z')} = \delta^i_j\bar{\mathcal{D}}^2\delta^8(z-z') \equiv \delta_+(z,z'),$$
$$\frac{\delta\bar{\sigma}^{\bar{i}}(z)}{\delta\bar{\sigma}^{\bar{j}}(z')} = \delta^{\bar{i}}_{\bar{j}}\mathcal{D}^2\delta^8(z-z') \equiv \delta_-(z,z')$$

Using the expansion (15) and (16) ones obtain the explicit form for the matrix of the second functional derivatives

$$\mathcal{H}_I^{\ J} = \begin{pmatrix} \mathcal{M}_i^{\bar{j}}\bar{\mathcal{D}}^2 & \delta_i^j\bar{\mathcal{D}}^2\mathcal{D}^2 \\ \delta_{\bar{i}}^{\bar{j}}\mathcal{D}^2\bar{\mathcal{D}}^2 & \bar{\mathcal{M}}_{\bar{i}}^j\mathcal{D}^2 \end{pmatrix} \times \begin{pmatrix} \delta^8(z-z') & 0 \\ 0 & \delta^8(z-z') \end{pmatrix}, \quad (30)$$

where we have used the covariant derivatives defined in the previous section and [33]

$$\mathcal{M}_i^{\ \bar{j}} = \bar{\mathcal{D}}^2K_{i;m}g^{m\bar{j}} + P_{,i;j}g^{j\bar{j}},$$
$$\bar{\mathcal{M}}_{\bar{i}}^j = \mathcal{D}^2K_{\bar{i};\bar{m}}g^{\bar{m}j} + \bar{P}_{,\bar{i};\bar{j}}g^{j\bar{j}}. \quad (31)$$

It is easy to show that the $\mathcal{M}$ and $\bar{\mathcal{M}}$ obey chirality properties $\bar{\mathcal{D}}_{\dot{\alpha}}\mathcal{M} = 0$, $\mathcal{D}_\alpha\bar{\mathcal{M}} = 0$. The matrix $\mathcal{H}_I^J$ can be rewritten as a product of two matrix

$$\begin{pmatrix} \mathcal{M} & \bar{\mathcal{D}}^2 \\ \mathcal{D}^2 & \bar{\mathcal{M}} \end{pmatrix}\begin{pmatrix} \bar{\mathcal{D}}^2\delta^8(z-z') & 0 \\ 0 & \mathcal{D}^2\delta^8(z-z') \end{pmatrix} = \begin{pmatrix} \mathcal{M} & \bar{\mathcal{D}}^2 \\ \mathcal{D}^2 & \bar{\mathcal{M}} \end{pmatrix}\begin{pmatrix} \delta_+(z-z') & 0 \\ 0 & \delta_-(z-z') \end{pmatrix}, \quad (32)$$

In further transformations we act as follows (see e.g. [6,30]). Using the definition (26), (27), and (32) we rewrite the one-loop correction in the form

$$-i\Gamma^{(1)} = \mathrm{Tr}\ln\begin{pmatrix} 0 & \bar{\mathcal{D}}^2 \\ \mathcal{D}^2 & 0 \end{pmatrix} + \mathrm{Tr}\ln\begin{pmatrix} 1 & \frac{1}{\Box_+}\bar{\mathcal{D}}^2\bar{\mathcal{M}} \\ \frac{1}{\Box_-}\mathcal{D}^2\mathcal{M} & 1 \end{pmatrix} = \frac{1}{2}\mathrm{Tr}\ln\begin{pmatrix} \Box_+ & 0 \\ 0 & \Box_- \end{pmatrix} + \frac{1}{2}\mathrm{Tr}\ln\left(1 - \begin{pmatrix} \frac{1}{\Box_+}\bar{\mathcal{D}}^2\bar{\mathcal{M}}\frac{1}{\Box_-}\mathcal{D}^2\mathcal{M} & 0 \\ 0 & \frac{1}{\Box_-}\mathcal{D}^2\mathcal{M}\frac{1}{\Box_+}\bar{\mathcal{D}}^2\bar{\mathcal{M}} \end{pmatrix}\right). \quad (33)$$

Then using the chiral d'Alambertian properties and the (anti)chirality properties of $\mathcal{M}$, $\bar{\mathcal{M}}$ one rewrites the the second term in a form which can be combined with the first term in (33) and obtains the expression

$$\Gamma^{(1)} = \frac{i}{2}\,\mathrm{Tr}\ln\begin{pmatrix} \Box_+ - \bar{\mathcal{D}}^2\mathcal{D}^2\bar{\mathcal{M}}\frac{1}{\Box_+}\mathcal{M} & 0 \\ 0 & \Box_- - \mathcal{D}^2\bar{\mathcal{D}}^2\mathcal{M}\frac{1}{\Box_-}\bar{\mathcal{M}} \end{pmatrix} = \frac{i}{2}\,\mathrm{Tr}_+\ln\Big(\Box_+ - \bar{\mathcal{D}}^2\mathcal{D}^2\bar{\mathcal{M}}\frac{1}{\Box_+}\mathcal{M}\Big) + \frac{i}{2}\,\mathrm{Tr}_-\ln\Big(\Box_- - \mathcal{D}^2\bar{\mathcal{D}}^2\mathcal{M}\frac{1}{\Box_-}\bar{\mathcal{M}}\Big). \tag{34}$$

Two independent (anti)chiral functional traces in the last expression can be treated separately by expanding the logarithm in the power series.

Further, we use the superspace Schwinger-De Witt techniques and explore the structure of the effective action superfunctional, including the analysis of divergences and finite contributions. For these goal we use the methods developed for SYM theory [6,7] (for recent development see e.g. [30,34,35]) and a covariant expansion of the corresponding propagator in powers of the superfield strengths $\mathcal{W}_\alpha$, $\bar{\mathcal{W}}_{\dot\alpha}$ and their covariant derivatives.

First of all we study a structure of divergences. One can show (analogous to SYM theory) that the divergences are given by the following expression

$$\Gamma^{(1)}_{\mathrm{div}} = \frac{i}{2}\,\mathrm{Tr}\int d^6z\,\ln\Box_+\,\delta^{(6)}_+(z-z') \times | - \frac{i}{2}\,\mathrm{Tr}\int d^6z\,\bar{\mathcal{D}}^2\bar{\mathcal{M}}\mathcal{M}\frac{1}{\Box_-^2}\mathcal{D}^2\delta^{(6)}_+(z-z')| + \text{c.c.} \tag{35}$$

Explicit evaluation of the divergences (35) is based on expansion of the logarithm of the operator in the second power in $\mathcal{D}$ derivatives and integrates by parts in order to release $\delta^4(\theta-\theta')$. Note that $\mathcal{W}$ should not be differentiated in the divergent terms because of dimensional reasons. We omit the the details of the calculations, just note that a heat kernel representation and a dimensional regularization scheme were used [6]. It leads to a simple and compact expression for the divergences

$$\Gamma^{(1)}_{\mathrm{div}} = -\frac{\Gamma(\omega)}{2(4\pi)^{2-\omega}}\left(\frac{m}{\mu}\right)^{-2\omega}\Big(\int d^6z\,\frac{1}{2}\mathcal{W}^{\alpha i}{}_k\mathcal{W}_{\alpha i}{}^k - \mathrm{tr}\int d^8z\,\bar{\mathcal{M}}\mathcal{M} + \text{c.c.}\Big), \tag{36}$$

where m, μ are IR and UV the mass scales, respectively, and $\omega = (4-d)/2$ is a regularizaton parameter. It should be noted that this result is valid for arbitrary Kähler potential and superpotential. Such form of for the one-loop effective action looks like a supersymmetric version of the known result [36] of Boulware and Brown.

Let us analyze the structure of the obtained divergent contributions. First of all we point out that the term in (36) which is given by integral over chiral subspace can be written in form of the $4D$, $\mathcal{N}=1$ supersymmetric ungauged WZNW action [16]:

$$\int d^8z\,\Gamma^i_{jk}(D^\alpha\phi^j)\Big[R^k_{li\bar m}\bar{D}^{\dot\alpha}\bar\phi^{\bar m}i\partial_{\alpha\dot\alpha}\phi^l + R^k_{li\bar m}\bar{D}^2\bar\phi^{\bar m}D_\alpha\phi^l + \frac{1}{2}\bar{D}^{\dot\alpha}R^k_{li\bar m}\bar{D}_{\dot\alpha}\bar\phi^{\bar m}D_\alpha\phi^l\Big] + \text{c.c.}. \tag{37}$$

Moreover, the first term in (37) has a form similar to the manifestly supersymmetric expression of WZNW term proposed by Nemeschansky and Rohm in Ref. [15]

$$S_{\mathrm{WZNW}} = ic\int d^8z(\beta_{ij\bar k}(\phi,\bar\phi)D^\alpha\phi^i\partial_{\alpha\dot\alpha}\phi^j\bar{D}^{\dot\alpha}\bar\phi^{\bar k} + \text{c.c.}), \tag{38}$$

where the bosonic parts consist of the bosonic WZNW term and an additional four-derivative term. Note that in contrast to the earlier analysis of supersymmetric WZNW term [15] where an infinite number of unspecified constants appeared in calculations of matrix elements based upon the N-R WZNW action (38) our action (37) is completely expressed only in terms of well defined geometric quantities.

It is known that for the higher derivative terms in this form of supersymmetric WZNW action a serious problem appears: the auxiliary fields became dynamical. In Ref. [37] a possibility to eliminate derivative terms of the auxiliary fields was examined and it was found that the condition for disappearance of these terms is equivalent to a condition of the term (38) vanishing $\beta_{ij\bar k,\bar l} - \beta_{\bar k\bar l i,j} = 0$. Another possibility to overcome this problem was considered by Gates and his collaborators who suggested a new nonconventional form of the supersymmetric WZNW term consisting in doubling the chiral superfields to chiral and complex linear superfields [17]. In the recent work [16] it was constructed the actual $4D$, $\mathcal{N}=1$ superspace WZNW action related to the non-Abelian consistent anomaly

$$S_{\mathrm{WZNW}} = C_0\Big(\frac{1}{4\pi^2}\Big)\mathcal{R}e\int d^8z(\mathcal{T}_{ij\bar k}D^\alpha\phi^i\partial_{\alpha\dot\alpha}\phi^j\bar{D}^{\dot\alpha}\bar\phi^{\bar k} + \mathcal{T}_{i\bar j\bar k}D^2\phi^i\bar{D}^{\dot\alpha}\bar\phi^{\bar j}\bar{D}_{\dot\alpha}\bar\phi^{\bar k} + \mathcal{T}_{ij\bar k\bar l}D^\alpha\phi^iD_\alpha\phi^j\bar{D}^{\dot\alpha}\bar\phi^{\bar k}\bar{D}_{\dot\alpha}\bar\phi^{\bar l}). \tag{39}$$

Comparison this action with the obtained expression (37) demonstrates one to one conformity. One can conclude that on a general Kähler manifold the one-loop counterterms have the form of supersymmetric WZNW term [16], while on the constant curvature superspace and on-shell $\mathcal{D}^2\phi^i = 0$ we get an expression analogous to N-R term [15]. The second term in (36) represents the fourth-order supersymmetric action for the nonlinear sigma model [5] with particular definition of the allowed tensors (G, A, T, H) [5] in terms of geometrical quantities and superpotential.

B. Finite contributions

In this section we present a method for calculation of next to leading Schwinger-De Witt coefficients for the one-loop effective action expansion on an arbitrary background.

Finding the superfield effective action is based on calculations of the chiral operator functional trace like

$$\mathrm{Tr}_{\pm}\mathcal{A} = \int d^6 z \mathcal{A}\,\delta_{\pm}(z, z'), \tag{40}$$

In the model under consideration one has

$$\begin{aligned}\Gamma^{(1)} = {}& \frac{i}{2}\mathrm{Tr}_+ \ln\Box_+ + \frac{i}{2}\mathrm{Tr}_- \ln\Box_- \\ &+ \frac{i}{2}\mathrm{Tr}_+ \ln\Big(1 - \frac{1}{\Box_+}\bar{\mathcal{D}}^2\bar{\mathcal{M}}\frac{1}{\Box_-}\mathcal{D}^2\mathcal{M}\Big) \\ &+ \frac{i}{2}\mathrm{Tr}_- \ln\Big(1 - \frac{1}{\Box_-}\mathcal{D}^2\mathcal{M}\frac{1}{\Box_+}\bar{\mathcal{D}}^2\bar{\mathcal{M}}\Big).\end{aligned} \tag{41}$$

The above expression contains four terms, two of them are chiral and two other are antichiral. Therefore it is sufficient to study only chiral terms and use conjugation to obtain others. Let us consider the contributions going from terms with Tr_+ for example. The first term can be rewritten via a proper-time integral

$$\begin{aligned}\mathrm{Tr}_+ \ln\Box_+ &= \int_0^\infty \frac{ds}{s} e^{-sm^2} \int d^6 z e^{s\Box_+}\delta_+(z, z')|_{z=z'} \\ &= \int_0^\infty \frac{ds}{s} e^{-sm^2} K_+(s),\end{aligned} \tag{42}$$

where we have introduced an IR cutoff m. The heat kernel has an asymptotic expansion in powers s and can be expressed as a series

$$K_+(s) = \frac{1}{(4\pi)^2 s^2}\sum_{n=2}^{\infty} a_n(z) s^n. \tag{43}$$

It is known that the coefficients $a_0 = a_1 = 0$ and the first nontrivial coefficient a_2 defines the divergences. So we know that finite contributions can be given by integral over full superspace. In particular it means that the contributions from any coefficient a_n with $n \geq 3$ are expressed as $\bar{\mathcal{D}}^2$ acting on field strengths and their covariant derivatives and, therefore, they can be transformed to a gauge invariant superfunctional on the full superspace. It is allows us to write the following differential equation for the kernel K_+

$$\begin{aligned}\frac{dK_+(s)}{ds} &= \frac{1}{(4\pi)^2}\sum_{n=3}^{\infty}(n-2)s^{n-3}a_n(z) \\ &= \int d^6 z \bar{\mathcal{D}}^2\mathcal{D}^2 e^{s\Box_+}\delta_+(z, z')|_{z=z'} \\ &= K^{\alpha}{}_{\alpha}(s) \\ &= \int d^8 z \mathcal{D}^2 e^{s\Box_+}\delta_+(z, z')|_{z=z'} \\ &= \frac{1}{(4\pi)^2 s^2}\sum_{n=0}^{\infty} s^n c_n(z).\end{aligned} \tag{44}$$

We see that it is convenient to redefine the coefficients in the series (43) in the form $a_n = \frac{1}{n-2}\bar{\mathcal{D}}^2 c_{n-1}$.

There are the various methods for evaluations of superfield heat kernels [38]. Here we adopt for our aims one of such methods [34]. First of all ones present the covariant chiral delta-function by integral

$$\begin{aligned}\delta_+(z, z') &= \bar{\mathcal{D}}^2\delta^8(z, z')I(z, z') \\ &= -\delta^4(\zeta)\delta^2(\zeta)I(z, z') \\ &= -\int d^6\eta\, e^{ip_{\alpha\dot\alpha}\zeta^{\alpha\dot\alpha} + \zeta^\alpha\pi_\alpha}I(z, z'),\end{aligned} \tag{45}$$

where $d^6\eta \equiv \frac{d^4p}{(2\pi)^4}d^2\pi$ and $I(z, z')$ is an operator of the parallel displacement [30]. Invariant superintervals are defined as

$$\begin{aligned}\zeta^{\alpha\dot\alpha} &= (x - x')^{\alpha\dot\alpha} - \frac{i}{2}\theta^\alpha\bar\theta^{\dot\alpha'} + \frac{i}{2}\theta^{\alpha'}\bar\theta^{\dot\alpha}, \\ \zeta^\alpha &= (\theta - \theta')^\alpha, \qquad \bar\zeta^{\dot\alpha} = (\bar\theta - \bar\theta')^{\dot\alpha}.\end{aligned} \tag{46}$$

The resulting $K^\alpha{}_\alpha$ from (44) is rewritten using integral over momenta

$$\begin{aligned}K^\alpha{}_\alpha(s) &= \int d^8 z \mathcal{D}^2 e^{s\Box_+}\delta_+(z, z')|_{z=z'} \\ &= \int d^8 z \int d^6\eta \frac{-1}{2}X^\alpha X_\alpha e^{s\Delta_+}I(z, z')|_{z=z'},\end{aligned} \tag{47}$$

where $\Delta_+ = \frac{1}{2}X^{\alpha\dot\alpha}X_{\alpha\dot\alpha} + i\mathcal{W}^\alpha X_\alpha + \frac{i}{2}(\mathcal{D}^\alpha\mathcal{W}_\alpha)$, and operators X_A defined as

$$\begin{aligned}X_{\alpha\dot\alpha} &= ip_{\alpha\dot\alpha} + \mathcal{D}_{\alpha\dot\alpha}, \\ X_\alpha &= \pi_\alpha - \frac{1}{2}p_{\alpha\dot\alpha}(\bar\theta - \bar\theta')^{\dot\alpha} + \mathcal{D}_\alpha.\end{aligned} \tag{48}$$

One can verify that an algebra of these operators has the same form as the algebra of covariant derivatives given in Subsec. III C:

$$\begin{aligned}\{X_\alpha, X_\beta\} = 0, \qquad [X_\beta, X^{\alpha\dot\alpha}] = \delta^\alpha_\beta\bar{\mathcal{W}}^{\dot\alpha}, \\ [X_{\beta\dot\beta}, X^{\alpha\dot\alpha}] = \frac{i}{2}G^{\alpha\dot\alpha}_{\beta\dot\beta},\end{aligned} \tag{49}$$

and $XA = (\mathcal{D}A) + (-1)^{|a||X|}AX$. Note that the shift $-\frac{1}{2}p_{\alpha\dot\alpha}(\bar\theta - \bar\theta')^{\dot\alpha}$ in X_α always vanishes in the coincidence limit. Since no any $\bar{\mathcal{D}}_{\dot\alpha}$ operators appear during calculations, we can consider all expressions in the coincidence limit from the very beginning.

Next, expanding the exponent in (47) and using the properties of the integral over bosonic and fermionic momenta as well as the action of the covariant derivatives on the parallel displacement operator in the coincidence limit [30]

$$\begin{aligned} I(z,z')|_{\zeta=0} &= 1, \qquad \mathcal{D}_\alpha I(z,z')|_{\zeta=0} = 0, \\ \mathcal{D}_\beta \mathcal{D}_\alpha I(z,z')|_{\zeta=0} &= 0, \qquad \mathcal{D}_{\alpha\dot\alpha} I(z,z')|_{\zeta=0} = 0, \end{aligned} \tag{50}$$

we obtain (after omitting contributions equal to total derivative) for the coefficient a_3 the following expression

$$\begin{aligned} a_3^{(1)} = -\frac{1}{8m^2}\int d^8z \Big(\frac{1}{6}(\mathcal{D}_\alpha \mathcal{W}^\beta)(\mathcal{D}_\beta \mathcal{W}^\alpha) \\ + \frac{1}{6}(\bar{\mathcal{D}}_{\dot\alpha}\bar{\mathcal{W}}^{\dot\beta})(\bar{\mathcal{D}}_{\dot\beta}\bar{\mathcal{W}}^{\dot\alpha}) + (\mathcal{D}\mathcal{W})(\mathcal{D}\mathcal{W})\Big). \end{aligned} \tag{51}$$

However this is only one part of the result. Other finite contributions having the same order on power s go from the second term in trace Tr_+ of logarithm expansion up to second order in (41). The results look like

$$a_3^{(2)} = \frac{1}{4m^2}\int d^8z \mathcal{D}^{\alpha\dot\alpha}\bar{\mathcal{M}}\mathcal{D}_{\alpha\dot\alpha}\mathcal{M}, \tag{52}$$

and

$$a_3^{(3)} = \frac{1}{4m^2}\int d^8z \bar{\mathcal{M}}\mathcal{M}\bar{\mathcal{M}}\mathcal{M}. \tag{53}$$

The final result is a sum $a_3^{(1)}$, $a_3^{(1)}$, $a_3^{(3)}$ obtained by substitution of (24) and (31) into (51)–(53). For the partial case $\mathcal{D}_A\mathcal{M} = 0$ and $\mathcal{D}_{\alpha\dot\alpha}\mathcal{W} = 0$ we obtain known in SYM theory G^2 term $\Gamma^{(1)} \sim \int d^8z \frac{1}{\mathcal{M}\bar{\mathcal{M}}} G^{\alpha\dot\alpha}_{\beta\dot\beta} G^{\beta\dot\beta}_{\alpha\dot\alpha}$ [35].

To conclude this section we point out that the theory under consideration can be treated as a fenomenological $\mathcal{N} = 1$ supersymmetric model following e.g. from some fundamental superstring theory [10] for description of low-energy effects. In such a case the forms of Kähler and (anti)chiral potentials are dictated by the fundamental theory. In particular, finite terms in the effective action stipulated by a_3-coefficient allow to find contributions to S-matrix of six order in momenta which are determined by the forms of Kähler and (anti)chiral potentials.

V. SUMMARY

In this paper we developed an approach for studying the quantum aspects of the $4D$ generic chiral superfield model. The model is given in terms of the Kähler potential and chiral and antichiral potentials. Effective action for the model under consideration is formulated on the base of background-quantum splitting and in one-loop approximation preserves all symmetries of the classical theory.

We introduced the reparametrization covariant derivatives acting on superfields and constructed their algebra in terms of commutators and anticommutators. It was proved that structure of this algebra coincides with ones for the covariant derivatives in SYM theory and the Kähler metric plays the role analogous to the prepotential in the SYM theory. We also constructed the chiral and antichiral d'Alambertians. These results open the possibilities to apply the methods, developed for evaluation of the effective action in SYM theory, for study of the effective action in $4D$ generic chiral superfield model.

We formulated the superfield proper-time techniques for covariant computations of the one-loop effective action. Both divergent and leading finite contributions to the one-loop effective action were found in an explicit form in geometric terms. It was showed that the divergent term reproduces the supersymmetric WZNW term [16] and fourth-order supersymmetric nonlinear sigma model [5].

ACKNOWLEDGMENTS

The work was supported in part by RFBR grant, project No. 06-02-16346, grant for LRSS, project No. 4489.2006.2 and INTAS grant. I. L. B. is grateful to DFG grant, project No. 436.RUS 113/669/0-3 and joint RFBR-DFG grant, project No. 06-02-04012 for partial support. The work of N. G. P. was supported in part by RFBR grant, project No. 05-02-16211.

[1] A. A. Belavin, A. M. Polyakov, and A. B. Zamolodchikov, Nucl. Phys. **B241**, 333 (1984).

[2] L. Alvarez-Gaume and D. Z. Freedman, Phys. Lett. B **94**, 171 (1980); Commun. Math. Phys. **80**, 443 (1981); L. Alvarez-Gaume, D. Z. Freedman, and S. Mukhi, Ann. Phys. (N.Y.) **134**, 85 (1981).

[3] L. Alvarez-Gaume and D. Z. Freedman, Phys. Rev. D **22**, 846 (1980); J. Bagger and E. Witten, Phys. Lett. B **118**, 103 (1982); L. Alvarez-Gaume and P. Ginsparg, Commun. Math. Phys. **102**, 311 (1985); L. Alvarez-Gaume, S. R. Coleman, and P. H. Ginsparg, Commun. Math. Phys. **103**, 423 (1986).

[4] B. Spence, Nucl. Phys. **B260**, 630 (1985); I. L. Buchbinder and S. V. Ketov, Theor. Math. Phys. (Engl. Transl.) **77**, 1032 (1988); Fortschr. Phys. **39**, 1 (1991).

[5] A. A. Deriglazov and S. V. Ketov, Theor. Math. Phys. (Engl. Transl.) **77**, 1160 (1988); S. V. Ketov, *Quantum Nonlinear Sigma Models: From Quantum Field Theory to Supersymmetry, Conformal Field Theory, Black Holes and Strings* (Springer, Berlin, 2000).

[6] I. L. Buchbinder and S. M. Kuzenko, *Ideas and Methods of Supersymmetry and Supergravity or a Walk Through Superspace*, (IOP Publ., Bristol and Philadelphia, 1998).

[7] S. J. Gates, Jr., M. T. Grisaru, M. Roček, and W. Siegel, *Superspace* (Benjamin Cummings, Reading, MA, 1983).

[8] A. S. Galperin, E. A. Ivanov, V. I. Ogievetsky, and E. S. Sokatchev, *Harmonic Superspace* (Cambridge University Press, Cambridge, 2001).

[9] B. Zumino, Phys. Lett. B **87**, 203 (1979).

[10] M. B. Green, J. H. Schwarz, and E. Witten, *Superstring Theory* (University Press, Cambridge, UK, 1987).

[11] I. L. Buchbider, M. Cvetić, and A. Yu. Petrov, Mod. Phys. Lett. A **15**, 783 (2000); Nucl. Phys. **B571**, 358 (2000); I. L. Buchbinder and A. Yu. Petrov, Phys. Lett. B **461**, 209 (1999); A. Brignole, Nucl. Phys. **B579**, 101 (2000); S. G. Nibberlink and T. S. Nyawelo, J. High Energy Phys. 01 (2006) 034.

[12] S. Weinberg, Physica A (Amsterdam) **96**, 327 (1979); *The Quantum Theory of Fields, II* (University Press, Cambridge, England, 2001); J. Gasser and H. Leutwyler, Ann. Phys. (N.Y.) **158**, 142 (1984); Nucl. Phys. **B250**, 4654 (1985).

[13] Mary K. Gaillard, Reports No. LBL-24114 and UCB-PTH-87/44, 1987 (to be published).

[14] J. Wess and B. Zumino, Phys. Lett. B **37**, 95 (1971); E. Witten, Nucl. Phys. **B223**, 422 (1985).

[15] D. Nemeschansky and R. Rohm, Nucl. Phys. **B249**, 157 (1985); T. E. Clark and S. T. Love, Phys. Lett. B **138**, 289 (1984).

[16] S. J. Gates, Jr., M. T. Grisaru, M. E. Knutt, and S. Penati, Phys. Lett. B **503**, 349 (2001); S. J. Gates, Jr., M. T. Grisaru, M. E. Knutt, S. Penati, and H. Suzuki, Nucl. Phys. **B596**, 315 (2001); S. J. Gates, Jr., M. T. Grisaru, and S. Penati, Phys. Lett. B **481**, 397 (2000).

[17] S. J. Gates, Phys. Lett. B **365**, 132 (1996); Nucl. Phys. **B485**, 145 (1997); S. J. Gates and S. M. Kuzenko, Nucl. Phys. **B543**, 122 (1999); Fortschr. Phys. **48**, 115 (2000).

[18] S. M. Kuzenko, Int. J. Mod. Phys. A **14**, 1737 (1999).

[19] S. J. Gates, Jr., S. Penati, and G. Tartaglino-Mazzucchelli, J. High Energy Phys. 05 (2006) 051; S. M. Kuzenko and W. D. Linch, III, J. High Energy Phys. 02 (**2006**) 038; G. Tartaglino-Mazzucchelli, Phys. Lett. B **599**, 324 (2004).

[20] J. Honerkamp, Nucl. Phys. **B36**, 130 (1972).

[21] P. S. Howe and K. S. Stelle, Int. J. Mod. Phys. A **4**, 1871 (1989).

[22] K. Stelle, Phys. Rev. D **16**, 953 (1977);E. Fradkin and A. Tseytlin, Phys. Rep. **119**, 233 (1985).

[23] B. Chandrasekhar, Phys. Rev. D **70**, 125003 (2004); Phys. Lett. B **614**, 207 (2005); L. Alvarez-Gaume and M. A. Vazquez-Mozo, J. High Energy Phys. 04 (2005) 007; K. Araki, T. Inami, H. Nakajima, and Y. Saito, J. High Energy Phys. 01 (2006) 109.

[24] O. D. Azorkina, A. T. Banin, I. L. Buchbinder, and N. G. Pletnev, Mod. Phys. Lett. A **20**, 1423 (2005); T. A. Ryttov and F. Sannino, Phys. Rev. D **71**, 125004 (2005); S. V. Ketov, hep-th/0602066.

[25] I. Jack, D. R. T. Jones, and L. A. Worthy, Phys. Rev. D **72**, 065002 (2005); O. D. Azorkina, A. T. Banin, I. L. Buchbinder, and N. G. Pletnev, Phys. Lett. B **633**, 389 (2006); **635**, 50 (2006); M. T. Grisaru, S. Penati, and A. Romagnoni, J. High Energy Phys. 02 (2006) 043.

[26] T. E. Clark and S. T. Love, Nucl. Phys. **B301**, 439 (1988).

[27] K. Higashijima and M. Nitta, Prog. Theor. Phys. **105**, 243 (2001); **108**, 185 (2002).

[28] I. L. Buchbinder and S. M. Kuzenko, Nucl. Phys. **B308**, 162 (1988).

[29] P. S. Howe, G. Papadopoulos, and K. S. Stelle, Phys. Lett. B **174**, 405 (1986).

[30] S. M. Kuzenko and I. N. McArthur, J. High Energy Phys. 05 (2003) 015; S. M. Kuzenko, Covariant supergraphs I–III, www.thphys.uni-heidelberg.de/hebecker/kuzenko2.pdf.

[31] I. A. Batalin and G. A. Vilkovisky, Phys. Lett. B **120**, 166 (1983); Phys. Rev. D **28**, 2567 (1983).

[32] S. Penati, A. Refolli, A. Van Proeyen, and D. Zanon, Nucl. Phys. **B514**, 460 (1998).

[33] In further manipulations it is useful to take into account the relations

$$\begin{aligned}\bar{\mathcal{D}}^2 K_{,i;j} &= R^l_{ij\bar{m}}(\bar{\mathcal{D}}^2\bar{\phi}^{\bar{m}})K_{,l}\\ &\quad + \frac{1}{2}(R^l_{ij\bar{m},\bar{n}}K_{,l} + R_{\bar{n}ij\bar{m}})\bar{D}^{\dot{\alpha}}\bar{\phi}^{\bar{n}}\bar{D}_{\dot{\alpha}}\bar{\phi}^{\bar{m}},\\ \mathcal{D}^2 K_{\bar{i};\bar{j}} &= R^{\bar{l}}_{\bar{i}\bar{j}m}(\mathcal{D}^2\phi^m)K_{,\bar{l}}\\ &\quad + \frac{1}{2}(R^{\bar{l}}_{\bar{i}\bar{j}m;n}K_{,\bar{l}} + R_{n\bar{i}\bar{j}m})D^{\alpha}\phi^m D_{\alpha}\phi^n,\end{aligned}$$

here semicolon subscript stands for repametrization covariant derivatives which were defined above.

[34] I. N. McArthur and T. D. Gargett, Nucl. Phys. **B497**, 525 (1997); N. G. Pletnev and A. T. Banin, Phys. Rev. D **60**, 105017 (1999).

[35] D. T. Grasso, J. High Energy Phys. 11 (2002) 012; 09 (2004) 054.

[36] D. Boulware and L. Brown, Ann. Phys. (N.Y.) **138**, 392 (1982).

[37] M. Nitta, Mod. Phys. Lett. A **15**, 2327 (2000).

[38] For earlier approaches to calculations of superfield De Witt coefficients see e.g. [39].

[39] I. L. Buchbinder, Yad. Fiz. **36**, No. 8, 509 (1982).