Brane models with a Ricci-coupled scalar field

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We consider the problem of a scalar field, nonminimally coupled to gravity through a $-\xi\phi^2 R$ term, in the presence of a brane. Exact solutions, for a wide range of values of the coupling parameter ξ, for both ϕ-dependent and ϕ-independent brane tension, are derived and their behavior is studied. In the case of a Randall-Sundrum geometry, a class of the resulting scalar field solutions exhibits a folded-kink profile. We go beyond the Randall-Sundrum geometry studying general warp factor solutions in the presence of a kink scalar. Analytic and numerical results are provided for the case of a brane or for smooth geometries, where the scalar field acts as a thick brane. It is shown that finite geometries with warp factors that asymptotically decrease exponentially are realizable for a wide range of parameter values. We also study graviton localization in our setup and find that the localizing potential for gravitons with the characteristic volcanolike profile develops a local maximum located at the origin for high values of the coupling ξ.

I. INTRODUCTION

The idea of realizing our Universe as a *defect* in a higher dimensional spacetime, although not new [1], has received a lot of attention in recent years in the framework of string theory where *D-branes* [2], i.e. membranes on which the fundamental string fields satisfy Dirichlet boundary conditions, play a significant role. In the framework of string/*M*-theory [3] or the AdS/CFT correspondence [4], brane models [5–8] have revealed new possibilities for the resolution of the hierarchy problem of particle physics as well as for the relation of gravity to the rest of fundamental interactions. In *D*-brane models, standard model fields are trapped on the brane, while gravitons propagate in the full higher dimensional space (*bulk*). In an interesting case of a brane model with an infinite extra dimension, gravitons are localized on the brane due to the curvature of the extra dimension [9]. A solution to Einstein's equations of motion with a flat metric on the brane and AdS₅ geometry in the bulk exists, provided the positive brane tension is finely tuned versus a negative bulk cosmological constant.

Although the standard model fields are assumed to be localized on the brane, gravity is not necessarily the only field propagating in the bulk. A number of brane models with bulk scalar fields have been constructed [10,11], either from a theoretical or phenomenological viewpoint [12]. Actually, the brane itself could be a defect substantiated by a bulk scalar field configuration (a *kink*) [13]. The presence of a bulk scalar field opens the possibility of a direct coupling of this field to the curvature scalar. A specific form of this coupling corresponds to the gravitational term appearing in the so-called tensor-scalar theory of gravity [14]. A bulk scalar field nonminimally coupled through a coupling of the form $\phi^2 R$ has also been considered in the Randall-Sundrum framework and numerical solutions have been discussed [15].

In the present article we consider a 3-brane embedded in 5D space endowed with a bulk scalar field ϕ, nonminimally coupled to gravity through a $-\xi\phi^2 R$ term. We investigate analytically the existence of solutions to the coupled system of equations of motion for the metric and the scalar field in the framework of a metric ansatz $\mathrm{diag}(e^{A(x_5)}\eta_{\mu\nu}, 1)$. In the case of the Randall-Sundrum form of the metric, we derive analytically a complete set of exact solutions for a range of values of the nonminimal coupling strength ξ, corresponding to specific choices of the scalar potential. Scalar fields, with or without nonminimal coupling, are often introduced against a given Randall-Sundrum background under the assumption that their effect on the background geometry will not be important. We do find exact nonsingular scalar field solutions compatible with an exact Randall-Sundrum background, taking into account the full backreaction of the field.

We show the existence of a class of solutions for a general warp function with an asymptotic Randall-Sundrum AdS₅ behavior. In all these considerations, we allow for a field-dependent brane tension. Furthermore, we discuss the existence of smooth AdS₅ solutions for which the role of the brane is played by a kink configuration of the bulk scalar field itself. Both numerical and an approximate analytic treatment of the problem are provided. In particular, we calculate the warp factors for smooth geometries in the presence of the kink for different boundary values at the origin and obtain various solutions. Although we concentrate on $\mathcal{Z}_2$ symmetric solutions, smooth asymmetric solutions are also possible. Through an analytical investigation we verify that, for a wide range of values of the parameters, we can get warp factors that decrease exponentially and thus provide us with finite geometries. We also find analytical solutions for certain special values of the parameters and different ranges of ξ. In the final section of this paper, we study graviton localization in our setup and check the form of the localizing potential for gravitons with the characteristic volcanolike profile. We find that for ξ higher than some specific value the potential develops a

local maximum at the origin, which gradually increases as we move towards higher values of the coupling parameter.

II. THE FRAMEWORK

Consider a general 5D theory of a real scalar field coupled to gravity. Allowing only for terms linear in the Ricci scalar, we may write the general action as

$$
S = \int d^5x \sqrt{-G} \Big\{ f(\phi)R - \frac{1}{2}(\nabla\phi)^2 - V(\phi) - \mathcal{L}_m \Big\},
\tag{1}
$$

where $f(\phi)$ is, for the moment, a general smooth positive-definite function of the scalar field ϕ. G_{MN} is the five-dimensional metric, not to be confused with the Einstein tensor. In the case of a constant f, we have the Einstein action. The last term corresponds to ϕ-independent matter. Note that, the above action can always be transformed through a *conformal transformation* $G_{MN} \to \tilde{G}_{MN} f(\phi)/2M^3$ into an action where the Ricci scalar enters in the Einstein fashion as $(2M^3)R$. Nevertheless, a ϕ-dependence will arise in the matter term giving a theory different than the one we would get in the absence of $f(\phi)$.

The equations of motion resulting from (1) are

$$
f(\phi)(R_{MN} - \tfrac{1}{2}G_{MN}R) - \nabla_M\nabla_N f(\phi) + G_{MN}\nabla^2 f(\phi)
$$
$$
= \tfrac{1}{2}T^{(\phi)}_{MN} + \tfrac{1}{2}T^{(m)}_{MN},
\tag{2}
$$

$$
\nabla^2\phi - \frac{dV}{d\phi} + R\frac{df}{d\phi} = 0,
\tag{3}
$$

with

$$
T^{(\phi)}_{MN} = \nabla_M\phi\nabla_N\phi - G_{MN}(\tfrac{1}{2}(\nabla\phi)^2 + V(\phi))
\tag{4}
$$

the energy-momentum tensor of the scalar field ϕ and $T^{(m)}_{MN}$ the energy-momentum tensor of (other) matter.

At this point we shall restrict the metric G_{MN} introducing the *warped ansatz*

$$
G_{MN} = \begin{pmatrix} e^{A(y)}\eta_{\mu\nu} & 0 \\ 0 & 1 \end{pmatrix},
\tag{5}
$$

where $x^M = (x^\mu, x^5) \equiv (x^\mu, y)$ and $\eta_{\mu\nu}$ is the 4D Minkowski metric with signature $(-1, 1, 1, 1)$. We can always choose $A(0) = 0$.

The presence of a *brane* introduces an extra term,

$$
-\int d^5x \sqrt{-G}\,\sigma(\phi)\delta(y) = -\int d^4x\,\sigma(\phi),
\tag{6}
$$

where the *brane tension* $\sigma(\phi)$ is, in general, ϕ-dependent. Introducing this term in the action modifies T_{MN} in Einstein's equations as

$$
\delta T_{MN} = -G_{\mu\nu}\delta^\mu_M\delta^\nu_N\sigma(\phi)\delta(y) = -\eta_{\mu\nu}\delta^\mu_M\delta^\nu_N\sigma(\phi)\delta(y).
$$

In what follows we shall ignore the presence of (extra) matter beyond the bulk scalar field. Substituting this metric ansatz into the equations of motion and assuming that the scalar field is just a function of the fifth coordinate, i.e. $\phi = \phi(y)$, we obtain

$$
\tfrac{3}{2}f(\dot{A})^2 + 2\dot{A}\,\dot{f} = \tfrac{1}{4}(\dot{\phi})^2 - \tfrac{1}{2}V,
\tag{7}
$$

$$
\tfrac{3}{2}f\ddot{A} + \tfrac{3}{2}f(\dot{A})^2 + \tfrac{3}{2}\dot{f}\dot{A} + \ddot{f} = -\tfrac{1}{4}(\dot{\phi})^2 - \tfrac{1}{2}V - \tfrac{1}{2}\sigma(\phi)\delta(y),
\tag{8}
$$

$$
\ddot{\phi} + 2\dot{A}\,\dot{\phi} - \frac{dV}{d\phi} + \frac{df}{d\phi}(-4\ddot{A} - 5(\dot{A})^2) - \frac{d\sigma}{d\phi}\delta(y) = 0.
\tag{9}
$$

The dot signifies differentiation with respect to the fifth coordinate y.

The *junction relations* at the point $y = 0$ where the brane is located are

$$
\Delta\dot{\phi}(0) \equiv \dot{\phi}(+0) - \dot{\phi}(-0) = \frac{f\sigma' - \tfrac{4}{3}\sigma f'}{f + \tfrac{8}{3}(f')^2},
\tag{10}
$$

$$
\Delta\dot{A}(0) \equiv \dot{A}(+0) - \dot{A}(-0) = -\left(\frac{\tfrac{1}{3}\sigma + \tfrac{2}{3}f'\sigma'}{f + \tfrac{8}{3}(f')^2}\right).
\tag{11}
$$

The prime signifies differentiation with respect to ϕ. By σ and σ' we indicate the values at $\phi(0)$.

Among the above three equations of motion in the *bulk*, only two are independent. They can be written as

$$
V(\phi) = -3f\dot{A}^2 - \tfrac{3}{2}f\ddot{A} - \tfrac{7}{2}\dot{A}\,\dot{f} - \ddot{f},
\tag{12}
$$

$$
\tfrac{1}{2}\dot{\phi}^2 = -\tfrac{3}{2}f\ddot{A} + \tfrac{1}{2}\dot{A}\,\dot{f} - \ddot{f}.
\tag{13}
$$

Let us now restrict the coupling function $f(\phi)$ to be a function quadratic[1] in ϕ. Introducing a dimensionless parameter ξ and normalizing it appropriately, we may write

$$
f(\phi) = 2M^3 - \frac{\xi}{2}\phi^2.
\tag{14}
$$

The scale M is related to the 5D Newton's constant $\tilde{G}$ as $2M^3 = (16\pi\tilde{G})^{-1}$. With this choice, we have

$$
\dot{f} = -\xi\dot{\phi}\phi, \qquad \ddot{f} = -\xi\dot{\phi}^2 - \xi\phi\ddot{\phi}
$$

and the equations of motion in the bulk become

---

[1]Even for a general coupling function, we may consider an expansion in even powers of the field $f(\phi) \approx f(0) + \tfrac{1}{2}f''(0)\phi^2 + \cdots$ and retain the lowest nontrivial term. Such an expansion would be valid for small field values $[\phi \ll (|2f(0)/f''(0)|)^{1/2}]$.

$$V(\phi) = -\frac{3}{2}\left(2M^3 - \frac{\xi}{2}\phi^2\right)(2\dot{A}^2 + \ddot{A}) + \frac{7\xi}{2}\dot{A}\dot{\phi}\phi$$
$$+ \xi\ddot{\phi}\phi + \xi\dot{\phi}^2, \tag{15}$$

$$\frac{1}{2}\dot{\phi}^2 = -\frac{3}{2}\left(2M^3 - \frac{\xi}{2}\phi^2\right)\ddot{A} - \frac{\xi}{2}\dot{A}\dot{\phi}\phi + \xi\ddot{\phi}\phi + \xi\dot{\phi}^2. \tag{16}$$

The junction relations take the form

$$\Delta\dot{\phi}(0) = \frac{(2M^3 - \frac{\xi}{2}\phi^2(0))\sigma' + \frac{4\xi}{3}\sigma\phi(0)}{2M^3 - \frac{\xi}{2}(1 - \frac{16}{3}\xi)\phi^2(0)}, \tag{17}$$

$$\Delta\dot{A}(0) = -\left(\frac{\frac{1}{3}\sigma - \frac{2\xi}{3}\phi(0)\sigma'}{2M^3 - \frac{\xi}{2}(1 - \frac{16}{3}\xi)\phi^2(0)}\right). \tag{18}$$

Again, σ and σ' are the corresponding values at $\phi = \phi(0)$. Note the simplification of the denominator at the $D = 5$ *conformal value*[2] $\xi_c = 3/16$.

III. RANDALL-SUNDRUM METRIC

In this section we shall make the definite choice of the warp function $A(y)$ to be the standard Randall-Sundrum warp function $A(y) = -\kappa|y|$ and impose $\mathcal{Z}_2$ symmetry on the scalar field $[\phi(-y) = \phi(y), \dot{\phi}(+0) = -\dot{\phi}(-0)]$. Substituting, we obtain the equation for $\phi(y)$ in the $y > 0$ bulk

$$\frac{1}{2}\dot{\phi}^2 = \kappa\frac{\xi}{2}\dot{\phi}\phi + \xi\ddot{\phi}\phi + \xi\dot{\phi}^2. \tag{19}$$

The values on the brane will have to obey $(1 - 2\xi)\dot{\phi}^2(0) = \xi\phi(0)(\kappa\dot{\phi}(0) + 2\ddot{\phi}(0))$. Thus, the boundary value $\phi(0) = 0$ is possible only with $\xi = 1/2$. We proceed distinguishing the two cases $[\phi(0) = 0$ and $\phi(0) \neq 0]$.

A. Special case with $\phi(0) = 0$

In this case, possible only for $\xi = 1/2$, we have the solution

$$\phi(y) = \frac{2\dot{\phi}(+0)}{\kappa}(1 - e^{-\kappa|y|/2}). \tag{20}$$

The junction relations give

$$\sigma = 12\kappa M^3, \qquad \sigma' = 2\dot{\phi}(+0). \tag{21}$$

Note that this solution is possible only with field-dependent brane tension. Note also that the first is the standard Randall-Sundrum relation. This special solution has the shape of a *folded kink* as plotted in Fig. 1. Beyond a small region near the brane it reaches a constant value $\phi(\pm\infty) = 2\kappa^{-1}\dot{\phi}(+0)$ (Fig. 1).

The required positivity of the coupling function

$$f(\phi) = 2M^3 - \frac{\xi}{2}\phi^2(y)$$
$$= 2M^3 - 2(\dot{\phi}(0)/\kappa)^2(1 - e^{-\kappa|y|/2})^2 > 0 \tag{22}$$

imposes the boundary value constraint

$$(\dot{\phi}(0))^2 < \kappa^2 M^3 \Rightarrow (\sigma')^2 < \frac{1}{3}\kappa\sigma. \tag{23}$$

The scalar potential corresponding to this solution can be obtained to be

$$V(\phi) = -2\kappa\sigma - \frac{1}{8}\sigma'^2 + \frac{1}{4}(\kappa\phi + \sigma')^2. \tag{24}$$

In the expression above, we have made use of the junction relations. If we were to start with a general quadratic potential $V(\phi) = \Lambda + C_1\phi + C_2\phi^2$, the solution (20) and $A(y) = -\kappa|y|$ is possible for $\Lambda = -2\kappa\sigma + \sigma'^2/8$, $C_1 = \kappa\sigma'/2$, and $C_2 = \kappa^2/4$.

B. General case

In the general case $\phi(0) \neq 0$, the equation of motion can be written as

$$\frac{1}{2}(1 - 2\xi)\frac{\dot{\phi}}{\phi} - \kappa\frac{\xi}{2} - \xi\frac{\ddot{\phi}}{\dot{\phi}} = 0 \tag{25}$$

and leads to the solution

$$\phi(y) = \phi(0)[1 + \alpha(4 - \xi^{-1})(1 - e^{-\kappa|y|/2})]^{(2\xi)/(4\xi-1)}, \tag{26}$$

where

$$\alpha \equiv \frac{\dot{\phi}(+0)}{\kappa\phi(0)}. \tag{27}$$

The junction relations take the form

$$2\kappa\alpha\phi(0) = \frac{(2M^3 - \frac{\xi}{2}\phi^2(0))\sigma' + \frac{4\xi}{3}\sigma\phi(0)}{2M^3 - \frac{\xi}{2}(1 - \frac{16}{3}\xi)\phi^2(0)},$$
$$2\kappa = \frac{\frac{1}{3}\sigma - \frac{2\xi}{3}\phi(0)\sigma'}{2M^3 - \frac{\xi}{2}(1 - \frac{16}{3}\xi)\phi^2(0)}. \tag{28}$$

These two constraints can be rewritten as

$$\sigma' = 2\kappa\phi(0)(\alpha - 4\xi),$$
$$\sigma = 6\kappa\left(2M^3 - \frac{\xi}{2}\phi^2(0)\right) + 4\xi\kappa\alpha\phi^2(0). \tag{29}$$

In order to study whether the positivity of the coupling function and the requirement of a positive tension brane ($\sigma > 0$) can be simultaneously satisfied, we consider the four possible sign choices of the nonminimal coupling strength parameter ξ and the boundary values parameter α. They are satisfied[3]

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[2]The conformal value in D dimensions is $\xi_c^{(D)} = \frac{(D-2)}{4(D-1)}$. See Ref. [16].
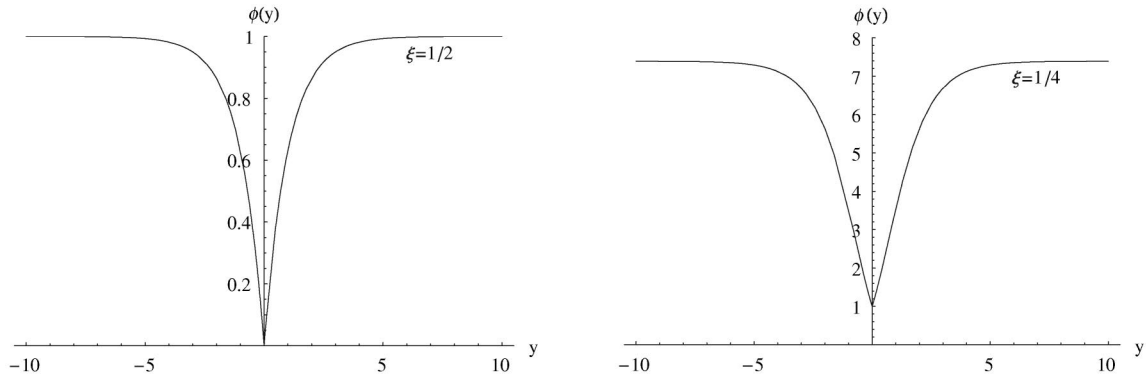
[3]We choose $\phi(0) > 0$.

FIG. 1. Solutions for a field-independent brane tension and different values of ξ.

(1) If $\xi > 0$, $\alpha > 0$, always.
(2) If $\xi < 0$, $\alpha < 0$, always.
(3) If $\xi < 0$, $\alpha > 0$, only if

$$0 < \alpha < \frac{3}{4} + \frac{3M^3}{|\xi|\phi^2(0)}. \tag{30}$$

(4) If $\xi > 0$, $\alpha < 0$, only if

$$\alpha > -\frac{3}{4} + \frac{3M^3}{\xi\phi^2(0)}. \tag{31}$$

1. Field-independent brane tension ($\sigma' = 0$)

To simplify our analysis we may consider separately the case of field-independent brane tension ($\sigma' = 0$). In this case the junction relations simplify to

$$6\kappa = \frac{\sigma}{[2M^3 - \frac{\xi}{2}(1 - \frac{16}{3}\xi)\phi^2(0)]} \tag{32}$$

and

$$\alpha = 4\xi. \tag{33}$$

Notice that the positivity of the brane tension is always satisfied, since $2M^3 - \xi\phi^2(0)/2 + 8\xi^2\phi^2(0)/3$ is positive if the coupling function is positive ($2M^3 - \xi\phi^2/2 > 0$).

The relation $\alpha = 4\xi$ simplifies the solution (26) to

$$\phi(y) = \phi(0)[1 + 4(4\xi - 1)(1 - e^{-\kappa|y|/2})]^{(2\xi)/(4\xi-1)}. \tag{34}$$

For $\xi > \xi_c = 3/16$, the quantity in brackets $16(\xi - \xi_c) - 16(\xi - \xi_c - 1/4)e^{-\kappa|y|/2}$ is positive.

Note that for the special value $\xi = 1/2$, if $\phi(y)$ is the solution, so is $\phi(y) + \text{const}$. Also, in the special case $\xi = 1/4$, the solution takes the form

$$\phi(y) = \phi(0)e^{2(1-e^{-\kappa|y|/2})}. \tag{35}$$

This is shown in Fig. 1.

The requirement of the positivity of the coupling function, in the allowed range $\xi_c < \xi$, corresponds to the inequality $2M^3 \geq \frac{\xi}{2}\phi^2(0)[1 + 4(4\xi - 1)]^{(4\xi)/(4\xi-1)}$. For the special value $\xi = 1/4$, this corresponds to $2M^3 \geq \frac{e^4}{8}\phi^2(0)$.

For the conformal value $\xi_c = 3/16$, the solution reduces to an increasing exponential

$$\phi_c(y) = \phi(0)e^{(3/4)\kappa|y|}. \tag{36}$$

For values of the coupling parameter in the range $0 < \xi < \xi_c$, the quantity in brackets vanishes at $y_0 = \pm\frac{2}{\kappa} \times \ln[1 + (\xi_c - \xi)^{-1}]$, while the exponent is negative, i.e. $\frac{2\xi}{4\xi-1} = -\frac{2\xi}{4(\xi_c - \xi)+\frac{1}{4}} < 0$. Thus, in this range the solution is singular.

For ξ negative, the solution

$$\phi(y) = \phi(0)[4(1 + 4|\xi|)e^{-\kappa|y|/2} - (3 + 16|\xi|)]^{(2|\xi|)/(4|\xi|+1)} \tag{37}$$

is characterized by an exponent between 0 and 1, while the expression in brackets vanishes at $y_0 = \pm\frac{2}{\kappa}\ln[1 + (3 + 16|\xi|)^{-1}]$. Note that $\dot{\phi}(y_0) = -\infty$. As we shall promptly see, these *solutions* are not acceptable since the scalar potential, possessing negative powers of the scalar field, is singular.

The scalar potential corresponding to the solutions found can be immediately obtained from Eq. (15). In order to do that it is useful to obtain the derivatives of the solution. They are

$$\dot{\phi}(y) = \phi(y)\left(\frac{\xi\kappa}{4\xi - 1}\right)$$
$$\times\left[(1 + 4(4\xi - 1))\left(\frac{\phi(y)}{\phi(0)}\right)^{(1-4\xi)/(2\xi)} - 1\right] \tag{38}$$

and

$$\ddot{\phi}(y) = -\dot{\phi}(y)\left(\frac{\xi\kappa}{4\xi - 1}\right)$$
$$\times\left[\frac{(2\xi - 1)}{2\xi}(1 + 4(4\xi - 1))\left(\frac{\phi(y)}{\phi(0)}\right)^{(1-4\xi)/(2\xi)} + 1\right].$$

$$(39)$$

Substituting the above into (15), we obtain

$$V(\phi) = -6\kappa^2 M^3 + \phi^2\left(C_1 + C_2\left(\frac{\phi}{\phi(0)}\right)^{(1-4\xi)/(2\xi)}\right.$$
$$\left. + C_3\left(\frac{\phi}{\phi(0)}\right)^{(1-4\xi)/\xi}\right),$$

$$(40)$$

where

$$C_1 = \frac{\xi\kappa^2}{2}\left(3 + 7\frac{\xi}{4\xi - 1} + 4\left(\frac{\xi}{4\xi - 1}\right)^2\right),$$

$$(41)$$

$$C_2 = -4\xi\kappa^2(1 + 4(4\xi - 1))\left(\frac{\xi}{4\xi - 1}\right),$$

$$C_3 = \frac{C_2^2}{32\xi^2\kappa^2}.$$

$$(42)$$

All powers are positive in the range $0 < \xi < \frac{1}{2}$. In the special case $\xi = 1/4$, the scalar potential includes logarithmic terms. It is

$$V(\phi) = -6\kappa^2 M^3 + \xi\kappa^2\phi^2[-\frac{1}{2} + \frac{3}{4}\ln(\phi/\phi(0))$$
$$+ \frac{1}{8}(\ln(\phi/\phi(0)))^2].$$

$$(43)$$

For the special value $\xi = 1/2$, the scalar potential has the quadratic form

$$V(\phi) = -6\kappa^2 M^3 + \frac{5}{8}\kappa^2(3\phi^2 - 8\phi\phi(0) + 25\phi^2(0)).$$

$$(44)$$

For the limiting conformal value $\xi_c = 3/16$, all of the above coefficients vanish and we obtain a constant potential $V = -6\kappa^2 M^3$. For negative values the appearing powers $\phi^{1/2\xi}$ and $\phi^{(1/\xi)-2}$ are negative and, since the solution $\phi(y)$ vanishes at a finite point, the potential is singular.

2. General case with field-dependent brane tension ($\sigma' \neq 0$)

In order to investigate the behavior of the scalar field solution (26), we first consider the case $\alpha > 0$. In this case we have a solution increasing near the origin, since $\kappa\alpha = \dot{\phi}(+0)/\phi(0) > 0$. The quantity in brackets is positive, provided $\alpha(4 - \xi^{-1}) > -(1 - e^{-\kappa|y|/2})^{-1}$. The lower limit of the right-hand side is -1, which corresponds to the range $\xi > \frac{1}{4+\alpha^{-1}}$. As examples, consider the cases $\xi = \alpha = 1$ and $\xi = \alpha = 1/8$. The first one corresponds to a positive exponent $2/3$, while the second corresponds to the exponent $-1/2$. They are both shown in Fig. 2.
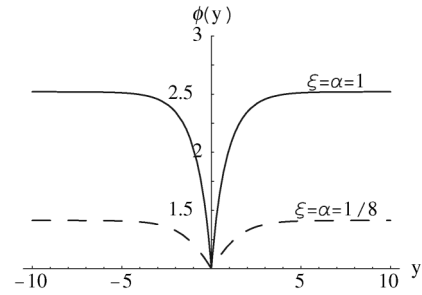

FIG. 2. Solution profiles for a field-dependent brane tension.

For values of ξ below this bound, there is a point for which the expression in brackets vanishes and, since the exponent is negative, there is a singularity. This point is $y_0 = -\frac{2}{\kappa}\ln[1 - \xi/\alpha(1 - 4\xi)]$. For the special value $\bar{\xi} = \frac{\alpha}{4\alpha+1}$, the singularity is pushed to infinity and we obtain a purely exponential form for the solution, namely

$$\bar{\phi}(y) = \phi(0)e^{\kappa\alpha|y|}.$$

$$(45)$$

Before we move to consider negative values, let us mention again the special value $\xi = 1/4$ which corresponds to

$$\phi(y) = \phi(0)e^{2\alpha(1 - e^{-\kappa|y|/2})}.$$

$$(46)$$

As we have remarked earlier, for $\xi > 0$ and $\alpha > 0$, the positivity of the brane tension ($\sigma > 0$) is always true. However, the requirement of a positive coupling function introduces a constraint on the parameters. It is sufficient to have $2M^3 > \frac{\xi}{2}\phi^2(0)[1 + \alpha(4 - \xi^{-1})]^{(4\xi)/(4\xi-1)}$. For the special value $\xi = 1/4$, this constraint has the form $2M^3 > \frac{e^{4\alpha}}{8}\phi^2(0)$.

For negative values $\xi < 0$ (and still $\alpha > 0$) the solution takes the form

$$\phi(y) = \phi(0)[1 + \alpha(4 + |\xi|^{-1})(1 - e^{-\kappa|y|/2})]^{(2|\xi|)/(4|\xi|+1)}.$$

$$(47)$$

Note that although the scalar potential has negative powers ($\phi^{-1/2|\xi|}$ and $\phi^{-2-1/|\xi|}$), there is no singularity, since the scalar field does not vanish anywhere. Note also that for $\xi < 0$ and $\alpha > 0$ the positivity of the brane tension introduces a constraint $\alpha < \frac{3}{4} + \frac{3M^3}{|\xi|\phi^2(0)}$.

Let us move now to consider the case $\alpha < 0$. Writing the solution as

$$\phi(y) = \phi(0)[1 - |\alpha|(4 - \xi^{-1})(1 - e^{-\kappa|y|/2})]^{(2\xi)/(4\xi-1)},$$

$$(48)$$

we see that, for $\xi > 1/4$, the exponent is positive. For $|\alpha| < \frac{1}{4}$ the quantity in brackets stays positive. However, for $|\alpha| > \frac{1}{4}$ it is necessary to limit the range of ξ to $\xi < \frac{1}{4 - \frac{1}{|\alpha|}}$. For the critical value $\bar{\xi} = \frac{|\alpha|}{4|\alpha|-1}$, the solution becomes a decreasing exponential, namely
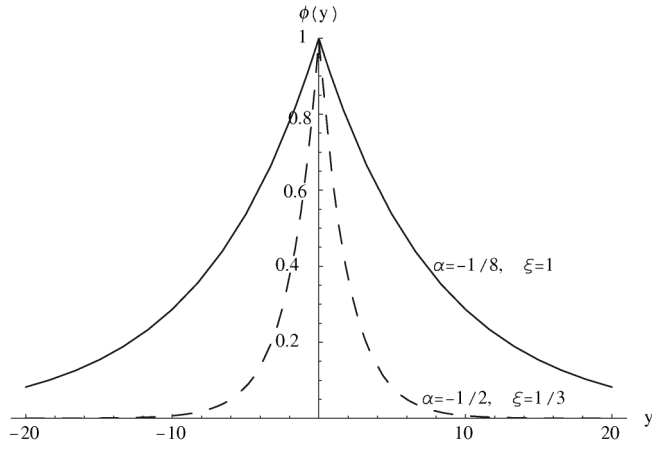
FIG. 3. Field-dependent brane-tension solutions with negative α.

$$\bar{\phi}(y) = \phi(0)e^{-\kappa|\alpha||y|}. \tag{49}$$

As examples of the solution in the above range, let us consider $\alpha = -1/8$, $\xi = 1$ and $\alpha = -1/2$, $\xi = 1/3$, shown in Fig. 3

For $|\alpha| > 1/4$ and ξ outside of the above range, i.e. $\xi > \frac{1}{4 - \frac{1}{|\alpha|}}$, we obtain solutions that vanish at a finite distance from the brane, namely $|y_0| = -\frac{2}{\kappa}\ln[1 - \xi/|\alpha|(4\xi - 1)]$. This, again, corresponds to a singular scalar potential due to the negative power $\phi^{1/\xi - 2}$ that appears in it.

For values below $1/4$ but positive, the solution takes the form

$$\phi(y) = \phi(0)[1 + |\alpha|(\xi^{-1} - 4)(1 - e^{-\kappa|y|/2})]^{-[(2\xi)/(1-4\xi)]} \tag{50}$$

and gives a smooth decreasing profile, just as seen above.

Before we move to consider negative values of ξ, let us consider the special case $\xi = 1/4$. In this case, we have

$$\phi(y) = \phi(0)e^{-2|\alpha|(1-e^{-\kappa|y|/2})} \tag{51}$$

shown for $\alpha = -1$ in Fig. 4

The positivity of the brane tension, for $\xi > 0$ and $\alpha < 0$, as was found earlier, introduces the constraint
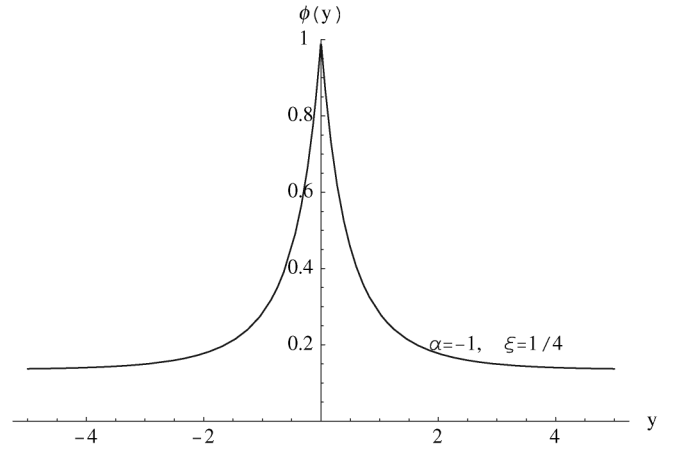
$$-\alpha > -\frac{3}{4} + \frac{3M^3}{\xi\phi^2(0)}. \tag{52}$$

On the other hand, the positivity of the coupling function, since $\phi(y)$ is a decreasing function, is covered by $2M^3 > \frac{\xi}{2}\phi^2(0)$.

For negative values $\xi < 0$, the solution can be written as

$$\phi(y) = \phi(0)$$
$$= [1 - |\alpha|(|\xi|^{-1} + 4)(1 - e^{-\kappa|y|/2})]^{(2|\xi|)/(4|\xi|+1)}. \tag{53}$$

It is easy to see that, for $|\alpha| > 1/4$, the scalar field vanishes

at a finite distance from the brane, namely $|y_0| = -\frac{2}{\kappa} \times \ln[1 - |\xi|/|\alpha|(4|\xi| + 1)]$. This, again, amounts to a singular scalar potential due to the negative powers $\phi^{-1/|\xi|}$ and $\phi^{-(1/|\xi|+2)}$.

The scalar potential has exactly the same form as in the $\sigma' = 0$ case ($\alpha = 4\xi$), the only difference being a slight change in the coefficients C_2 and C_3 which become

$$C_2 = -4\xi\kappa^2\left(1 + \frac{\alpha}{\xi}(4\xi - 1)\right)\left(\frac{\xi}{4\xi - 1}\right),$$
$$C_3 = \frac{C_2^2}{32\xi^2\kappa^2}. \tag{54}$$

IV. BEYOND RANDALL-SUNDRUM

Let us consider again our original set of the two independent equations (12) and (13) for the specific choice of coupling function $f(\phi) = 2M^3 - \xi\phi^2/2$. If we do not impose any restriction on the scalar potential function $V(\phi)$, we can consider the first equation as an equation that *determines* the scalar potential in terms of the functions $\phi(y)$ and $A(y)$. Concentrating on the second equation, we can view it as an equation for the warp factor, giving a different $A(y)$ for every different *choice* of $\phi(y)$ configuration. Motivated by the form of the solutions found in the Randall-Sundrum case, we may start by introducing a scalar field configuration in the form of a folded kink with Z_2 symmetry[4]

$$\phi(y) = \phi_0 \tanh(a|y|), \tag{55}$$

where $\phi_0 \equiv a^{-1}\dot{\phi}(+0)$. For the positivity of the coupling function it would be sufficient to require $\dot{\phi}^2(0) < 4M^3a^2/\xi$.

Substituting (55) into the junction relations, we obtain, since $\phi(0) = 0$,

FIG. 4. Profile of the solution for the special value $\xi = \frac{1}{4}$.

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[4] $\dot{A}(+0) = -\dot{A}(-0)$, $\dot{\phi}(+0) = -\dot{\phi}(-0)$.

$$\dot{A}(+0) = -\frac{\sigma}{12M^3}, \qquad \dot{\phi}(+0) = \frac{\sigma'}{2}. \qquad (56)$$

Note that the positivity of the brane tension requires $\dot{A}(+0) < 0$.

In the $y > 0$ bulk, we have $\dot{\phi} = a\phi_0(1 - \phi^2/\phi_0^2)$, $\ddot{\phi} = -2a^2\phi(1 - \phi^2/\phi_0^2)$. Substituting these into the equation of motion, we can write it in the form

$$\frac{3}{2}\phi_0\left(2M^3 - \frac{\xi}{2}\phi^2\right)X'(\phi) + \frac{\xi}{2}\phi_0\phi X(\phi) + 2a\xi\phi^2$$
$$+ a\phi_0^2\left(\frac{1}{2} - \xi\right)\left(1 - \frac{\phi^2}{\phi_0^2}\right) = 0, \quad (57)$$

where $X \equiv \dot{A}(y)$ and $\ddot{A} = \dot{X} = \dot{\phi}X'(\phi)$. This differential equation can be integrated to give

$$X(\phi) = C_0\left(1 - \frac{\xi\phi^2}{4M^3}\right)^{1/3}$$
$$+ \phi\left\{C_1\left(1 - \frac{\xi\phi^2}{4M^3}\right)^{1/3}{}_2F_1(1/2, 1/3, 3/2, \xi\phi^2/4M^3)\right.$$
$$\left. + C_2\right\}, \qquad (58)$$

where C_1 and C_2 are given by

$$C_1 = -\frac{a}{\phi_0}\left[\xi^{-1} - 6 + \frac{\phi_0^2}{12M^3}(2\xi - 1)\right],$$
$$C_2 = \frac{a}{\phi_0}\left[\xi^{-1} - 6 + \frac{\phi_0^2}{4M^3}(2\xi - 1)\right] \qquad (59)$$

and the integration constant $C_0 = \dot{A}(+0) = -\sigma/12M^3$ should be negative in order to have a positive brane tension.

The metric warp factor will be given by $e^{A(y)} = \exp[\int dy X(\phi)]$. Near the brane, i.e. for $y \to 0$ or $\phi \to 0$, we have

$$e^{A(y)} \approx \exp\left[-\frac{\sigma}{12M^3}y + O(y^2)\right], \qquad (60)$$

which is a pure Randall-Sundrum behavior.

In the asymptotic region $[y \to \infty \Rightarrow \phi \sim \phi_0 = a^{-1}\dot{\phi}(0)]$, we have

$$e^{A(y)} = \exp\left[(a\phi_0)^{-1}\int\frac{d\phi}{(1 - \phi^2/\phi_0^2)}X(\phi)\right]$$
$$\approx \exp\left[\frac{1}{2a\phi_0}\int\frac{d\phi}{(1 - \phi/\phi_0)}(X(\phi_0) + (\phi - \phi_0)\right.$$
$$\left. \times X'(\phi_0) + \cdots)\right] \approx e^{-\bar{\kappa}y}g(y), \qquad (61)$$

with $\bar{\kappa} \equiv -X(\phi_0)$ and

$$g(y) \equiv 2^{-[1/(2a)]X(\phi_0)}\exp\left[-\frac{\phi_0}{2a}X'(\phi_0)\right.$$
$$\left. + \frac{1}{2a}(X(\phi_0) + 2\phi_0X'(\phi_0))e^{-2ay} + \cdots\right]. \qquad (62)$$

Thus, in the asymptotic region we have a behavior exponentially close to a Randall-Sundrum behavior, provided the parameter $\bar{\kappa}$ is positive. This parameter is

$$\bar{\kappa} = \frac{\sigma}{2M^3}\left(1 - \frac{\xi\phi_0^2}{4M^3}\right)^{1/3} - af(\xi, \phi_0^2/M^3), \qquad (63)$$

where we have introduced a function $f(\xi, x)$ defined by

$$f(\xi, x) \equiv \left(6 - \xi^{-1} - \frac{x^2}{12}(2\xi - 1)\right)$$
$$\times (1 - x^2/4)^{1/3}{}_2F_1(1/2, 1/3, 3/2, \xi x^2/4)$$
$$+ \xi^{-1} - 6 + \frac{x^2}{4}(2\xi - 1). \qquad (64)$$

The coupling function is positive if $\xi\phi_0^2 < 4M^3$. The positivity of $\bar{\kappa}$ can always be satisfied with a large enough brane-tension parameter σ. Nevertheless, we can be more concrete by making a choice of $\dot{\phi}(+0)$. We can take $\phi_0^2 = M^3$. Then, the positivity of the coupling function restricts the values of ξ to $\xi < 4$. The function $f(\xi, 1)$ is negative for all allowed values of ξ. Thus, the warp factor will always be decreasing. The plot of this function of ξ is shown in Fig. 5.

We can construct numerical solutions for the function $A(y)$ and study the profile of the warp factor $e^{A(y)}$ for different values of the parameters of the model. In this case, the brane tension determines the value of the derivative of $A(y)$ at $y = 0$, so it fixes one of the two initial conditions needed for the numerical evaluation. The derivative of the brane tension with respect to the field, σ', is proportional to $a\phi_0$. Thus, changing it corresponds to a new value for $\dot{\phi}(0)$. Note that, although we have restricted
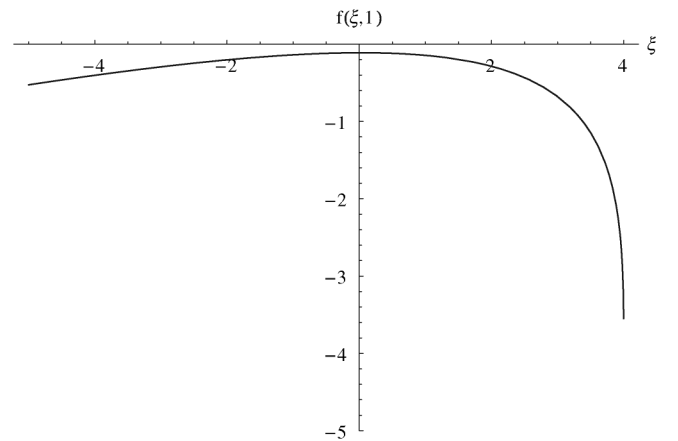

FIG. 5. A graph of the function $f(\xi, 1)$ plotted for all values of ξ. The function maintains a negative value.
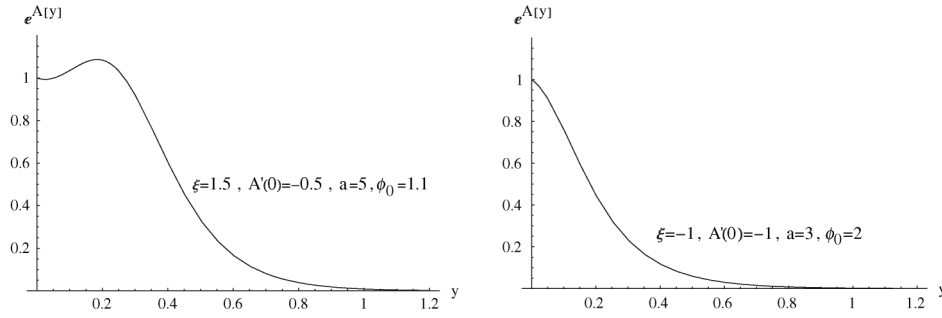
FIG. 6.　The warp factor for different choices of $A'(0)$, a, ξ, and ϕ_0 in units of $2M^3$.

ourselves on Z_2-symmetric solutions, asymmetric solutions are also possible. The profile of characteristic solutions is plotted in Fig. 6.

We see that, in general, the warp factor resembles the Randall-Sundrum decreasing exponential. Yet, for a range of values in the parameter space we get solutions which deviate slightly from this form. As the brane tension becomes smaller and ξ takes higher positive values, the warp factor exhibits a peak close to the brane, before it starts decreasing again. This peak is amplified as we approach the value of ξ for which the coupling function $f(\phi)$ tends to zero.

V. SMOOTH SPACES

As we saw in the last example, the presence of the brane was not essential to obtain a localized warp factor. In this section we shall consider solutions $A(y)$ when the *brane* is the scalar field configuration itself. Such an example is well known in the $\xi = 0$ case. Introducing a standard kink $[\phi = \phi_0 \tanh(ay)]$ into the $\xi = 0$ equation of motion, we obtain in the Z_2 symmetric case $[\dot{A}(0) = 0]$

$$e^{A(y)} = (\cosh(ay))^{-\gamma} e^{-(\gamma/4)\tanh^2(ay)}, \tag{65}$$

with $\gamma = \phi_0^2/9M^3$.

Smooth solutions of the bulk equations of motion are also present for $\xi \neq 0$. It is not difficult to see that the metric choice,

$$e^{A(y)} = (\cosh(ay))^{-\gamma}, \tag{66}$$

corresponds to the same scalar field solution $\phi(y) = \phi_0 \tanh(ay)$ with

$$\gamma = 2(\xi^{-1} - 6),$$
$$\phi_0 = a^{-1}\dot{\phi}(0) = (2M^3)^{1/2}\sqrt{\frac{6(1 - 6\xi)}{\xi(1 - 2\xi)}}. \tag{67}$$

This solution exists for $0 < \xi < 1/6$ and only for the above specially chosen value of ϕ_0[5] The curvature scalar of this

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[5]Note that this solution exists for the fine-tuned value of ϕ_0 in (67) and cannot be connected to the solution for $\xi = 0$ shown in (65). In fact $\xi \to 0$ corresponds to $\dot{\phi}(0) \to \infty$.

space is $R = 4a^2\gamma(1 - (1 + 5\gamma/4)\frac{\phi^2}{\phi_0^2})$. The scalar potential corresponding to this solution is a quartic function of the scalar field with tuned ξ-dependent coefficients. Note that this solution is a particular case of (58). The above choice of ϕ_0, together with the choice $C_0 = 0$, corresponds to

$$C_1 = 0, \qquad C_2 = -\frac{2a}{\sqrt{12M^3}}\sqrt{\xi^{-1}(1 - 2\xi)(1 - 6\xi)}. \tag{68}$$

It is interesting that the same metric choice corresponds also to the solution

$$\phi(y) = \phi(0)(\cosh(ay))^{-1} \tag{69}$$

with $\phi^2(0) = 12M^3(\xi^{-1} - 6)/(3 - 16\xi)$ defined in the same ξ-range.

Another interesting solution for the metric is defined by the choice $\phi_0 = 2\sqrt{\xi^{-1}M^3}$, for which (58) gives

$$\dot{A} = 2a\tanh(ay)\{-2 + \tfrac{1}{3}(8 - \xi^{-1})$$
$$\times \cosh^{-2/3}(ay)\,{}_2F_1(1/2, 1/3, 3/2, \tanh^2(ay))\}. \tag{70}$$

Integrating, we obtain

$$A(y) = -4\ln(\cosh(ay)) + \tfrac{1}{3}(8 - \xi^{-1})$$
$$\times \tanh^2(ay)F_{PFQ}(\{1, 1, 7/6\}, \{3/2, 2\}, \tanh^2(ay)). \tag{71}$$

The warp factor e^A is plotted in Fig. 7 for $\xi = 1/4$ and for $\xi = 1/9$.

For larger ξ the behavior does not change drastically. Note though that for values $\xi > 1/2$ the warp factor develops a maximum beyond the origin. In Fig. 8 we plot the cases $\xi = 1$ and $\xi = \infty$.

As already seen, the equation of motion for a kinklike scalar cannot be solved analytically for general boundary conditions. It is however possible to obtain numerical solutions. We expect to find a set of Z_2-symmetric solutions for $e^{A(y)}$ that reduce to the known $\xi = 0$ solution mentioned in [13]. As an example, we consider numerical solutions of the warp factor equation, imposing the bound-

ary values $A(0) = 1$, $A'(0) = 0$ and taking different values for ϕ_0. The resulting warp factors for $\xi = -2$ and $\xi = 0.8$ are shown in Fig. 9. We have taken $\phi_0 = a = 1$ in units of $2M^3$. For this choice of boundary values and units, $\xi = 2$ corresponds to the limiting value for which the function $f(\phi)$ becomes zero at the origin, so higher values of ξ are forbidden. Notice the peak beyond the origin in the second plot.
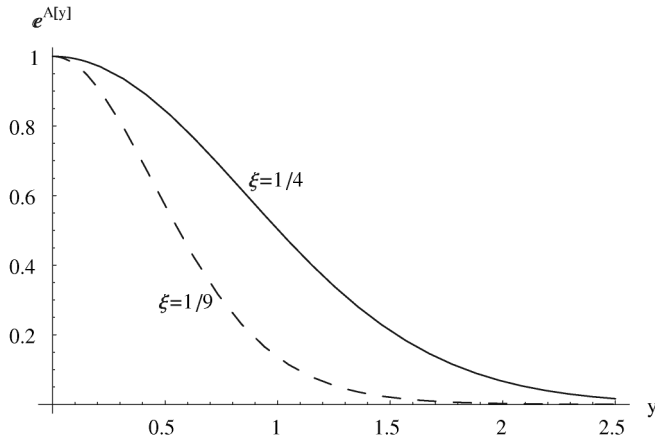

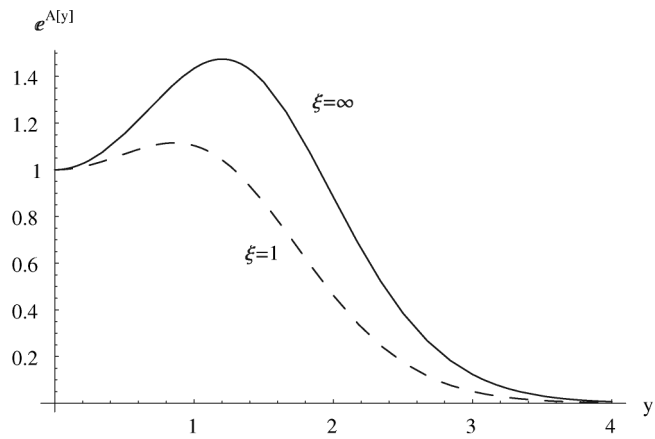FIG. 7. Warp factors for $\phi(y) = \phi(0)(\cosh(ay))^{-1}$, $\xi < 1/2$.
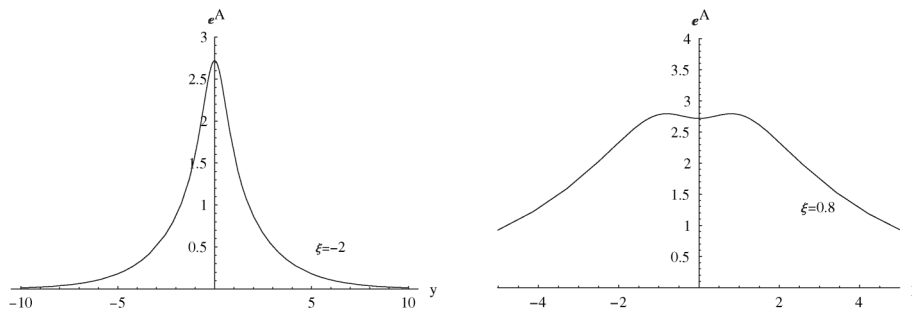

FIG. 8. Warp factors for $\phi(y) = \phi(0)(\cosh(ay))^{-1}$, $\xi > 1/2$.

VI. GRAVITON LOCALIZATION

It would be interesting to check if gravity is localized in the geometries we calculated and especially those that deviate from the original Randall-Sundrum. Let us quickly provide a general argument and then study some specific examples. We consider a perturbation

$$\delta G_{MN} = \delta_M^\mu \delta_N^\nu h_{\mu\nu}(x, y), \qquad \delta\phi = 0, \qquad (72)$$

for the gauge $h_{5M} = 0$. Imposing transversality ($h_\mu^\mu = \partial_\mu h^{\mu\nu} = 0$), we obtain to first order

$$\left(-\frac{d^2}{dy^2} - e^{-A(y)}\partial^2 + \ddot{A}(y) + \dot{A}^2(y)\right)h_{\mu\nu} = 0, \qquad (73)$$

where $\partial^2 = \eta_{\mu\nu}\partial^\mu\partial^\nu$. Notice that this result is independent of the coupling function $f(\phi)$. If we introduce a trial solution of the form of a product $h_{\mu\nu} \propto e^{ip\cdot x}\psi(y)$, we get a Schrödinger-like equation

$$\left(-\frac{d^2}{dy^2} + \ddot{A}(y) + \dot{A}^2(y)\right)\psi(y) = m^2 e^{-A(y)}\psi(y), \qquad (74)$$

where we have introduced the mass $m^2 = -p^2$. In order to study the spectrum of this equation, it is more convenient to transform it into a conventional Schrödinger equation. In order to eliminate the exponential, we may introduce the transformation

$$\frac{d}{dy} = e^{-A/2}\frac{d}{dz}, \qquad \psi(y) = e^{A/4}\bar{\psi}. \qquad (75)$$

The resulting equation is

$$\left(-\frac{d^2}{dz^2} + U(z)\right)\bar{\psi} = m^2\bar{\psi}, \qquad (76)$$

with the potential

$$U(z) = \frac{3}{4}\frac{d^2A}{dz^2} + \frac{9}{16}\left(\frac{dA}{dz}\right)^2. \qquad (77)$$

Note that this equation can be put into the form

$$\left(-\frac{d}{dz} - \frac{3}{4}\frac{dA}{dz}\right)\left(\frac{d}{dz} - \frac{3}{4}\frac{dA}{dz}\right)\bar{\psi} = m^2\bar{\psi}. \qquad (78)$$

This is supersymmetric quantum mechanics and the trans-

FIG. 9. Smooth numerical solutions.

formed graviton wave function (zero mode) corresponds to the supersymmetric ground state. This form also excludes the existence of tachyon modes. The zero mode is just $\psi_0(y) = Ne^{A(y)}$ and it is normalizable. We also have to know if there is a gap between the zero mode and the continuum of eigenstates. For this, we have to know the behavior of the potential $U(z)$. Although for most of the cases above the change of variable $z = \int dy e^{-A(y)/2}$ is not analytically integrable, we may draw some conclusions with the help of the asymptotic behavior (61). Since, for $y \rightarrow \infty$, we may have

$$z = \int dy e^{-A/2} \approx \int dy e^{\bar{\kappa}y/2}(g(y))^{-1/2} \quad (79)$$

or $z \propto e^{\bar{\kappa}y/2}$. As a result,

$$\lim_{y \rightarrow \infty} U(z) \propto e^A \rightarrow 0. \quad (80)$$

Therefore, the continuous spectrum starts from zero mass and there is no gap.

Next, we may check the profile of the localization potential $U(z)$ for various values of ξ. For $\xi = \frac{1}{8}$, we have $A(y) = -4\ln(\cosh(ay))$. In this case, the integration can be done analytically and the transformed coordinate is $z = \frac{1}{4a}\sinh(2ay) + \frac{y}{2}$. For $\xi = 0$, the warp function becomes $A(y) = -\frac{4}{9}(4\ln(\cosh(ay)) + \tanh^2(ay))$. We can only proceed numerically to perform the change of coordinates and calculate the potential. The resulting profiles are depicted in Fig. 10. The localizing potential has the familiar volcanolike shape we also encounter in standard Randall-Sundrum.

It turns out that the volcanolike profile of the potential is not maintained for all values of ξ. For the boundary value and unit choices made for the smooth numerical solutions of Sec. V depicted in Fig. 9, we find that at the value $\xi = (\sqrt{193} - 9)/16 \approx 0.306$ the global minimum at the origin $y = 0$ changes into a local maximum. Thus, as we move towards higher ξ's, a central spike is developed. For $\xi = 1/2$ the potential becomes zero on the *brane*, while as ξ goes to infinity the potential at that point approaches unity. The corresponding graphs are shown in Fig. 10.

VII. CONCLUSIONS

In the present article we investigated the existence of solutions for a nonminimally coupled bulk scalar field in a warped brane-world framework. For a scalar field coupling to gravity of the form $-\frac{1}{2}\xi\phi^2 R$, we derived a set of nonsingular solutions for a wide range of the coupling parameter ξ. We demonstrated the compatibility of the usual Randall-Sundrum warp factor with the presence of a nontrivial scalar field for a suitably chosen scalar potential. This was done for either a scalar field-dependent or independent brane tension. The profile of the scalar field solution in the field-independent brane-tension case is that of a folded kink [$\tanh(a|y|)$]. The scalar field acquires its minimum value on the brane, approaching a constant value as we move towards infinity in the y direction. The *conformal value* of the coupling parameter ξ_c separates the abovementioned solutions from singular *solutions*. Thus, for the Randall-Sundrum warp factor, nonsingular scalar field solutions exist only for $\xi > \xi_c$ and they are, in general, of the above folded-kink shape. For negative values of the coupling parameter ξ the scalar field *solutions* found correspond to a scalar potential with negative powers of the scalar field and, therefore, singular. A field-dependent brane tension allows for a more diverse range of behavior including scalar field solutions which exponentially decrease at infinity.

Guided by the scalar field set of solutions with a folded-kink type of profile, we investigated the existence of general warp factor solutions that are different from the exact Randall-Sundrum case but still localized. Thus, assuming a scalar field solution of the form $\tanh(a|y|)$, we derived corresponding warp factor solutions which we analyzed semianalytically and numerically. Our analytical treatment further showed that for a wide range of values in the parameter space of the model we get finite geometries which are well-behaved for large y, as long as the brane tension we introduce is large enough. Furthermore, we considered smooth warp factor solutions in which the role of the brane is played by the scalar field itself. In this setup we considered some special solutions and proceeded to study numerically general finite geometries. We concentrated on Z_2-symmetric solutions, although asymmetric ones are also possible. We considered a class of
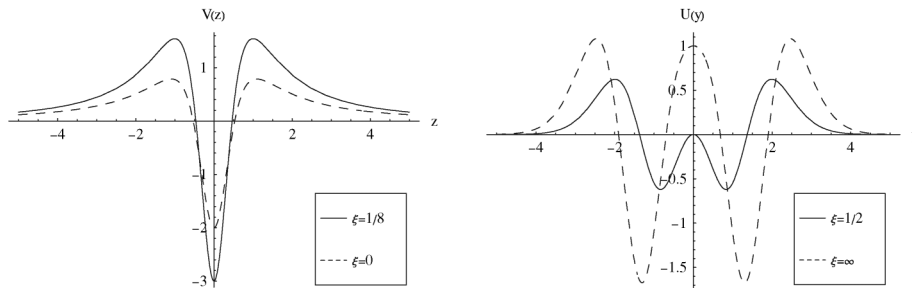

FIG. 10. Localizing potentials for various values of ξ and $\sigma = 0$.

solutions which asymptotically reduce to decreasing exponentials of the Randall-Sundrum type. These solutions exist for a coupling parameter ξ within a range of values. For a subset of these localized solutions, the warp factor is not a monotonous decreasing function but exhibits a second maximum close to but beyond the origin and subsequently decreases. We have also derived analytically special exact solutions existing for special choices of boundary values and for a range of the coupling parameter. For these solutions, the same warp factor corresponds to either a kink scalar solution or the solution $\phi = \phi(0) \times (\cosh(ay))^{-1}$.

Finally, we considered the localization of gravitons near the brane. Although the Schrödinger-like equation for gravitational perturbations is the same as in the minimal case, the warp factor detailed profile depends on the coupling parameter and the details of the localized spectrum should depend on it. Of course, again the spectrum has no mass gap and does not contain any tachyonic modes. The form of the localizing (volcano) potential depends on the detailed profile of the warp factor. It was studied numerically in a number of cases but also analytically in special cases. For a particular choice of boundary scalar field value and the special coupling parameter value $\xi = \frac{1}{8}$, the localizing potential has the typical $\xi = 0$ volcano profile. Nevertheless, for values of ξ larger than a certain value, the localizing potential develops a spike at the origin, that increases along with ξ. For $\xi = \frac{1}{2}$ the spike reaches zero, while it tends to one for very large values of ξ. This behavior is currently being studied and will be the subject of a future publication.

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