PHYSICAL REVIEW D **74,** 044036 (2006)

Signals from the brane-world black hole

Jianyong Shen and Bin Wang*
Department of Physics, Fudan University, Shanghai 200433, People's Republic of China

Ru-Keng Su†
China Center of Advanced Science and Technology (World Laboratory), P.O. Box 8730, Beijing 100080, People's Republic of China and Department of Physics, Fudan University, Shanghai 200433, People's Republic of China
(Received 10 July 2006; published 29 August 2006)

We have studied the wave dynamics and the Hawking radiation for a scalar field as well as a brane-localized gravitational field in the background of a brane-world black hole with a tidal charge containing information on the extra dimension. Comparing with four-dimensional black holes, we have observed the signature of the tidal charge which presents the signals of the extra dimension both in the wave dynamics and the Hawking radiation.

DOI: 10.1103/PhysRevD.74.044036 PACS numbers: 04.30.Nk, 04.70.Bw

I. INTRODUCTION

In the past years there has been growing interest in studying models with extra dimensions in which the standard model fields are confined on a three-brane playing the role of our four-dimensional (4D) world, while gravity can propagate both on the brane and in the bulk [1–3]. The extra dimensions need not be compact and, in particular, it was shown that it is possible to localize gravity on a three-brane when there is one infinite extra dimension [2,3]. One of the striking consequences of the theories with large extra dimensions is that the lowering of the fundamental gravity scale allows the production of mini black holes in the universe. Such mini black holes are centered on the brane and may have been created in the early universe due to density perturbations and phase transitions. Recently it was proposed that such mini black holes may also be produced in particle collision with the mass energy of TeV order or in the earth atmosphere due to the high-energy cosmic ray showers [4–6]. Once produced, these black holes will go through a number of stages in their lives [5–7], namely: i) the balding phase, where the black hole sheds the hair inherited from original ordinary object; ii) the spin-down phase, where the black hole loses its angular momentum; iii) the Schwarzschild quantum phase, where the black hole's mass will be decreased due to quantum process and finally iv) the Planck Phase. In the first phase, the black hole emits mainly gravitational radiation, while in the second and the third phases, the black hole will lose its energy mainly through the emission of the Hawking radiation.

It was argued that during a certain time interval the evolution of the perturbation around a black hole is dominated by damped single-frequency oscillation, called quasinormal modes (QNM), which carries a unique fingerprint of the black hole and is expected to be detected through gravitational wave observations in the near future (see reviews on this topic and references therein [8]). Recently the QNM of a brane-world black hole has been studied in [9]. The gravitational radiation of the mini black hole would be a characteristic sound and can tell us the existence of such a black hole. Another possibility of observing signatures of this kind of mini black hole exists in particle accelerator experiments, where the spectrum of Hawking radiation emitted by these small black holes can be detected [10–14]. Since these small black holes carry information of extra dimensions and have different properties compared to ordinary 4D black holes, these two tools of detecting mini black holes can help to read the extra dimensions.

The motivation of the present paper is to study signals of tiny black holes in the modern brane-world scenarios [15]. Under the assumptions of the theory, most standard matter fields are brane-localized, therefore from the observational point of view it is much more interesting to study the brane-localized modes in the QNM and Hawking radiation. In the Hawking radiation study, it was found that the emission on the brane is dominated compared to that off the brane [11]. We will investigate brane-world black holes in the 4D background and study the QNM and the emission of brane-localized Hawking radiation. By comparing the properties of QNM and Hawking radiation to those in ordinary 4D black holes, we will argue that the dependence of these properties on the extra dimension of spacetime could be read if the spectrums of QNM and Hawking radiation are detected. In our following study, we will employ the exact black hole solution to the effective field equations on the brane, which is of the Reissner-Nordstrom (RN) type given as [16]

$$ds_4^2 = -f dt^2 + f^{-1} dr^2 + r^2 d\Omega^2 \quad (1)$$

where $f = 1 - \frac{2M}{M_4^2 r} + \frac{q}{M_p^2 r^2}$, and q is not the electric charge of the conventional RN metric, but the 'tidal charge' arising from the projection onto the brane of the gravitational

*Electronic address: wangb@fudan.edu.cn
†Electronic address: rksu@fudan.ac.cn

field in the bulk. Thus q contains the information of the extra dimension. For $q > 0$, this metric is a direct analogy to the RN solution with two horizons, and both horizons lie inside the Schwarzschild horizon $2M/M_4^2$. For $q < 0$, the metric has only one horizon given by $r_+ = \frac{M}{M_4^2} \times (1 + \sqrt{1 - \frac{qM_4^4}{M^2 \tilde{M}_p^4}})$, which is larger than the Schwarzschild horizon. We will concentrate our attention on $q < 0$, since it was argued that this negative tidal charge is the physically more natural case [16].

Astrophysics limits the appearance of the RN black hole with macroscopic electric charge, while there is no exact constraint on the emergence of the effective RN black hole with tidal charge, especially negative tidal charge. However, the tidal charge affects the geodesics and the gravitational potential, so that its value should receive at least indirect limit from observations, which needs careful study. On the other hand, tidal charge contains the information of extra dimensions. It is of great interest to investigate its influence on the gravitational wave and Hawking radiation observations and whether it can leave us the signature of the extra dimension.

This paper is organized as follows. In Sec. II, we will investigate the QNM of the brane-world black hole with tidal charge. We will study the scalar as well as the gravitational perturbations in the background of this black hole. In Sec. III, we will examine the absorption and emission problems for the scalar and the brane-localized graviton in the background of the brane-world black hole. Our conclusions are summarized in Sec. IV. For simplicity we will adopt the nature units in the following discussion.

II. QUASINORMAL MODES

The effective 4D field equation on the brane, which is induced by the 5D Einstein equation in the bulk, is written as [17]

$$R_{\mu\nu} = -E_{\mu\nu}, \qquad R^{\mu}_{\mu} = 0 = E^{\mu}_{\mu}, \tag{2}$$

where $E_{\mu\nu}$ is the symmetric and tracefree limit on the brane of projected bulk Weyl tensor $E_{[\mu,\nu]} = 0 = E^{\mu}_{\mu}$, if we only take the vacuum solution into account and take a vanishing cosmological constant. The Bianchi Identity on the brane requires the constraint

$$\nabla^{\mu} E_{\mu\nu} = 0. \tag{3}$$

As mentioned in [18], Eq. (2) and (3) form a closed system of equations on the brane for static solutions. Dadhich *et al.* [16] investigated one of the vacuum static solutions on the brane and pointed out that Einstein-Maxwell solutions in general relativity provide a vacuum static brane-world solution, because both $E_{\mu\nu}$ from the influence of bulk and the energy momentum tensor of electromagnetic field satisfy Eq. (2) and (3) in spite of their different origins. Hence the 4D metric on the brane is given by

$$ds_4^2 = -f dt^2 + f^{-1} dr^2 + r^2 d\Omega^2, \qquad f = 1 - \frac{2M}{M_p^2}\frac{1}{r} + \frac{q}{\tilde{M}_p^2}\frac{1}{r^2}, \tag{4}$$

where q is a dimensionless tidal charge parameter. When $q > 0$, this solution has two horizons and has similar properties to that of the RN solution. The new situation occurs when $q < 0$ and the black hole has only one horizon which is bigger than the Schwarzschild horizon. So the negative tidal charge increases the entropy and decreases the temperature of the black hole. With the negative tidal charge the bulk influences tend to strengthen the gravitational field. This opinion was supported by the perturbative [19] and nonperturbative analysis [18]. We will concentrate our discussion on the brane-world black hole with the negative q. It is worth emphasizing that although the 4D metric on the brane is known as Eq. (4), the exact solution in the 5D bulk has not been given yet and is still a challenging task to find. Choosing negative q (in the following we express the negative tidal charge to be Q), for simplicity we rewrite the metric on the 4D brane

$$f = 1 - \frac{2M}{r} - \frac{Q}{r^2} \equiv \frac{1}{r^2}(r - r_+)(r - r_-), \tag{5}$$

where r_+ and r_- are two roots of $f = 0$ and

$$r_+ = M\left(1 + \sqrt{1 + \frac{Q}{M^2}}\right) \qquad r_- = M\left(1 - \sqrt{1 + \frac{Q}{M^2}}\right). \tag{6}$$

r_+ is the black hole horizon, while r_- is negative and without physical meaning. Now we start to consider the perturbations around this black hole. For massless scalar perturbations, the wave equation is given by the Klein-Gordon equation

$$\nabla^{\mu}\nabla_{\mu}\phi = 0, \tag{7}$$

which can be reduced to

$$\frac{\partial^2 \Psi_l}{\partial r_*^2} - \frac{\partial^2 \Psi_l}{\partial t^2} = V_s(r)\Psi_l, \tag{8}$$

where:

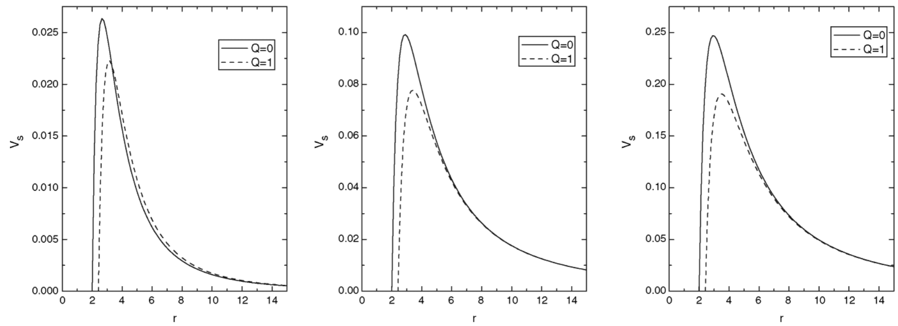

FIG. 1. The behavior of the effective potential of the massless scalar wave equation. We have taken $M = 1$, $l = 0, 1, 2$, respectively, from the left to the right.

$$V_s(r) = f\left[\frac{l(l+1)}{r^2} + \frac{f'}{r}\right]$$

is the effective potential. In deriving the above equation, we have used the variables separation $\phi = \sum_l \frac{\Psi_l(r,t)}{r} Y_{lm}(\theta, \varphi)$ and introduced the tortoise coordinate, given by

$$dr_* = dr/f \Rightarrow r_* = r + \frac{r_+^2}{r_+ - r_-}\ln(r - r_+) + \frac{r_-^2}{r_- - r_+}\ln(r - r_-). \tag{9}$$

The behaviors of the effective potential V_s are shown in Fig. 1. We see that just outside the black hole, there is a potential barrier. This barrier increases with the increase of the angular index l. While for the same l, the appearance of the negative tidal charge as a sequence of strengthening the gravitational field by the bulk effects suppressed the potential barrier.

Because of the potential barrier out of the black hole, the massless scalar perturbation will experience the quasinormal ringing. Using the null coordinates $u = t - r_*$ and $v = t + r_*$, we can rewrite Eq. (8) as

$$-4\frac{\partial^2 \Psi_l}{\partial u \partial v} = V(r)\Psi_l, \tag{10}$$

where r should be obtained by the inverse relation $r_*(r) = (v - u)/2$. We can numerically solve this equation by the finite difference method

$$\Psi_N = \Psi_E + \Psi_W - \Psi_S - \delta u \delta v V_s\left(\frac{v_N + v_W - u_N - u_E}{4}\right)\frac{\Psi_E + \Psi_W}{8}, \tag{11}$$

where points N, S, E, W form a null rectangle with the relative position as N: $(u + \delta u, v + \delta v)$, W: $(u + \delta u, v)$, E: $(u, v + \delta v)$ and S: (u, v). Employing this numerical method proposed in [20], we can observe the object picture of the quasinormal ringing and obtain the frequencies of the fundamental mode. Figure 2 shows the massless scalar wave outside the negative tidal charge black hole and Table I presents the frequencies. For the negative tidal charge, we observed that when the Q increases, the massless scalar perturbation will have less oscillations and decay more slowly. This behavior of the wave evolution is monotone with the increase of Q if the tidal charge $q = -Q$ is negative. For comparison, we have also calculated the QNM for a positive charged black hole; the result is similar to that of the RN black hole disclosed in [21–23]. Unlike the negative charged case, for the positive charge, the quasinormal ringing decays faster than the case when $Q = 0$. The QNM of the black hole for the $Q = 0$ case,

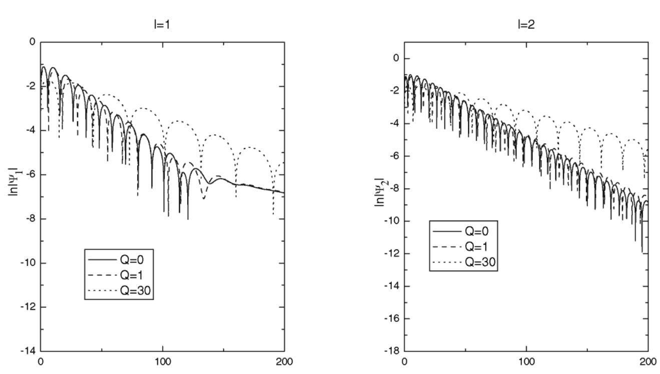

FIG. 2. The QNMs of the Schwarzschild black hole and the brane-world black hole. We have taken the mass $M = 1$.

TABLE I. Quasinormal frequencies of the massless scalar field in the Schwarzschild and tidal charge black hole backgrounds.

	$\omega = \omega_R + i\omega_I$ $l = 1$	$l = 2$
RN. $Q = -0.5$	$0.32 - i0.087$	$0.53 - i0.086$
Sch. $Q = 0$	$0.29 - i0.085$	$0.48 - i0.084$
Tdl. $Q = 1$	$0.26 - i0.079$	$0.42 - i0.079$
Tdl. $Q = 30$	$0.053 - i0.010$	$0.088 - i0.010$

which is the 4D Schwarzschild black hole, serves as a border separating the quasinormal ringing behaviors of the positive and negative charged black hole.

Now we start to consider the gravitational perturbation around the tidal charged black hole. The first order perturbation of the Eq. (2) should be $\delta R_{\mu\nu} = -\delta E_{\mu\nu}$. For the maximally symmetric black holes in high dimensions, a master equation can be obtained for gravitational perturbations [24]. However, due to the lack of knowledge of the exact 5D bulk metric in our case, we cannot determine the $\delta E_{\mu\nu}$ expression. In [9,12], a simple assumption $\delta E_{\mu\nu} = 0$ was adopted and justified in a region where the perturbation energy does not exceed the threshold of the Kaluza-Klein massive modes [25]. On the other hand, the similarity to the metric of the RN black hole sheds another light on the problem. Based on the fact that the characteristic of $E_{\mu\nu}$ on the brane is almost the same as the form of the energy-momentum tensor of electromagnetic field, i.e., its symmetry and tracelessness, we can apply the gravitational perturbation theory of the RN black hole to the brane-world black hole with tidal charge. Classifying the perturbation into axial and polar components and separating the angular dependence, as done in [26], we can obtain the axial perturbation which reads

$$\Lambda^2 Z_i^- = V_i^- Z_i^-, \qquad (i = 1, 2) \tag{12}$$

where

$$\Lambda^2 = \frac{\partial^2}{\partial r_*^2} - \frac{\partial^2}{\partial t^2}. \tag{13}$$

The effective potential reads:

$$V_i^- = \frac{\Delta}{r^5}\left[2(n+1)r - p_j\left(1 + \frac{p_i}{2nr}\right)\right] \quad (i, j = 1, 2; i \neq j) \tag{14}$$

where

$$n = \frac{1}{2}(l-1)(l+2) \tag{15}$$

$$p_1 = 3M + \sqrt{9M^2 - 8nQ} \qquad p_2 = 3M - \sqrt{9M^2 - 8nQ} \tag{16}$$

and $\Delta = r^2 - 2Mr - Q$.

The polar perturbations are described as

$$\Lambda^2 Z_i^+ = V_i^+ Z_i^+ \qquad (i = 1, 2) \tag{17}$$

and the effective polar potentials are

$$V_1^+ = \frac{\Delta}{r^5}\left[U + \frac{1}{2}(p_1 - p_2)W\right]$$
$$V_2^+ = \frac{\Delta}{r^5}\left[U - \frac{1}{2}(p_1 - p_2)W\right]. \tag{18}$$

Here U and W are given by

$$W = \frac{\Delta}{r\varpi^2}(2n + 3M) + \frac{1}{\varpi}(nr + M)$$
$$U = (2n + 3M)W + (\varpi - nr - M) - \frac{2n\Delta}{\varpi} \tag{19}$$

with $\varpi = nr + 3M + \frac{2Q}{r}$. It is noticed that when $Q > \frac{9M^2}{8n}$ ($l \neq 0$), p_1 and p_2 will be complex and the appearance of the imaginary part in the effective potentials will cause a series of interesting physical consequences. When $M = 1$, the upper limits $Q_{\rm up}$ are 0.5625 for $l = 2$ and 0.225 for $l = 3$. We will discuss it briefly in the next section; here we just consider the case when $Q \leq \frac{9M^2}{8n}$ ($l \neq 0$). Equations (12) and (17) can be written as

$$\frac{\partial^2 Z_i}{\partial r_*^2} - \frac{\partial^2 Z_i}{\partial t^2} = V_i Z_i \qquad (i = 1, 2, 3, 4) \tag{20}$$

by using Eq. (9) and the separation of time dependence e^{-ikt}. The potentials of V_i are V_1^-, V_2^-, V_1^+ and V_2^+ when i runs from 1 to 4, and those of Z_i are defined similarly. Although the mathematical forms of V_i are very different, the numerical calculation shows that the difference between the value V_1 and V_3 is extremely small, and the same situation happens for V_2 and V_4, as shown in Fig. 3. Actually, if we take $Q \to -Q^2$ in Eq. (5), V_1 (or V_3)

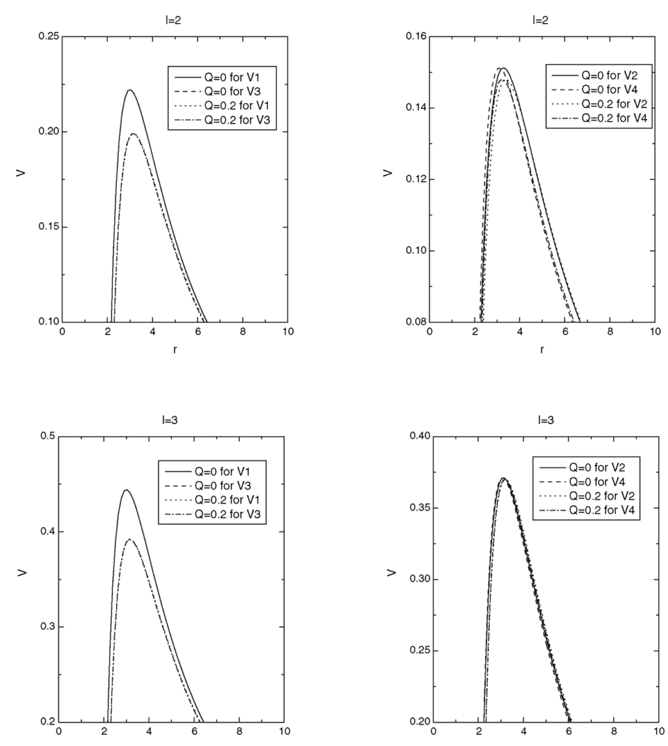

FIG. 3. The comparison of the effective potential of the gravitational perturbation between the Schwarzschild black hole and the brane-world black hole. We have taken black hole mass $M = 1$.

represents the potential of the pure electromagnetic perturbation and V_2 (or V_4) of gravitational perturbation in the RN black hole. In the stage of the brane-world black hole with a small Q, the role played by the static electrical charge as the pure electromagnetic perturbation in the RN black hole is now acted by the tidal charge as the perturbation closely related to that induced by the bulk effects, with the consideration that the exact bulk metric is still unknown and its effects are parameterized by Q on the brane. Therefore it is believed that V_2 (or V_4) stands for the gravitational perturbation mainly caused by the fluctuations of the brane, while V_1 (or V_3) for the potential contributed by the perturbation dominated by the fluctuations of the bulk effects projected on the brane.

Following the same steps as in the case of the scalar field perturbation, we can compute the QNMs of the gravitational perturbations, and the results are shown in Table II. Because of the similarity between the potentials of V_1, V_3 and V_2, V_4, their QNM results are extremely similar. For the negative tidal charged black hole, the quasinormal ringing behavior is separated from the positive charged black hole by the border when $Q = 0$, which is the 4D Schwarzschild black hole case. When the tidal charge becomes more negative, we observed the same behavior as that of the scalar perturbation; the gravitational perturbation will have less oscillation and last longer. This behavior is monotone with the increase of $|Q|$ when the tidal charge is negative. For the positive tidal charge, we observed the same QNM behavior as that of the RN black hole disclosed in [27].

TABLE II. Quasinormal frequencies of the gravitational perturbation around the Schwarzschild and tidal charge black holes.

$\omega = \omega_R + i\omega_I$	$l = 2$	$l = 3$
RN. $Q = -0.5$ for V_1, V_3	$0.54 - i0.086$	$0.76 - i0.086$
Sch. $Q = 0$ for V_1, V_3	$0.46 - i0.083$	$0.66 - i0.083$
Tdl. $Q = 0.2$ for V_1, V_3	$0.43 - i0.081$	$0.62 - i0.081$
RN. $Q = -0.5$ for V_2, V_4	$0.39 - i0.078$	$0.63 - i0.081$
Sch. $Q = 0$ for V_2, V_4	$0.37 - i0.077$	$0.60 - i0.080$
Tdl. $Q = 0.2$ for V_2, V_4	$0.37 - i0.077$	$0.60 - i0.080$

III. ABSORPTION AND EMISSION SPECTRA

In this section, we are going to discuss the emission of brane-localized scalars and gravitons. We will present the numerical results and comment on the extra dimensional influence on the particles emitted on the brane.

A. The massless scalar field

Taking $\Psi_l = rR_l(r)e^{-ikt}$ and substituting it into Eq. (8), we can write the radial part of KG equation as

$$(r - r_+)^2(r - r_-)^2R_l'' + [2r - (r_+ + r_-)](r - r_+) \times (r - r_-)R_l' + \{r^4k^2 - l(l + 1)(r - r_+)(r - r_-)\}R_l = 0. \tag{21}$$

Following the method of Persides [28], the analytic series solution of Eq. (21) near the horizon can be expressed as

$$R_l(r) = (r - r_+)^\rho \sum_{n=0}^{\infty} g_{l,n}(r - r_+)^n \qquad r \to r_+, \tag{22}$$

where the series of constants $g_{l,n}$ is determined by the iteration process substituting Eq. (22) into the radial equation Eq. (21). The index ρ is given by the index equation according to the Frobenius-Fuchs theorem in mathematics

$$\rho^2 + \frac{k^2 r_+^4}{(r_+ - r_-)^2} = 0. \tag{23}$$

In fact, the index equation depends only on the metric of the black hole and has nothing to do with the types of fields and their effective potentials. Hence, we take

$$\rho = -i\kappa_+ = -i\frac{kr_+^2}{r_+ - r_-} \tag{24}$$

and the negative sign indicates that there is only ingoing wave near the horizon. In the asymptotically flat region, the solution of Eq. (21) consists of the outgoing and ingoing waves

$$R_l(r) = (l + 1/2)[f_l^{(-)}F_{l(+)} + f_l^{(+)}F_{l(-)}], \tag{25}$$

where $f_l^{(-)}$ and $f_l^{(+)}$ are the coefficients. $F_{l(+)}$ and $F_{l(-)}$ correspond to the ingoing and outgoing waves, respectively, and have the form

$$F_{l(\pm)}(r) = (\pm i)^{l+1} \exp[\mp ikr_*] \sum_{n=0}^{\infty} \tau_{n(\pm)}(r - r_+)^{-(n+1)} \qquad r \to \infty \tag{26}$$

with $\tau_{0(\pm)} = 1$. Although the total series of the constants $g_{l,n}$ and $\tau_{n(\pm)}$ can be obtained in the iteration relations, the dominant terms are those of $n = 0$ in Eq. (22) and (26). The solutions of wave are rewritten as

$$R_l \approx g_{l,0}(r - r_+)^{-i\kappa_+}[1 + O(r - r_+)] \qquad r \to r_+$$
$$R_l \approx i^{l+1} f_l^{(-)} \frac{2l + 1}{2r}[e^{-ikr_*} - (-1)^l S_l e^{+ikr_*}] + O\left(\frac{1}{r^2}\right) \qquad r \to \infty, \tag{27}$$

where $S_l = \frac{f_l^{(+)}}{f_l^{(-)}}$ is the partial scattering amplitude. Introducing a phase shift δ_l in $S_l = e^{2i\delta_l}$ [12,14], the wave solution in the asymptotic region is

$$R_l \approx f_l^{(-)} \frac{2l + 1}{r} e^{i\delta_l} \sin\left[kr_* - \frac{\pi}{2}l + \delta_l\right] + O\left(\frac{1}{r^2}\right) \qquad r \to \infty. \tag{28}$$

In the spirit of scattering theory of quantum mechanics (QM), the ingoing wave coming from the infinity partly penetrates the potential barrier and is absorbed by the black hole, while the other part is scattered by the barrier and returned to the infinity. Therefore, the compatibility among the coefficients $g_{l,0}$, f_l^- and the scattering amplitude S_l must be found for the true solution of radial Eq. (21). Then the absorption cross section and the emission rate can be calculated. For this reason, it is necessary to discuss the Wronskians of Eq. (21). As done in the QM, it is easy to get the equation of Wronskians

$$W' + \frac{2r - r_+ - r_-}{(r - r_+)(r - r_-)} W = 0 \tag{29}$$

from Eq. (21) with the definition

$$W[R_l^*, R_l] = R_l^* \frac{\partial R_l}{\partial r} - R_l \frac{\partial R_l^*}{\partial r}. \tag{30}$$

The analytic solution of Eq. (29) is

$$W = \frac{C}{(r - r_+)(r - r_-)}. \tag{31}$$

Since the 'current' density is proportional to the Wronskian [29], the constant C plays a role of the 'flux' conversation analogous to what happens in QM, and the additional denominator $(r - r_+)(r - r_-)$ shows the effect of the curved spacetime background. Substituting the solution near the horizon Eq. (27) and that in the asymptotic region Eq. (28) into Eq. (30) and (31), we have

$$C = -2ik|g_{l,0}|^2 r_+^2 = -ik|f_l^{(-)}|^2 (2l + 1)^2 e^{-2\beta_l} \sinh(2\beta_l), \tag{32}$$

and

$$|g_{l,0}|^2 = |f_l^{(-)}|^2 \frac{(l + 1/2)^2}{r_+^2} (1 - e^{-4\beta_l}), \tag{33}$$

where $\beta_l = \mathrm{Im}[\delta_l]$. The partial absorption cross section σ_l is defined in the terms of $g_{l,0}$ [29] with unit amplitude $|f_l^{(-)}|^2 = 1$ of the ingoing wave as

$$\sigma_l = 4\pi r_+^2 \frac{|g_{l,0}|^2}{(2l + 1)k^2} = \frac{\pi}{k^2}(2l + 1)(1 - e^{-4\beta_l}). \tag{34}$$

The partial emission spectrum is given by the Hawking formula

$$\Gamma_l = \frac{dH_l}{dk} = \frac{k^3 \sigma_l}{2\pi^2 (e^{k/T} - 1)} \tag{35}$$

at the temperature of the black hole $T = \frac{r_+ - r_-}{4\pi r_+^2}$. The total absorption cross section and emission spectrum are $\sigma_{\rm tot} = \sum_l \sigma_l$ and $\Gamma_{\rm tot} = \sum_l \Gamma_l$ respectively.

Now we introduce the numerical method to obtain the imaginary part β_l in Eq. (34) of the phase shift, which is the key to the question of the absorption cross section and emission spectrum. This computation has also been done in [12,14]. The Wronskians of the ingoing and outgoing wave in the asymptotic region have the following property

$$\begin{aligned} W[F_{l(+)}, R_l] &= F_{l(+)} R_l' - R_l F_{l(+)}' \\ &= (l + 1/2) f_l^{(+)} W[F_{l(+)}, F_{l(-)}] \\ W[F_{l(-)}, R_l] &= F_{l(-)} R_l' - R_l F_{l(-)}' \\ &= -(l + 1/2) f_l^{(-)} W[F_{l(+)}, F_{l(-)}], \end{aligned} \tag{36}$$

where $W[F_{l(+)}, F_{l(-)}] = F_{l(+)} F_{l(-)}' - F_{l(-)} F_{l(+)}'$. The imaginary part β_l can be expressed as

$$\begin{aligned} \beta_l &= -\frac{1}{2} \mathrm{Im}[i \ln S_l] = -\frac{1}{2} \mathrm{Im}\left[i \ln \frac{f_l^{(+)}}{f_l^{(-)}} \right] \\ &= -\frac{1}{2} \mathrm{Im}\left[i \ln \left(-\frac{W[F_{l(+)}, R_l]}{W[F_{l(-)}, R_l]} \right) \right]. \end{aligned} \tag{37}$$

In practice, we take $g_{l,0} = 1$ for simplicity by considering that the partial scattering amplitude S_l is only determined by the ratio of $f_l^{(+)}$ and $f_l^{(-)}$ and the real part of δ_l is not important in this problem. Using the first equation in (27) as a boundary condition, one can integrate the radial equation Eq. (21), calculate numerically the ratio of the two Wronskians from Eq. (36) with the second equation of (27)

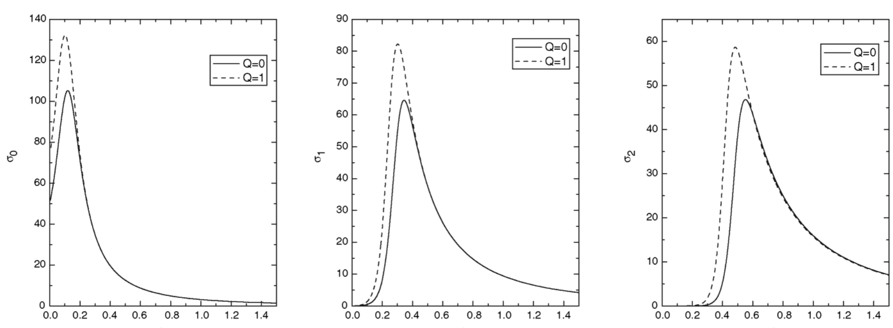

FIG. 4. The spectrum of the partial absorption cross sections σ_l for different angular indexes in the Schwarzchild black hole and the brane-world black hole. From the left to the right we choose $l = 0$, $l = 1$ and $l = 2$ respectively.

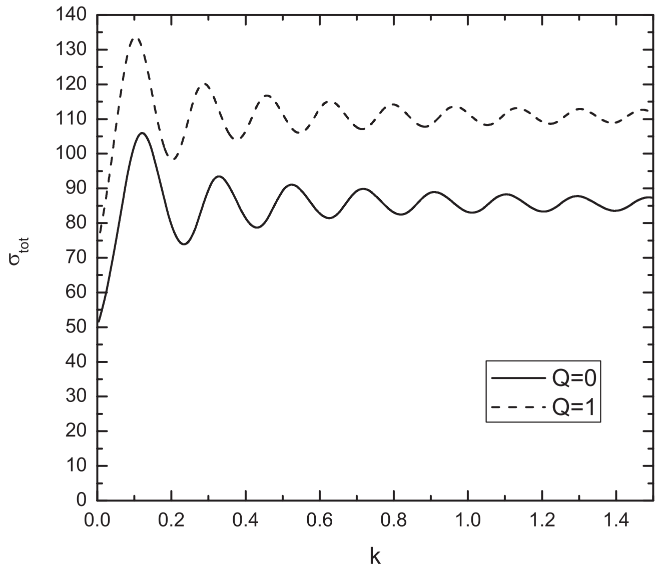

FIG. 5. The spectrum of the total absorption cross section σ_{tot} in the Schwarzchild black hole and the brane-world black hole.

in the asymptotic region, and finally get the value of β_l from the (37).

We have calculated the spectrum of absorption cross section in black holes with the mass $M = 1$ in both Schwarzschild and brane-world cases with the tidal charge. Figure 4 shows the spectrum of the partial absorption cross sections σ_l at different l. We can see that σ_0 equals to the surface area of the black hole $4\pi r_+^2$ as $k \to 0$, which is a universal property in low-energy region for the S-wave. All partial cross sections reach their peaks first and then decrease in the high-energy region. It is noticed that the σ_l on the brane-world black hole is always bigger than that in the Schwarzschild black hole because of its lower potential barriers with negative tidal charge as shown in Fig. 1. This indicates that the scalar field around the negative tidal charge brane-world black hole is absorbed by the black hole more easily due to the decrease of the potential barrier. The fact that more absorption is found around the black hole with negative tidal charge supports the argument that the bulk effect strengthens the gravitational field [16]. In Fig. 5, we show the spectrum of the total absorption cross section σ_{tot} which has the limiting value $27\pi r_+^2/4$ [12] as $k \to \infty$.

According to Eq. (35), we have done the calculation of the total emission spectrum as shown in Fig. 6. Unlike the oscillation pattern in the total absorption cross section in Fig. 5, the spectrum of emission reaches its peak and falls monotonely. The high l modes of σ_l seems suppressed in the Hawking radiation, since the peak of Γ_{tot} for high l is located at almost the same position of the peak of σ_0. The negative tidal charge Q enlarges the black hole horizon and decreases the temperature of the brane-world black hole compared to that of the 4D Schwarzschild black hole. However, the total emission spectrum does not decrease

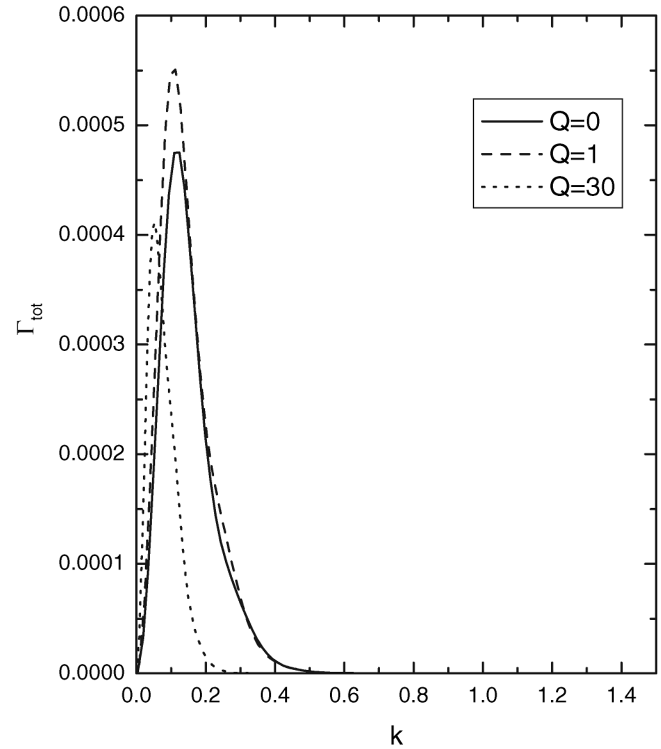

FIG. 6. The total emission spectrum Γ_{tot} in the Schwarzchild black hole and the brane-world black hole.

simply following the temperature as shown in the Fig. 6. There is a critical value $Q_h \approx 20$ for the black hole mass $M = 1$. When $Q > Q_h$ with the much lower temperature, the maximum value of the emission spectrum returns below that of Schwarzschild black hole. But when $0 < Q < Q_h$, the emission spectrum is enhanced with the increase of $|Q|$ and could be bigger than that of Schwarzschild black hole with the same mass. This behavior of the emission spectrum due to different values of negative tidal charge is the result of the competition between the enhancement of the absorption cross section in the numerator and the decreasing temperature in the denominator in (35). It changes the usual picture of the Hawking radiation that lower temperature leads to lower emission and shows the complexity of the influences of the bulk effects on the Hawking radiation.

B. The brane-localized gravitational perturbation

We would like to extend our discussion on the absorption cross section and emission spectrum to the gravitational perturbation in the brane-world black hole background. We will concentrate on the brane-localized perturbation, where the flux in the brane Eq. (40) is conserved. As happens in the scalar field, when the tidal charge becomes more negative, the heights of the potential barriers for V_1 (or V_3) and V_2 (or V_4) are suppressed. The decrease of the potential of V_2 (or V_4) is not as much as that of V_1 (or V_3). It is easier to influence the potential V_1 (or

V_3) than V_2 (or V_4) by the bulk effects because of their different origins.

By recovering the normal coordinates and separating out the time dependence, the perturbation Eq. (20) can be expressed as

$$f^2 Z_i'' + ff' Z_i' + k^2 Z_i = V_i Z_i \qquad (i = 1, 2, 3, 4). \quad (38)$$

Following the steps in the last subsection, we can get the same index ρ as in the Eq. (24) near the black hole horizon and in the asymptotic region $\rho = 0$

$$Z_i \approx g_{l,0} r_+ (r - r_+)^{-i\kappa_+}[1 + O(r - r_+)] \qquad r \to r_+$$
$$Z_i \approx f_l^{(-)}(2l + 1)e^{i\delta_l} \sin\left[kr_* - \frac{\pi}{2} l + \delta_l\right] + O\left(\frac{1}{r}\right) \quad r \to \infty. \quad (39)$$

The Wronskian of Eq. (20) as the definition of Eq. (29) is

$$f^2 W' + ff' W = 0 \Rightarrow W = \frac{C}{f} = \frac{r^2 C}{(r - r_+)(r - r_-)} \quad (40)$$

by combining the metric Eq. (5). C is still the constant with the meaning of the 'flux' conservation of the gravitational perturbation in the brane-world black hole and its value is

$$C = -2ik|g_{l,0}|^2 r_+^2 = -ik|f_l^{(-)}|^2 (2l + 1)e^{-2\beta_l} \sinh(2\beta_l) \quad (41)$$

according to Eq. (39). Expressions of the partial absorption cross section Eq. (34) and the emission spectrum Eq. (35) are still valid in the gravitational perturbation. Using the numerical method introduced above, we have calculated σ_l and the results are shown in Fig. 7. The absorption cross sections for V_1 and V_3 (or V_2 and V_4) overlap each other because of the identical numerical behaviors of the potentials. The suppression of the potential V_i barriers makes the absorption cross section larger. The influences of the tidal charge are much bigger for V_1(or V_3) than those for V_2(or V_4) due to their different perturbative origins, as discussed in the last section. Comparing to the 4D Schwarzschild black hole, the contrasts of the increase of the peak value due to the negative tidal charge with the bulk influence $|\sigma_l^{\rm Tdl} - \sigma_l^{\rm Sch}|/\sigma_l^{\rm Sch}$ are 11% and 2% for V_1 (or V_3) and V_2 (or V_4), respectively, when $l = 2$. At $l = 3$ the contrasts are 13% and 0.4% for V_1 (orV_3) and V_2 (or V_4), respectively.

Figure 8 illustrates the partial emission spectrum. The peak values of the emission spectrum at $l = 2$ are much larger than those at $l = 3$, approximately by $\sim 10^2$ times. Therefore the main emission spectrum for the gravitational perturbation can be represented by that of $l = 2$ in Fig. 8 due to the suppression for the higher l modes. But the total emission spectrum of the brane-localized gravitons is not well defined here, because the high l modes easily break down the constraint $Q < \frac{9M^2}{8n}(l \neq 0)$, where the flux on the brane is no longer conserved, but with the leakage to the bulk which makes our brane-localized investigation not

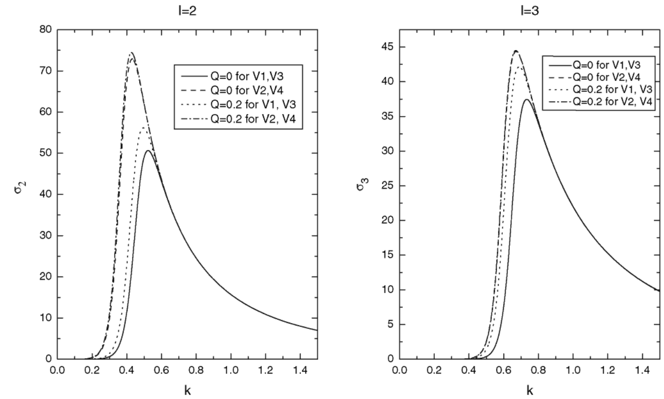

FIG. 7. The spectrum of the partial absorption cross section σ_l with $M = 1$ by taking $l = 2$ and $l = 3$ in the Schwarzschild black hole and the brane-world black hole.

appropriate. As in the case of scalar field, there also exist critical values Q_h for the potential V_2 (or V_4). If $Q_h < Q < \frac{9M^2}{8n}$, the peak of the emission spectrum is not beyond that in Schwarzschild black hole case and if $Q < Q_h$, the emission spectrum of the brane-world black hole is stronger than that of the 4D Schwarzschild case, despite its lower temperature. $Q_h \approx 0.48$ for $l = 2$ and $Q_h \approx 0.19$ for $l = 3$. But this critical value has not been observed in the Hawking radiation for V_1 (or V_3), where the emission of the brane-world black hole is always stronger than that of the 4D Schwarzschild black hole due to the emission of the brane-localized gravitons, although it has a lower temperature than that of the Schwarzschild hole. The mathematical reason is that the increase of the absorption cross section is not big enough to compensate for the change of temperature in the denominator of Eq. (35) for V_2 (or V_4), while V_1 (or V_3) creates the large enhancement of the absorption, which makes the numerator win the competition against the denominator. This unusual phenomenon can be considered as the pure effects of the bulk influence, which is hard to be imagined in 4D black holes. We have calculated the partial emission rate H_l by integrating Eq. (35) in the

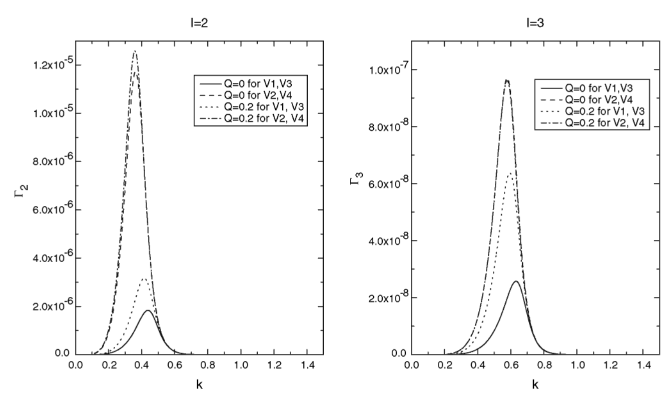

FIG. 8. The partial emission spectrum Γ_l with $M = 1$ by taking $l = 2$ and $l = 3$ in the Schwarzschild black hole and the brane-world black hole.

TABLE III. The partial emission rate H_l in the Schwarzschild and tidal charge black hole backgrounds.

	$l = 2$	$l = 3$
Sch. H_l for V_1, V_3	3.3×10^{-7}	5.0×10^{-9}
Sch. H_l for V_2, V_4	1.9×10^{-6}	1.8×10^{-8}
Tdl. H_l for V_1, V_3	5.5×10^{-7}	1.2×10^{-8}
Tdl. H_l for V_2, V_4	2.0×10^{-6}	1.8×10^{-8}

Schwarzschild black hole with $M = 1$ and also for the brane-world black hole with the same mass and $Q = 0.2$. The results are shown in Table III. The emission rates arise almost 2 times by the tidal parameter $Q = 0.2$ for V_1 (or V_3) but do not change much for V_2 (or V_4).

Now we discuss briefly the case for $Q > \frac{9M^2}{8n}$ ($l \neq 0$). The nonvanishing imaginary part of the potentials V_i leads to the nonzero right-hand side of the Wronskian equation

$$f^2 W' + f f' W = 2i Z_i^* Z_i \,\mathrm{Im}[V_i] \equiv i J_I, \qquad (42)$$

whose solution is easily obtained

$$W = \frac{iC}{f} + \frac{i}{f} \int \frac{J_I}{f} dr. \qquad (43)$$

If we still define the 'current density' as in [29], the 'flux' conservation on the brane can not be held due to the existence of the second term in the r.h.s of Eq. (43). This means that for a certain tidal charge parameter, the 'flux' of the gravitational perturbation in the higher modes satisfying $Q > \frac{9M^2}{8n}$ ($l \neq 0$) cannot stay on the brane, but leaves into the bulk inevitably (at least part of it does so), though the perturbation is created on the brane. In other words, when the tidal parameter $Q > 9/16$, the gravitational perturbation may leave the brane and could be very hard to detect even for the low angular index $l = 2$ in our method. This is an effect of the extra dimension. The detailed and precise analysis on the absorption and emission of the more negatively tidal charged brane-world black hole is not clear at this moment, since the exact solution of the metric in the 5D bulk is still unknown. On the other hand, many researches [11] show that the emission is denominated by that on the brane, hence the emission of the non brane-localized gravitons is beyond our discussion and cannot be analyzed appropriately in our present frame.

IV. CONCLUSION AND DISCUSSION

In this paper, we have studied the perturbations around the brane-world black hole whose projected bulk effects are represented by the tidal charge. The positive tidal charged black hole is of the RN type and the negative one has only one horizon, which is larger than that of the 4D Schwarzschild case surrounding the central singularity. The effect of the negative tidal charge strengthens the gravitational field and lowers the temperature of the black hole [16].

We have calculated the QNMs for both massless scalar field perturbation and gravitational perturbations. We have compared them to the case of the 4D Schwarzschild and RN black holes with the same mass. For the negative tidal charge case, unlike those observed in RN black hole, there is no critical charge where QNM differs below and above this critical charge [21–23]. When the negative tidal charge becomes more negative, the perturbations stay longer with less oscillations, regardless of what types of perturbations. The QNMs of the negative tidal charge and positive tidal charge black holes are separated by that of the 4D Schwarzschild black hole.

However, the influence of the tidal charge on the Hawking radiation is not as simple as that on the QNMs. In the scalar field, the negative tidal charge Q suppresses the effective potential barrier and the absorption cross section increases correspondingly as a consequence of its effect strengthening the gravitational field. It is interesting that the emission spectrum does not simply decrease with the temperature as that in the background of RN black hole [12]. There exist the critical values Q_h. When $Q > Q_h$, the peak of the emission spectrum is smaller than that of Schwarzschild black hole of the same mass. But if we take $Q < Q_h$, despite the lower temperature, the emission becomes stronger compared to the Schwarzschild black hole. This is the result of the enhancement of the absorption cross section and the decrease of the temperature according to Eq. (35) caused by the negative tidal charge, which changes our usual picture about the Hawking radiation in 4D black holes. We emphasize that it is a new behavior which has not been observed before.

As for the graviton, we find that when $Q > \frac{9M^2}{8n}$ ($l \neq 0$), the conservation of the graviton flux on the brane is broken and the leakage of the gravitons requires the exact solution of the brane-world black hole in the bulk which has not been known yet. We have investigated only the Hawking radiation of the brane-localized gravitons under the condition $Q < \frac{9M^2}{8n}$ ($l \neq 0$). As pointed out before, there are two kinds of gravitational perturbations: V_1 (or V_3) referring to those generated mainly by the fluctuation of the bulk effects and V_2 (or V_4) to those created by the factors on the brane. All these potential barriers are suppressed by the negative tidal charge. As a result, the absorption cross sections increase. For the potential V_2 (or V_4), the behaviors of the partial emission spectrums are much similar to those in the scalar field and there is a critical value Q_h, beyond which the peak of the emission spectrum is smaller and below this Q_h, the emission spectrum is stronger compared to that of the 4D Schwarzschild black hole, although with the negative tidal charge we always have lower black hole temperature. But for the potential V_1 (or V_3), no matter what the value of the tidal charge, the emission spectrum of the brane-world black hole is always stronger than that of the Schwarzschild black hole for the brane-localized gravitons. The phenomenon of increasing emission spectrum with decreasing temperature in the brane-world black hole has not been observed before. This is an effect of the negative tidal charge caused by the bulk effect.

We conclude that both the QNMs and the Hawking radiation have given signatures of negative tidal charge due to the bulk effects in the brane-world black hole, which differs from that of the 4D Schwarzschild black hole. We expect that these signatures can be observed in the future experiments, which could help us learn the properties of the extra dimensions.

ACKNOWLEDGMENTS

This work was partially supported by NNSF of China, Ministry of Education of China and Shanghai Science and Technology Commission. We would like to acknowledge E. Abdalla and E. Papantonopoulos for helpful discussions.

[1] N. Arkani-Hamed, S. Dimopoulos, and G. Dvali, Phys. Lett. B **429**, 263 (1998); I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos, and G.R. Dvali, Phys. Lett. B **436**, 257 (1998).
[2] L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 3370 (1999); **83**, 4690 (1999).
[3] K. Akama, Lect. Notes Phys. **176**, 267 (1982); V.A. Rubakov and M.E. Shaposhnikov, Phys. Lett. B **125**, 139 (1983); M. Visser, Phys. Lett. B **159**, 22 (1985); M. Gogberashvili, Europhys. Lett. **49**, 396 (2000).
[4] T. Banks and W. Fischler, hep-th/9906038.
[5] S.B. Giddings and S. Thomas, Phys. Rev. D **65**, 056010 (2002).
[6] S. Dimopoulos and G. Landsberg, Phys. Rev. Lett. **87**, 161602 (2001).
[7] C.M. Harris and P. Kanti, J. High Energy Phys. 10 (2003) 014.
[8] H.P. Nollert, Classical Quantum Gravity **16**, R159 (1999); K.D. Kokkotas and B.G. Schmidt, Living Rev. Relativity **2**, 2 (1999); B. Wang, Braz. J. Phys. **35**, 1029 (2005).
[9] E. Abdalla, B. Cuadros-Melgar, A.B. Pavan, and C. Molina, gr-qc/0604033; P. Kanti, R.A. Konoplya, and A. Zhidenko, gr-qc/0607048; P. Kanti and R.A. Konoplya, Phys. Rev. D **73**, 044002 (2006); D.K. Park, Phys. Lett. B **633**, 613 (2006).
[10] P. Kanti, hep-ph/0310162.
[11] C.M. Harris and P. Kanti, J. High Energy Phys. 10 (2003) 014; P. Kanti, Int. J. Mod. Phys. A **19**, 4899 (2004); P. Argyres, S. Dimopoulos, and J. March-Russell, Phys. Lett. B **441**, 96 (1998); T. Banks and W. Fischler, hep-th/9906038; R. Emparan, G.T. Horowitz, and R.C. Myers, Phys. Rev. Lett. **85**, 499 (2000).
[12] E. Jung and D.K. Park, Nucl. Phys. B **717**, 272 (2005); N. Sanchez, Phys. Rev. D **18**, 1030 (1978); E. Jung and D.K. Park, Classical Quantum Gravity **21**, 3717 (2004).
[13] E. Jung, S.H. Kim, and D.K. Park, Phys. Lett. B **615**, 273 (2005); **619**, 347 (2005); D. Ida, K. Oda, and S.C. Park, Phys. Rev. D **67**, 064025 (2003); C.M. Harris and P. Kanti, Phys. Lett. B **633**, 106 (2006); D. Ida, K. Oda, and S.C. Park, Phys. Rev. D **71**, 124039 (2005); G. Duffy, C. Harris, P. Kanti, and E. Winstanley, J. High Energy Phys. 09 (2005) 049; M. Casals, P. Kanti, and E. Winstanley, J. High Energy Phys. 02 (2006) 051; E. Jung and D.K. Park, Nucl. Phys. B **731**, 171 (2005); A.S. Cornell, W. Naylor, and M. Sasaki, J. High Energy Phys. 02 012 (2006); V.P. Frolov and D. Stojkovic, Phys. Rev. Lett. **89**, 151302 (2002); and , Phys. Rev. D **66**, 084002 (2002); D. Stojkovic, Phys. Rev. Lett. **94**, 011603 (2005); V. Cardoso, M. Cavaglia, and L. Gualtieri, Phys. Rev. Lett. **96**, 071301 (2006);J. High Energy Phys. 02 (2006) 021.
[14] D.K. Park, hep-th/0512021.
[15] A.S. Majumdar and N. Mukherjee, Int. J. Mod. Phys. D **14**, 1095 (2005) and reference therein; G. Kofinas, E. Papantonopoulos, and V. Zamarias, Phys. Rev. D **66**, 104028 (2002); Astrophys. Space Sci. **283**, 685 (2003); A.N. Aliev and A.E. Gumrukcuoglu, Phys. Rev. D **71**, 104027 (2005); S. Kar and S. Majumdar, hep-th/0510043; hep-th/0606026; S. Kar, hep-th/0607029.
[16] N. Dadhich, R. Maartens, P. Papadopoulos, and V. Rezania, Phys. Lett. B **487**, 1 (2000).
[17] T. Shiromizu, K. Maeda, and M. Sasaki, Phys. Rev. D **62**, 024012 (2000).
[18] R. Maartens, Phys. Rev. D **62**, 084023 (2000); A.N. Aliev and A.E. Gumrukcuoglu, Classical Quantum Gravity **21**, 5081 (2004).
[19] M. Sasaki, T. Shiromizu, and K. Maeda, Phys. Rev. D **62**, 024008 (2000).
[20] C. Gundlach, R.H. Price, and J. Pullin Phys. Rev. D **49**, 883 (1994).
[21] N. Anderson, Proc. R. Soc. A **442**, 427 (1993); K.D. Kokkotas and B.F. Schutz, Phys. Rev. D **37**, 3378 (1988).
[22] B. Wang, C.Y. Lin, and E. Abdalla, Phys. Lett. B **481**, 79 (2000).
[23] B. Wang, C. Molina, and E. Abdalla, Phys. Rev. D **63**, 084001 (2001).
[24] H. Kodama and A. Ishibashi, Prog. Theor. Phys. **110**, 701 (2003).
[25] Elcio Abdalla, Bertha Cuadros-Melgar, Sze-Shiang Feng, and Bin Wang, Phys. Rev. D **65**, 083512 (2002); Elcio Abdalla, Adenauer G. Casali, and Bertha Cuadros-Melgar, Int. J. Theor. Phys. **43**, 801 (2004); Nucl. Phys. B **644**, 201 (2002).
[26] F. Mellor and I. Moss, Phys. Rev. D **41**, 403 (1990).
[27] E. Berti and K.D. Kokkotas, Phys. Rev. D **67**, 064020 (2003).
[28] S. Persides, J. Math. Phys. (N.Y.) **14**, 1017 (1973).
[29] N. Sanchez, Phys. Rev. D **16**, 937 (1977).