Entropy and temperature of black holes in a gravity's rainbow

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The linear relation between the entropy and area of a black hole can be derived from the Heisenberg principle, the energy-momentum dispersion relation of special relativity, and general considerations about black holes. There exist results in quantum gravity and related contexts suggesting the modification of the usual dispersion relation and uncertainty principle. One of these contexts is the gravity's rainbow formalism. We analyze the consequences of such a modification for black hole thermodynamics from the perspective of two distinct rainbow realizations built from doubly special relativity. One is the proposal of Magueijo and Smolin and the other is based on a canonical implementation of doubly special relativity put forward recently by the authors. In these scenarios, we obtain modified expressions for the entropy and temperature of black holes. We show that, for a family of doubly special relativity theories satisfying certain properties, the temperature can vanish in the limit of zero black hole mass. For the Magueijo and Smolin proposal, this is only possible for some restricted class of models with bounded energy and unbounded momentum. With the proposal of a canonical implementation, on the other hand, the temperature may vanish for more general theories; in particular, the momentum may also be bounded, with bounded or unbounded energy. This opens new possibilities for the outcome of black hole evaporation in the framework of a gravity's rainbow.

I. INTRODUCTION

More than 30 years ago, Bekenstein argued that the entropy of a black hole is a linear function of the area of its event horizon [1]. He also proposed a value for the proportionality constant, deduced from a semiclassical calculation of the minimum increase in the area of a black hole when it absorbs a particle. Bekenstein's line of reasoning can in fact be generalized by considering the quantum nature of the particle and taking then into account the uncertainty principle and the energy-momentum dispersion relation [2]. This generalized argument leads essentially to the same conclusion about the linearity of the entropy with respect to the black hole area.

Subsequent works in different formalisms for quantum gravity (especially in string theory and loop quantum gravity) have not only provided an explanation to Bekenstein's result [3,4], but also revealed that the linear behavior of the entropy should be modified by a leading order correction that is logarithmic for large areas [5,6]. Similar results have been derived also by considering general properties of black holes [7].

As we have commented, the uncertainty principle and the dispersion relation play a key role in the quantum generalization of the Bekenstein argument. If one accepts the possibility that either (or both) of these elements suffers modifications, one will deduce a different result, with the appearance of terms additional to that proportional to the area. The effect of changes in the uncertainty principle was considered in Refs. [7–9], whereas a more general analysis including modified dispersion relations was recently presented by Amelino-Camelia *et al.*

[10,11], and by Ling and collaborators [12,13]. In particular, with a suitable modification of the dispersion relation and/or the uncertainty principle, a logarithmic term can be obtained.

Modifications to the standard dispersion relations and uncertainty principle arise indeed in several approaches to quantum gravity. For instance, one can find modified dispersion relations in quantum descriptions of spacetime that involve a discrete geometry, such as loop quantum gravity [14–16], and in schemes that adopt a noncommutative spacetime geometry [17]. On the other hand, generalized uncertainty principles have been derived in the context of string theory [18–20], in descriptions using noncommutative geometry [21], and in other kinds of analysis based on general considerations about the interplay between quantum mechanics and gravity [22].

Modified dispersion relations have also been studied from a phenomenological point of view, owing to the increasing interest in discussing their observational consequences [23–27]. In addition, one encounters deformed dispersion relations in the so-called doubly special relativity (DSR) theories [28–34]. The initial motivation for these theories was to solve the apparent inconsistency that exists between the relativity principle and the emergence of a fundamental scale (Planck scale), suggested by most approaches to quantum gravity [35]. The compatibility is regained in DSR theories by allowing a nonlinear action of the Lorentz symmetry. With this modification, not only energy and momentum cease to obey the standard dispersion relation, but also the conventional uncertainty relations (involving energy-momentum and spacetime) are

generically affected. In this way, the framework of DSR theories permits to deal simultaneously with both types of modifications in a quite general manner. Because of this reason, we will concentrate our attention on this framework from now on. Specifically, we will consider two different proposals to implement the consequences of DSR on the spacetime geometry. One of these proposals is the gravity's rainbow put forward by Magueijo and Smolin (MS) [36]. The other is based on a canonical implementation of the DSR theories recently suggested by the authors [37,38]. Actually, this second proposal can be regarded also as a sort of gravity's rainbow formalism inasmuch as it leads to spacetime metrics with an explicit dependence on the energy-momentum (corresponding to the test particles employed by the observer [36]).

The aim of this work is to discuss the effect that the modification of the dispersion relations and the uncertainty principle entail on black hole thermodynamics in the context of a gravity's rainbow. Many of the ideas of this discussion are inspired by those proposed in Ref. [11] (see also [7–10]), which provides the first detailed study of the combined effects of these types of modifications, and in the further elaboration of the arguments of that paper presented in Refs. [12,13]. In particular, Ref. [13] is the first discussion of the changes expected for black hole thermodynamics in a gravity's rainbow. Merit for the initial ideas must be granted to those works, though the contributions of our analysis are manifold: we depurate and systematize the arguments of those references for their application to gravity's rainbow schemes, we extend the conclusions to more general families of DSR theories (allowing not just a deformation of the energy, but also a generic deformation of the momentum), and we generalize the analysis to a gravity's rainbow formalism that differs from the MS one, proving that this alternative candidate leads to thermodynamics with specially appealing physical properties.

The paper is organized as follows. Section II reviews the (extended) Bekenstein argument including also quantum considerations. In Sec. III we summarize some aspects of DSR theories. We comment on the gravity's rainbow formalism introduced by Magueijo and Smolin and the one corresponding to our proposal for a canonical implementation of DSR. For each of these formalisms, we derive new spacetime coordinates, referred to as physical. In Sec. IV we make some considerations about the quantum description of the system and its "relation" to the two studied gravity's rainbow formalisms. We also obtain a modified lower bound for the change of black hole area. We deduce modified expressions for the black hole entropy in Sec. V and for the temperature in Sec. VI. Our conclusions are contained in Sec. VII. Finally, an appendix is added. We will use a set of units in which $\hbar = c = 1$, so that $L_P = E_P^{-1} = \sqrt{G}$ (with $\hbar$ being the Planck constant, c the speed of light, G the gravitational constant, L_P the Planck length, and E_P the Planck energy).

II. EXTENDED BEKENSTEIN ARGUMENT

In order to calculate the proportionality constant in the linear entropy-area relation, Bekenstein used a semiclassical argument [1] that generalizes the process of a particle falling into a black hole discussed previously by Christodoulou [39]. As a part of his argument, Bekenstein calculated the minimum growth of area that a black hole undergoes when it swallows a (neutral) classical particle with energy E and proper radius L [40] that crosses the horizon falling freely from a turning point in its orbit [1,2]. He concluded that $\Delta A \geq 8\pi L_P^2 E L$. Therefore, the area increase displays a fundamental lower bound that does not depend on the black hole properties.

In order to extend this analysis to the case of a quantum particle, one has to regard the radius of the particle as the uncertainty in its position, Δx, and introduce an appropriate correction factor in the coefficient of the above expression for ΔA. We will generically designate this corrected coefficient by a. One then has

$$\Delta A \geq a E \Delta x. \qquad (2.1)$$

Using the dispersion relation of special relativity and the usual uncertainty principle, one gets $E \geq 1/\Delta x$ [10,11] and, therefore, $\Delta A \geq a$.

On the other hand, Bekenstein also pointed out the existence of a universal upper bound on the entropy-to-energy ratio [41], $S/E \leq 2\pi L$, where L is the effective radius of the system. So, if a quantum system with entropy S_{mat} enters a black hole, the change of the ordinary matter entropy in the black hole exterior satisfies

$$-\Delta S_{\text{mat}} \leq b E \Delta x, \qquad (2.2)$$

where we have denoted the corresponding proportionality constant by b. This bound, together with Eq. (2.1), implies that $(b/a)\Delta A + \Delta S_{\text{mat}} \geq 0$. This inequality can be viewed as a generalized second law of thermodynamics, establishing that the first term in the expression represents the change in the black hole entropy, ΔS_{BH} [1]. By adjusting properly the coefficient b/a, one arrives at the well-known result:

$$S_{\text{BH}} = \frac{A}{4L_P^2}. \qquad (2.3)$$

For a Schwarzschild black hole, the case to which we restrict our attention from now on for simplicity, the associated temperature can be deduced then by employing the definition $T_{\text{BH}}^{-1} = \partial_m S_{\text{BH}}$ [1], where m is the mass of the black hole. If one considers the usual relation $A = 16\pi L_P^4 m^2$, one gets

$$T_{\text{BH}} = \frac{E_P^2}{8\pi m}, \qquad (2.4)$$

which reproduces the temperature that Hawking obtained in a different way [42].

The uncertainty principle and the dispersion relation have a key role in the derivation that we have presented. If either or both of them experienced modifications, as happens in the various frameworks that we have commented [14–22], the linearity in the expression of the entropy would also be modified. In the rest of the paper, we will analyze the implications that the modifications introduced by DSR theories have on black hole thermodynamics.

III. DOUBLY SPECIAL RELATIVITY AND GRAVITY'S RAINBOW

DSR theories are characterized by the inclusion of a Lorentz invariant energy and/or momentum scale, in addition to the fundamental scale provided by the speed of light [28–34]. The invariance of this new scale (supposed to be related to the Planck scale) is possible thanks to a nonlinear action of the Lorentz group in momentum space. A realization of this kind is obtained via an invertible nonlinear map U between the physical energy-momentum $P_a :=$ $(-E, p_i)$ and the original energy and momentum variables of standard relativity (in Minkowski space) $\Pi_a :=$ $(-\epsilon, \Pi_i)$, which are viewed as auxiliary variables [43] (lowercase Latin indices from the beginning and the middle of the alphabet represent Lorentz and flat spatial indices, respectively). Imposing that the action of rotations is not modified, the nonlinear map U is totally determined by two scalar functions g and f [33,44]. Following a notation similar to that of Refs. [37,38], the map U can be expressed

$$P_a = U^{-1}(\Pi_a) \Rightarrow \begin{cases} E = g(\epsilon, \Pi), \\ p_i = f(\epsilon, \Pi)\frac{\Pi_i}{\Pi}. \end{cases} \quad (3.1)$$

Here, Π denotes the magnitude of the auxiliary momentum. We get different DSR theories depending on the choice of functions f and g. On the other hand, to recover the standard linear action of the Lorentz group in the limit of small energies and momenta compared to the scale of the DSR theory, one must impose that the functions (g, f) tend to the identity [i.e., behave like (ϵ, Π)] in that limit.

In order to determine the corresponding transformation rules in position space and the deformed spacetime geometry, there exist different proposals in the literature [36–38,44–46]. In this paper, we will focus on the MS proposal of a gravity's rainbow [36] and on a variant of it based on a canonical implementation of DSR [37,38].

The MS proposal rests on the requirement that the contraction between the energy-momentum and an infinitesimal spacetime displacement be a linear invariant in DSR. In contrast, our proposal demands the invariance of the symplectic form $\mathbf{d}q^a \wedge \mathbf{d}\Pi_a$, where q^a represents the (asymptotically) flat spacetime coordinates, that we will refer to as auxiliary. Both proposals lead to geometries that depend explicitly on the energy and momentum of the system (namely, the test particle used by the observer

[36]). This fact explains the name *gravity's rainbow* [36] given to this type of formalisms. Unlike what happens with the MS proposal, ours leads to modified spacetime coordinates x^a that are conjugate to the physical energy-momentum P_a. Namely, the relation between (q^a, Π_a) and (x^a, P_a) is a canonical transformation. Similar canonical proposals for the implementation of DSR have been suggested by other authors [46,47].

We will concentrate our discussion on a family of DSR theories that, without being completely generic, is in fact rather general (at least compared with the cases studied so far in connection with black hole thermodynamics). In these theories, the physical energy depends only on the auxiliary one, i.e. $E = g(\epsilon)$. In addition, we require the ratio of the physical and auxiliary momenta to be well defined when the latter of these momenta tends to zero. This is a minor restriction on f [48], since in any case it must be approximately equal to Π for small energies and momenta. Our condition guarantees that $f(\epsilon, \Pi)$ vanishes when the auxiliary momentum does. Finally, since we are only considering spherically symmetric black holes for simplicity, we will impose spherical symmetry also on (the test particle) phase space, restricting to physical momenta that are parallel or antiparallel to x^i. As a consequence, $\varepsilon^i_{jk} p_i dx^j = 0$ (for given momentum), where ε_{ijk} denotes the Levi-Civita symbol. Since $p_i/p = \Pi_i/\Pi$, this condition can be rewritten as $dx^i = dx^j \Pi_j \Pi^i/\Pi^2$ (with the usual sum convention in repeated indices).

In these circumstances, one gets the following scaling with our proposal of a canonical implementation of DSR (see [37]):

$$\begin{aligned} dq^0 &= \partial_\epsilon g\left(dx^0 \pm \frac{\partial_\epsilon f}{\partial_\epsilon g} dx\right) := \partial_\epsilon g\, d\tilde{x}^0_\pm, \\ dq^i &= \partial_\Pi f\, dx^i. \end{aligned} \quad (3.2)$$

Here $x = \sqrt{x^i x^i}$, and $\tilde{x}^0_\pm$ is a new coordinate that, although not canonically conjugate to the energy (in the sense that $\{\tilde{x}^0_\pm, x^i\}$ is not a set conjugate to the physical energy-momentum), differs from the canonical time only in a shift that is constant in spacetime.

Following the MS proposal, on the other hand, one arrives at [36]:

$$dq^0 = \frac{g}{\epsilon} dx^0, \qquad dq^i = \frac{f}{\Pi} dx^i. \quad (3.3)$$

Therefore we see that, in the two considered cases, the effect on the geometry consists essentially of two independent scalings: a conformal transformation of the spatial components and a time dilation, both of them constant in spacetime. For instance, one can obtain the modified Schwarzschild solution for the gravity's rainbow following the steps explained in detail in Ref. [36]. This solution reproduces formally the familiar one for general relativity (with a suitable identification of coordinates) except for the

commented scaling of the spatial metric and the diagonal time component.

We can analyze simultaneously the two gravity's rainbow formalisms by denoting the corresponding scale factors with the abstract notation $G(\epsilon)$ and $F(\epsilon, \Pi)$:

$$dq^0 = G(\epsilon)d\tilde{x}^0_\pm, \qquad dq^i = F(\epsilon, \Pi)dx^i, \qquad (3.4)$$

with $\tilde{x}^0_\pm$ designating x^0 for the MS proposal. Note that the time and spatial scale factors are given in one case by the partial derivatives of the functions g and f with respect to the auxiliary energy and momentum, respectively, whereas in the other case they are simply the ratios of those quantities, namely:

$$G(\epsilon) := \begin{cases} \frac{g(\epsilon)}{\epsilon} & \text{MS proposal,} \\ \partial_\epsilon g(\epsilon) & \text{Canonical proposal,} \end{cases} \qquad (3.5)$$

$$F(\epsilon, \Pi) := \begin{cases} \frac{f(\epsilon, \Pi)}{\Pi} & \text{MS proposal,} \\ \partial_\Pi f(\epsilon, \Pi) & \text{Canonical proposal.} \end{cases} \qquad (3.6)$$

By canonical proposal, we understand here our proposal for a canonical implementation of DSR.

Expressions (3.1) and (3.4) lead to deformations of the dispersion relation and to generalized uncertainty principles (because the commutators of the momentum with the auxiliary spatial coordinates vary). In this way, the gravity's rainbow formalisms incorporate in fact the two types of modifications whose consequences for black holes we want to discuss.

IV. MODIFIED BOUND ON THE CHANGE OF BLACK HOLE AREA

A. Quantum description of the system

Expression (2.1) provides a lower bound for the increase of black hole area in general relativity. The magnitude E is the energy of the particle which is going to be absorbed, measured at infinity in the asymptotically flat spacetime, and Δx is the uncertainty in the position of the particle. When relativity is modified, it seems natural to assume that the bound continues to apply with E being the energy measured by an asymptotic observer[1] and Δx the position uncertainty. However, in DSR, E and x no longer correspond to the standard energy and position variables for (asymptotically) flat spacetime: they are the variables that we have called physical and transform (in the asymptotic region) according to a nonlinear action of the Lorentz group. In this way, the expression for the area increase incorporates modifications with respect to Eq. (2.1) arising from the DSR deformation.

In the following, we will call ΔA the change of area obtained for DSR, whereas $\Delta A_0 := a\epsilon\Delta q$ denotes the undistorted lower bound for standard general relativity

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[1]See nonetheless Subsection IV B concerning the case of a perturbative quantization.

[see Eq. (2.1)]. In order to relate these two quantities, we have to take into account first the kind of quantum description adopted. We can consider two possibilities. In one case, the quantization of the system is carried out choosing as time parameter and position variables the auxiliary coordinates corresponding to (the asymptotically) flat spacetime, which can be seen as a background. This is the typical philosophy of a perturbative approach. In the other case, on the contrary, the quantization is constructed in terms of the physical time and position variables. For this reason, we will refer to these two types of descriptions as perturbative and nonperturbative quantization, respectively [37,38].

More specifically, for the perturbative quantization a complete set of elementary variables are the auxiliary variables (q^i, Π_i), while the auxiliary time q^0 plays the role of evolution parameter. Hence, the uncertainty principle corresponding to this perturbative description is $\Delta q\Delta\Pi \geq 1/2$ (with $q = \sqrt{q^i q^i}$) [49]. On the other hand, the nonperturbative quantization can be built by regarding the physical time x^0 as the evolution parameter and the canonically conjugate physical variables (x^i, p_i) as the elementary ones. The uncertainty principle for this nonperturbative description is then $\Delta x\Delta p \geq 1/2$. Our next task consists of expressing the quantities that appear in ΔA in terms of the elementary variables that correspond to each of these types of quantization and apply the uncertainty principle associated with them.

B. Bound on the change of area

For the case of the nonperturbative quantization, the bound on the change of area for DSR can be expressed

$$\Delta A \geq aE\Delta x = \frac{E}{\epsilon} \frac{\Delta\Pi}{\Delta p} \frac{\Delta x\Delta p}{\Delta q\Delta\Pi} \Delta A_0. \qquad (4.1)$$

In accord with the absorption process described in Sec. II, we restrict to particles that are away from rest an amount $\Delta\Pi$, interpretable as the uncertainty that affects their momenta. Employing Eq. (3.1) and the fact that the function f vanishes when Π does, we conclude

$$\Delta A \geq \frac{g/\epsilon}{f/\Delta\Pi} \widehat{\Delta A}_0, \qquad (4.2)$$

where we have defined

$$\widehat{\Delta A}_0 := \frac{\Delta x\Delta p}{\Delta q\Delta\Pi} \Delta A_0. \qquad (4.3)$$

The factor $\Delta x\Delta p/(\Delta q\Delta\Pi)$ in $\widehat{\Delta A}_0$ compensates the change of basic uncertainty relations with respect to those for flat spacetime in the Bekenstein argument. Taking into account this change, the line of reasoning of Sec. II would lead to the result $(b/a)\widehat{\Delta A}_0 + \Delta S_{\text{mat}} \geq 0$. Note that the ratio of the scale factors encountered in Eq. (3.3) appears now in the right-hand side of expression (4.2). According

to this fact, the MS gravity's rainbow formalism seems to match with the nonperturbative description.

Let us consider now the perturbative description. Recalling Eq. (3.2) and treating Δx and Δq as spatial distances, one gets $\Delta x = \Delta q / \partial_\Pi f$. In addition, if the auxiliary energy ϵ is viewed quantum mechanically as the generator of time translations in the evolution parameter q^0 and one identifies the quantity E in ΔA with the corresponding generator of translations in x^0 obtained with a straight application of the chain rule, rather than with the exact physical energy, one concludes that the role of E must be played by $\epsilon \partial_\epsilon g$. This same assignation was in fact made in Ref. [11] when studying the leading order corrections caused by modified dispersion relations in black hole thermodynamics. With these considerations,

$$\Delta A \geq \partial_\epsilon g a \epsilon \Delta x = \frac{\partial_\epsilon g}{\partial_\Pi f} \Delta A_0. \qquad (4.4)$$

In the above expression, the multiplicative factor is the ratio of the scale factors obtained in Eq. (3.2). Thus, the same type of connection that seems to exist between the nonperturbative quantum description and the MS proposal appears now between the perturbative description and our proposal for the canonical implementation of DSR. In both cases, the bound on the change of area for general relativity is corrected by a factor that depends on the auxiliary energy and momentum of the particle. In the case of the nonperturbative quantization, there is an additional modification coming from the change in the uncertainty relations, which has been absorbed in the definition of $\widehat{\Delta A_0}$. In the regime of low energies, the functions g and f tend to the identity and the physical and auxiliary variables coincide, so that the standard result is recovered.

We can deal simultaneously with formulas (4.2) and (4.4) by employing the notation introduced in Eqs. (3.5) and (3.6) and calling both $\widehat{\Delta A_0}$ in the nonperturbative description and ΔA_0 in the perturbative one by $\overline{\Delta A_0}$. We can then write

$$\Delta A \geq \frac{G(\epsilon)}{F(\epsilon, \Delta \Pi)} \overline{\Delta A_0} := H(\epsilon, \Delta \Pi) \overline{\Delta A_0}. \qquad (4.5)$$

Explicitly

$$H(\epsilon, \Pi) = \begin{cases} \frac{\Pi g(\epsilon)}{\epsilon f(\epsilon, \Pi)} & \text{MS proposal,} \\ \frac{\partial_\epsilon g(\epsilon)}{\partial_\Pi f(\epsilon, \Pi)} & \text{Canonical proposal.} \end{cases} \qquad (4.6)$$

We will use this relation to derive modified expressions for the black hole entropy and temperature in the two considered gravity's rainbow formalisms. In principle, this inequality is valid for all possible values of the energy-momentum of the particle absorbed by the black hole. At least for the simple case of a modified Schwarzschild black hole, it turns out that when the function H satisfies certain conditions, the set of inequalities obtained with different energies and momenta amounts just to a single inequality.

Furthermore, the factor arising from H in that inequality depends only on the area radius $r_s = \sqrt{A/(4\pi)}$ (i.e., the Schwarzschild radius in general relativity) [50]. The conditions on H follow from the next arguments.

First, since the auxiliary energy and momentum satisfy the usual dispersion relation of special relativity, $\epsilon \geq \Delta \Pi$ (with the equality attainable for massless particles). Therefore, if $H(\epsilon, \Delta \Pi)$ is an increasing function of the variable $\Delta \Pi$, we can maximize it to $H(\epsilon, \epsilon)$. In addition, $2\pi/\epsilon$ cannot exceed the diameter of the black hole, $2r_s$, because otherwise the particle would be scattered instead of absorbed (see also Ref. [11]). We notice that $2\pi/\epsilon$ is the wavelength in the case of a massless particle, and it is smaller than or equal to the Compton wavelength if the particle is massive. So, $\epsilon \geq \pi/r_s$. If $H(\epsilon, \epsilon)$ is a decreasing function of ϵ, it then reaches its maximum at π/r_s.

In this situation, it is easy to see that expression (4.5) is satisfied for all allowed values of the auxiliary energy-momentum if and only if it is satisfied for $\epsilon = \Delta \Pi = \pi/r_s$ [51], namely

$$\Delta A \geq H\left(\frac{\pi}{r_s}, \frac{\pi}{r_s}\right) \overline{\Delta A_0}. \qquad (4.7)$$

In the next section, we will use this inequality as the key element to derive modified expressions for the entropy and temperature of the black hole. Let us check now if the conditions demanded on the function H are fulfilled in some of the DSR models that are more often found in the literature (see the appendix for details).

The first of these DSR models can be considered the prototype of the DSR2 family [32,33] (i.e., theories with bounded physical energy and momentum). It is seen in the appendix that $H(\epsilon, \Delta \Pi) = g(\epsilon) \Delta \Pi / [\epsilon f(\epsilon, \Delta \Pi)] = 1$ in this model for the MS proposal. Therefore, both requirements about the function H are trivially satisfied, but there is no modification of the thermodynamics. For our canonical implementation, on the other hand, $H(\epsilon, \Delta \Pi) = \partial_\epsilon g / \partial_\Pi f = 1/(1 + \lambda \epsilon)$. Since this function is constant in $\Delta \Pi$ and decreasing in ϵ, the conditions are satisfied.

Our next example is a DSR analogue of the Einstein-Rosen gravitational waves [52], which is of the DSR3 class (i.e., only the physical energy is bounded). It is shown in the appendix that in this case one gets for the MS proposal $H(\epsilon, \Delta \Pi) = (1 - e^{-\lambda \epsilon})/(\lambda \epsilon)$, and for our proposal $H(\epsilon, \Delta \Pi) = e^{-\lambda \epsilon}$. Both functions are independent of $\Delta \Pi$ and decreasing in ϵ. Hence the requirements are fulfilled.

Finally, we consider the model of DSR1 class (i.e., with bounded physical momentum) introduced in Refs. [28,29]. In fact, our analysis cannot be applied in this model because the physical energy depends on the auxiliary momentum, $E = g(\epsilon, \Delta \Pi)$, unless one restricts all considerations to fixed Casimir invariant $\epsilon^2 - \Delta \Pi^2$. For the MS proposal, given the complexity of the involved functions, we study exclusively the case of massless parti-

cles ($\epsilon = \Delta\Pi$), for which we show in the appendix that $H = (1 + \lambda\epsilon)\ln(1 + \lambda\epsilon)/(\lambda\epsilon)$. This is an increasing function of ϵ. Thus, the conditions are not satisfied. For our proposal, on the other hand, one obtains $H = 1$ with any fixed value of the Casimir invariant, and hence the thermodynamics remain unaltered.

V. BLACK HOLE ENTROPY

Let us discuss now the modification of the entropy-area relation (2.3). As we have seen, in standard general relativity we have $(b/a)\overline{\Delta A}_0 + \Delta S_{\mathrm{mat}} \geq 0$ (once the proper basic uncertainty relations have been taken into account). In the passage to deformed relativity, the change of black hole area satisfies relation (4.7) provided that the function H fulfills certain conditions. In this passage, the entropy of ordinary matter is not modified (assuming that one has already adopted a correct quantum description), because it simply reflects the number of physical degrees of freedom of the system. Combining this information, one concludes

$$\frac{b}{a}\frac{\Delta A}{H(\frac{\pi}{r_s}, \frac{\pi}{r_s})} + \Delta S_{\mathrm{mat}} \geq 0. \tag{5.1}$$

If one accepts the reasonable hypothesis that the (Schwarzschild) black hole entropy is a function of its area only, the above relation can be understood as a modified generalized second law, with a natural identification of the change in black hole entropy:

$$\Delta S_{\mathrm{BH}} = \frac{b}{a}\frac{\Delta A}{H(\frac{\pi}{r_s}, \frac{\pi}{r_s})}. \tag{5.2}$$

Notice that, to recover the Bekenstein-Hawking law (2.3) for large black holes ($r_s \to \infty$), the limit in which H tends to the unity, the constant b/a must be fixed equal to the usual factor $1/(4L_P^2)$.

Finally, in order to obtain the functional form of the entropy with the area, we substitute $A = 4\pi r_s^2$. Integrating Eq. (5.2) we get [12,13]

$$S_{\mathrm{BH}}(A) = \frac{1}{4L_P^2}\int_{A_1}^{A}\frac{d\tilde{A}}{H(\sqrt{4\pi^3/\tilde{A}}, \sqrt{4\pi^3/\tilde{A}})}. \tag{5.3}$$

Here, A_1 is a reference area where the entropy is fixed to vanish. It is natural to choose it equal to zero, but we note that the integral might then diverge. In that case, a nonzero area (e.g. the Planck area) should be given as the reference. The convergence of the integral for $A_1 = 0$ is ensured at least for those theories in which one can find a constant $\delta > 0$ so that $\lim_{r_s \to 0} r_s^{2-\delta}/H(\pi/r_s, \pi/r_s) = 0$.

Let us study now the behavior of the entropy for large values of the black hole area. With this aim, we expand the function $1/H$ around zero (in both of its arguments) and keep only up to quadratic terms. Remembering that $H(0, 0) = 1$, we obtain after integration (up to an irrelevant

additive constant)

$$S_{\mathrm{BH}} \approx \frac{1}{4L_P^2}\left[A + 4C_1\sqrt{\pi^3 A} + 2\pi^3 C_2\ln\frac{A}{L_P^2}\right], \tag{5.4}$$

with

$$C_1 = -(\partial_\epsilon H + \partial_\Pi H)|_0,$$
$$C_2 = [-\partial_\epsilon^2 H - \partial_\Pi^2 H - 2\partial_\Pi\partial_\epsilon H + 2(\partial_\epsilon H + \partial_\Pi H)^2]|_0. \tag{5.5}$$

Here, the symbol $|_0$ denotes evaluation at vanishing arguments.

If one imposes that the leading order correction to the Bekenstein-Hawking law be logarithmic, in agreement with several analyses in the literature [5–7], the constant coefficient C_1 must vanish. This can be understood as a restriction on the allowed DSR theories. In that case, the coefficient C_2 becomes

$$C_2 = -[\partial_\epsilon^2 H + \partial_\Pi^2 H + 2\partial_\Pi\partial_\epsilon H]|_0. \tag{5.6}$$

VI. BLACK HOLE TEMPERATURE

A. Derivation from the entropy

We turn now to analyze the modification of the black hole temperature. With this purpose, we associate a mass $m = E_P^2 r_s/2$ to the black hole. This expression reproduces the definition of Schwarzschild mass in general relativity and is recovered in the gravity's rainbow formalisms [36]. It is worth pointing out that the mass m runs in principle from zero to infinity (if so does A). From the definition $T_{\mathrm{BH}}^{-1} := \partial_m S_{\mathrm{BH}}$ and Eq. (5.2) [with $b/a = 1/(4L_P^2)$], one arrives then at the modified temperature

$$T_{\mathrm{BH}} = H(4\pi^2 T_0, 4\pi^2 T_0)T_0. \tag{6.1}$$

Here, $T_0 = E_P^2/(8\pi m)$ represents the Hawking temperature in standard general relativity.

The Hawking temperature tends to zero as $m \to \infty$ (or equivalently as $r_s \to \infty$). So, in general relativity the black hole radiates with a negligible temperature when its mass is very large. On the contrary, T_0 diverges when m (and r_s) approaches zero. As a consequence, in general relativity, the amount of radiation emitted by a tiny black hole is enormous. The evaporation accelerates explosively when the mass becomes small, in the final stages of the black hole lifetime. We want to explore whether the modifications that arise in the context of a gravity's rainbow can significantly change this behavior of the temperature. Specifically, we want to investigate whether the modified temperature can vanish in the limit of zero black hole mass. This would open the possibility that the black hole evaporation eventually stops or takes an infinite time, providing a radically different scenario for the resolution of the information paradox.

Let us call $z := 4\pi^2 T_0 = \pi E_P^2/(2m)$. Employing Eq. (4.2), one obtains for the MS proposal

$$T_{\rm BH} = \frac{1}{4\pi^2} \frac{g(z)}{f(z,z)} z. \tag{6.2}$$

We remember that g is a positive and increasing function, because it is approximately the identity at low energies and is invertible. For the temperature $T_{\rm BH}$ to vanish in the zero mass limit, it is then necessary (though not sufficient) that $\lim_{z\to\infty} f(z,z)/z = \infty$. One can realize that this precludes the existence of an invariant momentum scale. Therefore, the temperature cannot vanish for theories that belong to the DSR1 and DSR2 classes.

On the other hand, from Eq. (4.4) the modified temperature for our proposal is

$$T_{\rm BH} = \frac{1}{4\pi^2} \frac{\partial_\epsilon g(z)}{\partial_\Pi f(z,z)} z. \tag{6.3}$$

In theories of the DSR2 and DSR3 classes, $\partial_\epsilon g(z)$ tends to zero at infinity. If the decrease of this derivative dominates over the possible increase of $z/\partial_\Pi f(z,z)$, the temperature vanishes for zero mass. In the DSR1 class, $\partial_\epsilon g(z)$ does not tend to zero. So, one must necessarily have $\lim_{z\to\infty} \partial_\Pi f(z,z)/z = \infty$. Since the existence of a bound on the physical momentum implies only that $\lim_{z\to\infty} f(z,z)$ must be finite, it is not impossible that the temperature vanishes with the mass in models of the DSR1 class. Therefore, in comparison with the MS proposal, our proposal of a canonical implementation of DSR leads to a richer variety of options for the asymptotic vanishing of the modified temperature.

We can study the behavior of the modified temperature in the DSR models considered in the appendix. In the DSR2 model, the temperature coincides with T_0 for the MS proposal, whereas $T_{\rm BH} = E_P^2/[4\pi(\pi\lambda E_P^2 + 2m)]$ for our proposal. In this latter case, although the temperature does not vanish for zero mass, the situation is much better than in standard general relativity, because $T_{\rm BH}$ tends to the constant $1/(4\pi^2\lambda)$. On the other hand, using the expression of the function H obtained in the appendix for the DSR3 model (Einstein-Rosen gravitational waves) we get $T_{\rm BH} = [1 - e^{-\pi\lambda E_P^2/(2m)}]/(4\pi^2\lambda)$ for the MS proposal. Thus, the temperature tends also to a constant in the limit of vanishing black hole mass. For our alternative proposal, $T_{\rm BH} = E_P^2 e^{-\pi\lambda E_P^2/(2m)}/(8\pi m)$ and the temperature does indeed vanish when $m \to 0$. Finally, in the case of the DSR1 model, the temperature suffers no modifications when the conditions that allow the application of our analysis are satisfied [see end of Subsection IV B and Eq. (A10)].

B. Derivation from the surface gravity

Expression (6.1) for the modified temperature and our subsequent discussion are only applicable if the function H satisfies certain conditions spelled out in Subsection IV B.

These conditions allow us to pass from a set of inequalities involving $H(\epsilon, \Delta\Pi)$ for a whole range of values of the auxiliary energy-momentum to the single inequality (4.7). This latter inequality leads to a generalized second law that depends only on the area of the black hole and the entropy of the ordinary matter. For arbitrary DSR theories, however, the conditions on H will not be fulfilled. Although one can find alternative conditions on H that allow to arrive at Eq. (4.7), it is possible to generalize the study of the modification of the temperature and the entropy in the following way.

First, in the spirit of the gravity's rainbow [36], one can tentatively admit a black hole temperature $\mathcal{T}_{\rm BH}$ that depends on the energy-momentum of the test particles. Then, as pointed out by Ling, Li, and Zhang [13], the expression for the corresponding temperature can be derived by analyzing the behavior of the gravity's rainbow metric near the horizon. The temperature is $\mathcal{T}_{\rm BH} = \kappa/(2\pi)$, where κ is the surface gravity on the horizon:

$$\kappa = -\frac{1}{2} \lim_{r\to r_s} \sqrt{\frac{-g^{rr}}{g^{00}}} \frac{(g^{00})'}{g^{00}}. \tag{6.4}$$

The prime denotes the derivative with respect to r. For the Schwarzschild solution, the introduction of the gravity's rainbow results in the time and spatial scalings (3.4), which produce the following transformation of the metric components:

$$g^{00} \to \frac{g^{00}}{[G(\epsilon)]^2}, \qquad g^{rr} \to \frac{g^{rr}}{[F(\epsilon, \Pi)]^2}. \tag{6.5}$$

With these transformation laws, one straightforwardly obtains

$$\mathcal{T}_{\rm BH} = H(\epsilon, \Pi)T_0. \tag{6.6}$$

This expression depends on the energy and momentum of the test particles. It seems reasonable to consider as natural test particles those provided by the black hole itself by means of its radiation. In this way, the gravity's rainbow would take into account (to a certain extent) the back-reaction of the geometry. This radiation would be dominated by massless particles with average energy proportional to the Hawking temperature, at least for sufficiently large black holes. This would justify identifying the test particles as massless ones with $\epsilon = \Pi = \xi T_0$, where ξ is a constant approximately of order unity.

Nonetheless, one should expect this to be only an approximation, with quantum corrections to the value of the average energy and fluctuations around the typical test particle becoming increasingly important for smaller black holes. One could try to mimic the effect of those corrections and departures from the proposed approximation by evaluating H at $\epsilon = \xi T_0\{1 + O[(T_0/E_P)^{n_1}]\}$ and $\Pi = \xi T_0\{1 + O[(T_0/E_P)^{n_2}]\}$, with n_1 and n_2 two positive constants and the symbol O denoting the order of the uncon-

trolled terms. The resulting temperature could then be interpreted as the genuine modified temperature of the black hole, T_{BH}. Assuming that H is analytic in the region of small arguments and expanding it around ξT_0, one would obtain

$$T_{BH} = T_0 + [H(\xi T_0, \xi T_0) - 1]T_0\left\{1 + O\left[\frac{T_0^n}{E_P^n}\right]\right\}. \quad (6.7)$$

We have used that $H(0, 0) = 1$. Here, n is the minimum of n_1 and n_2 and may in principle be any positive constant. As a consequence, in the above expression one would generally be sure only of the significance of the leading order correction to the Hawking temperature, a correction that arises from the first nonconstant term in the Taylor series of H around zero. This contrasts with the situation found for theories that satisfy the requirements introduced in Subsection IV B, for which a full expression for the modified temperature has been derived.

VII. SUMMARY AND CONCLUSIONS

We have studied the modified entropy and temperature of (Schwarzschild) black holes in the framework of a gravity's rainbow. We have considered two different formalisms of this type. One is the original gravity's rainbow proposed by Magueijo and Smolin. The other is a related formalism based on a proposal of the authors about a canonical implementation of DSR. This implementation leads to a set of spacetime coordinates that are canonically conjugate to the physical energy and momentum. In both formalisms, the metric depends (explicitly) on the energy and momentum of the particle that is supposed to test the geometry. We have discussed the implications that both modified dispersion relations and generalized uncertainty principles have on black hole thermodynamics. Gravity's rainbow formalisms incorporate these two kinds of modifications and, in this sense, provide a suitable arena to carry out the desired discussion. In this context, we have focused our attention on the rather general case when the DSR theory associated with the gravity's rainbow has a physical energy that depends only on the undistorted energy of standard special relativity (and the physical momentum vanishes if the undistorted one does).

As the starting point for our discussion we have employed Bekenstein's calculation about the minimum change of area that a black hole suffers when it absorbs a particle. More specifically, we have taken into account the quantum nature of the particle in that calculation. For both the MS proposal and our proposal, we have motivated the introduction of a modified lower bound on the change of area that reproduces the one derived by Bekenstein in general relativity, except for a factor that depends on the (undistorted) energy-momentum of the absorbed particle. We have shown that, for a certain set of DSR theories, the different bounds obtained with the allowed range of ener-

gies and momenta for the particle can be captured in a single bound whose corrective factor is just a function of the black hole area. For such DSR theories, the energy-momentum dependent factor is an increasing function of the momentum but becomes a decreasing function once its two arguments (energy and momentum) are made to coincide. In particular, these conditions of monotony are satisfied by the DSR3 analogue of the Einstein-Rosen waves and by the familiar representative of the DSR2 models. For the usual representative of the DSR1 models the situation is more complicated; only with our proposal for a canonical implementation of DSR and under certain circumstances, the model satisfies the commented conditions.

In addition, we have explored the consequences of adopting two different types of quantization for the system. In one of them, the elementary position and momentum variables are those associated with flat space, and the time of the flat background is taken as the evolution parameter of the quantum dynamics. In the other case, the elementary position and momentum variables are the physical variables of the DSR theory, and the evolution is given in terms of the physical time of the system. At least as far as the analyzed bound on the change of black hole area (and the corresponding modified temperature) is concerned, we have shown that the former of these quantum descriptions is connected with our canonical formalism, whereas the latter is related to the MS proposal.

By employing the modified bound obtained for the area change together with the Bekenstein bound for the entropy-to-energy ratio, we have derived an inequality that can be interpreted as a (modified) generalized second law of thermodynamics. In this manner, we have been able to identify a modified expression for the black hole entropy, given as a function of the area. Using this expression we have deduced the modified temperature of the black hole. We have shown that, for the gravity's rainbow of Magueijo and Smolin, this temperature can vanish in the limit of zero black hole mass only in the case of some particular DSR3 models. With our proposal for a canonical implementation, on the other hand, the temperature may vanish for a much ampler family of models. In particular, models of the three distinct types of DSR families are allowed. This result suggests that black holes might stop their evaporation or expend an infinite time in the process, opening an avenue for the resolution of the information loss problem in black hole physics. This issue deserves further research along the lines that have already been proposed in Refs. [11,12].

Our analysis provides a systematic elaboration of the arguments based on the Bekenstein bound for its application to DSR theories and gravity's rainbow formalisms, assuming the validity of these formalisms as extensions of DSR that incorporate the effect of curvature [36]. An important influence must be attributed to Refs. [10–13]. Comparing our study with them, we have extended the

discussions that were available in the literature to DSR models whose physical momentum depends nonlinearly on the auxiliary one corresponding to special relativity. In addition, our study covers not only the MS proposal for a gravity's rainbow, but also a proposal whose distinctive feature is the canonical implementation of DSR [37,38]. As we have seen, the modified black hole thermodynamics arising from this alternative proposal has very appealing physical properties, nicer in general than those deduced with the MS construction.

The black hole metrics that we have considered are solutions to the modified Einstein equations that arise in the gravity's rainbow formalisms obtained with these two different proposals. In both cases, the geometry and the gravitational constant generally depend on the energy-momentum of the test particle that is used as a probe by the observer: employing the language of renormalization theory, geometry "runs" [36]. Nonetheless, it is worth clarifying that, for each given energy-momentum, these modified equations possess the same invariance under changes of coordinates as in general relativity. In particular, instead of using coordinates of the Schwarzschild-type like in Ref. [36], one can describe the modified Schwarzschild solutions in any other set of coordinates (e.g. the generalization of the Eddington-Finkelstein or the Kruskal-Szekeres coordinates) without changing the conclusions. Similarly, for each energy-momentum of the test particle, one can introduce well-defined notions of spatial and null infinity, showing that the black hole solution is asymptotic flat [53].

Finally, an interesting line of research for future investigations is the possible existence of *modified* black hole analogues in condensed matter physics. It is known that, under certain approximations, the description of some condensed matter systems can be split into a background configuration, interpretable as a black hole geometry, and some relativistic fields propagating on it [54]. Beyond the geometric regime in which these approximations are valid, the relativistic fields adopt modified dispersion relations, with corrections that become important for large frequencies. This situation presents a suggestive parallelism with that encountered in the gravitational theories that we have considered. In this respect, the fact that the underlying physics and the emergence of modifications are well understood in analogue models can be an important plus.

ACKNOWLEDGMENTS

APPENDIX: DSR MODELS

In this appendix, we give details about three specific DSR models that have been analyzed in the literature.

DSR2. The first of these models is taken as the prototype of the so-called DSR2 theories, in which both the physical energy and the physical momentum present an upper bound. In this model the nonlinear action of the Lorentz group in momentum space is generated by combining each boost with a dilatation. The model is characterized by the functions [32,33]

$$g(\epsilon) = \frac{\epsilon}{1 + \lambda_2 \epsilon}, \qquad f(\epsilon, \Pi) = \frac{\Pi}{1 + \lambda_2 \epsilon}. \qquad (A1)$$

For the two rainbow realizations analyzed in the main text, namely, the MS proposal and our proposal for a canonical implementation of DSR (that we will call the canonical proposal in the rest of this appendix), expressions (A1) lead to the following functions F, G, and H:

(i) MS proposal:

$$G(\epsilon) = \frac{g(\epsilon)}{\epsilon} = \frac{1}{1 + \lambda_2 \epsilon},$$

$$F(\epsilon, \Pi) = \frac{f(\epsilon, \Pi)}{\Pi} = \frac{1}{1 + \lambda_2 \epsilon}, \qquad (A2)$$

$$H(\epsilon, \Pi) = \frac{G(\epsilon)}{F(\epsilon, \Pi)} = 1.$$

(ii) Canonical proposal:

$$G(\epsilon) = \partial_\epsilon g(\epsilon) = \frac{1}{(1 + \lambda_2 \epsilon)^2},$$

$$F(\epsilon, \Pi) = \partial_\Pi f(\epsilon, \Pi) = \frac{1}{1 + \lambda_2 \epsilon}, \qquad (A3)$$

$$H(\epsilon, \Pi) = \frac{G(\epsilon)}{F(\epsilon, \Pi)} = \frac{1}{1 + \lambda_2 \epsilon}.$$

DSR3. The second example is a DSR version of the Einstein-Rosen waves. These are vacuum solutions to general relativity that describe cylindrical gravitational waves with linear polarization (i.e., spacetimes with an axial spacelike Killing vector and a translational one that commute and are hypersurface orthogonal). The connection between cylindrical gravity and DSR has been analyzed in Ref. [55]. For these waves, the physical energy turns out to be given by a nonlinear function of a different, auxiliary energy that is defined via quantum field theory in flat spacetime [52]. For each angular frequency and wave number (ϵ, Π) in this auxiliary theory, the nonlinear relation is

$$g(\epsilon) = \frac{1 - e^{-\lambda_3 \epsilon}}{\lambda_3}, \qquad f(\Pi) = \Pi. \qquad (A4)$$

This model can be regarded as a DSR3 theory, with bounded physical energy but unbounded momentum. The corresponding functions F, G, and H have the following form:

(i) MS proposal:

$$G(\epsilon) = \frac{g(\epsilon)}{\epsilon} = \frac{1 - e^{-\lambda_3 \epsilon}}{\lambda_3 \epsilon},$$

$$F(\epsilon, \Pi) = \frac{f(\epsilon, \Pi)}{\Pi} = 1, \tag{A5}$$

$$H(\epsilon, \Pi) = \frac{G(\epsilon)}{F(\epsilon, \Pi)} = \frac{1 - e^{-\lambda_3 \epsilon}}{\lambda_3 \epsilon}.$$

(ii) Canonical proposal:

$$G(\epsilon) = \partial_\epsilon g(\epsilon) = e^{-\lambda_3 \epsilon},$$

$$F(\epsilon, \Pi) = \partial_\Pi f(\epsilon, \Pi) = 1, \tag{A6}$$

$$H(\epsilon, \Pi) = \frac{G(\epsilon)}{F(\epsilon, \Pi)} = e^{-\lambda_3 \epsilon}.$$

DSR1. The third example was actually the first DSR model that appeared in the literature [28]. For this reason, all models that share with it the property of possessing a bounded physical momentum but unbounded physical energy are said to belong to the DSR1 class. The functions that determine this model are [28,29,43]

$$g(\epsilon|\eta) = \frac{1}{\lambda_1} \ln\left[1 + \lambda_1 \epsilon \sqrt{1 + \frac{\lambda_1^2 \eta^2}{4}} + \frac{\lambda_1^2 \eta^2}{2} \right],$$

$$f(\epsilon, \Pi|\eta) = \frac{\Pi \sqrt{1 + \frac{\lambda_1^2 \eta^2}{4}}}{1 + \lambda_1 \epsilon \sqrt{1 + \frac{\lambda_1^2 \eta^2}{4}} + \frac{\lambda_1^2 \eta^2}{2}}, \tag{A7}$$

where $\eta^2 := \epsilon^2 - \Pi^2$ is the Casimir invariant.

We restrict our attention to the case of fixed η, because the function g would otherwise depend on the auxiliary momentum, contradicting the assumptions of our analysis in the main text. Moreover, in the case of the MS proposal, we will further concentrate our study exclusively on massless particles (e.g., photons) in order to simplify the dis-

cussion. Substituting $\eta = 0$ directly in Eq. (A7), one obtains

$$g_0(\epsilon) := g(\epsilon|\eta = 0) = \frac{1}{\lambda_1} \ln(1 + \lambda_1 \epsilon),$$

$$f_0(\epsilon, \pi) := f(\epsilon, \Pi|\eta = 0) = \frac{\Pi}{1 + \lambda_1 \epsilon}. \tag{A8}$$

Using these functions for the MS proposal and, more generally, the functions (A7) for our canonical proposal, it is a simple exercise to deduce the following expressions for F, G, and H:

(i) MS proposal for massless particles:

$$G(\epsilon) = \frac{g_0(\epsilon)}{\epsilon} = \frac{\ln(1 + \lambda_1 \epsilon)}{\lambda_1 \epsilon},$$

$$F(\epsilon, \Pi) = \frac{f_0(\epsilon, \Pi)}{\Pi} = \frac{1}{1 + \lambda_1 \epsilon},$$

$$H(\epsilon, \Pi) = \frac{G(\epsilon)}{F(\epsilon, \Pi)} = \frac{1 + \lambda_1 \epsilon}{\lambda_1 \epsilon} \ln(1 + \lambda_1 \epsilon). \tag{A9}$$

(ii) Canonical proposal:

$$G(\epsilon) = \partial_\epsilon g(\epsilon|\eta) = \frac{\sqrt{1 + \frac{\lambda_1^2 \eta^2}{4}}}{1 + \lambda_1 \epsilon \sqrt{1 + \frac{\lambda_1^2 \eta^2}{4}} + \frac{\lambda_1^2 \eta^2}{2}},$$

$$F(\epsilon, \Pi) = \partial_\Pi f(\epsilon, \Pi|\eta)$$

$$= \frac{\sqrt{1 + \frac{\lambda_1^2 \eta^2}{4}}}{1 + \lambda_1 \epsilon \sqrt{1 + \frac{\lambda_1^2 \eta^2}{4}} + \frac{\lambda_1^2 \eta^2}{2}},$$

$$H(\epsilon, \Pi) = \frac{G(\epsilon)}{F(\epsilon, \Pi)} = 1. \tag{A10}$$

The invariant scales λ_n (with $n = 1, 2, 3$) for the different DSR models do not necessarily coincide. Nonetheless, we have obviated this difference in the main text for simplicity, adopting the notation λ for all of them.

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[50] Conversely, the relation $A = 4\pi r_s^2$ can be seen as the definition of A in terms of the radial position of the horizon.

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