

**Quantum phantom cosmology**

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We apply the formalism of quantum cosmology to models containing a phantom field. Three models are discussed explicitly: a toy model, a model with an exponential phantom potential, and a model with phantom field accompanied by a negative cosmological constant. In all these cases we calculate the classical trajectories in configuration space and give solutions to the Wheeler-DeWitt equation in quantum cosmology. In the cases of the toy model and the model with exponential potential we are able to solve the Wheeler-DeWitt equation exactly. For comparison, we also give the corresponding solutions for an ordinary scalar field. We discuss, in particular, the behavior of wave packets in minisuperspace. For the phantom field these packets disperse in the region that corresponds to the big-rip singularity. This thus constitutes a genuine quantum region at large scales, described by a regular solution of the Wheeler-DeWitt equation. For the ordinary scalar field, the big-bang singularity is avoided. Some remarks on the arrow of time in phantom models as well as on the relation of phantom models to loop quantum cosmology are given.

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**I. INTRODUCTION**

It is a striking fact that our universe is currently accelerating [1]. A major open problem is to provide a fundamental type of matter which may be responsible for this, since any of the forms of matter we know from our experience cannot explain this phenomenon. This matter is not visible, but provides a dominant fraction of the energy density in the universe and was therefore given the name “dark energy” (for a current review see for example [2]). The best known, and perhaps the simplest candidate for such a matter is the cosmological constant, but theoretical physics provides more options. One of them is an evolving scalar field with appropriate kinetic and potential energies. In general, it may mimic various types of matter during different periods of the cosmological evolution.

Dark energy is characterized by negative pressure which causes the repulsion of matter in the universe and, as a consequence, its acceleration. In terms of the standard energy conditions in general relativity [3], dark energy must violate the strong energy condition  $\rho + 3p > 0$ ,  $\rho > 0$ . Assuming a barotropic equation of state of the matter in the universe,  $p = w\rho$  ( $w = \text{constant}$ ), where  $p$  and  $\rho$  are the pressure and the density of dark energy, respectively, it requires that  $w < -\frac{1}{3}$ .

However, according to more recent observations [4,5], dark energy is even more biased towards larger negative values of the barotropic index  $w \lesssim -1$ . This means that it

would have to violate the null energy condition  $\rho + p > 0$  [3] and, consequently, all the remaining energy conditions such as: the weak energy condition  $\rho > 0$ ,  $\rho + p > 0$ , and the dominant energy condition  $\rho > 0$ ,  $-\rho < p < \rho$ . Dark energy of this type was dubbed *phantom* [6,7]. A phantom may be represented by an evolving scalar field which possesses negative kinetic energy (often called “ghost field”). Although phantom fields lead to various problems [8], they are observationally supported as a possible source of dark energy and deserve thorough investigation (but see [9] for an alternative view). Moreover, there may exist phantom fields without pathologic behavior in the ultra-violet regime [10]. Phantom models of the universe admit a new type of singularity called a big-rip singularity [6,7,11]. At the big rip, energy density and pressure diverge as the scale factor  $a(t)$  goes to infinity at a finite time. This is different from an ordinary big crunch singularity, which leads to a blowup of the energy density and pressure as the scale factor approaches zero at a finite time. Another possible singularity is the big brake where the expansion rate is zero and the acceleration rate approaches minus infinity [12]. Besides, more exotic types of singularities may appear such as the sudden future singularity [13], the generalized sudden future singularity [14] where there is a blowup of the higher-order derivatives of the scale factor with smooth evolution of the scale factor and the energy density, the type III singularity [15] and the type IV singularity [16] where the evolution of the scale factor is smooth. These singularities have weaker properties than a big rip [17].

In Ref. [18], classical phantom cosmologies were studied and a large variety of possible cosmological sce-

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narios were found. Also, the duality relation between standard matter and phantom matter models was revealed (see also [19,20]) which has an analogon in the duality symmetry present in superstring cosmology [21].

It is worth mentioning that, once the supernova data are analyzed in a prior-free manner, an evolving equation of state with a time-dependent barotropic index  $w = w(t)$  for the dark energy is favored (cf. [22–24]). Such models were also studied in the quantum context in [25] where a canonical momentum was attached to a time-dependent barotropic index. Some authors have also studied the thermodynamical properties of phantom models [26,27].

In all these investigations, an evolving universe was described by classical cosmology. Quantum effects were only studied in certain phases of the evolution, for example, close to a singularity, cf. [28], without applying quantum theory to the universe as a whole. It is the purpose of this paper to accomplish the latter goal, that is, to discuss quantum cosmology with phantom fields. The interest in this is due to the fact that for both experimental and theoretical reasons it seems that quantum theory is universally valid [29]. Therefore, the universe as a whole has to be described by quantum theory. If phantom fields play a dominant role, it has to be investigated whether this causes deviations from the standard formalism of quantum cosmology and whether there are interesting physical consequences.

Quantum cosmology must be based on a theory of quantum gravity [30]. Candidates for such a theory include string theory, loop quantum gravity and quantum geometrodynamics. Our present analysis will, like most investigations of quantum cosmology, be based on the Wheeler-DeWitt equation of quantum geometrodynamics. Independently of the correct theory of quantum gravity, this framework should yield an adequate description at least on the energy scales below the Planck scale (if not on all scales). If one approaches the Planck scale, modifications such as loop quantum cosmology [31] might become necessary. The investigations in our paper are independent of such modifications and will be discussed in a future paper.

A central feature of the Wheeler-DeWitt equation is its local hyperbolic signature [30,32]. In regions of configuration space near closed Friedmann cosmologies, it is globally hyperbolic, that is, there is only one minus sign in the kinetic term [33]. The negative part of the kinetic term is related to the scale factor of the Friedmann model, which in a certain sense thus plays itself the role of a phantom field. The presence of an indefinite kinetic term is intimately connected with the attractive nature of gravity [34].

Besides its hyperbolic character, the most important feature of the Wheeler-DeWitt equation is its independence of an external time parameter [30,32]. This holds, in fact, for every system that is reparametrization invariant at the classical level. Consistent discussions of quantum

cosmology must thus be based on the intrinsic structure of this equation and avoid the use of an intuitive but wrong picture of an external Newtonian time. For this purpose it is necessary to study the classical trajectories in a *configuration space* where the classical time parameter  $t$  is eliminated.

The structure of the Wheeler-DeWitt equation is important for the imposition of boundary conditions in quantum cosmology. In the hyperbolic case one has a wave equation whose form suggests imposing boundary conditions at constant values of the scale factor,  $a$ . This is of importance, for example, if one attempts to construct wave packets that follow the classical trajectories in configuration space like standing tubes [35–37]. It is also crucial for an understanding of what pre- and post big-bang phases mean in quantum string cosmology [38,39]. The origin of the arrow of time can in principle be traced back to the structure of this wave equation [32,40,41].

The presence of a phantom field changes the structure of the Wheeler-DeWitt equation: If only the phantom is present besides the scale factor (“phantom dominance”), its structure becomes elliptic, while in the general case it becomes of a mixed (ultrahyperbolic) nature. This has implications for the imposition of boundary conditions. A change of signature in the Wheeler-DeWitt equation has hitherto been noticed in the presence of nonminimally coupled fields [42]. In our paper we shall present the formalism of quantum phantom cosmology and some of its main physical consequences.

Our paper is organized as follows. In Sec. II we shall study and solve the classical equations of motion for the phantom field in a Friedmann universe. After the presentation of the necessary equations, we give the solutions for the classical trajectories in configuration space for three models: A toy model with vanishing phantom potential (Sec. II B), a model with exponential phantom potential (Sec. II C), and a model with cosh-potential and a negative cosmological constant (Sec. II D). For comparison, in all these cases, we give the results for a nonphantom scalar field. Section III contains in the same order the discussion of the quantum theory for these models, both for a phantom field and a corresponding ordinary scalar field. We are able to solve the Wheeler-DeWitt equation exactly for the toy model and the model with exponential potential. In particular, we discuss wave packet solutions and find that quantum effects dominate in the region of the classical big-rip singularity. Therefore, *quantum effects occur at large scales*. Since the solutions of the Wheeler-DeWitt equation are regular there, the big-rip singularity has vanished in the quantum theory. Furthermore, in the realistic scalar field models, the wave function vanishes at the big bang. Thus, this singularity is likewise excluded in the quantum theory. In Sec. IV we give a summary of the results and the outlook of the problem of the arrow of time and possible modifications of the obtained picture due to loop quantum cosmology.

## II. CLASSICAL PHANTOM COSMOLOGIES IN CONFIGURATION SPACE

### A. Classical equations

We consider a Friedmann universe with scale factor  $a(t)$  and a homogeneous scalar field  $\phi(t)$ . We assume here that the  $\phi$ -field dominates over other matter degrees of freedom, so that it is the only degree of freedom besides the scale factor. The action reads

$$S = \frac{3}{\kappa^2} \int dt N \left( -\frac{a\dot{a}^2}{N^2} + \mathcal{K}a - \frac{\Lambda a^3}{3} \right) + \frac{1}{2} \int dt N a^3 \left( \ell \frac{\dot{\phi}^2}{N^2} - 2V(\phi) \right). \quad (1)$$

Here,  $\kappa^2 = 8\pi G$ ,  $N$  is the lapse function,  $\Lambda$  is the cosmological constant,  $V(\phi)$  is a potential of the field  $\phi$ ,  $\mathcal{K} = 0, \pm 1$  is the curvature index, and we have set  $c = 1$ . The parameter  $\ell$  distinguishes between a phantom field (where  $\ell = -1$ ) and an ordinary scalar field (where  $\ell = +1$ ).

We set  $N = 1$ , so the time parameter is the standard Friedmann cosmic time. The action then becomes

$$S = \frac{3}{\kappa^2} \int dt \left( -a\dot{a}^2 + \mathcal{K}a - \frac{\Lambda}{3}a^3 \right) + \frac{1}{2} \int dt (a^3 \ell \dot{\phi}^2 - 2a^3 V(\phi)). \quad (2)$$

The canonical momenta are given by

$$\pi_a = -\frac{6a\dot{a}}{\kappa^2}, \quad \pi_\phi = \ell a^3 \dot{\phi}. \quad (3)$$

The canonical Hamiltonian  $\mathcal{H}$ , which is constrained to vanish, reads

$$\mathcal{H} = -\frac{\kappa^2}{12a} \pi_a^2 + \frac{\ell}{2} \frac{\pi_\phi^2}{a^3} + a^3 \frac{\Lambda}{\kappa^2} + a^3 V - \frac{3\mathcal{K}a}{\kappa^2} = 0. \quad (4)$$

Expressed in terms of the ‘‘velocities,’’ see (3), this constraint becomes identical to the Friedmann equation,

$$\left( \frac{\dot{a}}{a} \right)^2 \equiv H^2 = \frac{\kappa^2}{3} \left( \ell \frac{\dot{\phi}^2}{2} + V(\phi) \right) + \frac{\Lambda}{3} - \frac{\mathcal{K}}{a^2}. \quad (5)$$

The term in parentheses is the energy density of the scalar field,

$$\rho \equiv \ell \frac{\dot{\phi}^2}{2} + V(\phi). \quad (6)$$

We recognize that for the standard scalar field ( $\ell = 1$ ), no classically forbidden regions in configuration space exist due to the indefiniteness of the total kinetic term. This is different from the phantom case ( $\ell = -1$ ), where only the region

$$V(\phi) + \frac{\Lambda}{\kappa^2} - \frac{3\mathcal{K}}{\kappa^2 a^2} \geq 0 \quad (7)$$

is classically allowed (this restriction is due to the negative definiteness of the total kinetic term).

The field  $\phi$  obeys the second-order equation of motion

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} + \ell V'(\phi) = 0, \quad (8)$$

which is equivalent to the conservation equation  $\dot{\rho} + 3H(\rho + p) = 0$ , provided the standard perfect-fluid energy-momentum tensor is introduced. This equation is trivially fulfilled by the cosmological constant  $\Lambda$  ( $\rho = \text{constant}$ ,  $p = -\rho$ ). In (8) we recognize a formal reversal of the potential in the phantom case compared to an ordinary scalar field case, since the sign in front of the  $V'$ -term changes. With the help of (5), the second-order equation for  $a$  can be put into the form

$$\frac{\ddot{a}}{a} - \frac{\Lambda}{3} + \frac{\kappa^2}{3} (\ell \dot{\phi}^2 - V(\phi)) = 0. \quad (9)$$

Again, assuming the perfect-fluid energy-momentum tensor, the scalar field exerts the pressure

$$p \equiv \ell \frac{\dot{\phi}^2}{2} - V(\phi). \quad (10)$$

Note that the case of a cosmological constant is included by having the additional equation of state

$$p_\Lambda = -\rho_\Lambda = -\frac{\Lambda}{\kappa^2}. \quad (11)$$

Assuming a constant barotropic index  $w$  for the scalar field, we can use (6) and (10) to find the relation between the scalar field and its potential [43],

$$V(\phi(t)) = \frac{\ell}{2} \frac{1-w}{1+w} \dot{\phi}^2(t), \quad w \neq -1. \quad (12)$$

This is analogous to the virial theorem in which the kinetic energy is proportional to the potential energy of the field. However, as has already been mentioned above, it may be more physical to assume a time-dependent barotropic index [22–24].

### B. Classical phantom trajectory for vanishing phantom potential and vanishing cosmological constant

In this section we shall consider a simple model with field potential  $V(\phi) = 0$  and cosmological constant  $\Lambda = 0$ . This leads to an equation of state for stiff matter,  $p = \rho$ ,  $w = 1$ , in contrast to the current observational status [1,4]. However, such an evolution may perhaps be valid in ekpyrotic/cyclic scenarios where this matter dominates the collapsing phase of the cosmological evolution [44]. Moreover, in such a case the energy density  $\rho < 0$ , and thus this model does not seem to represent dark energy which is usually assumed to have positive energy density. However, it captures interesting ‘‘phantom features,’’ since it violates all energy conditions, and it has the merit that it is easily

manageable. More realistic models will be discussed below.

In order to get classical solutions in the phantom case ( $\ell = -1$ ) we have to choose  $\mathcal{K} = -1$  in (7). Since we are interested in constructing wave packets from the Wheeler-DeWitt equation, we want to find a classical trajectory in *configuration space*, where the classical time  $t$  is eliminated. This is motivated by the fact that no such parameter is present in the Wheeler-DeWitt equation.

Since  $\phi$  is a cyclic variable,  $\pi_\phi$  is constant, so from (3) one has  $\dot{\phi}^2 = C_p^2/a^6$  with a constant  $C_p$ . From (5) one then has (we choose  $C_p > 0$ )

$$\frac{d\phi}{da} = \pm \frac{C_p}{a\sqrt{a^4 - \frac{\kappa^2 C_p^2}{6}}}, \quad (13)$$

which can easily be integrated to yield

$$\phi(a) = \pm \frac{1}{\kappa} \sqrt{\frac{3}{2}} \arccos \frac{\kappa C_p}{\sqrt{6} a^2}. \quad (14)$$

For convenience, we choose  $\kappa^2 = 6$ . Then the solution reads

$$\phi(a) = \pm \frac{1}{2} \arccos \frac{C_p}{a^2}. \quad (15)$$

The classical trajectory (15) has a minimum value of the scale factor,  $a_{\min} = \sqrt{C_p}$ , and reaches infinite values of  $a$  at finite values of  $\phi = \pm\pi/4$ . In this sense it resembles a big rip solution. However, with respect to  $t$  the scale factor reaches infinity only at  $t = \pm\infty$  and, moreover,  $\rho \propto a^{-6}$  which is the density scaling appropriate to a stiff fluid. Nonetheless, in configuration space the trajectory has some features of a big rip, and this is why this toy model is of interest.

For an ordinary scalar field ( $\ell = 1$ ) and for  $\mathcal{K} = -1$ , one gets instead of (15),

$$\phi(a) = \pm \frac{1}{2} \operatorname{arcsinh} \frac{C_f}{a^2}. \quad (16)$$

There is no turning point; Eq. (16) just describes two branches for which  $a \rightarrow \infty$  if  $\phi \rightarrow 0$ , and  $a \rightarrow 0$  if  $\phi \rightarrow$

$\pm\infty$ . The two solutions (15) and (16) are depicted in Fig. 1. For  $\mathcal{K} = 1$  one obtains the solution with a turning point (arccosh instead of arcsinh) that was discussed in [35].

### C. Classical trajectories for exponential scalar field potential and vanishing cosmological constant

A model with phantom equation of state and a true big-rip singularity for the phantom model appears if the potential in (2) is chosen to be exponential [45,46]

$$V(\phi) = V_0 e^{-\lambda\kappa\phi}, \quad (17)$$

and  $\Lambda = 0$ . Interest in this type of scalar field potentials in cosmology arose when it became clear that the classical model has an attractor solution with scalar field domination [47,48]. This alleviates the fine-tuning problem of the initial energy of the scalar field [49]. Such an attractor exists not only in the case of a conventional scalar field, but also for the phantom field [45,46]. Exponential potentials for scalar fields arise in the context of Kaluza-Klein theories [50,51], higher-derivative gravity in ( $D + 4$ ) dimensions [52,53], higher-order gravity [54], supergravity and superstring theories [55,56], see also [49] for an overview.

In the following, we shall consider the case of a flat universe,  $\mathcal{K} = 0$ . From Eq. (7) one sees immediately that for this choice of parameters neither the ordinary scalar field nor the phantom field model possesses classically forbidden regions. The classical equations of motion (8) and (9) can be transformed into a dynamical system with the Friedmann Eq. (5) as a constraint [45–47,49]. For  $\ell = -1$  and arbitrary values of  $\lambda$ , as well as for  $\ell = +1$  and  $\lambda < \sqrt{6}$ , this system has an attractor solution given by [19]:

$$\phi(t) = \frac{2}{\lambda\kappa} \ln \left[ 1 + \ell \frac{\lambda^2 H_0}{2} (t - t_0) \right], \quad (18)$$

$$\frac{a}{a_0} = \left[ 1 + \ell \frac{\lambda^2 H_0}{2} (t - t_0) \right]^{\ell(2/\lambda^2)}. \quad (19)$$

Introducing  $\alpha \equiv \ln(a)$  for later convenience, one obtains the following simple trajectory in configuration space,

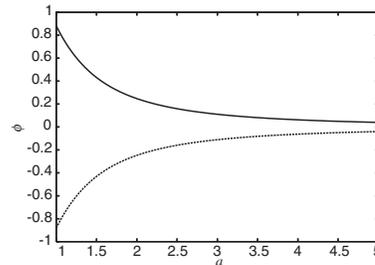
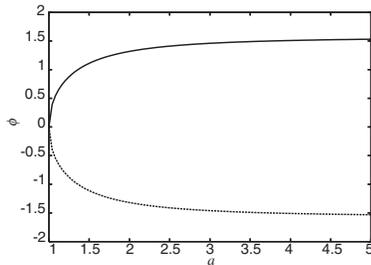


FIG. 1. The classical trajectory in configuration space for the toy model with vanishing scalar field potential and vanishing cosmological constant. The diagram on the left-hand side shows the trajectory for the phantom field model. On the right-hand side the trajectory for the  $\ell = 1$  scalar field model is plotted.

$$\phi(\alpha) = \ell \frac{\lambda}{\kappa} \alpha. \quad (20)$$

For this attractor solution, the “kinetic energy” defined from (5)—writing this equation in the form  $\ell E_{\text{kin}} + E_{\text{pot}} = 1$ —is given by (using  $\kappa^2 = 6$  in the second step)

$$E_{\text{kin}} \equiv \frac{\kappa^2}{6} \left( \frac{d\phi}{d\alpha} \right)^2 = \frac{\lambda^2}{6}$$

and thus constant. Therefore, also the “potential energy” of the scalar field is constant,

$$E_{\text{pot}} \equiv \frac{\kappa^2 V}{3H^2} = 1 - \frac{\ell\lambda^2}{6}.$$

The equation of state parameter of the field is

$$w = -1 + \ell \frac{\lambda^2}{3}. \quad (21)$$

Thus for  $\ell = -1$ ,  $\phi$  indeed describes a phantom field with  $w < -1$ , whereas the scalar field with  $\ell = 1$  covers the range  $w > -1$ . Accordingly, the energy density scales as  $\rho = \rho_0 \left( \frac{a}{a_0} \right)^{-\ell\lambda^2}$ . As expected, this yields a big-rip singularity for  $\ell = -1$ , since in the limit  $t \rightarrow t_1 \equiv t_0 - 2\ell/(\lambda^2 H_0)$  the energy density and the scale factor diverge. For  $t \rightarrow \infty$ ,  $a$  and  $\rho$  vanish. This is in contrast to the  $\ell = 1$  model: In the limit  $t \rightarrow t_1$ ,  $a$  goes to zero and  $\rho$  diverges, yielding a big bang, while for  $t \rightarrow \infty$ ,  $a$  diverges and  $\rho$  goes to zero.

#### D. Classical trajectories for scalar field fluid and negative cosmological constant

It is easy to obtain a simple set of classical solutions for cosmological models with a negative cosmological constant [18]. In contrast to a positive cosmological constant which supports cosmological repulsion, the negative cosmological constant is a source of attraction and can overcome the influence of repulsion from dark energy with negative pressure such as cosmic strings, domain walls, and phantom, see for example [18]. This allows models with a negative cosmological constant and other fluids to evolve symmetrically between two singularities with an extremum in between. In particular, *it is possible to have an evolution between the two big rips which appear at finite cosmic time*, as will be shown below.

We assume a flat universe,  $\mathcal{K} = 0$ , with a negative cosmological constant  $\Lambda < 0$  and a cosmological fluid with barotropic equation of state  $p = w\rho$ ; the latter will be mimicked by a scalar field  $\phi$  of either standard or phantom type. In this case, the energy conservation equation gives

$$\rho = Ca^{-3(w+1)}, \quad (22)$$

and the energy conservation equation (11) for the cosmological constant remains valid. This can be used to solve

the system (5) and (9) in terms of the scale factor as

$$a(t) = \left[ A \sin\left(\frac{|D|}{\sqrt{3}}(-\Lambda)^{1/2}t\right) \right]^{1/D}, \quad (23)$$

where

$$D = \frac{3}{2}(1+w), \quad \frac{6C}{A^2} = -\Lambda > 0. \quad (24)$$

Using (24), we can rewrite (12) in the form

$$V(\phi(t)) = \frac{\ell}{2} \frac{3-D}{D} \dot{\phi}^2(t), \quad (25)$$

which allows us to write (6) as (note that as in Sec. II B we have assumed  $\kappa^2 = 6$ ),

$$\rho = \frac{3\ell}{2D} \dot{\phi}^2 = \frac{\dot{a}^2}{2a^2} - \frac{\Lambda}{6}. \quad (26)$$

With all these assumptions we are able to calculate the evolution of the scalar field as

$$\phi(t) = \pm \frac{1}{\sqrt{3}|D|} \sqrt{\frac{D}{\ell}} \ln \left| \tan\left(\frac{|D|}{2\sqrt{3}}(-\Lambda)^{1/2}t\right) \right|. \quad (27)$$

Let us note that  $\ell = +1$ ,  $D > 0$  for an ordinary scalar field, while  $\ell = -1$ ,  $D < 0$  for the phantom. Then, the above expressions make sense since  $D/\ell = |D| > 0$ . For  $D > 0$  (negative cosmological term plus  $w > -1$  fluid), the evolution of the universe based on (23) begins with a big bang at  $t = 0$ , reaches a maximum  $a_{\text{max}} = A^{1/D}$ , and terminates with a big crunch at  $t = \pi$ . For  $D = -|D| < 0$  (the phantom case), the evolution starts with a big rip at  $t = 0$ , reaches a minimum  $a_{\text{min}} = A^{-1/|D|}$ , and terminates with a big rip at  $t = \pi$ . The latter case is of special interest because it allows a symmetric evolution of the scale factor in the presence of a phantom field. This model may also be of interest to study the cosmological arrow of time, see Sec. IV.

A similar type of symmetric evolution appears in configuration space. Using (23) and (27) to eliminate the classical time coordinate, we obtain the trajectory

$$\phi(a) = \pm \frac{1}{\sqrt{3}|D|} \sqrt{\frac{D}{\ell}} \ln\left(\frac{a^D}{A + \sqrt{A^2 - a^{2D}}}\right). \quad (28)$$

From this we can see that there are two branches. For  $\ell = -1$  each of them extends to infinity, that is,  $\phi \rightarrow \pm\infty$ , for  $a \rightarrow \infty$  and reaches a minimum  $\phi(a) = 0$ , for  $a_{\text{min}} = A^{-1/|D|}$ . For  $\ell = 1$  one recognizes the presence of the maximum  $a_{\text{max}}$ . The trajectories in configuration space are depicted in Fig. 2. From (27) and (26) one can reconstruct the potential of the scalar field,

$$V(\phi) = V_0 \cosh^2\left(\frac{\phi}{F}\right), \quad (29)$$

where

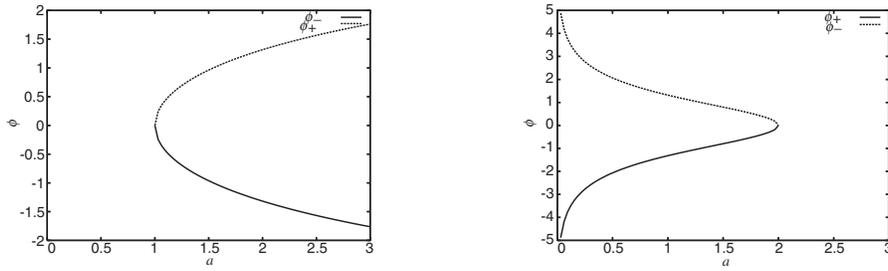


FIG. 2. The classical trajectories in configuration space for the models with cosh-potential and negative cosmological constant. On the left-hand side, the trajectory for the phantom field model is shown. The classical trajectory for the scalar field model  $\ell = 1$  is shown on the right-hand side. The similarity to the classical trajectories in the toy model in Sec. II B is obvious.

$$V_0 = -\frac{\Lambda}{3} \frac{\ell}{2} (3 - D) F^2 D, \quad F = \frac{1}{\sqrt{3|D|}} \sqrt{\frac{D}{\ell}}. \quad \left( \frac{\hbar^2}{2} \bar{a} \frac{\partial}{\partial \bar{a}} \bar{a} \frac{\partial}{\partial \bar{a}} + \ell \frac{\hbar^2}{2} \frac{\partial^2}{\partial \bar{\phi}^2} + \frac{1}{\bar{a}^6} \left( V(i\bar{\phi}) + \frac{\Lambda}{6} \right) \right) \psi(\bar{a}, \bar{\phi}) = 0. \quad (34)$$

Note that for  $\ell = 1$  the potential is positive only for  $D < 3$  (i.e.,  $w < 1$ ). This restriction is similar to the restriction  $\lambda < \sqrt{6}$  in Sec. II C.

### III. QUANTUM COSMOLOGY FOR PHANTOM AND ORDINARY FIELD

#### A. Wheeler-DeWitt equation and phantom duality

Quantization of the Hamiltonian constraint (4) leads to the Wheeler-DeWitt equation. Choosing the Laplace-Beltrami factor ordering and again the convention  $\kappa^2 = 6$ , it reads

$$\left( \frac{\hbar^2}{2} a \frac{\partial}{\partial a} a \frac{\partial}{\partial a} - \ell \frac{\hbar^2}{2} \frac{\partial^2}{\partial \phi^2} + a^6 \left( V(\phi) + \frac{\Lambda}{6} \right) - \frac{\mathcal{K} a^4}{2} \right) \psi(a, \phi) = 0. \quad (30)$$

Let us note that under the phantom duality [19]

$$a \rightarrow \frac{1}{\bar{a}}, \quad (31)$$

$$\phi \rightarrow -i\bar{\phi}, \quad (32)$$

for  $\mathcal{K} = 0$  the Wheeler-DeWitt equation for  $a, \phi$ ,

$$\left( \frac{\hbar^2}{2} a \frac{\partial}{\partial a} a \frac{\partial}{\partial a} - \ell \frac{\hbar^2}{2} \frac{\partial^2}{\partial \phi^2} + a^6 \left( V(\phi) + \frac{\Lambda}{6} \right) \right) \psi(a, \phi) = 0, \quad (33)$$

transforms into the Wheeler-DeWitt equation for  $\bar{a}, \bar{\phi}$ , that is,

The transformation for  $\phi$  is thus just a Wick rotation.

On the other hand, Eq. (33) can conveniently be rewritten in terms of the scale factor  $\alpha \equiv \ln(a)$  as

$$\left( \frac{\hbar^2}{2} \frac{\partial^2}{\partial \alpha^2} - \ell \frac{\hbar^2}{2} \frac{\partial^2}{\partial \phi^2} + e^{6\alpha} \left( V(\phi) + \frac{\Lambda}{6} \right) \right) \Psi(\alpha, \phi) = 0. \quad (35)$$

It is this form of the Wheeler-DeWitt equation with which we shall work in the following.

#### B. Quantum phantom cosmology - no phantom potential

For vanishing potential  $V = 0$ ,  $\Lambda = 0$ , and  $\mathcal{K} = -1$  the solution to the phantom ( $\ell = -1$ ) Wheeler-DeWitt equation (30) is found by a separation ansatz,

$$\psi_k(a, \phi) = C_k(a) \varphi_k(\phi). \quad (36)$$

We choose

$$\varphi_k(\phi) = e^{-ik\phi/\hbar}, \quad (37)$$

because real exponentials would lead to exponentially increasing wave functions for  $\phi \rightarrow \pm\infty$  that would not reflect classical behavior. From (30) one then gets the following equation for the  $C_k$  (primes denote derivatives with respect to  $a$ ),

$$a^2 C_k'' + a C_k' + \frac{1}{\hbar^2} (a^4 - k^2) C_k = 0. \quad (38)$$

Solutions of this equation are Bessel functions  $Z_{k/2\hbar}(a^2/2\hbar)$ . However, we have to impose the boundary condition that  $\psi(a, \phi) \rightarrow a^{-\infty}$  in order to reflect the behavior of the classical trajectories (15) which have in configuration space a minimum with respect to  $a$ . We therefore have to choose the Bessel function  $J_{k/2\hbar}(a^2/2\hbar)$  with  $k > 0$ .

The connection to the classical solution (15) should be performed through a formal WKB limit “ $\hbar \rightarrow 0$ .” We thus

have to look for an asymptotic expansion of  $J$  where both the argument and the index are large. We use the expression [57]

$$J_\nu(\nu z) = \left(\frac{4\zeta}{1-z^2}\right)^{1/4} \left(\frac{\text{Ai}(\nu^{2/3}\zeta)}{\nu^{1/3}} + \frac{\exp(-\frac{2}{3}\nu\zeta^{3/2})}{1+\nu^{1/6}|\zeta|^{1/4}} \mathcal{O}\left(\frac{1}{\nu^{4/3}}\right)\right) \quad (39)$$

and set  $\nu = k/2\hbar$ ,  $z = a^2/k$ . The choice for  $\zeta$  depends on whether  $z^2 \geq 1$  or  $z^2 < 1$ . Let us consider first the case  $z^2 \geq 1$  which corresponds to  $a^4/k^2 \geq 1$ . From [57] one sees that then

$$-\zeta = \left(\frac{3}{2}\sqrt{\frac{a^4}{k^2} - 1} - \frac{3}{2}\arccos\frac{k}{a^2}\right)^{2/3}. \quad (40)$$

We also use the asymptotic expression for the Airy function occurring in (39), see [58],

$$\text{Ai}\left(\left[\frac{k}{2\hbar}\right]^{2/3}\zeta\right) \sim \pi^{-1/2}\left[-\left(\frac{k}{2\hbar}\right)^{2/3}\zeta\right]^{-1/4}\sin\theta_k, \quad (41)$$

where

$$\theta_k = -\frac{k}{3\hbar}\zeta^{3/2} + \frac{\pi}{4}. \quad (42)$$

The classical trajectory is then recovered through the principle of constructive interference: We look for the extremum of the phase

$$S_k \equiv \theta_k \pm \frac{k\phi}{\hbar} \quad (43)$$

of the total wave function with respect to  $k$ . One then easily finds that the requirement  $\partial S_k/\partial k = 0$  at  $k = \bar{k}$  leads to (15). One can thus identify  $C_p = \bar{k}$ .

What happens for  $z^2 < 1$ ? As one can easily see from the corresponding expression in [58],  $\zeta < 0$  and the Airy function decays exponentially. This is as expected, since  $a^4/k^2 < 1$  corresponds to the classically forbidden region.

One can also easily check that  $S_k$ , Eq. (43), is a solution of the Hamilton-Jacobi equation arising from (4) through the substitutions  $\pi_a \rightarrow \partial S_k/\partial a$  and  $\pi_\phi \rightarrow \partial S_k/\partial \phi$ .

In the case of the conventional scalar field, one gets a change of sign for the  $k^2$ -term in (38). The solutions for  $C_k(a)$  are then the Bessel functions  $J_{ik/2\hbar}(a^2/2\hbar)$  and  $J_{-ik/2\hbar}(a^2/2\hbar)$ . Since there are no classically forbidden regions, both solutions seem to be allowed. It can again easily be checked that the classical solution (16) follows in the formal limit “ $\hbar \rightarrow 0$ ” from the principle of constructive interference: One gets the two branches of (16) from the two Bessel functions. This suggests to use one *or* the other Bessel function if one wants to avoid interferences (and thus nonclassical behavior) at large  $a$ . Since (30) is

hyperbolic for  $\ell = 1$ , one is free to impose boundary conditions at constant  $a$ , that is, one can either impose one packet or two packets there, depending on whether one wants one branch of the classical solution to be represented or both.

In the phantom case discussed above, the Wheeler-DeWitt equation is elliptic; one there only imposes the boundary condition that  $\psi$  goes to zero at  $a \rightarrow 0$  and that it is at most oscillating at the other boundaries. This fixes the solution to be  $J_{k/2\hbar}(a^2/2\hbar)$  or a superposition thereof. Explicitly, one would consider the following superposition for the construction of a wave packet,

$$\psi(a, \phi) = \int_0^\infty dk A(k) e^{-ik\phi/\hbar} J_{k/2\hbar}(a^2/2\hbar), \quad (44)$$

where  $A(k)$  is a function of  $k$  that is peaked around a particular value  $\bar{k}$  (e.g. a Gaussian). One would not expect the packet to exhibit dispersion near the minimum of the classical trajectory, since the phase of the Bessel function is not rapidly varying with respect to  $k$ , in contrast to the case of a massive scalar field discussed in [35]. We shall, however, expect the occurrence of a dispersion at large values of  $a$ . We shall discuss this explicitly for the more realistic case in Sec. III C below.

Making an analogy to ordinary quantum mechanics, one would compare the solution in the elliptic case to an “initial wave function”  $\psi(t=0, x)$ , whereas the hyperbolic case would correspond to the time evolution  $\psi(t, x)$ , since one would have for (30) a distinguished set of foliations with respect to an intrinsic time defined by the scale factor. This intrinsic time could be used as a physical time with respect to which, for example, further degrees of freedom could be evolved, cf. Sec. IV.

### C. Quantum phantom cosmology—exponential phantom potential

For nonzero, exponential potential as in Sec. II C, the Wheeler-DeWitt equation is most conveniently solved after a transformation to new variables in such a way that the potential cancels in front of  $\Psi$ . This is obtained by first transforming to light-cone type coordinates  $z_1 \equiv \alpha + \sqrt{\ell}\phi$ ,  $z_2 \equiv \alpha - \sqrt{\ell}\phi$ . For  $\ell = 1$ , these are just the characteristics of the Wheeler-DeWitt equation. The equation now takes the form

$$\hbar^2 \frac{\partial^2 \Psi}{\partial z_1 \partial z_2} + f(z_1, z_2) \Psi = 0, \quad (45)$$

from which a transformation to new variables can be made such that  $f(z_1, z_2)$  is canceled. This corresponds to a transformation to the variables

$$\begin{aligned}
 u_\ell(\alpha, \phi) &= \frac{\sqrt{2V_0}}{3} \frac{e^{3\alpha - (\lambda\sqrt{6}/2)\phi}}{1 - \ell\left(\frac{\lambda}{\sqrt{6}}\right)^2} \\
 &\quad \times \left( \cosh(X) + \frac{1}{\sqrt{\ell}} \frac{\lambda}{\sqrt{6}} \sinh(X) \right), \\
 v_\ell(\alpha, \phi) &= \frac{\sqrt{2V_0}}{3} \frac{e^{3\alpha - (\lambda\sqrt{6}/2)\phi}}{1 - \ell\left(\frac{\lambda}{\sqrt{6}}\right)^2} \\
 &\quad \times \left( \frac{1}{\sqrt{\ell}} \sinh(X) + \ell \frac{\lambda}{\sqrt{6}} \cosh(X) \right),
 \end{aligned}$$

where  $X \equiv \sqrt{\ell}(3\phi - \ell\frac{\lambda\sqrt{6}}{2}\alpha)$ . For both the phantom and the ordinary field,  $u_\ell$  and  $v_\ell$  are real. The Wheeler-DeWitt equation in these variables takes the simple form

$$\hbar^2 \left( \frac{\partial^2 \Psi}{\partial u_\ell^2} - \ell \frac{\partial^2 \Psi}{\partial v_\ell^2} \right) + \Psi = 0. \quad (46)$$

Making a WKB-approximation ansatz,  $\Psi = C e^{\pm iS}$ , one obtains at lowest order the Hamilton-Jacobi equation

$$\left( \frac{\partial S_0}{\partial u_\ell} \right)^2 - \ell \left( \frac{\partial S_0}{\partial v_\ell} \right)^2 = 1. \quad (47)$$

This is solved via a separation ansatz by  $S_{0k} = ku_\ell - \sqrt{\ell(k^2 - 1)}v_\ell$ . Of course, the Hamilton-Jacobi equation is also solved by actions carrying different signs in front of  $u_\ell$  and  $v_\ell$ . These are obtained from the one chosen above by rotations in the  $(u_\ell, v_\ell)$ -plane. For  $\ell = -1$ , all solutions can be mapped onto each other in this way. This is an obvious consequence of the rotational symmetry of Eq. (47) for  $\ell = -1$ . As  $u_1 > 0$  (recall that  $\lambda < \sqrt{6}$  for  $\ell = 1$ ) for the conventional scalar field, here only two solutions can be mapped onto each other.

From the classical action  $S_{0k}$ , the equations of motion are obtained via  $\frac{\partial S_{0k}}{\partial k} \big|_{k=\bar{k}} = c$ . (Note that  $S_{0k}$  evaluated at  $k = \bar{k}$  is always real.) For the special case  $c = 0$  and

$$\bar{k}^2 = 1/E_{\text{pot}} = \left( 1 - \frac{\ell\lambda^2}{6} \right)^{-1}$$

one obtains the classical trajectories

$$\phi(\alpha) = \ell \frac{\lambda}{\sqrt{6}} \alpha, \quad (48)$$

cf. (20).

Plugging this lowest-order ansatz into the Wheeler-DeWitt equation, one finds that the equation is already satisfied *exactly*. Thus we get the following exact wave packet solution to the Wheeler-DeWitt equation,

$$\begin{aligned}
 \Psi(u_\ell, v_\ell) &= \int dk A(k) \left( C_1 e^{(i/\hbar)(ku_\ell - \sqrt{\ell(k^2-1)}v_\ell)} \right. \\
 &\quad \left. + C_2 e^{-(i/\hbar)(ku_\ell - \sqrt{\ell(k^2-1)}v_\ell)} \right). \quad (49)
 \end{aligned}$$

By construction, the classical trajectories can be recovered from this equation through the principle of constructive interference. We choose for the amplitude a Gaussian with width  $\sigma$  centered around  $\bar{k}$ ,

$$A(k) = \frac{1}{(\sqrt{\pi}\sigma\hbar)^{1/2}} e^{-((k-\bar{k})^2/2\sigma^2\hbar^2)}.$$

Taking  $C_1 = C_2$  for definiteness, one obtains wave packets of the form

$$\begin{aligned}
 \psi(u_\ell, v_\ell) &\approx C_1 \pi^{1/4} \sqrt{\frac{2\sigma\hbar}{1 - i\sigma^2\hbar S''_0}} \\
 &\quad \times \exp\left( \frac{iS_0}{\hbar} - \frac{S_0^2}{2(\sigma^{-2} - i\hbar S''_0)} \right) + \text{c.c.}, \quad (50)
 \end{aligned}$$

where a Taylor expansion of  $S_{0k}$  has been carried out around  $\bar{k}$  (primes denoting derivatives with respect to  $k$ ) and the terms of the order  $(k - \bar{k})^3$  in the exponent have been neglected. (For simplicity, in this expression  $S_{0k}(\bar{k}) \equiv S_0$ .) This can be done if the Gaussian is strongly peaked around  $\bar{k}$ , that is, if  $\sigma$  is sufficiently small. Since  $S'_{0k}(\bar{k}) = 0$  gives the classical trajectory, the packet is peaked around it. For the conventional scalar field as well as for the phantom field, the wave packet thus follows the classical trajectory but *spreads* as  $v_\ell^2 \rightarrow \infty$ . This can be recognized from (50), since the term proportional to  $[S''_{0k}(\bar{k})]^2$  in the width of the Gaussian increases without limit,

$$S''_{0k}(\bar{k}) = \frac{v_\ell}{(\ell(\bar{k}^2 - 1))^{3/2}}. \quad (51)$$

It is even more obvious from the absolute square of the wave packet [neglecting for simplicity the complex conjugate part in (50)],

$$\begin{aligned}
 |\psi(u_\ell, v_\ell)|^2 &\approx |C_1|^2 \sqrt{\pi} \frac{2\sigma\hbar}{\sqrt{1 + \sigma^4\hbar^2(S''_0)^2}} \\
 &\quad \times \exp\left( -\frac{S_0^2}{\sigma^{-2} + \sigma^2\hbar^2(S''_0)^2} \right). \quad (52)
 \end{aligned}$$

The spreading occurs due to the nontrivial dispersion relation, that is, due to the fact that  $S_{0k}$  depends nonlinearly on  $k$ . The semiclassical approximation is thus not valid throughout configuration space.

For the phantom field we have  $u_{-1} \rightarrow -\infty$ ,  $v_{-1} \rightarrow \infty$  when we approach the big-rip singularity. This singularity thus lies in a genuine quantum region. Since for  $\ell = -1$  one has

$$v_\ell^2 \sim e^{6\alpha - \lambda\sqrt{6}\phi} \equiv e^{6\alpha} V(\phi),$$

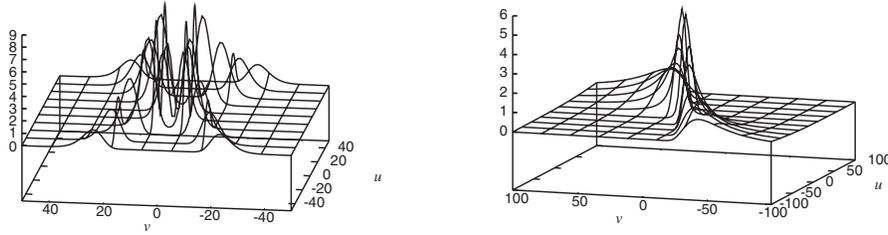


FIG. 3. The amplitude of the wave packet (50) for an exponential potential solution of the WDW equation for  $\ell = 1$  (left) and  $\ell = -1$  (right). Here,  $\hbar$  was set to unity and parameters  $\sigma = 0.1$  and  $\lambda = \kappa/2$  have been chosen. The wave packet for the phantom field model is seen to spread near the classical singularity. For the scalar field model one has  $\Psi \rightarrow 0$  at the origin. In each sector corresponding to one copy of the  $(\alpha, \phi)$  plane, the same wave packet propagates.

it is obvious that the occurrence of the nontrivial potential is responsible for the dispersion.

The big-rip singularity is thus “smoothed out”—when the wave packets disperse, we can no longer use an approximate time parameter; time and the classical evolution come to an end, and one is just left with a stationary quantum state. This corresponds to quantum gravity effects at very large scales. Hitherto such a case has only been encountered near the turning point of a classically recollapsing universe, as a consequence of the demand that the wave function go to zero for large scale factor [35,41].

Because of the fact that  $u_1 > 0$  for the conventional scalar field model, here two inequivalent actions exist. Apart from the wave packet constructed from the function  $S_{0k} = ku_1 - \sqrt{k^2 - 1}v_1$ , one gets a second wave packet constructed from  $S_{0k} = -ku_1 - \sqrt{k^2 - 1}v_1$ . Moreover, the entire  $(\alpha, \phi)$ -plane is mapped into only one quarter of the  $(u_1, v_1)$ -plane. One would therefore require the wave packet to vanish at the boundary of the physical region. The only solution satisfying this requirement is naturally the trivial one. To get a nontrivial solution, one has to lessen the boundary condition and require  $\Psi = 0$  only at the origin of the  $(u_1, v_1)$ -plane. The fact that the wave packet does not vanish at the  $u_1 = 0$  and  $v_1 = 0$  line is due to the non-normalizability of the wave packet in both  $\alpha$  and  $\phi$ , which in turn has its origin in the fact that the classical trajectory has no turning point.

The implementation of this condition results in a wave packet which vanishes at the big-bang singularity,  $\Psi \rightarrow 0$  as  $\alpha \rightarrow -\infty$ , and spreads for large  $\alpha$ . The big-bang singularity does therefore not exist in the quantum theory. In the phantom field model, no such restriction occurs due to the fact that the entire  $(u_{-1}, v_{-1})$ -plane represents the entire  $(\alpha, \phi)$ -plane. The wave packets for both the phantom and the ordinary scalar field are depicted in Fig. 3.

#### D. Quantum phantom cosmology—scalar field fluid and negative cosmological constant

For the model discussed in Sec. IID, the classical solutions require a potential of the form  $V(\phi) = V_0 \cosh^2(\phi/F)$ , cf. (29). The Wheeler-DeWitt equation therefore reads

$$\frac{\hbar^2}{2} \left( \frac{\partial^2}{\partial \alpha^2} - \ell \frac{\partial^2}{\partial \phi^2} \right) \Psi(\alpha, \phi) + e^{6\alpha} \left( V_0 \cosh^2 \left( \frac{\phi}{F} \right) + \frac{\Lambda}{6} \right) \Psi(\alpha, \phi) = 0. \quad (53)$$

The classical singularities lie in a region of large  $|\phi|$ . In order to study the quantum behavior there, it is thus sufficient to approximate the potential for large  $|\phi|$ ,

$$\tilde{V}(\phi) \approx \frac{V_0}{4} e^{\pm 2\phi/F}, \quad (54)$$

where in the following the upper sign refers to positive  $\phi$ , and the lower sign to negative  $\phi$ . This makes the problem very similar to the one of Sec. III C. The Wheeler-DeWitt equation is here simplified by a transformation on the variables

$$u_\ell(\alpha, \phi) \equiv \frac{\sqrt{V_0}}{3\sqrt{2}} \frac{e^{3\alpha \pm (\phi/F)}}{1 - \frac{\ell}{9F^2}} \left( \cosh(X) \mp \frac{1}{3F\sqrt{\ell}} \sinh(X) \right),$$

$$v_\ell(\alpha, \phi) \equiv \frac{\sqrt{V_0}}{3\sqrt{2}} \frac{e^{3\alpha \mp (\phi/F)}}{1 - \frac{\ell}{9F^2}} \left( \frac{1}{\sqrt{\ell}} \sinh(X) \pm \frac{1}{3F} \cosh(X) \right),$$

where  $X \equiv \sqrt{\ell}(3\phi \pm \ell\alpha/F)$ . In these variables, we recover the form

$$\hbar^2 \left( \frac{\partial^2 \Psi}{\partial u_\ell^2} - \ell \frac{\partial^2 \Psi}{\partial v_\ell^2} \right) + \Psi = 0. \quad (55)$$

Again, one obtains a solution from a WKB ansatz. The Hamilton-Jacobi equation is again given by (47) (this equivalence is, of course, only formal, since  $u_\ell$  and  $v_\ell$  are defined differently). It is again solved by  $S_{0k} = ku_\ell - \sqrt{\ell(k^2 - 1)}v_\ell$ , where the remarks of Sec. III C concerning the choice of action apply here as well. The equations of motion obtained for  $\frac{\partial S_{0k}}{\partial k} \Big|_{k=\bar{k}} = 0$  are

$$\phi(\alpha) = \mp \frac{\ell}{\sqrt{3}} \sqrt{\frac{D}{\ell}} \alpha + C_{\bar{k}, \ell}. \quad (56)$$

This solution coincides approximately with the classical solutions (28): If one approximates (28) for  $\ell = -1$  for large  $a$ , one gets ( $\pm$  label the different branches of the

classical solution)

$$\phi_{\pm}(\alpha) = \pm \frac{1}{\sqrt{3}} \sqrt{\frac{D}{\ell}} \alpha \pm F \ln(2A), \quad (57)$$

where  $\alpha \geq 0$ . Therefore, the limit of large positive  $\phi$  is obtained on the  $\phi_+$ -branch, while the limit for large negative  $\phi$  is reached on the  $\phi_-$ -branch. On the other hand, an approximation for *small*  $a$  in the case  $\ell = 1$  yields

$$\phi_{\pm}(\alpha) = \pm \frac{1}{\sqrt{3}} \sqrt{\frac{D}{\ell}} \alpha \mp F \ln(2A), \quad (58)$$

where  $\alpha \leq 0$ . Because of this, the limit of large positive  $\phi$  is obtained on the  $\phi_-$ -branch, and for large negative  $\phi$  on the  $\phi_+$ -branch. Thus the solution to the approximated Hamilton-Jacobi equation (47) coincides with the approximation of Eq. (28). Of course, a special choice for  $k$  has to be made to fix the onset. The fact that for  $\ell = -1$  large  $\phi$  correspond to large  $a$ , and for  $\ell = 1$  large  $\phi$  correspond to small  $a$  is due to phantom-scalar field duality.

With the help of the classical action  $S_{0k}$ , the approximate Wheeler-DeWitt equation can be solved. Again, the WKB ansatz satisfies the equation exactly. The wave packet is of the same form as in Sec. III C, with a different definition of  $u_{\ell}$  and  $v_{\ell}$  and another choice of the center of the Gaussian,  $\bar{k}$ . As in the case of vanishing cosmological constant, the wave packet spreads for  $v_{\ell}^2 \rightarrow \infty$ . The big-rip singularity in these variables occurs at  $v_{-1}^2 \rightarrow \infty$ ,  $u_{-1} \rightarrow \infty$ . Thus, again, the singularity is hidden in a quantum regime and the semiclassical approximation is not valid throughout configuration space. Because of the restriction  $D < 3$  for the  $\ell = 1$  model, the same remarks concerning the range of the coordinates as in Sec. III C apply here. So at the big bang,  $\Psi \rightarrow 0$ . In analogy to [41] one would expect quantum effects to occur also in the region of the classical turning point. This will be addressed in a future publication.

#### IV. DISCUSSION AND OUTLOOK

In our paper we have applied the formalism of standard quantum cosmology (using the Wheeler-DeWitt equation) to a situation where phantom fields are present. This is of interest because there are novel features with regard to both the structure of the equation (elliptic or ultrahyperbolic instead of hyperbolic) as well as the presence of new scenarios (big-rip singularity at large scale factors in the classical model). In fact, one of the most intriguing features is the possible *occurrence of quantum effects for large scale factors*.

For various models we have determined and discussed the classical trajectories in configuration space. We have then considered the corresponding Wheeler-DeWitt equations; we have given various solutions and addressed the classical limit as well as the behavior of wave packets following the classical trajectories in configuration space. We have found that the packets disperse in the region of the

classical big-rip singularity. This singularity is thus “smeared out” by quantum effects at large scale factor. Once the wave packets disperse, no approximate time parameter can be defined [30] and the classical evolution terminates in a singularity-free way.

For the conventional scalar field model we have found that the wave packet vanishes at the big-bang singularity due to the implementation of appropriate boundary conditions. In this way, the big-bang singularity is removed from the quantum theory. This is similar to the avoidance of the singularity in models of loop quantum cosmology [31] and shell collapse [59]. Without this boundary condition the wave packet would just have approached the region  $\alpha \rightarrow -\infty$  without spreading; this lack of dispersion is a result of the Wheeler-DeWitt equation taking the form of a free wave equation in this limit.

The present work can be extended in various directions. The next step would be to add other “conventional” scalar fields and to investigate the full quantum dynamics. In particular, this would be of importance for a discussion of the arrow of time [32]. In order to define an appropriate entropy it is necessary to introduce a set of inhomogeneous degrees of freedom. In the case of a classically recollapsing universe it has been found that the arrow of time is correlated with the scale factor of the universe, that is, the arrow of time must formally reverse at the maximal expansion [32,41]. (The reversal is formal because quantum effects near the classical turning point do not allow classical observers to survive this region.) It is of interest to investigate whether a similar behavior occurs here. Regarding the classical evolution of the phantom fields depicted on the left-hand sides of the Figs. 1 and 2, one would again expect a correlation of the entropy with increasing scale factor, that is, there would be no collapse followed by an expansion but only two separate branches of expansion separated by a quantum region. The classical evolution would then start out of the quantum phase at small scales and describe an expansion of the universe ended by a quantum phase near the “big rip region.” We shall address this scenario in a forthcoming paper. This will also deal with a possible quantum phase near weak singularities (sudden future singularity, generalized sudden future singularity, type III and type IV) as mentioned in the Introduction.

We have based our discussion on the Wheeler-DeWitt equation of quantum geometrodynamics. More recently, an alternative formulation of canonical quantum gravity called loop quantum gravity has gained considerable attention, cf. [30]. A major prediction of this approach is the presence of a discrete structure for geometric operators. This formalism was applied to cosmology where it led to new features [31]. Instead of the usual Wheeler-DeWitt equation one gets a difference equation for the scale factor. While near the Planck scale this equation gives different results from the (differential) Wheeler-DeWitt equation

(and therefore can prevent the occurrence of the classical singularities), it coincides with it for higher values of  $a$ . It thus seems that near the classical big-rip singularity the same scenario emerges that has been discussed in the present paper. However, it is of interest to investigate quantitatively the differences and similarities of ordinary quantum phantom cosmology and loop quantum phantom cosmology. A first paper in this direction has studied the *effective* dynamics from loop quantum cosmology and its consequences [60]. The results indicate that the big rip can be avoided. We hope to return to these and related issues in a future publication.

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