Two-dimensional quantum black holes, branes in Banados-Teitelboim-Zanelli spacetime, and holography

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We solve semiclassical Einstein equations in two dimensions with a massive source and we find a static, thermodynamically stable, quantum black hole solution in the Hartle-Hawking vacuum state. We then study the black hole geometry generated by a boundary mass sitting on a nonzero tension 1-brane embedded in a three-dimensional Banados-Teitelboim-Zanelli (BTZ) black hole. We show that the two geometries coincide and we extract, using holographic relations, information about the conformal field theory (CFT) living on the 1-brane. Finally, we show that the quantum black hole has the same temperature of the bulk BTZ, as expected from the holographic principle.

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I. INTRODUCTION

In the framework of the proposed duality between gravity on $(d+1)$-dimensional anti-de Sitter (AdS_{d+1}) spaces and d-dimensional conformal field theory (CFT_d) living on the AdS_{d+1} boundary, first proposed by Maldacena [1], Witten [2] suggested a duality between Schwarzschild-AdS black holes (SAdS) and a conformal field theory (CFT) at high temperature (TCFT) on the SAdS boundary. This idea can be naively understood thinking that very massive black holes, although stable, emit a black body radiation [3]. However, as the black body spectrum does not carry information, the Hawking mechanism is usually associated to a nonunitary process [4]. A TCFT is a unitary theory therefore SAdS black holes cannot be fully dual to a TCFT. Indeed this is the case [5]. We can easily understand why by considering the three-dimensional Banados-Teitelboim-Zanelli (BTZ) black hole [6].

The metric for the BTZ black hole is

$$ds^2 = -F(r)dt^2 + \frac{dr^2}{F(r)} + r^2 d\theta^2, \tag{1}$$

where

$$F(r) = \frac{r^2}{L^2} - m, \qquad \theta \equiv \theta + 2\pi, \qquad 0 \le r < \infty. \tag{2}$$

The horizon of this black hole is in $r_h = \sqrt{m}L$, m is its mass, and L the AdS length. This spacetime is a solution of Einstein equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{1}{L^2}g_{\mu\nu}.$$

For a scalar field Φ propagating in this background the action is given by

$$\mathcal{A}(\Phi) = \int_{\rm BTZ}\sqrt{-g}|d\Phi|^2 + \int_{\partial\rm BTZ}\sqrt{-h}\Phi_0 d\Phi|_{\partial\rm BTZ},$$

where the first integral represents the bulk action and the second the boundary action, h and Φ_0 are the induced metric and the value of the scalar field on the BTZ boundary (here denoted by ∂BTZ), respectively. The AdS/CFT correspondence relates the above boundary action to the partition function of a scalar operator in the dual conformal field theory, in this case a TCFT, in the following formal way

$$\left\langle \exp\int_{\partial\rm BTZ}\Phi_0\mathcal{O}\right\rangle_{\rm TCFT} = \mathcal{A}_b(\Phi_0).$$

Φ_0 now represents the source of a scalar operator $\mathcal{O}$ of conformal dimension $\Delta = (1+\sqrt{1+\mu^2L^2})/2$, where μ are the Kaluza-Klein masses of the solutions of $\Box_{\partial\rm BTZ}\Phi_0 = -\mu^2\Phi_0$ [7]. With this prescription, one can calculate correlation functions of the scalar operator $\mathcal{O}$ in the usual way. For the two-point correlation function we have

$$\langle\mathcal{O}(x^a)\mathcal{O}(x_0^a)\rangle_{\rm TCFT} = \frac{\delta^2}{\delta\Phi_0(x^a)\delta\Phi_0(x_0^b)}\left\langle \exp\int_{\partial\rm BTZ}\Phi_0\mathcal{O}\right\rangle_{\rm TCFT}\Bigg|_{\Phi_0=0} = \frac{\delta^2}{\delta\Phi_0(x^a)\delta\Phi_0(x_0^b)}\mathcal{A}_b(\Phi_0)\Bigg|_{\Phi_0=0} = G(x^a, x_0^a), \tag{3}$$

where x^a, x_0^a are the boundary coordinates and $G(x^a, x_0^a)$ is called the bulk to boundary correlator. At very large time (3) is given by [7]

$$G(x^a, x_0^a) \sim e^{-2\sqrt{m}\Delta(t+t_0)/L}. \tag{4}$$

We note that the correlator (4) is exponentially decaying and this is a signal that information has been lost for this background. Indeed, for a conformal field theory at finite temperature, we expect that its two-point correlation functions oscillate in a quasiperiodic manner with the quasi-

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periodicity dictated by the Poincarè recurrence [5,8,9]. To solve this puzzle, Maldacena in [10] and then Hawking in [11] suggested that the correct bulk to boundary operator describing the boundary theory should be of the form

$$G = \sum_{\mathcal{M}_i} G_i(\mathcal{M}_i), \tag{5}$$

where the sum is over all the possible topologies $\mathcal{M}_i$ satisfying the same boundary conditions. This is very reminiscent of the path integral sum over histories. Taking account of only the BTZ background is similar to coarse graining the phase space in the Feynman path integral. In this way the apparent information loss as seen from the black hole perspective is very similar to the information loss in the collapse of wave functions in ordinal quantum mechanics.

However bulk black hole solutions might be very useful to understand the behavior of semiclassical black holes. In fact if we consider a conformal field theory at finite temperature with a UV cutoff (equivalent to a coarse grained process), this will induce a classical gravitational field and will not necessarily need to be unitary. Obviously this approximation will break down at the quantum gravity regime where we should restore the unitarity. In order to obtain this theory as a boundary of some asymptotically AdS spacetime we need to truncate the boundary from spatial infinity to a finite point, as suggested by [12,13] (see [14] for the zero temperature case) to holographically explain the no-go theorem of [15]. The d-dimensional gravitational effective theory on the boundary will be governed by the semiclassical Einstein equations

$$R_{ab} - \tfrac{1}{2}Rg_{ab} + \Lambda g_{ab} = k^2\langle T_{ab}\rangle. \tag{6}$$

R_{ab} is the Ricci tensor projected onto the brane, $k^{-2} = M_*^{d-2}$ with M_* being the fundamental mass scale (for $d = 2$ $k^2 = 1$), and $\langle T_{ab}\rangle$ is the expectation value of the energy momentum tensor of the conformal field theory with a UV cutoff σ related with the boundary tension, i.e. the trace of the boundary extrinsic curvature $K^\alpha{}_\alpha = K$. The duality has been checked for the case of three-dimensional brane black holes embedded in a four-dimensional bulk spacetime [12] and evidences were given in the case of a four-dimensional black hole formation in a five-dimensional embedding [16].

If the bulk is a three-dimensional space, as in the BTZ case, the boundary is a two-dimensional surface and (6) reduces to [17,18]

$$\Lambda g_{ab} = \langle T_{ab}\rangle. \tag{7}$$

This case is very interesting as the dynamics of the gravitational field is of purely quantum origin. Note also that the presence of a cosmological constant is necessary because of the trace anomaly $\langle T^a{}_a\rangle \neq 0$.

In this paper we will find a solution to (7) in the presence of a boundary mass. We will find a static, thermodynamically stable quantum black hole solution in thermal equilibrium in the Hartle-Hawking state. We will then consider a three-dimensional braneworld model with a BTZ bulk and we will show how our quantum solution can also be obtained by slicing this three-dimensional spacetime with a nonzero tension, asymptotically AdS, 1-brane. In this way we prove that the conjectured duality between classical bulk black holes and quantum brane black holes of [12,13] applies to our case and we extract, using holographic relations, information about the CFT living on the 1-brane.

II. QUANTUM BLACK HOLE IN TWO DIMENSIONS

We start by considering the two-dimensional action

$$I = \frac{1}{2}\int d^2x(R - 2\Lambda)\sqrt{-h} + \int d^2x\sqrt{-h}\mathcal{L}_{\rm CFT},$$

where $\mathcal{L}_{\rm CFT}$ is the Lagrangian of a conformal field theory. At this action we add a Gibbons-Hawking term [19]

$$I_b = -\int dx^a b_a(K + \mathcal{L}_b), \tag{8}$$

where b^a is the normal to the boundary, $\mathcal{L}_b$ is a boundary Lagrangian, and K_{ab} is the extrinsic curvature of the boundary. Since the boundary (8) is unidimensional the only possible boundary Lagrangians are either of a point particle or a world sheet of mass μ for a timelike or spacelike worldline. As we show in the appendix, the variation of the boundary action (8) is trivial. We therefore have the choice of setting the boundary action to vanish on the semiclassical solution. In this way the boundary term will be irrelevant in the semiclassical calculations and therefore the techniques used in [18] straightforwardly apply to our case.

In the semiclassical approximation this theory is described by the set of equations

$$\Lambda g_{ab} = \langle T_{ab}\rangle, \tag{9a}$$

$$K = \mu, \tag{9b}$$

where we consider negligible the quantum correction to the boundary Lagrangian.

The trace anomaly of the conformal field theory can be determined by the only knowledge of the background geometry and it is [20]

$$\langle T^a{}_a\rangle = -\frac{\hbar\gamma}{24\pi}R. \tag{10}$$

γ is proportional to the number of fields in the theory where matter fields are counted with opposite signs with respect to the graviton contribution.

Using the gauge freedom in fixing the coordinates we can write the spacetime metric as

$$ds^2 = -\Omega^2(u, v)dudv. \tag{11}$$

Conservation equations $\nabla^b\langle T^a{}_b\rangle = 0$, Eqs. (9a) and (10) give the following equations

$$-\tfrac{1}{2}\Lambda\Omega^2 = \langle T_{uv}\rangle, \tag{12a}$$

$$0 = \langle T_{uu}\rangle = \langle T_{vv}\rangle, \tag{12b}$$

$$\langle T_{uu}\rangle = -\frac{\gamma}{12\pi}\Omega\partial_u^2\Omega^{-1} + \tilde{U}(u), \tag{12c}$$

$$\langle T_{vv}\rangle = -\frac{\gamma}{12\pi}\Omega\partial_v^2\Omega^{-1} + \tilde{V}(v), \tag{12d}$$

$$\hbar^{-1}\langle T^a{}_a\rangle = -\frac{\gamma}{24\pi}R, \tag{12e}$$

where $\tilde{U}$ and $\tilde{V}$ set the vacuum state in which Eq. (7) is solved.

We now set $\tilde{U}$ and $\tilde{V}$ constant and proportional to the number of fields γ by writing

$$\tilde{U} = \tilde{V} = \frac{q}{48\pi}\gamma, \tag{13}$$

where q is a constant. As we will see later this choice corresponds to setting the vacuum to be the Hartle-Hawking state.

Equations (12) can now be solved for Ω and give

$$\Omega^2 = \frac{4q}{\lambda^2}\frac{e^{(v-u)\sqrt{q}}}{(1+e^{(v-u)\sqrt{q}})^2}, \tag{14}$$

where

$$\lambda^2 = \frac{48\pi\Lambda}{\hbar\gamma}. \tag{15}$$

In order to understand the physical meaning of the two constants q and λ we rewrite our metric in the Schwarzschild gauge

$$ds^2 = -f(x)dt^2 + \frac{dx^2}{f(x)} \tag{16}$$

by setting

$$q = \lambda^2 N + M^2, \qquad t = \frac{v+u}{2}$$

and

$$x_+ = -\frac{\sqrt{q}}{\lambda^2}\frac{(1-e^{(v-u)\sqrt{q}})}{(1+e^{(v-u)\sqrt{q}})} + \frac{M}{\lambda^2}, \tag{17a}$$

$$f_+(x_+) = \lambda^2 x_+^2 + 2Mx_+ - N \tag{17b}$$

with

$$M < x_+\lambda^2 < \sqrt{q} + M. \tag{18}$$

Alternatively, we can use, in place of x_+, the following coordinate

$$x_- = -\frac{\sqrt{q}}{\lambda^2}\frac{(1-e^{(v-u)\sqrt{q}})}{(1+e^{(v-u)\sqrt{q}})} - \frac{M}{\lambda^2}, \tag{19a}$$

$$f_-(x_-) = \lambda^2 x_-^2 - 2Mx_- - N \tag{19b}$$

with

$$-M < x_-\lambda^2 < \sqrt{q} - M. \tag{20}$$

The interval (20) can also be restricted to

$$-M < x\lambda^2 < 0. \tag{21}$$

We can analytically extend the ranges of values for x_+ in (18) and x_- in (21) to $0 \le x_+ < \infty$ and $-\infty < x_- \le 0$, respectively. We now implement the boundary condition (9b) and we set the boundary at $x = 0$. The manifold satisfying the boundary conditions (9b) can be constructed by matching the two patches x_+ and x_- in $x_+ = x_- = 0$ and defining

$$f(x) = \lambda^2 x^2 + 2M|x| - N, \tag{22}$$

where $-\infty < x < \infty$ and $\mu = M/\sqrt{N}$ and so, for a positive mass μ, it follows that M and N must be both non-negative. The black hole horizon is at

$$|x_h| = \frac{-M + \sqrt{q}}{\lambda^2} \tag{23}$$

and it is purely quantum. In fact, using Eqs. (15) and (23) it becomes

$$|x_h| = \hbar\frac{\gamma}{48\pi\Lambda}(-M + \sqrt{q}), \tag{24}$$

and $\lim_{\hbar\to 0} x_h = 0$.

Rescaling the coordinates, by defining $\tilde{t} = \sqrt{N}t$ and $\tilde{x} = x/\sqrt{N}$, we can rewrite our metric as

$$ds^2 = -\tilde{f}(\tilde{x})d\tilde{t}^2 + \frac{d\tilde{x}^2}{\tilde{f}(\tilde{x})}$$

with

$$\tilde{f}(\tilde{x}) = \lambda^2\tilde{x}^2 + 2\mu|\tilde{x}| - 1.$$

It is now clear that the metric depends only on the two physical quantities λ^2 and μ and it is asymptotically AdS with AdS length $l = 1/\lambda$. The black hole mass, the spacetime mass when the AdS contribution is subtracted, is therefore only determined by the boundary mass and it is $E = \mu$ [21].

The black hole metric (16) with (22) was first found in [22] (where, due to different boundary conditions, the boundary mass is $\mu = M$) as a solution of a different two-dimensional gravity theory [23] and also here the black hole mass is proportional to the mass on the boundary [24].

A. Conformal properties

In this section we explore the conformal properties of our solution. Although the $(\tilde{x}, \tilde{t})$ coordinates seem more natural, we will continue to use (x, t) coordinates because, as we will see later, these are the ones in which the correspondence with the BTZ brane black hole is clearly manifest.

Being $x = 0$ a physical boundary we can just consider the patch $0 < x < \infty$. The black hole horizon is defined in (23). We introduce a tortoise coordinate r_* as

$$r_* = \int \frac{dx}{\lambda^2 x^2 + 2Mx - N} = \frac{1}{2\sqrt{q}} \ln\left(\left| \frac{\sqrt{q} - \lambda^2 x - M}{\sqrt{q} + \lambda^2 x + M} \right| \right), \quad (25)$$

and the horizon is now moved to $r_* \to \infty$. Introducing null coordinates u and v, such that

$$u = t - r_*, \quad (26)$$

$$v = t + r_*, \quad (27)$$

the conformal factor becomes the one in Eq. (14). Note that the null coordinates u and v cover the full spacetime $(-\infty < u,\ v < \infty)$ and at the horizon $u \to \infty$ and $v \to -\infty$. These coordinates do not anyway represent a continuous and complete set of coordinates across the horizon. In order to have a global set of coordinates we introduce

$$U = -\frac{1}{\sqrt{q}} e^{-\sqrt{q}u}, \quad (28)$$

$$V = \frac{1}{\sqrt{q}} e^{\sqrt{q}v}, \quad (29)$$

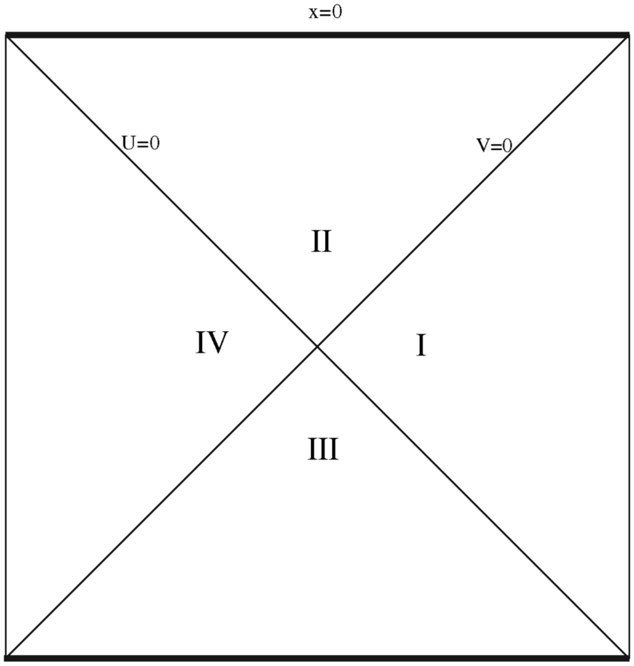

FIG. 1. Conformal diagram of the maximally extended black hole solution of metric (16) with (22). The lines in bold represent the boundary $x = 0$.

and the metric becomes

$$ds^2 = -\frac{4q}{\lambda^2} \frac{1}{1 - qUV} dU dV. \quad (30)$$

In this coordinate system the horizon is at $U \to 0$ and $V \to 0$ and $-\infty < U < 0$ and $0 < V < \infty$. The spacetime can be now analytically extended to the whole plane $-\infty < U$, $V < \infty$. We can also define the new Cartesian coordinates

$$T = \frac{U + V}{2}, \quad (31)$$

$$R = \frac{V - U}{2}. \quad (32)$$

The Penrose diagram of this maximally extended spacetime is shown in Fig. 1.

B. Temperature

In this section we show that at our black hole is associated with a physical quantum temperature due to the presence of a boundary in $x = 0$. Similar features are studied in [25] for the dilatonic case.

We consider the quantization of a massless scalar field ϕ in our two-dimensional spacetime, in this section we will use units such that $\hbar = 1$. The wave equation

$$\Box \phi = 0$$

has solutions, with respect to the extended coordinates defined in (28) and (29), given by the orthonormal modes

$$\Phi_k = \frac{1}{\sqrt{4\pi\omega}} e^{i(kX - \omega T)},$$

where $\omega = |k| > 0$ and $-\infty < k < \infty$. These modes are positive frequency with respect to the timelike Killing vector ∂_T, in fact they satisfy

$$\mathcal{L}_{\partial_T} \Phi_k = -i\omega \Phi_k.$$

The modes with $k > 0$ consist of right-moving waves $(4\pi\omega)^{-(1/2)} e^{-i\omega U}$ along the rays $U =$ constant, and they are analytic functions of U and bounded in the upper-half V-plane. The modes with $k < 0$ consist of left-moving waves $(4\pi\omega)^{-(1/2)} e^{-i\omega V}$ along the rays $V =$ constant, and are analytic functions of U and bounded in the lower-half U-plane.

The general solution of the wave equation may be expanded as

$$\phi = \sum_{k=-\infty}^{\infty} (a_k \Phi_k + \hat{a}_k^\dagger \Phi_k^*). \quad (33)$$

Upon quantization, a_k and $\hat{a}_k^\dagger$ become annihilation and creation operators and the vacuum state for the inertial observer is defined, as usual, by

$$a_k |0_A\rangle = 0. \quad (34)$$

We can now also adopt an alternative quantization prescription based on modes defined using the null coordinates of Eqs. (26) and (27). The wave equation is conformally invariant and we have mode solutions given by

$$\phi_k = \frac{1}{\sqrt{4\pi\omega}} e^{i(kx \pm \omega t)}, \tag{35}$$

where $\omega = |k| > 0$ and $-\infty < k < \infty$. The upper sign in (35) applies in region IV of the Penrose diagram in Fig. 1 and the lower sign in region I. The presence of this sign change can be regarded as due to the fact that a right-moving wave in region I moves towards increasing values of x, while in region IV it moves towards decreasing values of x, or simply due to the time reversal we have in region IV. These modes are positive frequency modes with respect to the timelike Killing vector ∂_t in region I and $-\partial_t$ in region IV, satisfying

$$\mathcal{L}_{\pm\partial_t}\phi_k = -i\omega\phi_k,$$

in region I and IV, respectively.

We can now define

$$\phi_k^{(1)} = \begin{cases} \frac{1}{\sqrt{4\pi\omega}} e^{i(kx-\omega t)}, & \text{in region I;} \\ 0, & \text{in region IV,} \end{cases} \tag{36}$$

and

$$\phi_k^{(4)} = \begin{cases} 0, & \text{in region I;} \\ \frac{1}{\sqrt{4\pi\omega}} e^{i(kx+\omega t)}, & \text{in region IV.} \end{cases} \tag{37}$$

The set in Eq. (36) is complete in region I while the set in (37) is complete in region IV, but neither set separately is complete on all our spacetime. However both sets together are complete, and lines $t =$ constant taken across both region I and IV are Cauchy surfaces for the whole spacetime. Therefore, these modes can also be analytically continued into the regions II and III and so can be used as a base for quantizing the field ϕ that can be then expanded as

$$\phi = \sum_{k=-\infty}^{\infty} (b_k^{(1)}\phi_k^{(1)} + b_k^{(1)\dagger}\phi_k^{(1)*} + b_k^{(2)}\phi_k^{(4)} + b_k^{(2)\dagger}\phi_k^{(4)*}), \tag{38}$$

and the vacuum state can be now defined as the one satisfying

$$b_k^{(1)}|0_B\rangle = b_k^{(2)}|0_B\rangle = 0. \tag{39}$$

This vacuum state is obviously not equivalent to the one defined in (34) as we can easily see by analyzing the different modes. To derive the Bogolubov transformations relating the operators $b_k^{(1)}$ and $b_k^{(2)}$ to the operators a_k of the inertial observer we will follow an argument due to Unruh [26].

Note that the solution (36) with support in region I can be extended to region II and the solution (37) with support in region IV can be extended to region III and we can define the new following modes

$$\psi_k^{(1)} = \phi_k^{(1)} + e^{-\pi\omega/\sqrt{q}}\phi_{-k}^{(4)*}, \tag{40}$$

$$\psi_k^{(4)} = \phi_{-k}^{(1)*} + e^{\pi\omega/\sqrt{q}}\phi_k^{(4)}, \tag{41}$$

that are all defined in the entire spacetime (all four regions) and that represent a set of positive-energy solutions of the wave equation. Therefore, an inertial observer may expand a general solution as

$$\phi = \sum_{k=-\infty}^{\infty} (C_k^{(1)}\psi_k^{(1)} + C_k^{(1)\dagger}\psi_k^{(1)*} + C_k^{(2)}\psi_k^{(4)} + C_k^{(2)\dagger}\psi_k^{(4)*}), \tag{42}$$

and the vacuum state can be now defined as the one satisfying

$$C_k^{(1)}|0_A\rangle = C_k^{(2)}|0_A\rangle = 0. \tag{43}$$

We can now easily relate these modes to the $b_k^{(1)}$ and $b_k^{(2)}$ by using Eqs. (40)–(42) and we obtain

$$b_k^{(1)} = C_k^{(1)} + e^{-\pi\omega/\sqrt{q}}C_k^{(2)\dagger}, \tag{44}$$

$$b_k^{(2)} = C_k^{(2)} + e^{-\pi\omega/\sqrt{q}}C_k^{(1)\dagger}. \tag{45}$$

The C-modes are not properly normalized. From the commutation relations for the b-modes

$$[b_k^{(r)}, b_{k'}^{(s)\dagger}] = \delta^{rs}\delta_{kk'}$$

we deduce

$$[C_k^{(r)}, C_{k'}^{(s)\dagger}] = \frac{e^{\pi\omega/\sqrt{q}}}{2\sinh(\pi\omega/\sqrt{q})}\delta^{rs}\delta_{kk'}$$

and so we define the normalized creation and annihilation operators by

$$c_k^{(r)} = e^{-\pi\omega/\sqrt{q}}\sqrt{2\sinh(\pi\omega/\sqrt{q})}C_k^{(r)}$$

so that

$$[c_k^{(r)}, c_{k'}^{(s)\dagger}] = \delta^{rs}\delta_{kk'}.$$

The $b_k^{(r)}$ operators can now be written in terms of the $c_k^{(r)}$ as follows

$$b_k^{(1)} = \frac{1}{\sqrt{2\sinh(\pi\omega/\sqrt{q})}}\{e^{\pi\omega/2\sqrt{q}}c_k^{(1)} + e^{-\pi\omega/2\sqrt{q}}c_k^{(2)\dagger}\}, \tag{46}$$

$$b_k^{(2)} = \frac{1}{\sqrt{2\sinh(\pi\omega/\sqrt{q})}}\{e^{\pi\omega/2\sqrt{q}}c_k^{(2)} + e^{-\pi\omega/2\sqrt{q}}c_k^{(1)\dagger}\}, \tag{47}$$

and these are the Bogolubov transformation relating the states $|0_A\rangle$ and $|0_B\rangle$.

Now suppose the system is the state $|0_A\rangle$, the number operator for the observer associated to $|0_B\rangle$, is simply given by

$$N(k) = b_k^{(1)\dagger}b_k^{(1)}$$

since $b_k^{(2)\dagger}$ excites modes which vanish in region I and are therefore nonaccessible to the observer whose trajectory is in region I. Using the Bogolubov transformation (46) and (47) and the definition (43) of the vacuum state $|0_A\rangle$, we obtain the expectation value of the number operator

$$\langle 0_A|N(k)|0_A\rangle = \frac{e^{-\pi\omega/\sqrt{q}}}{2\sinh(\pi\omega/\sqrt{q})} = \frac{1}{e^{2\pi\omega/\sqrt{q}} - 1},$$

and this is precisely the Planck spectrum for radiation at temperature, replacing $\hbar$,

$$T = \hbar\frac{\sqrt{q}}{2\pi k_B}, \tag{48}$$

where k_B is the Boltzman constant. Note that we get the same result for the temperature by considering the Wick rotation in imaginary time as it was done in [22] for the black hole of the two-dimensional Jakiw Teitelboim (JT) gravity theory [23].

The heat capacity of our black hole is given by

$$C = \frac{d\mu}{dT} = \frac{4\pi^2k_B^2}{\mu\hbar N}T. \tag{49}$$

We can see that the black hole has a positive heat capacity and therefore it can reach thermal equilibrium with the thermal bath due to the Hawking radiation.

From Eqs. (12) and (13) we have that the vacuum expectation value of the normal ordered stress tensor operator is simply given by

$$\langle : T_{uu} :\rangle = \tilde{U} = \frac{\gamma}{48\pi}q, \qquad \langle : T_{vv} :\rangle = \tilde{V} = \frac{\gamma}{48\pi}q$$

after transforming to extended coordinates U and V via the Schwarzian derivative we get [27]

$$\langle : T_{UU} :\rangle = \langle : T_{VV} :\rangle = 0$$

and so, as we stated before, our semiclassical equation is actually solved in the Hartle-Hawking vacuum state [28].

III. A BRANE IN BTZ

As we previously discussed, the two-dimensional black hole described above is purely quantum, by means that the presence of the horizon is due only to quantum mechanical effects. The holographic conjecture of [12,13] implies that boundaries of some asymptotically AdS spaces should correspond to our semiclassical solution. The only known (asymptotically AdS) black hole in three dimensions is the BTZ one [6]. We then expect our solution to be a slice of a BTZ black hole.

A boundary solution with nonzero vacuum energy (equivalent to a UV cutoff on the brane) is equivalent to a braneworld solution [29]. A braneworld is a slice (brane) of a given bulk once a Z_2 symmetry with respect to the brane is introduced. In our case the system is governed by the following action

$$\mathcal{A}_g = \frac{1}{2k_3^2}\int d^3x\left(R + \frac{2}{L^2}\right)\sqrt{-g} - \frac{2\sigma}{k_3^2}\int_\Sigma d^2x\sqrt{-h}, \tag{50}$$

where n^α is the normal to the brane Σ and L is the AdS_3 length; σ and $h_{\alpha\beta}$ represent the vacuum energy of and the induced metric on the brane and k_3^2 is the inverse of the three-dimensional Planck mass.

The vanishing of the variation of the action (50) implies the Einstein equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{1}{L^2}g_{\mu\nu},$$

with the boundary condition [30]

$$K_{\alpha\beta} = \sigma h_{\alpha\beta}, \tag{51}$$

where K is the extrinsic curvature of Σ.

A. 1-brane

We want to introduce a static 1-brane in the 3D BTZ black hole spacetime. In order to do that we consider the surface

$$\Sigma:\ \theta - \Psi(r) = 0 \tag{52}$$

whose normal is given by

$$n_\alpha = \pm A(0, -\Psi', 1), \tag{53}$$

where $' = \partial_r$ and the normalization factor $A = r(\Psi' r^2 F(r) + 1)^{-1/2}$. The $\pm$ sign is related to the orientation of Σ as we shall see later.

Equation (51) is solved, in the BTZ background, by the function $\Psi(r)$

$$\Psi_\pm(r) = \pm\frac{\ln\left(\frac{2\sigma^2L^4m + 2\sigma L^2\sqrt{m}\sqrt{r^2(1-\sigma^2L^2)+\sigma^2L^4m}}{Lr}\right)}{\sqrt{m}}. \tag{54}$$

As we can see there are two different branches of the solution. We will call these two branches the + and − branch and we will indicate them with Ψ_+ and Ψ_-, respectively. The periodicity condition of θ in (2) implies

$$\Psi_\pm(r) \equiv \Psi_\pm(r) + 2\pi.$$

To have a lighter notation we introduce the following quantities

$$\alpha = 2\sigma^2 L^3 m, \qquad \beta = 4\sigma^2 L^2 m(1 - \sigma^2 L^2), \tag{55}$$

so that

$$\Psi_\pm(r) = \pm \frac{1}{\sqrt{m}} \ln\left(\frac{\alpha + \sqrt{\beta r^2 + \alpha^2}}{r}\right).$$

The induced metric on the brane is given by

$$ds_\pm^2 = -\left(\frac{r^2}{L^2} - m\right)dt^2 + \left(\frac{1}{\frac{r^2}{L^2} - m} + r^2\Psi_\pm'^2\right)dr^2, \tag{56}$$

and so the Ricci scalar is given by

$$R_\pm = -\frac{2m\beta}{mL^2\beta + \alpha^2} = -\frac{2}{L^2}(1 - \sigma^2 L^2).$$

We can easily see that the two-dimensional brane is indeed asymptotically AdS (if $\sigma^2 \neq L^{-2}$) with cosmological constant

$$\Lambda_2 = -\frac{1}{L^2}(1 - \sigma^2 L^2).$$

We now turn our attention to the properties of the slice and we consider only the + branch $\Psi_+(r)$ of (54) that from now on we will simply indicate as $\Psi(r)$ (the analysis of the − branch is completely analogous to the following one). We have

$$\Psi' = \frac{\partial \Psi}{\partial r} = -\frac{\alpha}{\sqrt{\beta r^2 + \alpha^2}\sqrt{m} r} < 0,$$

so our function $\Psi(r)$ is always decreasing and also

$$(r^2\Psi'^2)' = -\frac{2\alpha^2 \beta r}{(\beta r^2 + \alpha^2)^2 m}. \tag{57}$$

The right-hand side of (57) is equal to zero in $r = 0$ and always decreasing after that. Considering this, since

$$h_{tt}(0)h_{rr}(0) = -1 < 0, \tag{58}$$

we have that $h_{tt}h_{rr} < 0$ always, avoiding Euclidean patches on the brane.

Given the periodicity of Ψ we are now interested in the points $r = r_n$ in which the brane makes a full loop (i.e. where the two branches intersect on the Cartesian x axis, see Fig. 2). These points are defined by the equation

$$\Psi(r = r_n) = n\pi$$

with n integer. A solution is $r_0 = 0$ and the others are

$$r_n = \frac{2\alpha e^{n\pi\sqrt{m}}}{e^{2n\pi\sqrt{m}} - \beta}. \tag{59}$$

Note that

$$\frac{dr_n}{dn} < 0$$

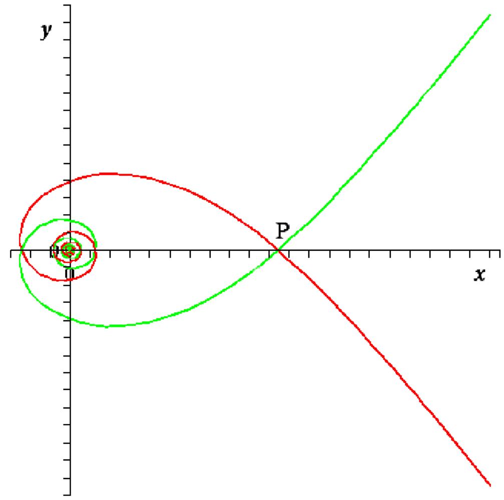

FIG. 2 (color online). Plot of Ψ_+ and Ψ_- in a Cartesian plane (x, y), $x = r\cos\theta$ and $y = r\sin\theta$ with $\theta = \Psi_\pm$. P is the point $(r_{n_{\max}}, 0)$ and it is the last point in which the two branches intersect.

and r_n blows up if there exists an integer $n = n_c$ such that

$$e^{2n_c\pi\sqrt{m}} = 4L^2\sigma^2 m(1 - \sigma^2 L^2)$$

if such an integer does not exist, the value of r at which the two branes intersect is $r = r_{n_{\max}}$, where

$$n_{\max} = \left[\frac{1}{2\pi\sqrt{m}} \ln(4L^2\sigma^2 m(1 - \sigma^2 L^2))\right]_\rightarrow > 0, \tag{60}$$

and with $[a]_\rightarrow$ we mean the next integer after a if a is not an integer. So if $\frac{1}{2\pi\sqrt{m}} \ln(4L^2\sigma^2 m(1 - \sigma^2 L^2))$ is an integer the brane will wrap around an infinite number of times. This will also happen in the asymptotically flat case in which $\sigma^2 = L^{-2}$. In the more likely case in which this is not an integer, the brane will wrap around only for a finite number of times and will then reach infinity with a defined asymptotic angle as is shown in Fig. 2.

B. Black hole

The induced metric (56) does not represent yet a black hole, the presence of the horizon is indeed only due to an accelerated coordinate system. In fact (56) can be easily transformed to the AdS_2 metric [25]. In order to find a black hole solution, we consider a positive mass μ localized on the brane which acts as a boundary of the brane. The global three-dimensional Z_2-symmetric solution implementing this scenario, will therefore be constructed by considering the portion of the spacetime whose boundary is given by the two profiles $\Psi_\pm$ from the last intersection point ($r = r_{n_{\max}}$) in the increasing r direction (see Fig. 2). In doing this we need to choose the sign of the normal in Eq. (53). We must impose that, in Cartesian coordinates, the n^y component of the normal is negative and, since we have

$$n^y = -\sin\theta n^r - r\cos\theta n^\theta,$$

with $\theta = \Psi(r)$, we can just impose that

$$\lim_{r\to\infty} n^y < 0 \Rightarrow \frac{\alpha}{\sqrt{\beta m L^2}} \sin(\theta_\infty) < 0,$$

where

$$\theta_\infty = \lim_{r\to\infty} \Psi(r) = \frac{1}{2\sqrt{m}} \ln(4L^2\sigma^2 m(1 - \sigma^2 L^2)),$$

and so the sign is related to the sign of $\sin\theta_\infty$. Note that when $\theta_\infty = n\pi$, where n is an integer, the sign of the normal cannot be defined and this is in complete agreement with what we saw above (see discussion after Eq. (60)).

The conical singularity formed by the intersection of Ψ_+ and Ψ_- describes a spacelike particle sitting on our 1-brane. From the action point of view this is equivalent to add to the action (50) the boundary Lagrangian (8).

To make the above discussion more concrete we will use a different coordinate gauge (we will again consider only the + branch). By making the coordinate change

$$\rho = \frac{\sqrt{r^2(1 - \sigma^2 L^2) + \sigma^2 L^4 m}}{1 - \sigma^2 L^2},$$

the transformed metric will verify the property $g_{tt} = g_{\rho\rho}^{-1}$. As we said we would like to truncate the range of r to be $r_{n_{\max}} \le r < \infty$. This implies a minimum value for the new coordinate ρ, given by

$$\rho_m = \sqrt{\frac{r^2_{n_{\max}}}{1 - \sigma^2 L^2} + \frac{\sigma^2 L^4 m}{(1 - \sigma^2 L^2)^2}}.$$

We now shift this point to the origin by setting

$$x = \rho - \rho_m,$$

so that $0 \le x < \infty$. We now copy and paste the branch Ψ_- in $x = 0$. Equivalently we extend x to the range $-\infty < x < \infty$ and we require that $g_{\alpha\beta}(-x) = g_{\alpha\beta}(x)$. By setting

$$\lambda^2 = \frac{1 - \sigma^2 L^2}{L^2}, \tag{61}$$

and

$$M = \frac{1}{L}\sqrt{\frac{r^2_{n_{\max}}}{L^2}(1 - \sigma^2 L^2) + \sigma^2 L^2 m}, \qquad N = m - \frac{r^2_{n_{\max}}}{L^2},$$

we obtain the induced metric on the brane to be

$$ds^2 = -(\lambda^2 x^2 + 2M|x| - N)dt^2 + \frac{dx^2}{(\lambda^2 x^2 + 2M|x| - N)}. \tag{62}$$

The metric (62) is equivalent to the metric (16) with the function f given by (22) and therefore represents a black hole surrounding a boundary mass μ.

Given that boundary mass $\mu = M/\sqrt{N}$ is not negative, we find again that $N > 0$. This condition now implies that our brane must cross the BTZ horizon and therefore that brane and bulk black hole must share the same horizon. In fact from the condition $N > 0$ we get that

$$r_{n_{\max}} < r_h = \sqrt{m}L, \tag{63}$$

so

$$\frac{2\alpha e^{n_{\max}\pi\sqrt{m}}}{e^{2n_{\max}\pi\sqrt{m}} - \beta} < \sqrt{m}L.$$

By setting $x = e^{n_{\max}\pi\sqrt{m}}$ we have that

$$x - \left(\frac{x^2}{4L^2\sigma^2 m} - (1 - \sigma^2 L^2)\right)\sqrt{m} < 0,$$

and so being $x > 0$ we need

$$x > 2\sqrt{m}L\sigma(1 + L\sigma),$$

or

$$n_{\max} > \frac{\ln(2\sqrt{m}L\sigma(1 + L\sigma))}{\pi\sqrt{m}}. \tag{64}$$

We can always write

$$n_{\max} = 1 + \frac{1}{2\pi\sqrt{m}} \ln(4L^2\sigma^2 m(1 - \sigma^2 L^2)) - \epsilon,$$

where $0 < \epsilon < 1$. With this (64) reduces to

$$e^{2(1-\epsilon)\pi\sqrt{m}} \frac{1 - \sigma L}{1 + \sigma L} > 1.$$

As $\frac{1-\sigma L}{1+\sigma L} < 1$, to satisfy (63) we need a massive enough BTZ black hole. This is in line with the discussion of [2] which requires the bulk black hole to have a large mass in order to be quantum mechanically stable and to correspond to a CFT in thermal equilibrium.

IV. CONCLUSIONS

Motivated by the conjectured duality between brane-world bulk black holes and semiclassical black holes of [12,13] we studied two-dimensional quantum black holes and 1-brane slices of a three-dimensional BTZ bulk black hole.

We found a new static two-dimensional quantum black hole solution surrounding a boundary mass, in thermodynamical equilibrium in the Hartle-Hawking vacuum state. This solution exists only if the two-dimensional cosmological constant is nonzero, as the conformal field theory relates the trace anomaly to the cosmological constant.

The proposed duality would imply the existence of a static asymptotically AdS two-dimensional brane black hole with nonzero tension as a slice of an asymptotically AdS thermodynamically stable [2] three-dimensional space. Studying slices of the BTZ black hole we found that indeed, for a massive enough bulk black hole, such a solution does exist only in the nonvanishing cosmological constant case and we showed that it shares the same geometry of our quantum solution. We also found a resonance between the BTZ parameters and the 1-brane tension for which such a construction is impossible. It would be interesting to reinterpret it, in the holographic prospective, from the point of view of a deformed conformal field theory living on the spacial infinity of BTZ, extending [31] to the finite temperature case, however this is beyond the scope of this paper.

In any case, we can go a bit further with the duality between the two black holes, expressing the temperature of our two-dimensional quantum black hole in terms of the parameter M and N obtained from the slicing of BTZ. We find that

$$T = \hbar \frac{\sqrt{q}}{2\pi k_B} = \hbar \frac{\sqrt{m}}{2\pi k_B L}. \tag{65}$$

The temperature (65) is the same temperature of the bulk BTZ black hole [7]. The boundary theory must therefore be a TCFT (with a UV cutoff) and temperature given by the bulk black hole as we would expect [2]. This also fixes the choice of the time coordinate to be t instead of $\tilde{t}$. It therefore seems that the conjectured duality between classical bulk black holes and quantum brane black holes [12,13] applies to our case.

In three dimensions the holographic relation [32] reads

$$\gamma = \frac{12\pi L}{k_3^2} > 0.$$

As we said γ is proportional to the sum of the number of matter fields and the number of gravitons. Matter fields are counted positively and gravitons negatively [18]. It is then clear that the theory describing our black hole has to be a matter dominated one.

Equating (15) with (61) we obtain

$$\Lambda = \hbar \frac{1 - \sigma^2 L^2}{\sigma L} > 0.$$

The configuration $\Lambda > 0$ and $\gamma > 0$ cannot be obtained classically, indeed if $\Lambda > 0$ (positive energy density in the Universe) one expects, classically, to have a positive curvature (this is the case in dilatonic gravity). In our case instead, starting from a positive cosmological constant, we get a negative curvature. This is one of the possibilities envisaged in [18] absent in the classical theory.

The fact that our black hole solution is due to the presence of matter might imply that our solution should correspond to the ending state of a gravitational collapse. A very interesting question is therefore if a classical collapse of a brane can be also holographically described as a quantum gravitational collapse in the semiclassical theory we considered, however this is beyond the scope of this paper and we leave it for future work.

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APPENDIX: BOUNDARY ACTION

We here show that the variation of the boundary extrinsic curvature with respect to the boundary induced metric is trivial. Consider in fact

$$\int_{\partial\Sigma} dt \sqrt{h} K,$$

standard calculations show that its variation reads (see for example [33])

$$\int_{\partial\Sigma} dt \sqrt{h} (K_{\alpha\beta} - h_{\alpha\beta} K) \delta h^{\alpha\beta},$$

and since in one dimension $K_{\alpha\beta} - h_{\alpha\beta} K \equiv 0$, the above variation is zero. This result implies that in two dimensions the extrinsic curvature of a given boundary can be freely fixed to a value μ, where μ represents the mass associated with the boundary. In particular, in this paper, we would like to interpret the boundary mass μ as the mass of a spacelike particle.

The action associated with a spacelike point particle is

$$I_b = \mu \int d\tau \sqrt{u_\alpha u_\beta g^{\alpha\beta}}, \tag{A1}$$

where $dx^\alpha u_\alpha = d\tau$ is the proper length of the particle world sheet and u^α is the two-velocity of the particle. In two dimensions, the boundary metric defined by the particle worldline is $h_{\alpha\beta} = u_\alpha u_\beta$. Therefore the action (A1) can be rewritten as an explicit boundary action

$$I_b = \mu \int_{\partial\Sigma} dt \sqrt{h} \sqrt{u_\alpha u_\beta h^{\alpha\beta}},$$

where $\mu > 0$ is the positive boundary mass [22] and $\partial\Sigma$ is the boundary defined by the particle world sheet.

The variation of this action with respect to $\delta h^{\alpha\beta}$ is

$$\delta I_b = -\frac{\mu}{2} \int_{\partial\Sigma} dt \sqrt{h} \delta h^{\alpha\beta} \left(\frac{u_\alpha u_\beta}{\sqrt{u_\mu u_\nu h^{\mu\nu}}} - h_{\alpha\beta} \sqrt{u_\mu u_\nu h^{\mu\nu}} \right).$$

Imposing now the normalization of the worldline vector $u^\alpha u_\alpha = 1$ we find that the above variation is zero. Therefore the only dynamical equation is obtained from the variation of x^α, i.e. the equation of motion of the particle world sheet. In order to interpret the boundary mass μ as the particle mass we therefore need to show that the particle can sit on the boundary chosen, given a spacetime metric.

We consider our spacetime in the natural coordinates $(\tilde{t}, \tilde{x})$, so that

$$ds^2 = -\tilde{f}(\tilde{x})d\tilde{t}^2 + \frac{d\tilde{x}^2}{\tilde{f}(\tilde{x})},$$

where $\tilde{f}(\tilde{x}) = \lambda^2\tilde{x}^2 + 2\mu|\tilde{x}| - 1$. The geodesic equation is solved for

$$u^{\tilde{t}} = \frac{C}{\tilde{f}}, \qquad u^{\tilde{x}} = \sqrt{-\tilde{f} + C^2}, \tag{A2}$$

where C is the energy per unit mass of the particle. For physical reasons $|C| \geq 1$, as the total energy of the particle cannot be smaller than the mass of the particle itself. In particular the particle is at "rest" [34] for $|C| = 1$. From (A2) we can therefore see that the only point in which the particle is at rest is in $\tilde{x} = 0$, so that $u^{\tilde{t}} = 1$ and $u^{\tilde{x}} = 0$. This point is of an (unstable) equilibrium as the potential $V = -\tilde{f}$ has a maximum in $\tilde{x} = 0$. We wish to comment here that since $\tilde{x} = 0$ represents a point of unstable equilibrium for the world sheet, the point particle approximation of a totally collapsed body can no longer be used under perturbations and therefore, in this case, a more detailed model for the collapsed matter has to be introduced to study the stability of our system. However this study is beyond the scope of the current paper and it is postponed for future research.

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