

Merger transitions in brane–black-hole systems: Criticality, scaling, and self-similarity

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We propose a toy model for studying merger transitions in a curved spacetime with an arbitrary number of dimensions. This model includes a bulk N -dimensional static spherically symmetric black hole and a test D -dimensional brane ($D \leq N - 1$) interacting with the black hole. The brane is asymptotically flat and allows a $O(D - 1)$ group of symmetry. Such a brane–black-hole (BBH) system has two different phases. The first one is formed by solutions describing a brane crossing the horizon of the bulk black hole. In this case the internal induced geometry of the brane describes a D -dimensional black hole. The other phase consists of solutions for branes which do not intersect the horizon, and the induced geometry does not have a horizon. We study a critical solution at the threshold of the brane–black-hole formation, and the solutions which are close to it. In particular, we demonstrate that there exists a striking similarity of the merger transition, during which the phase of the BBH system is changed, both with the Choptuik critical collapse and with the merger transitions in the higher dimensional caged black-hole–black-string system.

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I. INTRODUCTION

Caged (Kaluza-Klein) black holes, black strings, and possible transitions between them are subjects which have recently attracted a lot of attention (see e.g. reviews in [1,2] and references therein). Kol [3] demonstrated that during the black-hole–black-string transition the Euclidean topology of the system changes. These transitions were called *merger* transitions. One of the interesting features of the merger transitions [3] is their close relation to the Choptuik critical collapse phenomenon [4]. The merger transitions are, in many aspects, similar to the topology change transitions in classical and quantum gravity (see e.g. [5] and references therein). In particular, one can expect that during both types of transitions the spacetime curvature can infinitely grow. This means that the classical theory of gravity is not sufficient for their description and a more fundamental theory (such as string theory) is required. In these circumstances, it might be helpful to have a toy model for the merger and topology changing transitions, based on the physics which is well understood. In this paper we would like to propose such a toy model.

The model consists of a bulk N -dimensional black hole and a test D -dimensional brane in it ($D \leq N - 1$). We assume that the black hole is spherically symmetric and static. It can be neutral or charged. As we shall see, the detailed characteristics of the black hole are not important for our purposes. We assume that the brane is infinitely thin and use the Dirac-Nambu-Goto [6,7] equations for its description. We consider the brane which is static and spherically symmetric, so that its world sheet has the $O(D - 1)$ group of the symmetry. We assume also that the brane reaches the asymptotic infinity where it has the

form of the $(D - 1)$ plane. The brane world sheet gets one more dimension because of its time evolution. To keep the brane static one needs to apply a force to it. The detailed mechanism is not important for our consideration, so we do not discuss it here.

As a result of the gravitational attraction, the brane, which is flat at infinity, is deformed. There are two different types of possible equilibrium configurations. A brane either crosses the black-hole horizon, or it lies totally outside the black hole. In the former case, the internal metric of the brane, induced by its embedding, describes a geometry of a D -dimensional black hole. This happens because the time-like at infinity Killing vector of the bulk geometry being restricted to the brane is the Killing vector for the induced geometry. Thus the brane spacetime has the Killing horizon (and hence the event horizon) which is located at the intersection of the brane with the bulk–black-hole horizon. A case when the brane lies within the equatorial plane of the bulk black hole is an example of such a configuration. Since the equatorial plane of the spherically symmetric static black hole is invariant under the reflection mapping the upper half space onto the lower one, the equatorial plane is a geodesic surface, and hence, it is a minimal one. Thus the “equatorial” plane automatically satisfies the Dirac-Nambu-Goto equations.

An example of a configuration of the second type is a brane located at a far distance from the black hole. Its geometry is a plane which is slightly deformed by the gravitational attraction of the bulk black hole. One can use the weak field approximation to calculate this deformation [8].

Let us consider a one-parameter family of the branes which are asymptotically flat and parallel to the chosen equatorial plane. This family can be naturally split into two parts (phases). One of them is formed by subcritical solutions which do not intersect the black-hole horizon, while

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the other one is formed by solutions crossing the horizon. We shall show that there exists a *critical* solution separating these two phases.

In this paper we study the critical solution at the threshold of the brane–black-hole formation, and the solutions which are close to it. Our goal is to study a transition between the subcritical and supercritical phases. In particular, we demonstrate that there exists a striking similarity of this transition both with the Choptuik critical collapse [4] and with the merger transitions in the black-hole–black-string system [2,3].

The results presented in this paper are a natural generalization of [9,10] adopted to the higher dimensional case [11].

II. BRANE EQUATIONS

Let us consider a static test brane interacting with a bulk static spherically symmetrical black hole. For brevity, we shall refer to such a system (a brane and a black hole) as the BBH system. We assume that the metric of the bulk N -dimensional spacetime is

$$dS^2 = g_{\mu\nu} dx^\mu dx^\nu = -F dT^2 + F^{-1} dr^2 + r^2 d\Omega_{N-2}^2, \quad (1)$$

where $F = F(r)$, and $d\Omega_{N-2}^2$ is the metric of the $(N-2)$ -dimensional unit sphere S^{N-2} . We define the coordinates θ_i ($i = 1, \dots, N-2$) on this sphere by the relations

$$d\Omega_{i+1}^2 = d\theta_{i+1}^2 + \sin^2 \theta_{i+1} d\Omega_i^2. \quad (2)$$

In what follows the explicit form of F is not important. We assume only that the function $F = F(r)$ has a simple zero at $r = r_0$, where the horizon of the black hole is located, and it grows monotonically from 0 at $r = r_0$ to 1 at the spatial infinity $r \rightarrow \infty$, where it has the following asymptotic form:

$$F - 1 \sim r^{-(N-3)}. \quad (3)$$

For the vacuum (Tangherlini [12]) solution of the Einstein equations,

$$F = 1 - (r_0/r)^{N-3}. \quad (4)$$

We denote by X^μ ($\mu = 0, \dots, N-1$) the bulk spacetime coordinates and by ζ^a ($a = 0, \dots, D-1$) the coordinates on the brane world sheet. The functions X^μ (ζ^a) determine the brane world sheet describing the evolution of the $(D-1)$ -dimensional object (brane) in a bulk N -dimensional spacetime. We assume that $D \leq N-1$. A test brane configuration in an external gravitational field $g_{\mu\nu}$ can be obtained by solving the equations which follow from the Dirac-Nambu-Goto action [6,7],

$$S = \int d^D \zeta \sqrt{-\det \gamma_{ab}}, \quad (5)$$

where

$$\gamma_{ab} = g_{\mu\nu} X_{,a}^\mu X_{,b}^\nu \quad (6)$$

is the D -dimensional induced metric on the world sheet. Usually the action contains the brane tension factor μ . This factor does not enter into the brane equations. For simplicity we put it equal to 1.

We assume that the brane is static and spherically symmetric, so that its world sheet geometry possesses the group of the symmetry $O(D-1)$. If $D < N-1$ we choose the brane surface to obey the equations

$$\theta_{N-2} = \dots = \theta_D = \pi/2. \quad (7)$$

The brane world sheet with the above symmetry properties is defined by the function $\theta_{D-1} = \theta(r)$. We shall use $\zeta^a = (T, r, \theta_i)$ ($i = 1, \dots, n = D-2$) as the coordinates on the brane. For this parametrization the induced metric on the brane is

$$\begin{aligned} ds^2 &= \gamma_{ab} d\zeta^a d\zeta^b \\ &= -F dT^2 + [F^{-1} + r^2 (d\theta/dr)^2] dr^2 + r^2 \sin^2 \theta d\Omega_n^2, \end{aligned} \quad (8)$$

and the action (5) reduces to

$$S = \Delta T \mathcal{A}_n \int dr \mathcal{L}, \quad (9)$$

$$\mathcal{L} = r^n \sin^n \theta \sqrt{1 + Fr^2 (d\theta/dr)^2}. \quad (10)$$

Here $n = D-2$, ΔT is the interval of time, and $\mathcal{A}_n = 2\pi^{n/2}/\Gamma(n/2)$ is the surface area of a unit n -dimensional sphere. A brane configuration is determined by solutions of the following Euler-Lagrange equation:

$$\frac{d}{dr} \left(\frac{d\mathcal{L}}{d\theta} \right) - \frac{d\mathcal{L}}{dr} = 0, \quad (11)$$

which for the Lagrangian (10) is of the form

$$\frac{d^2 \theta}{dr^2} + B_3 \left(\frac{d\theta}{dr} \right)^3 + B_2 \left(\frac{d\theta}{dr} \right)^2 + B_1 \frac{d\theta}{dr} + B_0 = 0, \quad (12)$$

$$B_0 = -\frac{n \cot \theta}{Fr^2}, \quad B_1 = \frac{n+2}{r} + \frac{1}{F} \frac{dF}{dr}, \quad (13)$$

$$B_2 = -n \cot \theta, \quad B_3 = r \left[(n+1)F + \frac{r}{2} \frac{dF}{dr} \right]. \quad (14)$$

For the brane crossing the horizon, Eq. (12) has a regular singular point at $r = r_0$. A solution regular at this point has the following expansion near it:

$$\theta(r) \approx \theta_0 + \theta'_0 (r - r_0) + \dots, \quad \theta'_0 = \frac{n \cot \theta_0}{2\kappa r_0^2}. \quad (15)$$

This supercritical solution is uniquely defined by the initial value θ_0 .

If the brane does not cross the horizon, a radius r on the brane surface reaches its minimal value $r = r_1 > r_0$. For the symmetry reason, it occurs at $\theta = 0$. A regular solution of (12) near this point has the following behavior:

$$\theta \approx \pm \sqrt{\frac{2(n+1)}{B_3(r_1)}} \sqrt{r-r_1} + \dots \quad (16)$$

Such a subcritical solution is uniquely determined by the value of the parameter r_1 .

III. FAR DISTANCE SOLUTIONS

Consider a brane located in the equatorial plane $\theta = \pi/2$. Since the bulk metric is invariant under the discrete transformation $\theta \rightarrow \pi - \theta$, the surface $\theta = \pi/2$ is geodesic, and hence minimal. This means that $\theta = \pi/2$ is a solution of the test brane equations. This can also be easily checked by using Eq. (12).

Let us consider now a solution which asymptotically approaches $\theta = \pi/2$. We write this solution in the form

$$\theta = \frac{\pi}{2} + q(r). \quad (17)$$

Assuming that q is small and using (3), one can write Eq. (12) in the region $r \rightarrow \infty$ as

$$\frac{d^2 q}{dr^2} + \frac{n+2}{r} \frac{dq}{dr} + \frac{n}{r^2} q = 0. \quad (18)$$

For $n > 1$ this equation has a solution

$$q = \frac{p}{r} + \frac{p'}{r^n}. \quad (19)$$

The case $n = 1$ is a degenerate one and the corresponding solution is

$$q = \frac{p + p' \ln r}{r}. \quad (20)$$

This case was considered in detail earlier in [9,10]. In this paper we focus on the higher dimensional BBH systems and assume that $n > 1$. In this case the first term in (19) is the leading one. The brane surface is asymptotically parallel to the equatorial plane, and $p = \lim_{r \rightarrow \infty} r q$ is the distance of the brane from it. We shall call this quantity p a *shift parameter*.

If the brane crosses the horizon (a supercritical solution) it is uniquely determined by the angle θ_0 , and the asymptotic data $\{p, p'\}$ are well-defined continuous functions of θ_0 . In a general case, the function $p(\theta_0)$ may be nonmonotonic in the interval $(0, \pi/2)$, so that for two different values of θ_0 one has the same value of the shift parameter p . If the brane does not cross the horizon (a subcritical solution), it is uniquely determined by the minimal radius r_1 , and the asymptotic data $\{p, p'\}$ are continuous functions of r_1 . In a general case, it may also happen that for the same shift parameter p one has two (or more) solutions, and, for

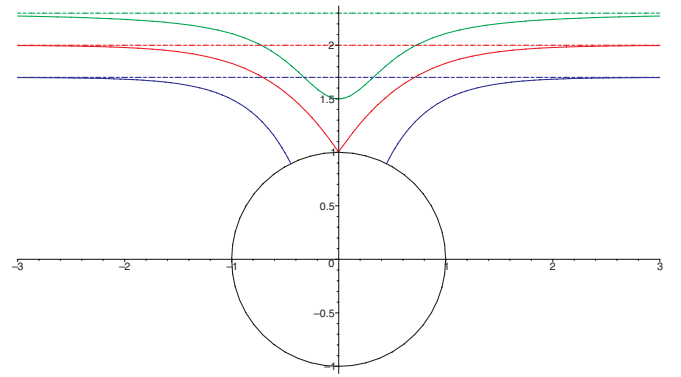


FIG. 1 (color online). A brane interacting with a black hole: The round circle is the black-hole horizon. We schematically show the profiles of the brane for three different cases. The brane which touches the horizon at one point and has a conelike profile near it is critical. Two other solid lines show branes close to the critical one. One of them (subcritical) does not intersect the horizon, while the other (supercritical) one crosses the horizon at the spherical surface with a small radius R_0 . At far distance the brane surfaces are parallel to the equatorial plane of the black hole.

example, one of them is subcritical and the other supercritical [9,13].

We consider subcritical and supercritical solutions as two different phases of the BBH system. Let us consider a continuous deformation of solutions during which a solution transits from one phase to another. If we parametrize these solutions by a parameter λ , there exists a special value of it, $\lambda = \lambda_*$, where the phase is changed. We call the corresponding solution *critical*. (Figure 1 schematically shows the critical and near-critical solutions for this process.) We focus our attention on the solutions close to the critical one and study the properties of the BBH system during the *merger transition* when the solution changes its phase.

IV. BRANE CONFIGURATIONS NEAR THE HORIZON

Near the horizon the coefficient F of the bulk metric has the form

$$F \approx 2\kappa(r - r_0) + O((r - r_0)^2), \quad (21)$$

where $\kappa = \frac{1}{2}(dF/dr)|_{r_0}$ is the surface gravity. To study the near-horizon behavior of the brane, it is convenient to introduce the proper distance coordinate

$$Z = \int_{r_0}^r \frac{dr}{\sqrt{F}}. \quad (22)$$

In the vicinity of the horizon, one has

$$r - r_0 \approx \kappa Z^2/2, \quad F \approx \kappa^2 Z^2. \quad (23)$$

We are interested in the case when a “central” part of the brane is located in close vicinity to the horizon or crosses it.

In the latter case, we assume that the radius R_0 of the surface of the intersection of the brane with the bulk horizon is much smaller than the size of the horizon r_0 . Under these conditions one can approximate the spacetime close to the bulk–black-hole horizon by the Rindler space where the horizon is a $(n + 1)$ -dimensional plane. In this approximation the metric (1) takes the form

$$dS^2 = -\kappa^2 Z^2 dT^2 + dZ^2 + dL_{N-2}^2, \quad (24)$$

where dL_{N-2}^2 is the metric of $(N - 2)$ -dimensional Euclidean space \mathbb{E}^{N-2} . We write dL_{N-2}^2 in the form

$$dL_{N-2}^2 = dY_D^2 + \dots + dY_{N-2}^2 + dR^2 + R^2 d\Omega_{D-2}^2, \quad (25)$$

and choose the Cartesian coordinates Y so that Eq. (7) takes the form $Y_D = \dots = Y_{N-2} = 0$, while the brane equation is

$$F(Z, R) = 0. \quad (26)$$

We write a solution of this equation in a parametric form,

$$Z = Z(\lambda), \quad R = R(\lambda). \quad (27)$$

Then the induced metric on the brane is

$$dS^2 = -\kappa^2 Z^2 dT^2 + [(dZ/d\lambda)^2 + (dR/d\lambda)^2] d\lambda^2 + R^2 d\Omega_n^2. \quad (28)$$

Here, as earlier, $n = D - 2$. The action (5) for this induced metric takes the form

$$S = \kappa \Delta T \mathcal{A}_n \mathcal{S}, \quad (29)$$

$$\mathcal{S} = \int d\lambda Z R^n \sqrt{(dZ/d\lambda)^2 + (dR/d\lambda)^2}. \quad (30)$$

This action is evidently invariant under a reparametrization $\lambda \rightarrow \tilde{\lambda}(\lambda)$. In the regions where either Z or R is a monotonic function of λ , these functions themselves can be used as parameters. As a result, one obtains two other forms of the action which are equivalent to \mathcal{S} :

$$\mathcal{S} = \int dZ \mathcal{L}_R = \int dR \mathcal{L}_Z, \quad (31)$$

where

$$\mathcal{L}_R = Z R^n \sqrt{1 + R'^2}, \quad \mathcal{L}_Z = Z R^n \sqrt{1 + \dot{Z}^2}. \quad (32)$$

Here the prime means the derivative with respect to Z , while the dot stands for the derivative with respect to R . The corresponding Euler-Lagrange equations are

$$Z R R'' + (R R' - n Z)(1 + R'^2) = 0, \quad (33)$$

$$R Z \ddot{Z} + (n Z \dot{Z} - R)(1 + \dot{Z}^2) = 0. \quad (34)$$

It is easy to check that the form of Eq. (33) is invariant under the following transformation:

$$R(Z) = k \tilde{R}(\tilde{Z}), \quad Z = k \tilde{Z}. \quad (35)$$

Similarly, the transformation

$$Z(R) = k \tilde{Z}(\tilde{R}), \quad R = k \tilde{R} \quad (36)$$

preserves the form of (34).

To obtain the boundary conditions for Eqs. (33) and (34) we require that the induced metric on the brane is regular. This implies that the curvature invariants are regular as well. Let us consider the scalar curvature of the induced metric, which we denote by \mathcal{R} . For the brane parametrization $R = R(Z)$, the induced metric is

$$dS^2 = -\kappa^2 Z^2 dT^2 + [1 + R'^2] dZ^2 + R^2 d\Omega_n^2, \quad (37)$$

and the corresponding scalar curvature takes the form

$$\mathcal{R} = \mathcal{R}^{(2)} + \frac{n(n-1)}{R^2} - 2n \frac{\square^{(2)} R}{R} - n(n-1) \frac{(\nabla R)^2}{R^2}. \quad (38)$$

The quantities on the right-hand side of this relation are calculated for the two-dimensional metric which is the metric of the $(T - Z)$ sector of (28). In particular, $\mathcal{R}^{(2)}$ is the two-dimensional curvature of this metric. Calculating \mathcal{R} and using the brane equations (33) to exclude the second derivatives $d^2 R/dZ^2$, one gets

$$\mathcal{R} = \frac{2R^2 R'^2 - 6n Z R R' + n(3n-1) Z^2}{Z^2 R^2 (1 + R'^2)}. \quad (39)$$

Simple analysis shows that, if the brane crosses the horizon of the bulk black hole, then the regularity of \mathcal{R} on the horizon requires

$$R|_{Z=0} = R_0, \quad \left. \frac{dR}{dZ} \right|_{Z=0} = 0. \quad (40)$$

By using (33) one obtains, for this solution near the horizon,

$$R = R_0 + \frac{n Z^2}{4 R_0} + \dots \quad (41)$$

For a brane which does not cross the horizon, one has $Z(R=0) > 0$, and at this point one has $R' \rightarrow \infty$. This gives the following boundary conditions:

$$Z|_{R=0} = Z_0, \quad \left. \frac{dZ}{dR} \right|_{R=0} = 0. \quad (42)$$

These conditions can also be obtained from the regularity of \mathcal{R} in the parametrization $Z = Z(R)$. Using (34) one gets

$$Z = Z_0 + \frac{R^2}{2(n+1)Z_0} + \dots \quad (43)$$

One can check that when the condition of regularity of the

Ricci scalar is satisfied, the other curvature invariants are also finite.

A solution near the horizon [either (40) or (42)] determines uniquely a global solution of the brane equation (12). Thus, for a given small value of R_0 (Z_0) the asymptotic data of the corresponding solution, $\{p, p'\}$, are uniquely determined and are continuous functions of R_0 (Z_0).

V. CRITICAL SOLUTIONS AS ATTRACTORS

Equations (33) and (34) have a simple solution

$$R = \sqrt{n}Z \quad (44)$$

which plays a special role. We call it a *critical solution*. It describes a critical brane which touches the horizon of the bulk black hole at one point, $Z = R = 0$. The critical solution separates two different families of solutions (phases), supercritical and subcritical. Let us show that the critical solution is an *attractor*, and solutions of both families are attracted to the critical solution asymptotically.

To do this we, following [10], introduce new variables:

$$x = R', \quad y = Z^{-1}RR', \quad ds = dZ/(yZ). \quad (45)$$

In these variables Eq. (33) can be written in the form of the first order regular autonomous system,

$$\frac{dx}{ds} = x(n - y)(1 + x^2), \quad (46)$$

$$\frac{dy}{ds} = y[n - 2y + x^2(n + 1 - y)]. \quad (47)$$

The critical points on the phase space (x, y) are the points where the right-hand side of both Eqs. (46) and (47) vanish simultaneously. These critical points are as follows: a node, $(0,0)$; a saddle point, $(0, n/2)$; and two focus points, $(\pm\sqrt{n}, n)$. For different n the phase portraits are similar. Figure 2 shows the phase portraits for $n = 2$ and $n = 4$, which have the focus points at $(\sqrt{2}, 2)$ and $(2,4)$, respectively. The focus points are attractors, and they correspond to the critical solutions $R = \sqrt{n}Z$.

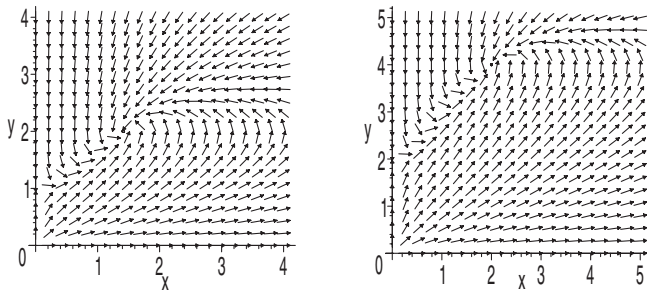


FIG. 2. Phase portraits of the system (46) and (47) for $n = 2$ (left plot) and for $n = 4$ (right plot).

VI. NEAR-CRITICAL SOLUTIONS IN THE VICINITY OF THE HORIZON

Let us consider now solutions which are close to the critical one,

$$R = \sqrt{n}Z + \rho(Z). \quad (48)$$

Substituting this expression into (33), and keeping linear in ρ terms, we obtain

$$Z^2\rho'' + (n+1)Z\rho' + (n+1)\rho = 0. \quad (49)$$

A solution of this equation is of the form

$$\rho = Z^\beta, \quad \beta_\pm = \frac{1}{2}(-n \pm \sqrt{n^2 - 4n - 4}). \quad (50)$$

The parameter n is a positive integer. For $n \leq 4 < 2(1 + \sqrt{2})$ the exponent β is complex, while for $n \geq 5$ it is real.

Let us consider first the case of complex β , that is, when the number of the brane spacetime dimensions is $D \leq 6$. We write a near-critical solution (48) in the form

$$R = \sqrt{n}Z + Z^{-n/2}\Re(CZ^{i\alpha}), \quad (51)$$

where C is a complex number and $\alpha = \frac{1}{2}\sqrt{4 + 4n - n^2}$. The initial data $R(Z=0) = R_0$ determines the factor C . We denote the corresponding values of it by $C(R_0)$. Under the scaling transformation (35) the surface $Z=0$ is invariant, while R_0 is transformed into $\tilde{R}_0 = k^{-1}R_0$. Using (51) we obtain

$$C(kR_0) = k^{-1-n/2+i\alpha}C(R_0). \quad (52)$$

For $D \geq 7$ both the exponents β_\pm are real and one has $\beta_+\beta_- = 4(n-1)$, so that both β_\pm are negative. The critical solution (44) is again the attractor and one has

$$R = \sqrt{n}Z + C_+Z^{\beta_+} + C_-Z^{\beta_-}. \quad (53)$$

Using again the arguments based on the scaling properties of Eq. (33), one can conclude that C_\pm considered as functions of R_0 obey the properties

$$C_\pm(kR_0) = k^{\beta_\pm-1}C_\pm(R_0). \quad (54)$$

Similarly, one can consider near-critical solutions of Eq. (34). Let us write

$$Z = \frac{1}{\sqrt{n}}R + \zeta(R). \quad (55)$$

Then, keeping linear in ζ terms, one obtains

$$R^2\zeta'' + (n+1)R\zeta' + (n+1)\zeta = 0. \quad (56)$$

It has the same form and the same coefficients as Eq. (49). For this reason, as can be expected, the critical dimension and the scaling property (35) are the same for both linearized equations.

VII. SCALING AND SELF-SIMILARITY

Since near the horizon Eq. (12) reduces to (33), one can uniquely define a special global solution of (12) which near the horizon reduces to the critical solution (44). This critical solution near the horizon is of the form [14]

$$\theta \approx \sqrt{\frac{2n(r-r_0)}{\kappa r_0^2}}. \quad (57)$$

Being traced to infinity, the critical solution determines uniquely the asymptotic data $\{p, p'\}$ which we denote by $\{p_*, p'_*\}$, respectively. If the function $F(r)$ in the metric is given, one can find $\{p_*, p'_*\}$, for example, by solving Eq. (12) numerically.

Consider a supercritical solution. For this solution the radius R_0 of the horizon of the brane black hole is connected with θ_0 as follows: $R_0 = r_0 \sin \theta_0$. For small θ_0 one has $R_0 = r_0 \theta_0$. We denote by $\{p, p'\}$ the asymptotic data for a solution with a given R_0 . For the critical solution, $R_0 = 0$, the asymptotic data are $\{p_*, p'_*\}$. We also write $\Delta p = \sqrt{(p-p_*)^2 + (p'-p'_*)^2}$. Similarly, one can define $\Delta \bar{p}$ for subcritical solutions. In both cases $\Delta p = 0$ if and only if the solution is critical. One can use Δp as a measure indicating how close a solution is to the critical one. Our goal is to demonstrate that $R \rightarrow 0$ (or Z_0), considered as a function of Δp , has a self-similar behavior. We shall show that there exists a critical dimension, D_* , such that for $D > D_*$ this symmetry is continuous, while for $D \leq D_*$ it is discrete. This critical dimension for the BBH system is $D_* = 6$. It should be emphasized that in the definition one can use any positive definite quadratic form of $p - p_*$ and $p' - p'_*$ of Δp instead of $(p - p_*)^2 + (p' - p'_*)^2$. It can be checked that this does not change the result.

A. $D \leq 6$ case

Since in the solution (44) is the attractor, both subcritical and supercritical solutions which are close to the critical solution in the vicinity of the horizon remain close to it everywhere. This implies that the difference between the near-critical and critical solutions is small and it can be obtained by solving a linear equation. For this equation both $(p - p_*, p' - p'_*)$ and the complex coefficient C in (51) are uniquely defined by R_0 , so that there exists a linear relation between these coefficients,

$$\begin{aligned} C &= a(p - p_*) + b(p' - p'_*), \\ \bar{C} &= \bar{a}(p - p_*) + \bar{b}(p' - p'_*), \end{aligned} \quad (58)$$

where a and b are complex numbers. Substituting $\{p, p'\}$ obtained from these equations into the definition of Δp , one gets

$$(\Delta p)^2 = P^2 [C\bar{C} - \bar{A}C^2 - \mathcal{A}\bar{C}^2], \quad (59)$$

$$\mathcal{A} = |\mathcal{A}|e^{iA} = \frac{a^2 + b^2}{2(|a|^2 + |b|^2)}, \quad (60)$$

$$P^2 = -\frac{2(|a|^2 + |b|^2)}{(ab - \bar{a}\bar{b})^2}. \quad (61)$$

It is easy to show that P^2 is non-negative, and

$$|\mathcal{A}| \leq 1/2. \quad (62)$$

In the latter relation the equality takes place when $a/\bar{a} = b/\bar{b}$. This is a degenerate case and, by a small change of the bulk metric, this condition will be violated. Thus we assume that $|\mathcal{A}| < 1/2$.

We denote by $\Delta \tilde{p}$ and \tilde{C} quantities which correspond to the solution with $R = \tilde{R}_0$. Then one has

$$\frac{(\Delta p)^2}{(\Delta \tilde{p})^2} = \frac{C\bar{C} - \bar{A}C^2 - \mathcal{A}\bar{C}^2}{\tilde{C}\bar{\tilde{C}} - \bar{A}\tilde{C}^2 - \mathcal{A}\bar{\tilde{C}}^2}. \quad (63)$$

The scaling transformation (35) maps $Z = 0$ onto itself, and hence it preserves the position of the horizon, while it changes the value of R_0 . Let us choose $k = R_0/\tilde{R}_0$; then, using (52), one has

$$C = (R_0/\tilde{R}_0)^{(n+2)/2 - i\alpha} \tilde{C}, \quad \gamma = 1 + n/2. \quad (64)$$

Let us write

$$B = \mathcal{A} \frac{\tilde{C}}{\bar{C}} = |\mathcal{A}|e^{iB}. \quad (65)$$

Then one has

$$\frac{(\Delta p)^2}{(\Delta \tilde{p})^2} = \frac{R_0^{n+2} [1 - 2|\mathcal{A}| \cos(2\alpha \ln R_0 + B')]}{\tilde{R}_0^{n+2} [1 - 2|\mathcal{A}| \cos(B)]}, \quad (66)$$

where $B' = B - 2\alpha \ln \tilde{R}_0$. It follows from this relation that

$$\begin{aligned} \ln(\Delta p) &= \frac{n+2}{2} \ln R_0 + H \\ &+ \frac{1}{2} \ln [1 - 2|\mathcal{A}| \cos(2\alpha \ln R_0 + B')], \end{aligned} \quad (67)$$

where H and B' do not depend on R_0 .

For small R_0 the leading part of this equation gives

$$\ln R_0 \sim \frac{2}{n+2} \ln(\Delta p). \quad (68)$$

Using this relation and iterating (67), one obtains

$$\ln R_0 \sim \gamma \ln(\Delta p) + f(\ln(\Delta p)) + C. \quad (69)$$

Here C is a constant,

$$\gamma = \frac{2}{n+2}, \quad (70)$$

and $f(z)$ is a periodic function of z , $f(z + \omega) = f(z)$, with the period ω ,

$$\omega = \frac{\pi(n+2)}{\sqrt{4+4n-n^2}}. \quad (71)$$

For $n = 1$ relations (70) and (71) reproduce the result obtained in [9,10].

B. $D > 6$ case

In this case the real coefficients C_{\pm} are connected with the asymptotic data $\{p, p'\}$ as follows:

$$\begin{aligned} p - p_* &= a_+ C_+ + a_- C_-, \\ p' - p'_* &= a'_+ C_+ + a'_- C_-, \end{aligned} \quad (72)$$

and one has

$$\begin{aligned} (\Delta p)^2 &= (a_+^2 + a_-^2) C_+^2 + (a_+^2 + a_-^2) C_-^2 \\ &\quad + 2(a_+ a_- + a'_+ a'_-) C_+ C_-. \end{aligned} \quad (73)$$

Equation (54) implies

$$C_{\pm} = \left(\frac{R_0}{\tilde{R}_0} \right)^{1-\beta_{\pm}} \tilde{C}_{\pm}. \quad (74)$$

Since $a_+^2 + a_-^2$ does not vanish and $1 - \beta_{\pm} > 0$, $1 - \beta_+ < 1 - \beta_-$, the first term on the right-hand side of the relation (73) is the leading one for small R_0 . $(\Delta p)^2$ as a function of R_0 has the following asymptotic form for small R_0 :

$$\ln(\Delta p)^2 \sim 2(1 - \beta_+) \ln R_0. \quad (75)$$

Or, equivalently, one has

$$\ln R_0 \sim \gamma \ln(\Delta p), \quad \gamma = \frac{n+2 + \sqrt{n^2 - 4n - 4}}{4(n+1)}. \quad (76)$$

Hence, for the number of dimensions higher than the critical one, one has the scaling law (76). The subleading oscillations are absent and the symmetry is continuous.

The relations (69)–(71) and (76) demonstrate the scaling law for the supercritical solutions close to the critical one. It is easy to check that similar relations are valid for subcritical solutions as well.

VIII. MERGER TRANSITIONS: INSIDE BRANE STORY

Let us imagine that there is an observer “living” within the brane who is not aware of the existence of the bulk extra dimensions. Let us discuss what happens from his or her point of view when the BBH system changes its phase. As before, we consider a one-parameter family and assume that the merger transition occurs at the special value of the parameter $\lambda = \lambda_*$. From the point of view of the brane observer, this family describes a process in which a brane black hole is either created or destroyed. The surface gravity of the brane black hole always coincides with the surface gravity κ of the bulk black hole, and hence it

remains constant during this process. For the description of the transition, it is convenient to make the Wick’s rotation $T \rightarrow i\tau$ and to choose the Euclidean time τ to have the period $\kappa/(2\pi)$. In such an (“canonical” by the terminology adopted in [3]) approach the Euclidean topology of the BBH system in two different phases is different. Namely, for the family of subcritical solutions the topology is $S^1 \times R^{D-1}$, while for the supercritical solutions the topology is $R^2 \times S^{D-2}$. Following [3], we call such a topology change transition the *merger transition*.

The internal geometry of the brane is determined by its embedding into the bulk spacetime. But if the brane observer does not know about the existence of extra dimensions, he/she may try to use the “standard” D -dimensional Einstein equations in order to interpret the observed brane geometry. Such an observer would arrive at the conclusion that the brane spacetime is not empty and there exists a distribution of the matter in it. By using the D -dimensional Einstein equations,

$$G_{ab} = R_{ab} - \frac{1}{2}\gamma_{ab}R = \mathcal{T}_{ab}, \quad \mathcal{T}_{ab} = 8\pi T_{ab}, \quad (77)$$

one can obtain the corresponding effective stress-energy tensor T_{ab} . (We use units in which the D -dimensional gravitational coupling constant is 1.) To calculate G_{ab} we write the induced metric on the brane (8) in the form $[A, B = (0, 1)]$

$$ds^2 = \gamma_{ab} d\xi^a d\xi^b = h_{AB} dy^A dy^B + \rho^2(y) d\Omega_n^2. \quad (78)$$

The symmetry of the metric (78) implies that \mathcal{T}_a^b has the following nonvanishing components:

$$\mathcal{T}_a^b = \begin{pmatrix} \mathcal{T}_0^0 & \mathcal{T}_1^0 & 0 & \dots & 0 \\ \mathcal{T}_1^0 & \mathcal{T}_1^1 & 0 & \dots & 0 \\ 0 & 0 & \hat{\mathcal{T}} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \hat{\mathcal{T}} \end{pmatrix}. \quad (79)$$

The calculations give

$$\begin{aligned} \mathcal{T}_{AB} &= n \left[-\frac{\rho_{:AB}}{\rho} + \left(\frac{\square^{(2)}\rho}{\rho} + \frac{n-1}{2\rho^2} \mathcal{P} \right) h_{AB} \right], \\ \hat{\mathcal{T}} &= (n-1) \left[\frac{\square^{(2)}\rho}{\rho} + \frac{n-2}{2\rho^2} \mathcal{P} \right] - \frac{1}{2} \mathcal{R}^{(2)}, \end{aligned} \quad (80)$$

where $\mathcal{P} = (\nabla\rho)^2 - 1$.

For the supercritical solutions, substituting (41) into (80), one obtains at the horizon ($Z = 0$)

$$\mathcal{T}_T^T = \mathcal{T}_Z^Z = \frac{n}{2R_0^2}, \quad \hat{\mathcal{T}} = \frac{n^2 + 2n - 4}{4R_0^2}. \quad (81)$$

Similarly, for the subcritical solutions, substituting (43) into (80), one obtains at the top of the brane ($R = 0$)

$$\mathcal{T}_T^T = -\frac{n}{2(n+1)Z_0^2}, \quad \mathcal{T}_R^R = \hat{\mathcal{T}} = \frac{n(n+3)}{2(n+1)^2 Z_0^2}, \quad (82)$$

Equations (81) and (82) show that in both phases the tensor T_a^b calculated at the horizon respects the symmetries of the induced metric.

IX. DISCUSSION

We discussed merger transitions in the brane–black-hole system. We would like to emphasize that there exists a striking similarity between this phenomenon and the merger transitions in the black-hole–black-string system [2,3]. Namely, close to the horizon the near-critical solutions of the BBH system have the same “double cone” structure as the solutions for the merger transition of the caged black holes. The equations defining the near-critical solutions, following from the Einstein action for the latter system, and the equations for the BBH system are very similar. In both cases there is a critical dimension of the spacetime where the scaling parameter α becomes real. For $D = 3$, the parameter α is the same in both cases.

Kol [3] discussed a possible relation between merger transitions and the Choptuik phenomena. This similarity with the critical collapse takes place in the BBH system as well. To demonstrate this let us consider a slow evolution of the BBH system in time which starts with no-brane black hole, so that a black hole is formed as a result of time evolution. As before, one has a one-parameter family of quasistatic configurations, and at the moment of brane–black-hole formation the asymptotic data $(\{p, p'\})$ reaches the critical value $(\{p_*, p'_*\})$. The relations (82) show that the curvature at the center of the brane is growing until it formally becomes infinite at the moment of the black-hole formation. For $D \leq 6$ the growing of the curvature is periodically modulated, as it happens in the critical collapse case [15,16]. For the number of dimensions higher than the critical one (for $D > 6$), this oscillatory behavior disappears. The scaling laws, relating the size R_0 of the formed brane–black-hole with Δp , also have the same scaling and self-similar behavior as in the critical collapse case [4].

In our analysis we focused on the static brane solutions and neglected the effects connected with the brane tension. Let us discuss what happens when the brane is moving. If the brane approaches the bulk black hole, the BBH will be created. In the inverse case when the brane, initially crossing the bulk–black-hole horizon, is moving away from the black hole, the BBH can disappear. In a general case, for finite velocity of the brane one cannot use the adiabatic approximation and describe the brane motion as a set of static configurations. To study dynamics of this process one needs to include a time variable explicitly and to solve a two-dimensional problem. This can be done numerically. In many aspects this problem is similar to (but more

complicated than) numerically solving equations for moving cosmic strings interacting with the bulk black hole [17,18]. Let us emphasize that the dynamics of the process of BBH disappearance is not a time reversed version of the BBH formation process.

The reason for this time asymmetry is connected with the presence of the bulk black hole. Consider a bulk black hole formed as a result of the gravitational collapse of some matter. By definition, the black hole is a region of the bulk spacetime causally disconnected from the future null infinity. A solution obtained by inversion of the direction of time $t \rightarrow -t$ describes a completely different physical process, an expansion of the matter from the white black hole (see e.g. [19]). Physical processes in the spacetime of a black hole obey the natural condition of regularity at the future event horizon. To study such processes in the spacetime of a black hole long after its formation one can “forget” about the details of the gravitational collapse and use the eternal black-hole approximation imposing the same regularity conditions at the future event horizon. For static solutions these regularity conditions imply regularity at the past horizon of the eternal black hole. For time-dependent configurations, in a general case, this is not true.

Interaction of a moving brane with the black hole was studied numerically in [20–22]. A main motivation for this study is connected with the following question: under which conditions can a black hole moving with some velocity escape from the brane [23,24]? By numerically solving test brane equations in a spacetime of a static black hole, it was shown [20] that in the considered cases a moving brane bends and eventually the radius of the pinched part goes to zero. This indicates that the extraction of the test brane from the bulk black hole is accompanied by its reconnection [20]. After the reconnection, a piece of the brane near the horizon (“a baby brane”) is absorbed by the black hole, while the other part, located outside of the black hole (“a mother brane”), keeps moving away from the black hole. In such a process a brane observer registers that the BBH disappears at the moment of the reconnection.

In the limit of small velocity of the brane, effects connected with the brane tension cannot be ignored and they can significantly change the dynamics of the system. Consider a D -dimensional brane with the tension σ in the spacetime of an N -dimensional black hole with the gravitational radius r_0 . For these parameters one can construct the following dimensionless combination: $\epsilon = G^N \sigma / r_0^{N-D-2}$, where G^N is the Newtonian coupling constant in the bulk spacetime. It is argued in [20,22] that the escape velocity for such a system is of the order of $\epsilon^{1/2}$.

In the above discussion we assumed that the brane is infinitely thin. In the field theory with spontaneous symmetry breaking, the branelike objects (e.g. domain walls) arise as a solution of nonlinear equations, and they have finite thickness. Static thick domain walls interacting with a black hole were studied in [25–29].

One can expect that for an infinitely thin brane in the process of its reconnection, during which the BBH disappears, the curvature of the induced metric infinitely grows, as it happens for the critical solutions discussed above. Under these conditions one cannot neglect the finite-thickness effect. One can expect that formal singularities in the infinitely thin brane description would disappear when one uses a (more fundamental) field theory description. In this connection the recent numerical calculations performed in [30] are very stimulating. It is interesting to analyze in more detail the dynamics of the destruction of BBHs in the framework of the field theory models. For a finite-thickness brane, a choice of the surface, which represents it, is not unique. Thus for the thick brane the meaning of the induced geometry is not well defined. If the gravity is an emergent phenomenon, the merger and topology change transitions in the physical spacetime might have similar features as those of the discussed toy

model. Namely, in the vicinity of these transitions one cannot use the metric for the description of the details of the transition anymore. Instead, a more detailed microscopic description in terms of constituents (e.g. strings) is required. The toy model proposed in this paper and its field theory analogues might be useful for modeling these transitions.

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