Artifacts and $\langle A^2 \rangle$ power corrections: Reexamining $Z_\psi(p^2)$ and Z_V in the momentum-subtraction scheme

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The next-to-leading-order (NLO) term in the operator product expansion (OPE) of the quark propagator vector part Z_ψ and the vertex function g_1 of the vector current in the Landau gauge should be dominated by the same $\langle A^2 \rangle$ condensate as in the gluon propagator. On the other hand, the perturbative part has been calculated to a very high precision thanks to Chetyrkin and collaborators. We test this on the lattice, with both clover and overlap fermion actions at $\beta = 6.0, 6.4, 6.6, 6.8$. Elucidation of discretization artifacts appears to be absolutely crucial. First *hypercubic artifacts* are eliminated by a powerful method, which gives results notably different from the standard democratic method. Then, the presence of unexpected, very large, nonperturbative, $O(4)$ *symmetric* discretization artifacts, *increasing* towards small momenta, is demonstrated by considering Z_V^{MOM}, which should be constant in the absence of such artifacts. They impede in general the analysis of OPE. However, in two special cases with overlap action—(1) for Z_ψ; (2) for g_1, but only at large p^2—we are able to identify the $\langle A^2 \rangle$ condensate; it agrees with the one resulting from gluonic Green functions. We conclude that the OPE analysis of quark and gluon Green function has reached a quite consistent status, and that the power corrections have been correctly identified. A practical consequence of the whole analysis is that the renormalization constant Z_ψ ($= Z_2^{-1}$ of the momentum-subtraction (MOM) scheme) may differ sizably from the one given by democratic selection methods. More generally, the values of the renormalization constants may be seriously affected by the differences in the treatment of the various types of artifacts, and by the subtraction of power corrections.

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I. INTRODUCTION

The study of the quark propagator and vertex functions in momentum space has been extensively pursued in the literature starting in the 70's with analytical considerations [1–3], later completed by the discovery of the presence of a contribution of the A^2 operator, due to gauge fixing [4]. Numerical lattice QCD has more recently treated this issue [5–13]. The scalar part of the quark propagator is related via axial Ward identities to the pseudoscalar vertex function. The role of the Goldstone boson pole in the latter has been thoroughly discussed [14–17].

In our paper Ref. [18], we have initiated a new study of the propagator and of the vector vertex function, which we pursue here. This paper has been devoted to an improved treatment of hypercubic artifacts. Now we present the whole of the work, which also includes the analysis of the other types of artifacts and the extraction of the power corrections.

We mainly study the vector part of the inverse quark propagator, the one which is proportional to $\not{p}$, Z_ψ, and the vector vertex function g_1 related to it by the Ward identity, or, equivalently, on the ratio $Z_V^{\mathrm{MOM}}(p) = \frac{Z_\psi(p^2)}{g_1(p^2)}$. One of the reasons is to check for the effect of the $\langle A^2 \rangle$ condensate which has been discovered via power corrections to the gluon propagator and three point Green functions at large momenta [19–22] (let us recall that, in order to perform trustable perturbative calculations, we take always $p > 2.6$ GeV). OPE shows that the effect of the $\langle A^2 \rangle$ condensate should have almost the same magnitude in the $Z_\psi(p^2)$ of the quark as in the gluon propagator. Moreover, the perturbative-QCD corrections are known to be varying especially slowly (the anomalous dimension being zero at one loop in the Landau gauge), as we shall recall later. This should give a favorable situation to display the power corrections (recall that in the gluon case, it was difficult to disentangle the power and the logarithmic corrections, which were moreover very sensitive to the value of Λ_{QCD}). Now, for the clover action, the crude values plotted in literature (with only a selection of democratic points) for $Z_\psi(p^2)$ are extremely flat above 2 GeV [23]. This was formerly considered as being natural, as a consequence of the vanishing of the one-loop anomalous dimension. However, thinking more about it, it must be considered on the contrary as very worrying since it means that the decrease predicted both by the perturbative-QCD corrections and the $\langle A^2 \rangle$ condensate *would not be seen*. This leads us to start a very systematic study of the problem, with the following series of improvements on earlier works:

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(i) We reach an energy of 10 GeV by matching several lattice spacings so that we are in a better position to eliminate lattice artifacts and also to identify power corrections, which requires a large momentum range.

(ii) We make use of a very efficient way of eliminating hypercubic artifacts. A first version was used to study gluon Green functions [24,25]. In Ref. [18], this method has been improved for the specific case of the quark Green functions, where such artifacts are huge, especially in Z_ψ and for the overlap action. Hypercubic artifacts have often been cured by the ''democratic'' method which considers only momenta with equilibrated values of the components. This method points in the right direction but, as we shall discuss in more details, is by far insufficient when the hypercubic artifacts reach such a level as illustrated in this article. In fact, we are then led to qualitatively different conclusions.

(iii) We make systematic use of Ward-Takahashi identities relating the quark propagator and the vertex function. According to these Ward identities, Z_V^{MOM} should be independent of p^2 up to artifacts. Then, Z_V^{MOM} is a very sensitive test of the presence of artifacts. The fact that we observe a strong dependence of the lattice Z_V^{MOM} on p^2 shows unambiguously the existence of large remaining discretization artifacts, this time respecting $O(4)$ symmetry, and *decreasing* with negative power of momentum. We can also check the consistency of Z_V^{MOM} with other determinations of Z_V.

(iv) The above mentioned $O(4)$ symmetric artifacts could constitute a very serious problem, since we do not know how to eliminate them, and they would hide the OPE power corrections. Fortunately, in the case of overlap fermions (with $s = 0$), we find that for Z_ψ, there remains only very small $O(4)$ artifacts. Then, we can obtain a satisfactory estimate of the $1/p^2$ correction due to the $\langle A^2 \rangle$ condensate. For this reason, we consider mainly overlap fermions (with $s = 0$), although they have some specific inconveniences (for example very large perturbative corrections).

This work, as well as the preceding one, clearly shows that lattice artifacts are overwhelming at the start, both the hypercubic and $O(4)$ ones. The hypercubic ones have been shown to be cleanly eliminated by our method. The remaining $O(4)$ ones, on the other hand, cannot, and we do not foresee the possibility of a similarly efficient method, wherefrom we have to rely on situations where they are small for reasons which are not known, so that their smallness appears accidental. In that respect, it may seem worrying; but, on the other hand, we are very happy to have found that the OPE can be checked to a good accuracy and power corrections are found consistent with the gluon analysis. It must be noted that on the whole, the main body of the analysis relies on the overlap data, because we have found that they exhibit smaller $O(4)$ artifacts.

Another obvious interest of the study is then to improve the determination of the momentum-subtraction (MOM) renormalization constants, by taking into account both continuum power corrections and artifacts; and, indeed it indicates that much care must be exerted in using the MOM renormalization approach when high precision will be required, a point which has already been illustrated by the Goldstone contribution to Z_P^{MOM}.

In Sec. II, we will recall theoretical premises; in Sec. III we will indicate the lattice conventions and the simulations which we have performed, as well as recall the general problems of the improvement; the three following sections are devoted to artifacts, with a large development justified by the importance of the topic: in Sec. IV, we will recall the method to eliminate hypercubic lattice artifacts, and emphasize the difference with the democratic method; in Secs. V and VI, we discuss the other artifacts (Lorentz scalar artifacts, volume effects); in Sec. VII, we will give the results of the overlap action; in Sec. VIII, we perform several consistency checks: chiral symmetry of the overlap action VIII A; comparison with clover action VIII B; finally, we note the agreement with the previous gluon analysis, Subsection VIII C. In Sec. IX, we will give our conclusions and further discussions.

II. THEORETICAL PREMISES

We work in the Landau gauge. Let us first fix the notations that we will use in the continuum, the precise lattice definitions being given later. We will use all along the Euclidean metric. The continuum quark propagator is a 12×12 matrix $S(p_\mu)$ for 3-color and 4-spinor indices. The inverse propagator is expanded according to:

$$S^{-1}(p) = \delta_{a,b} Z_\psi(p^2)(i\not{p} + m(p^2)), \tag{1}$$

where a, b are the color indices. $Z_\psi(p^2)$ being a standard lattice notation (for the precise lattice definition, see below, Sec. III). Obviously, one has in the continuum, with trace on spin and color:

$$Z_\psi(p^2) = 1/12\,\mathrm{Tr}(S(\mu)\gamma_\mu p_\mu)/p^2. \tag{2}$$

Sometimes one uses the alternative quantity:

$$b(p^2) = Z_\psi(p^2) m(p^2) \tag{3}$$

to describe the scalar part of the propagator.

Let us consider a colorless vector current $\bar{q}\gamma_\mu q$. The three point Green function G_μ is defined by

$$G_\mu(p, q) = \int d^4x d^4y e^{ip\cdot y + iq\cdot x} \langle q(y)\bar{q}(x)\gamma_\mu q(x)\bar{q}(0)\rangle. \tag{4}$$

The vertex function is then defined by

$$\Gamma_\mu(p, q) = S^{-1}(p) G_\mu(p, q) S^{-1}(p + q). \tag{5}$$

In the whole paper, we will restrict ourselves to the case where the vector current carries a vanishing momentum transfer q_μ. In the following we will omit to write $q_\mu = 0$ and we will understand $\Gamma_\mu(p_\mu)$ as the bare vertex function computed on the lattice.

From Lorentz covariance and discrete symmetries

$$\Gamma_\mu(p) = \delta_{a,b}[g_1(p^2)\gamma_\mu + ig_2(p^2)p_\mu + g_3(p^2)p_\mu \not{p} + ig_4(p^2)[\gamma_\mu, \not{p}]] \tag{6}$$

which should be obeyed approximately on the lattice, as we checked.

A. Renormalization and Ward-Takahashi (WT) identities

The renormalized vertex function is then $Z_2 Z_V \Gamma_\mu(p)$. Here we must say something of conventions for renormalization constants. The standard definition of renormalization constants has been to *divide* the bare quantity by the renormalization constant to obtain the renormalized quantity (except for photon or gluon vertex renormalizations Z_1 which we do not use). Z_2 is the standard renormalization of fermions $\psi_{\text{bare}} = \sqrt{Z_2}\psi_R$, $S(p) = Z_2(\mu^2) S_R(p)$. In principle renormalization of composite operators, for instance Z_V, should be defined similarly. We have followed this convention in our works on gluon fields. But, in the case of *quark* operators, an opposite convention has become standard in lattice calculations: $(\bar\psi \mathcal{O} \psi)_{\text{bare}} = Z_{\mathcal{O}}^{-1} (\bar\psi \mathcal{O} \psi)_R$; we feel compelled to maintain this convention for the sake of comparison with parallel works on the lattice. This explains our writing of the renormalized vertex function. In the continuum $Z_V = 1$ (conserved current). We keep Z_V since the local vector current on the lattice is not conserved, and the discrepancy, which is of course an artifact, generates however finite effects in graphs due to additional divergencies multiplying the a terms (which have higher dimension).

The Ward identity in the renormalized form tells us that at infinite cutoff:

$$(\Gamma_R)_\mu(p) = -i \frac{\partial}{\partial p^\mu} S_R^{-1}(p). \tag{7}$$

After multiplying both sides by Z_2^{-1} to return to bare quantities

$$Z_V \Gamma_\mu(p) = -i \frac{\partial}{\partial p^\mu} S^{-1}(p), \tag{8}$$

which from (1)–(6) implies

$$\begin{aligned} Z_V g_1(p^2) &= Z_\psi(p^2), \qquad Z_V g_3(p^2) = 2 \frac{\partial}{\partial p^2} Z_\psi(p^2), \\ -Z_V g_2(p^2) &= 2 \frac{\partial}{\partial p^2} b(p^2), \qquad g_4(p^2) = 0. \end{aligned} \tag{9}$$

We note that the first Eq. (9) implies that Z_V is independent of the renormalization scheme *up to artifacts* (it is a ratio of bare quantities). Of course, this will hold up to terms vanishing as inverse powers of the cutoff at infinite cutoff, which are called *artifacts* in the lattice language. It must be recalled that, on the lattice, the Ward identity is not exact, but holds only up to artifacts, because we work at finite cutoff, and the deviation will be found very large in some cases. A very important consequence of the Ward identity for our study is that the ratio $g_1(p^2)/Z_\psi(p^2)$ is constant up to artifacts, or that deviations of this ratio from a constant are pure artifacts.

Defining analogously the vertex function of the pseudoscalar $(\bar q \gamma_5 q)$ density

$$\Gamma_5(p_\mu) = g_5(p^2)\gamma_5, \tag{10}$$

the axial Ward identity implies

$$\frac{Z_P}{Z_S} m_q g_5(p^2) = b(p^2), \tag{11}$$

where m_q is $1/(2a)(1/\kappa - 1/\kappa_c)$.

B. MOM renormalization; radiative corrections

To perform renormalization on the lattice, we appeal as usual to the convenient MOM schemes, which does not refer to a specific regularization. To speak technically, the precise renormalization scheme that we use is the one called RI' by Chetyrkin, Eq. (26) in Ref. [26]. This is in fact the most standard MOM scheme in the continuum, developed a long time ago by Georgi, Politzer, and Weinberg. It consists of setting the renormalized Green functions to their *tree approximation* at the renormalization point $p_i^2 = \mu^2$, in the chiral limit. The inverse bare propagator $S^{-1}(p)$ is normalized through:

$$S_R^{-1}(p)|_{p^2=\mu^2} = \delta_{a,b}(i\not{p} + m(p^2)). \tag{12}$$

Making $p^2 = \mu^2$ shows that

$$Z_2^{\text{MOM}}(p^2) = Z_\psi(p^2)^{-1} \tag{13}$$

up to artifacts. To renormalize the bare vertex function $g_1(p^2)$, we multiply it by the factor $Z_2^{\text{MOM}}(\mu^2) Z_V^{\text{MOM}}(\mu^2)$. The MOM renormalization for g_1 must be chosen so that the renormalized WT identity (Eq. (7)) $g_1^R(p^2) = Z_\psi^R(p^2)$ holds, therefore $g_1^R(p^2 = \mu^2) = 1$; we deduce that $Z_V^{\text{MOM}}(\mu^2)$ is the ratio $Z_\psi(\mu^2)/g_1(\mu^2)$ *up to artifacts*, but as remarked above, this must be nothing but the scheme independent Z_V: therefore this ratio $Z_V^{\text{MOM}}(\mu^2)$ is independent of μ^2, up to artifacts. From now on, we define Z_V^{MOM} as the ratio $Z_\psi(p^2)/g_1(p^2)$ (measured in fact on the lattice); we write

$$Z_V^{\mathrm{MOM}}(a^{-1}, p^2) = Z_\psi(p^2)/g_1(p^2) \tag{14}$$

which recalls that Z_V^{MOM}, which should be in fact *independent of* p^2 in the limit where the cutoff a^{-1} is infinite, is not so at finite a^{-1}—i.e. there are artifacts.

The Ward identity implies that $Z_\psi(p)$ and $g_1(p)$ have, in particular, the same perturbative p scale dependence. From the calculations of Chetyrkin *et al.* [26,27], we may express, for example, the perturbative running of Z_ψ at large p as a function of the running $\alpha_{\mathrm{MOM}}(p)$. This is our main choice throughout this paper, although we discuss the effect of substituting an expansion in $\alpha_{\overline{\mathrm{MS}}}(p)$. More precisely, we will always choose to use the definition of $\alpha_{\mathrm{MOM}}(p)$ by the symmetric three-gluon vertex. The advantage of quarks is that we can reach an accuracy of four loops in the renormalization group (RG) expansion, because we do not need β_3^{MOM} (for symmetric $\alpha_{\mathrm{MOM}}(p)$), since the dimension of the fermion is 0 at lowest order in the Landau gauge. Since the expression is lengthy and not necessary for present understanding, we refer the reader to Appendix A. Of course, even with such accuracy, such an expression cannot be expected to hold for too small p; we esteem the lower bound to be $p_{\mathrm{min}} = 2.6$ GeV, from our experience in the case of gluons; indeed, we must avoid going down too close to the "bumps" which manifest clearly that the gluon Green functions become nonperturbative.

The perturbative calculation requires a value of Λ_{QCD}; in order to compare with existing values from a different origin, it is convenient to reexpress all values in terms of the equivalent $\Lambda_{\overline{\mathrm{MS}}}$, whatever the scheme employed to make the perturbative expansion of the Green functions. One advantage of $Z_\psi(p)^2$ is that it is not as sensitive to the Λ_{QCD} value as the gluon quantities, because of the vanishing of the LO fermion anomalous dimension in the Landau gauge; we choose $\Lambda_{\overline{\mathrm{MS}}} = 0.237$ GeV from a previous analysis [19], not far from the ALPHA estimate [28], $\Lambda_{\overline{\mathrm{MS}}} = 0.238(19)$ GeV; we will discuss in the end the sensitivity of our results to this choice.

$Z_\psi(p^2)$ and $g_1(p^2)$ should have also the same nonperturbative power corrections, up to a constant. We consider them now.

C. Power correction from the $\langle A^2 \rangle$ condensate

An OPE analysis as those performed in Refs. [19–22] leads to consider a $\langle A^2 \rangle$ condensate coupled to the quark propagator and vertex in Landau gauge. Let us recall that such a condensate could not contribute to gauge-invariant Green functions, and is present only in (gauge fixed) gauge-noninvariant Green functions. The meaning and magnitude of such a condensate has been extensively discussed in the recent literature. Our aim here is to detect its effect on the Green functions through OPE, which provides a way of testing theoretical ideas on its existence and magnitude.

For the propagator, we can write

$$S^{-1}(p) = S_{\mathrm{pert}}^{-1}(p) + i\not{p}\frac{d_{\mathrm{bare}}(\alpha_{\mathrm{bare}})}{p^2}\frac{\langle : A^2_{\mathrm{bare}} : \rangle}{4(N_C^2 - 1)}\delta_{a,b} + \cdots, \tag{15}$$

where we only keep the leading term in $\not{p}$. The calculation of the coefficients of the OPE has been performed in the chiral limit, and therefore one has as far as possible to stay near this limit.

In the renormalization prescription denoted by "RI'" in Chetyrkin papers (which amounts to the standard MOM of Georgi and Politzer in the chiral limit), and expanding everywhere in terms of α_s^{MOM}, we obtain from Eq. (15) (see Appendix B)

$$\frac{Z_\psi(p^2)}{Z_\psi^{\mathrm{pert}}(\mu^2)} = \frac{Z_\psi^{\mathrm{pert}}(p^2)}{Z_\psi^{\mathrm{pert}}(\mu^2)} + \frac{32\pi}{3}\alpha(p)\left(\frac{\alpha(p)}{\alpha(\mu)}\right)^{-((\gamma_{A^2}^{(0)} - \gamma_0)/\beta_0)} \times \frac{\langle (A^2)_R(\mu)\rangle}{4(N_C^2 - 1)}\frac{1}{p^2}, \tag{16}$$

where $S^{-1} = i\not{p}\delta_{a,b}Z_\psi$ and $S_{\mathrm{pert}}^{-1} = i\not{p}\delta_{a,b}Z_\psi^{\mathrm{pert}}$ up to $O(m_q\not{p}/p^2)$-terms. The condensate $\langle (A^2)_R(\mu)\rangle$ is renormalized at the scale μ. Z_ψ^{pert} is given in Eq. (2) and the coefficients (γ_0 being the fermion anomalous dimension to lowest order) are

$$\beta_0 = 11, \qquad \gamma_0 = 0, \qquad \gamma_{A^2}^{(0)} = \frac{35N_c}{12} = \frac{35}{4}. \tag{17}$$

As we have noted, $Z_V^{\mathrm{MOM}}(p^2)$ should be constant in p^2 from the Ward identity (9), up to artifacts; then it cannot receive any power correction from A^2, and therefore g_1 receives exactly the same contribution from the condensate as Z_ψ. We will use this as a very useful test.

The essential step is then to fit this formula on the lattice data to extract $\langle A^2 \rangle$. The renormalization constants at each β, $Z_\psi(\mu^2, \beta)$ will enter in the fit as free parameters to be determined, although they would be expected *a priori* to be close to lattice perturbation theory predictions. Of course, in general, we have to add lattice artifacts to Eq. (16), and one of the main problems we will discuss is how to determine them accurately.

An important warning must be made here, concerning the low accuracy in the perturbative calculation of the Wilson coefficient of A^2 written above, namely, it is only tree order with renormalization group improvement. Expanding in terms of α_s^{MOM}, although it may seem natural, is completely arbitrary, and one would wish the results to be the same with $\alpha_s^{\overline{\mathrm{MS}}}$. While this is the case to a good precision for Z_ψ^{pert}, this is obviously not the case here, due to the low order of the expansion: $\alpha_s(p)$ is quite different in the two schemes. At $p = 2.6$ GeV, the ratio MOM/$\overline{\mathrm{MS}}$ is around 2 and decreases slowly down to 1.4 at 10 GeV; taking into account the anomalous dimension amounts roughly to replace $\alpha_s(p)$ by $\alpha_s(\mu)$, then the ratio of

coefficients in terms of the two coupling constants is only slightly closer to 1: it is 1.5 in average over the whole range. This means that the coefficient is reduced by 50% when using $\alpha_s^{\overline{\rm MS}}$. This is due to the fact that the ratio of coupling constants decreases only very slowly up to the largest available momenta. As a consequence, the determination of $\langle A^2 \rangle$ obtained by fitting the lattice data will be automatically affected by the same amount. We will give the results with the convention of using everywhere $\alpha_s^{\rm MOM}$, as we have done for gluons. We shall first show that the power correction is indeed present and well determined, and then express it in terms of the condensate value, which suffers from the above uncertainty. We also note that the ratio of condensates fitted from gluons and quark Green functions, which should be 1 ideally, is not affected by this uncertainty, since the Wilson coefficients relative to the various Green functions differ mainly by purely algebraic numbers (the anomalous dimensions differ only slightly).[1]

III. LATTICE CALCULATIONS

A. Actions and parameters

We have first used Sheikholeslam-Wohlert improved Wilson quarks (here called clover) with the $c_{\rm SW}$ coefficients computed in [30]. 100 quenched gauge configurations have been computed at $\beta = 6.0, 6.4, 6.6, 6.8$ with volumes 24^4, 16^4, and 8^4. We have performed the calculation for five quark masses but in practice, for what is our concern in this paper, the quark mass dependence has not surprisingly proven to be negligible; anyway, since the theoretical calculations are performed in the chiral limit, we have to work as close as possible to the chiral limit; then, we present only for simplicity the results for the *lightest quark mass*, about 30 MeV, i.e.

$$\kappa = 0.1346, 0.135\,38, 0.135\,15, 0.134\,89$$
$$\text{for } \beta = 6.0, 6.4, 6.6, 6.8. \tag{18}$$

It should also be mentioned that all the results presented for clover action refer to the 24^4 lattices unless stated otherwise.

In addition to clover fermions, the use of overlap fermions [31] has been revealed to be necessary, and even crucial, to obtain a good determination of the power correction, because the most embarrassing type of artifacts, the $O(4)$ invariant ones, have been found to be smaller in certain cases with the overlap action (see Sec. VII). We have used approximately the same physical masses, i.e. as in the clover case

$$am_0 = 0.03, 0.016\,67, 0.0125, 0.01$$
$$\text{for } \beta = 6.0, 6.4, 6.6, 6.8 \tag{19}$$

[1]While finishing the article, we have become aware, thanks to D. Becirevic, of the calculation of the two-loop anomalous dimension of A^2 by [29], in the $\overline{\rm MS}$ scheme.

TABLE I. Lattices spacings.

β	6.0	6.4	6.6	6.8
a^{-1} (GeV)	1.966	3.66	4.744	6.1
a (fm)	0.101	0.055	0.042	0.033

with $s = 0$ and volumes of 16^4. The bare mass m_0 and s are defined from

$$D_{\rm over} = (1 + s + am_0/2) + (1 + s - am_0/2) \times \frac{D_w(-(1+s))}{\sqrt{D_w(-(1+s))^\dagger D_w(-(1+s))}}, \tag{20}$$

where $D_w(-(1+s))$ is the Wilson-Dirac operator with a (negative) mass term $-1-s$,

$$D_w(-1-s) \equiv \tfrac{1}{2}\gamma_\mu(\nabla_\mu + \nabla^*_\mu) - \tfrac{1}{2}a\nabla^*_\mu\nabla_\mu - 1 - s. \tag{21}$$

We use $s = 0$. $s = 0.4$ is considered preferable from locality requirements [32], however the difference is slight as soon as β is larger than 6.0. The reason for using only the small lattice 16^4 is well known; it is due to limitation in the special treatment needed for small eigenvalues of the Neuberger operator. In practice, as for clover action, we discuss only the lightest quark mass, roughly corresponding to the same 30 MeV.

We express everywhere quantities in physical units. Throughout this paper we will use for the lattice spacings the values in the following Table I which follow the β dependence found in Ref. [33], Appendix C, Formule C.1.

B. Improvements

The propagators $S(x, 0)$ from the origin to point x have been computed and their Fourier transform is defined as:

$$\tilde{S}(p) = \sum_x e^{-ip\cdot x} S(x, 0). \tag{22}$$

They have been averaged among all configurations and all momenta p_μ within one orbit of the hypercubic symmetry group of the lattice, exactly as for gluon Green functions in [24].

The two actions are $O(a)$ improved; however, this is not sufficient to improve the propagator itself, which is an off shell quantity. In the case of overlap quarks, the propagator and other Green functions can be improved according to a standard and exact procedure [34] which should eliminate $O(a)$ discretization errors in Green functions, at large p, in the perturbative regime[2]:

[2]Indeed, in our opinion, the argument on the vanishing of artifacts uses chiral symmetry of vacuum matrix elements, which holds only when spontaneous symmetry breaking can be neglected. Therefore, it should hold only at large p although this is not stated explicitly.

$$\tilde{S}_*(p) = \frac{\tilde{S}(p) - \frac{1}{2}}{1 - am_0/2}. \quad (23)$$

On the other hand, such a well-defined improvement recipe for the propagator with the clover action does not exist: we would have to fix the counterterms necessary to eliminate the $O(a)$ terms from the numerical study of the propagator itself. Note that the standard contact term with a perturbative evaluation of the coefficient leads to subtract from $Z_\psi(p)$ a term of the type a/p^2. Indeed, it subtracts a term of the type $a\ m(p^2)$, where $m(p^2)$ is the continuum mass term, vanishing at large p, with a rather small absolute value less than 30 MeV for $p > 2.6$ GeV, i.e. in the region where we are working. But this term happens to be much smaller than the observed artifacts. We indeed find a decreasing artifact: a 1.9 GeV$^3/p^2$ Eq. (40), which amounts to a 280 MeV at $p = 2.6$ GeV. This suggests that the standard perturbative improvement is rather ineffective. Anyway, the analysis of the artifacts is included in our discussion of remaining $O(4)$ invariant artifacts, and in fact, as we shall see, this is not feasible with accuracy.

From now on, the notation $S(p)$ will represent the improved quark propagator in the case of overlap quarks and the standard one in the case of clover quarks. Now there are still various ways to define the $Z_\psi(p)$ and $m(p)$ of Eq. (1) on the lattice.

In both cases we fit the inverse quark propagator by

$$\tilde{S}^{-1}(p) = \delta_{a,b} Z_\psi(p)(i\bar{\not{p}} + m(p^2)) \quad (24)$$

according to Eq. (1) and where $\bar{p}_\mu \equiv \frac{1}{a}\sin(ap_\mu)$. We write $Z_\psi(p)$ and $m(p)$ because of the loss of the Lorentz invariance. $Z_\psi(p)$ can then be written as:

$$Z_\psi(p) = \tfrac{1}{12}\mathrm{Tr}[\gamma_\mu \bar{p}_\mu S^{-1}(p)]/(\bar{p})^2. \quad (25)$$

We could have chosen instead of $\bar{p}_\mu$ any other definition equivalent to p_μ up to terms of order $\mathcal{O}(a^2)$, for example p_μ or $2/a\sin(ap_\mu/2)$. It is important to stress that after eliminating the hypercubic artifacts as explained in Sec. IV, the different definitions give similar results; however, the definition we choose here permits to reach the final result with smaller hypercubic corrections. Note that in our first gluon papers, we did not extract the hypercubic artifacts in this manner, and therefore the choice of an appropriate definition for the lattice gluon propagator was very important, as well as the one of the gluon vertex function.

The three point Green functions with vanishing momentum transfer are computed by averaging analogously over the thermalized configurations and the points in each orbit

$$G_\mu(p, q = 0) = \langle \gamma_5 \tilde{S}(p)^\dagger \gamma_5 \gamma_\mu \tilde{S}(p)\rangle, \quad (26)$$

where the identity $S(0, x) = \gamma_5 S^\dagger(x, 0)\gamma_5$ has been used. The vertex function is then computed according to Eq. (5) and we choose for the lattice form factor g_1

$$g_1(p^2) = \frac{1}{36}\mathrm{Tr}\left[\Gamma_\mu(p, q = 0)\left(\gamma_\mu - \bar{p}_\mu \frac{\bar{\not{p}}}{\bar{p}^2}\right)\right], \quad (27)$$

where the trace is understood over both color and Dirac indices.

Finally, according to the Ward identity (9) we compute Z_V^{MOM} simply from Eq. (14) where any effective p^2 dependence of Z_V should come only from lattice artifacts.

C. Classification of remaining artifacts

The question of eliminating lattice artifacts has been perhaps our main difficulty in this work. Let us stress that the discretization artifacts we will discuss are all due to the *QCD interaction*. Indeed, our definition of $Z_\psi(p^2)$ is such that it is equal to 1, as in the continuum limit, when interaction is switched off, and we have also $g_1 = 1$ in that case. This illustrates that, in general, it may not be sufficient, by far, to extract the free case artifacts, or to use such prescriptions as replacing p_μ by $\bar{p}_\mu$.

Note that we have used $\mathcal{O}(a)$ improved *actions*. But of course, there still remains artifacts of higher order in the lattice spacing, and furthermore, the Green functions are not thereby improved. In the overlap case, a simple definite recipe exists, as we have explained above, to improve exactly the propagator and the other Green functions at large p, but obviously $\mathcal{O}(a)$ artifacts at small p and $\mathcal{O}(a^2)$ remain.

Since we will have a detailed discussion, it is useful first to remind of the main species of artifacts which are expected. First, we have *discretization* artifacts,[3] which themselves split into two:

(i) hypercubic artifacts, treated in Sec. IV which are the most visible because they break the elementary $O(4)$ symmetry; they are seen, as we plot invariants of Green functions as a function of p^2, as a large discrepancy between the value for different orbits at the same p^2. With some simple democratic treatment, it is easy to get a relatively regular function of p^2. Nevertheless, in general, there remain nonanalytic oscillations, and to eliminate them will be shown to be very important to obtain the final physical result, and requires sophisticated methods.

(ii) $O(4)$ invariant discretization artifacts, which remain after elimination of the hypercubic ones, and which will be discussed in Sec. V. Let us say that this is the weakest point, because we do not have theoretical principles to determine their form, neither is there a systematical empirical method to determine them.

Second, there may be *finite volume* artifacts which will be discussed in Sec. VI, and very shortly since they do not seem sizable.

[3]We will use the term "discretization artifact" preferably to the other common one, "ultraviolet artifact" because, as we shall find, these artifacts may show up at small p as well, due to nonperturbative effects.

IV. ELIMINATION OF HYPERCUBIC LATTICE ARTIFACTS

A. Generalities

Since we use hypercubic lattices our results are invariant under a discrete symmetry group H_4, a subgroup of the continuum Euclidean $O(4)$, but not under $O(4)$ itself. This implies that lattice data for momenta which are not related by an H_4 transformation but are by a $O(4)$ rotation will in principle differ. Of course this difference must vanish in the continuum limit, i.e. when $a \to 0$. It must be considered as one species of the so-called discretization effects, or ultraviolet artifacts. They also may be termed as ''anisotropy artifacts.'' The hypercubic artifacts turned out to be particularly striking for Z_ψ. The results, including overlap-computed quantities, have already been presented in [18]. The raw lattice data for Z_ψ and Z_V^{MOM} exhibit dramatically the ''half-fishbone'' structure which is a symptom of strong hypercubic artifacts, and we recall that these effects are especially strong in the overlap case.

Altogether we would like to recall the following hierarchy: first, the hypercubic artifacts are 1 order of magnitude larger for overlap quarks than for clover ones. Second, for both types of quarks the hypercubic artifacts for Z_ψ are 1 order of magnitude larger than those for g_1.

These hypercubic artifacts have been of course a long standing problem in lattice calculations, and methods have been devised for a long time to handle them. A general idea has been the so-called democratic one: the hypercubic effects are minimal when the four components of the momentum for a given p^2 do not differ too much (this is democracy between components; ideal democracy is for diagonal $p \propto (1, 1, 1, 1)$). Then the question is how to make this criterion quantitative, the rationale being to find a compromise between two contradictory requirements: (1) to be as democratic as possible, which tends to reduce the number of points and (2) to retain enough points to have a real curve.

The precise criterion is often something of a secret recipe, not communicated in papers. On studying the gluon propagator, the authors of Ref. [35] have made explicit a selection method, keeping only the orbits having a point within a cylinder around the diagonal. Several other similar criteria have been written. As it can be seen, the compromise is never very good, leaving rather large oscillations in Green functions. Moreover, it is not suitable for a precision treatment.

The alternative idea which we propose, on the contrary, relies on the use of all the orbits, and a method to extract the physical point from an extrapolation of the different orbits.[4] A first successful application of this idea was for the gluon propagator [36].

[4]The initial idea is due to Claude Roiesnel.

B. Our method: the $\mathbf{p}^{[2n]}$ extrapolation method

We use here the final refined form of the method, the so-called $\mathbf{p}^{[2n]}$ extrapolation method, presented in [18], and which has been shown to be necessary to obtain satisfactory results for Z_ψ.

In order to perform a global fit, we start from the remark that we are dealing with dimensionless quantities g_1 and Z_ψ. It is thus natural to expect that hypercubic artifacts contribute via dimensionless quantities times a constant.[5] Next we assume that there is a regular continuum limit. We denote the generic Green function as Q; it depends *a priori* on p^2, but also $a^2 p^{[4]}$, $a^4 p^{[6]}$: $Q(p^2, a^2 p^{[4]}, a^4 p^{[6]}, a^4 p^{[4]}, \cdots)$, where $p^{[2n]} \equiv \sum_{\mu=1,4} p_\mu^{2n}$; we Taylor expand it around the $O(4)$ symmetric limit $Q(p^2, 0, 0, 0, \cdots)$. Of course we must truncate this Taylor expansion of Q in a and we choose to expand it up to a^4. Note that at this stage, the function $Q(p^2, 0, 0, 0, \cdots)$ may still depend on a through terms of the form $a^2 p^2$, etc., but we do not consider presently this further dependence; it will be discussed in the next sections.

As a conclusion, we have fitted our results over the whole range of p^2 according to the following formula, where the c_i's are constants independent of p^2:

$$Q(p^2, a^2 p^{[4]}, a^4 p^{[6]}, \cdots) = Q(p^2, 0, 0) + c_1 \frac{a^2 p^{[4]}}{p^2} + c_2 \left(\frac{a^2 p^{[4]}}{p^2} \right)^2 + c_3 \frac{a^4 p^{[6]}}{p^2} + c_4 a^4 p^{[4]}, \tag{28}$$

with indeed small χ^2's. We have also checked the validity of this expansion for the free propagator.

The functional form used for $Q(p^2, 0, 0)$ does not influence significantly the resulting artifact coefficients. We can even avoid using any assumption about this functional form by taking the value for $Q(p^2, 0, 0)$ at each p^2 as an independent parameter to be fitted.[6]

C. Quantitative comparison with the democratic method

We would like now to recall the quantitative comparison of our '' $\mathbf{p}^{[2n]}$ extrapolation method'' Eq. (28) with the more common ''democratic selection'' methods; the latter method is carefully defined in [35]. This comparison is important, since almost all works up to now are using some variant of the democratic method, and since the difference with this method is crucial, as we show, to extract power corrections.

Let us consider the overlap case. If we try to select [35] the orbits which are in a cylinder around the diagonal with a radius $2\pi/L$, this is too restrictive anyway for our 16^4

[5]We neglect a possible logarithmic dependence on p^2.
[6]We have enough data for that.

overlap case, where we have only 22 orbits at the start. In order to have a less restrictive democratic criterion and to make a bridge with our own method, we will use the $p^{[2n]}$'s defined above. In our language, democracy can be translated as having a small enough ratio $p^{[4]}/(p^2)^2$. Momenta proportional to (1, 1, 1, 1) and (1, 0, 0, 0) have ratios $1/4$ (minimum ratio, maximally democratic) and 1 (maximum, totally undemocratic), respectively. We then retain the democratic orbits defined by an intermediate $p^{[4]}/(p^2)^2 \leq .5$. This leaves already only 7 orbits out of 22 for every β.

In Fig. 1 we plot for Z_ψ the result of this selection, as compared with our own method. Figure 1 clearly shows visible oscillations in the democratic curve demonstrating that the hypercubic artifacts have not been totally eliminated, while after applying the $\mathbf{p}^{[2n]}$ extrapolation method, Eq. (28), the curves are perfectly smooth; they do not either exhibit the oscillations which remain in previous methods.

We prefer our own method for this reason and also because of the loss of information due to the rejection of ''undemocratic'' points, which leaves one with very few points. This appears crucial in the overlap case, where one is constrained to use small lattices.

Then another very important aspect appears: while completely eliminating the hypercubic oscillations, we also considerably modify the mean value of the curve, as defined by an analytic fit; for the present case, as seen in the figure, our curve is considerably lower, with a quite steeper descent than the democratic one.

We will return later to the fact that Z_V^{MOM} remains not at all constant in p after this treatment.

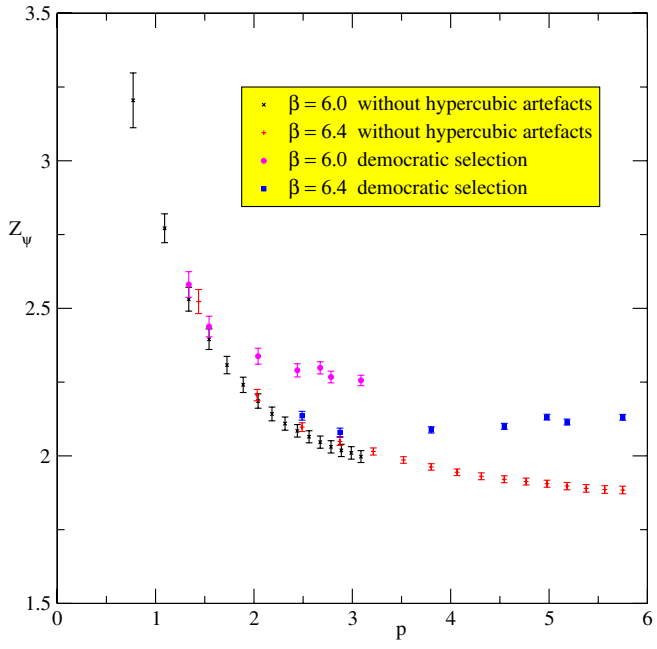

FIG. 1 (color online). Comparison between the democratic method and our refined treatment for the overlap $Z_\psi(p^2)$ at 6.0 and 6.4. Above 2 GeV, precisely where one must study the power corrections, the difference is striking.

D. The importance of optimizing the elimination of hypercubic artifacts

Let us stress then that, as is particularly visible from this comparison, the difference of the $\mathbf{a}^2 p^{[2n]}$ method with the standard democratic method is not at all academic in the context of the study of power corrections and renormalization constants. The difference with the previous versions of our method is also not negligible, as we have found.

(1) It is obvious that with the standard democratic method, we would obtain quite different results for power corrections and $O(4)$ symmetric artifacts, and therefore for the resulting perturbative contribution. In fact, it has not even been really considered that Z_ψ could be affected by such power corrections. (2) Moreover, we observe that the power corrections, as well as the residual $O(4)$ symmetric discretization artifacts, extracted by the previous variants of our method for treating the hypercubic artifacts are not the same; indeed, when using a previous cruder treatment for overlap action, we have found an important a^2p^2 artifact, which disappears with the more refined treatment, and we were also finding different power corrections (with a weaker condensate). The clover case shows similar spurious issues. This means that for a too crude treatment, some hypercubic artifacts can be spuriously mimicked as part of $O(4)$ symmetric discretization artifacts or continuum power corrections. This *does not* imply that the determination of power corrections is uncertain in this respect, but rather that it is very important to push hypercubic artifact elimination to the best to obtain the genuine continuum power corrections.

Let us finally mention an interesting consequence; as is well known, the values of Green functions at different momentum points in a Monte Carlo lattice calculation are highly correlated, which should lead to very small χ^2/d.o.f. for fits describing the p dependence by smooth analytic expressions (by small, we mean well below one). It is not found so with too crude treatments of the hypercubic artifacts, because of the erratic oscillations which always remain in the latter methods mimicking statistical deviations. We observe however an impressive decrease of the χ^2 down to its expected small value when we improve the treatment of the data, showing that we are now obtaining indeed very smooth functions as physically expected.

Another encouraging fact is that, after having eliminated consistently hypercubic artifacts, the different definitions of Z_ψ on the lattice converge to the same final result (see the comments around Eq. (24)).

V. PROOF OF THE PRESENCE OF $O(4)$ SYMMETRIC "NONCANONICAL" DISCRETIZATION ARTIFACTS

We now start, in the rest of the discussion, from the data obtained through the above treatment of the hypercubic artifacts. They still differ from the continuum by renor-

malization and by $O(4)$ symmetric discretization artifacts. It happens that the determination of these artifacts is still harder in general than for hypercubic ones (as for finite volume artifacts, which we estimate to be weak, see the corresponding short section below).

Indeed, in practice, there is no similar unambiguous method to determine the $O(4)$ symmetric discretization artifacts. True, they manifest themselves by a residual variation with β, and, in principle, we could study the variation with β at each momentum, and then extrapolate to the continuum. However, this requires too many momenta and too much accuracy, if we want really to extract the power correction from the extrapolation to the continuum.

(a) A possible general method for treating $O(4)$ scalar artifacts.
Then, a more practical method consists in assuming a prescribed analytical form for both the continuum and the artifacts, with some unknown parameters to be determined by the χ^2 adjustment.
One has therefore to appeal to our *a priori* knowledge (a) of the continuum, as a function of p; (b) of the structure of $O(4)$ symmetric artifacts, as a function of p and a, so that we could make fits with prescribed functions depending on a limited number of free parameters. For the continuum, this is exactly what is provided by the OPE, with the renormalization constants z_b and $\langle A^2 \rangle$ as free parameters. For $O(4)$ symmetric artifacts, a standard idea is to recourse to *lattice perturbation theory*, in which the structure of artifacts is easily explicited. After all, this is what we have invoked for the hypercubic artifacts. As for $O(4)$ symmetric artifacts, the result is quite simple: in the case of a scalar function, and in the chiral limit, where there is no other dimensioned parameters than a and p, there can be no other artifacts than $a^n p^n$, with $n > 0$. Then we could work with a few parameters only.

(b) Why it fails.
But now, there appears a very unfortunate circumstance. It is seen that this usual assumption of a perturbative structure of artifacts does not work at all, at least in general. We observe undoubtedly $O(4)$ symmetric artifacts decreasing with p, i.e. for instance of the form $1/p^n$ and $n > 0$ times some positive power of a. To show it, the best way is to consider the p dependence of $Z_V^{\mathrm{MOM}} = g_1/Z_\psi$, which should be momentum independent close to the continuum limit. Any p dependence is therefore to be attributed to artifacts. Now, the lattice data show quite clearly a very strong p dependence of Z_V^{MOM} except at large p. This is true for clover action, and for overlap action as well, see Fig. 2 for overlap fermions.
The artifacts are similar; they decrease monotonously with increasing p for $\beta = 6.4$, 6.6, 6.8; they could be fitted by negative powers of the momentum squared; for the overlap case, this is also true for $\beta = 6.0$.

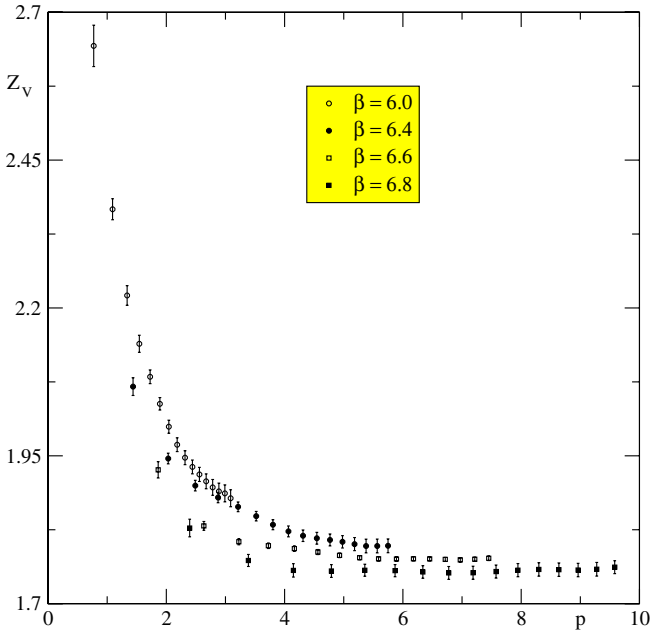

FIG. 2 (color online). Overlap Z_V^{MOM} as a function of momentum, respectively, for $\beta = 6.0$ and 6.4, and $\beta = 6.6$ and 6.8, for the overlap action. The huge momentum dependence which is displayed must be entirely attributed to discretization artifacts.

(c) Consequences for determination of power corrections.
The necessity of taking into account such negative powers of p is very embarrassing because, as we explained, we cannot distinguish the $O(4)$ symmetric discretization artifacts by the sole β dependence; we have to rely on the p dependence. Now it is clear that we will have much difficulty in distinguishing the power corrections and the artifacts decreasing with p, and therefore to determinate the condensate. One of the problems is that when increasing the number of negative power terms in the description of the artifact, the tendency is to get alternating signs, and therefore rather unstable results. There is no rationale as to where we should stop. In addition, we must consider that we have also as parameters in the fit the renormalization constants z_b, which are practically chosen as an independent parameter at each β. To add to the uncertainty, we observe that within the precision of the data, we can obtain equivalently good solutions by modifying the $O(4)$ artifacts with a correlative change in the z_b's. Then, in spite of many efforts, we have in fact not been able to extract *stable* and accurate values of $\langle A^2 \rangle$ by this method, although we have gotten a clear signal that it is *positive* and *sizable*. Then we have been able to fix the continuum power corrections only by

exploiting particular circumstances, to which we devote the rest of the analysis, after a few words on finite volume artifacts.

VI. FINITE VOLUME ARTIFACTS

A priori, finite volume artifacts are not expected to be important in this study because we work at large momentum with respect to the inverse lattice size. Let us recall that in the clover case [18], we have found only very small volume artifacts, after a careful study with 8^4, 16^4, 24^4 lattices. Only the first points with smallest n^2 were showing some effect.

We have not performed the same tests for the overlap action because of the slowness of numerical calculation; we have only ran on a 16^4 lattice. We are conscious that this is a possible weak point, since this volume is small, and we rely mainly on the analysis of overlap results for the determination of the condensate. So we are thinking of extending this analysis to larger volume as soon as possible. It would, in particular, be interesting to test the possibility of a chiral-symmetry restoration at small physical volumes (see further comments in Subsection VIII A).

We stress however that the consistency which is obtained for the continuum power corrections between the overlap results and the clover ones with a greater volume 24^4 (see Subsection VIII B), is a further proof that the volume effects are not crucial for our purpose of determining the *condensate*.

We must also observe that we can determine the condensate mainly from our large p data ($p > 4$–5 GeV) (see VII A), with large Lp, where volume artifacts are expected to be even smaller.

A. Discretization or volume artifacts?

The fact that the above $O(4)$ symmetric artifacts are observed at small p—and only at small p—is deserving of a special discussion, since it is so counter to usual expectation. Our ears are indeed accustomed to the dictum: ''artifacts at small p, finite volume artifacts,'' ''artifacts at large p, finite spacing artifacts.'' Why are they not volume artifacts? Since it is one important finding of our study, we collect here our arguments.

Let us stress that, in the overlap case, the use of a small volume 16^4 at large β, to which we are constrained, is not the reason for our finding of large $O(4)$ symmetric artifacts at small p. The first argument is that they are also seen in the clover case where we have tested in detail the smallness of volume artifacts; to reiterate, we have seen that only one or two of the smallest momenta seem affected by a volume dependence, while the artifacts we are discussing now are present over a large number of points.

Moreover, let us stress that they have the typical behavior of discretization artifacts, i.e. they decrease at large β and fixed p: indeed, Z_V (which should be flat in the continuum limit) is flatter and flatter as the beta increases at a fixed number of sites 16^4, i.e. as the physical volume decreases. More precisely, we observe that, as the volume decreases, we have more and more points where it is flat, i.e. where it has small artifacts: the respective number of points is 0 at $\beta = 6.0$, 3 at $\beta = 6.4$, 9 at $\beta = 6.6$, 14 at $\beta = 6.8$. This is exactly counter to a volume effect, for which the flatness should be obtained for n^2 larger that some *fixed number*. In other terms, for a volume effect, we would expect the artifacts to vanish at smaller and smaller values of p as β decreases (i.e. when the physical volume increases). Instead, we find that they vanish around 4–5 GeV irrespective of $\beta = 6.4, 6.6, 6.8$; even more, at $\beta = 6.0$, with the largest volume, Z_V never becomes flat.

On the other hand, we have no understanding of why Z_V^{MOM} seems devoid of sizeable artifacts at large p: it must be regarded as an accident. But it must be underlined that this is not an accident specific to our particular problem. It is a well-known fact, on which any MOM practitioner is relying without being able to explain it: one measures the Z's in regions of large p, assuming that artifacts are small, although they would be expected to be large precisely there from lattice perturbation theory.

VII. CASES WITH SMALL $O(4)$ SYMMETRIC ARTIFACTS WITH THE OVERLAP ACTION—DETERMINATION OF THE CONDENSATE

As we have underlined as the conclusion of Sec. V, paragraph (c), the $O(4)$ symmetric artifacts we have found, when present, impede a clear determination of the power correction. It is fortunate that they seem absent or small in some cases. All correspond to overlap action; then the results are convincingly consistent. Therefore, we concentrate on the overlap fermions in the present and following Secs. VII and VIII A. As concerns the clover action, we will not be able to have such compelling conclusions, but we show that they are at least compatible (see Sec. VIII B).

A. Large p ($p \gtrsim 5$ GeV)

At this point, we notice that the above difficulties may disappear first at large p, above roughly 5 GeV (therefore at $\beta = 6.4, 6.6, 6.8$), because Z_V is constant in this region, especially for the overlap action; therefore it is suggestive that the artifacts of Z_ψ and g_1 are small there, barring for the unprobable eventuality that Z_ψ and g_1 would happen to have exactly the same artifacts. We then fit Z_ψ and g_1 with a formula containing no $O(4)$ artifact, i.e. we take only the continuum expression times the renormalization factor z_b, at fixed β, and we obtain then a very encouraging conclusion. Choosing the window of p within which Z_V is well constant, we obtain at each $\beta = 6.4, 6.6, 6.8$, and for Z_ψ and g_1, i.e. for *six* independent data, almost the same condensate value; we quote a common fit (see Fig. 3) to the three Z_ψ, with $p_{\min} = 5, 5, 4$ GeV, respectively, corresponding to the respective p where Z_V is beginning to be

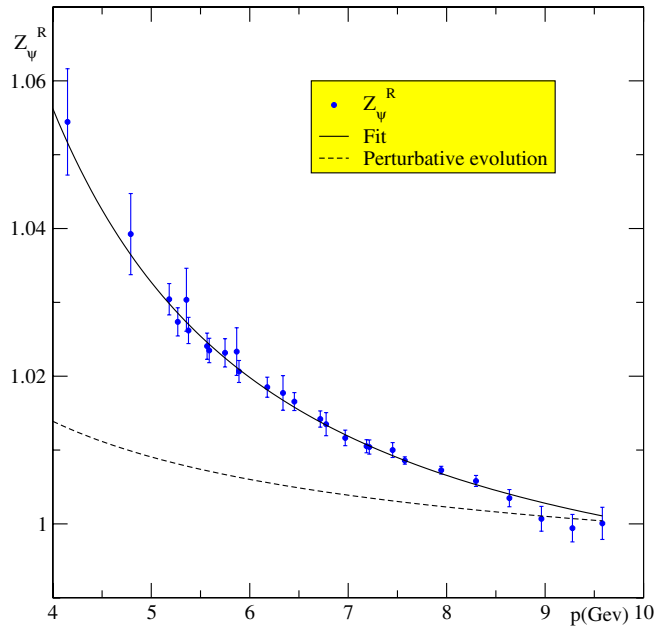

FIG. 3 (color online). Renormalized overlap Z_ψ at $\beta = 6.4$, 6.6, 6.8, and large p, compared with the purely perturbative result. The fit of the solid line is made with the perturbative part and only the condensate term, in the windows where the respective $Z_V^{\rm MOM}$ become flat (solid line). One finds $\langle A^2 \rangle = 3.1 \pm 0.3$. Note that Z_ψ is still far from being flat. It has a monotonous decrease, of around 6% over the range, mainly due to the condensate. The dotted line is the purely perturbative result, clearly inconsistent with the data. Normalization is made at 10 GeV. g_1 is described by a similar fit.

flat:

$$\langle A^2 \rangle \simeq (3.1 \pm 0.3)\ \text{GeV}^2 \tag{29}$$

large and positive, with a rather small error, and quite consistent with what is found in the gluon sector (see below). The χ^2/d.o.f. $= 0.07$ is very small as expected from very correlated data. Let us recall that this value of $\langle A^2 \rangle$ is obtained with the convention that the Wilson coefficient of the operator is expressed in terms of $\alpha_s^{\rm MOM}$.

A remarkable feature of this region of momentum is that both Z_ψ and g_1, and therefore Z_V too, are almost independent of β. We take it as a further indication that artifacts are accidentally small there.

Let us reinsist that the value of the condensate given in Eq. (29) corresponds to the convention, always followed in this paper, that the Wilson coefficient of the operator, calculated only at leading order, is expressed in terms of $\alpha^{\rm MOM}$. The choice of $\alpha^{\overline{\rm MS}}$, would lead to an appreciably higher value (larger by around 70%). However, what is important is that the power correction by itself is well determined by our analysis, almost independently of such a change. Indeed, let us replace the OPE expression for the power correction by a simple power, without logarithms corresponding to α_s, while maintaining the full perturbative expressions for the perturbative part. The fit then gives for the coefficient c of the power term c/p^2:

$$c = (0.767 \pm 0.083)\ \text{GeV}^2 \tag{30}$$

when using $\alpha^{\rm MOM}$ in the perturbative part, and:

$$c = (0.844 \pm 0.083)\ \text{GeV}^2 \tag{31}$$

when using $\alpha^{\overline{\rm MS}}$ instead, a small change indeed, of only 10%, reflecting the small change of the perturbative part, which is calculated by theory to a great accuracy.

B. Z_ψ over the whole allowed range of p

In addition, we observe another fact: Z_ψ—but not g_1—is strikingly independent of β over the full range of p. We have no explanation for that, but we can at least interpret this as meaning that Z_ψ is free of $O(4)$ symmetric artifacts over all this range. And indeed, we can fit Z_ψ on the full range $p > 2.6$ GeV allowed for the perturbative calculation, and the four β's with:

$$\langle A^2 \rangle = (2.73 \pm 0.21)\ \text{GeV}^2, \tag{32}$$

see Fig. 4.

We thus obtain a remarkable similarity of the condensate with the previous value. We also obtain similar results by varying the window of the momenta, provided the lower limit is not pushed beyond 5 GeV, and also by selecting various triplets of β values. This seems to support the consistency of our assumptions about artifacts.

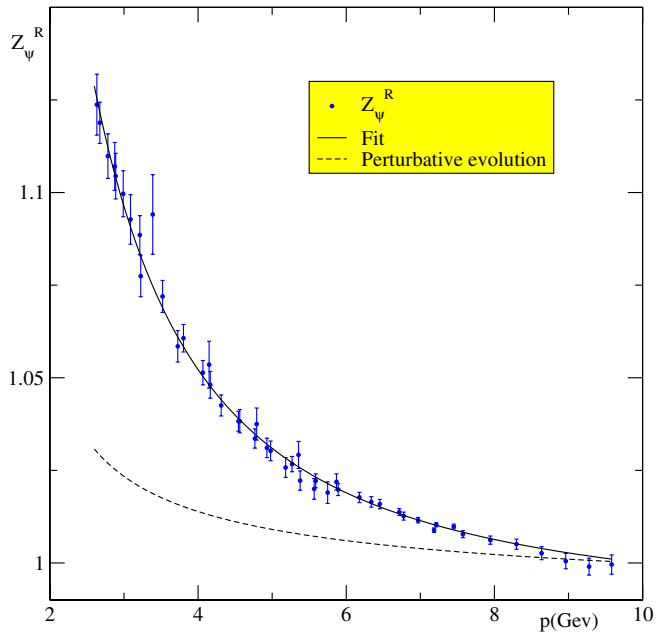

FIG. 4 (color online). Overlap Z_ψ at $\beta = 6.0$, 6.4, 6.6, 6.8, renormalized at 10 GeV. The fit for Z_ψ extends to the full allowed range of momentum, with $\beta = 6.0$ included, and a close value of the condensate $\langle A^2 \rangle = 2.75 \pm 0.2$. The dotted line is the purely perturbative result.

If we want to still improve the agreement with the large p analysis, we can introduce a further $1/p^4$ term, accounting for the possibility that we may not be sufficiently asymptotic to have a good description with the $1/p^2$ condensate alone. For the full window, and the four β's, we obtain a very good fit with:

$$\langle A^2 \rangle = (3.2 \pm 0.3)\ \mathrm{GeV}^2 \tag{33}$$

and with a small subleading term:

$$(-1.1 \pm 0.4)\ \mathrm{GeV}^4/p^4 \tag{34}$$

with a minus sign which explains that the value of $\langle A^2 \rangle$ is slightly higher than in the preceding fits. It is then closer to the large p fit.

C. Z_V^{MOM}

From our analysis, we can deduce values of Z_V^{MOM} standardly defined, i.e. free from artifacts:

$$\begin{aligned} Z_V^{\mathrm{MOM,\ large\ } p}(6.4) &= 1.798 \pm 0.012, \\ Z_V^{\mathrm{MOM,\ large\ } p}(6.6) &= 1.776 \pm 0.003, \\ Z_V^{\mathrm{MOM,\ large\ } p}(6.8) &= 1.756 \pm 0.011, \end{aligned} \tag{35}$$

''*large p*'' superscript is to recall that we have selected the value at large p, where we hope to have small $O(4)$ artifacts, in view of the observed flatness; the precise value is obtained by a fit. For $\beta = 6.0$, we quote $Z_V^{\mathrm{MOM,\ large\ } p}(6.0) = 1.878 \pm 0.014$, but Z_V^{MOM} is not yet flat at the highest momentum. Let us repeat that it is remarkable, and perhaps surprising, to observe such a constancy with β, while one-loop perturbation theory would predict a strong variation with β.

D. Comparison with lattice perturbation theory

The fact that Z_ψ and Z_V are very different from 1 (in fact not far from 2) in the overlap case with $s = 0$ may seem surprising. But in fact, already in lattice perturbation theory, the tendency is that the one-loop corrections to Z_ψ and Z_V are large because of a very large tadpole contribution to the self-energy [37,38] (while g_1 remains close to 1 as we find nonpertubatively). The net effect is already large at $\beta = 6.0$, for the usual $s = 0.4$: $Z_V = 1.247$, and still larger for our $s = 0$: $Z_V = 1.444$; we use Table 1 of Ref. [37] for the analytical expressions (the definition of Z_ψ is different, but by a negligible amount); the numbers are quoted assuming a boosted perturbation theory (BPT) boosted coupling with $g^2_{\mathrm{BPT}} = 1.68$ (see Ref. [38] under Eq. (44)). Of course, what is surprising is first that the nonperturbative determination (35) is still much larger, and, second, that it is almost independent of β over a large range.

Note that the value found by Ref. [39] for Z_A at $\beta = 6.0$, $Z_A = 1.55$, which should equate Z_V, is also much larger than BPT; however, this is not Z_A^{MOM} as measured directly; it is the Z_A deduced from hadronic WT identities. One may think of large artifacts which render different the results from various definitions of Z_A, Z_V; our study below, Sec. VIII A 2, shows indeed the presence of such effects, very large at $\beta = 6.0$; but they are decreasing rapidly at larger β, and our finding is that, at $\beta = 6.8$, Z_V^{MOM} is still much larger than the boosted lattice perturbation theory.

VIII. CONSISTENCY CHECKS

A. Z_V and Z_A with the overlap action—complementary studies on chiral symmetry and artifacts

To ascertain the soundness of our analysis, which may be surprising in several respects, we also perform several consistency checks of general nature in the overlap case; indeed, some strong statements deriving from chiral symmetry can be formulated.

1. Z_A/Z_V in the nonperturbative MOM scheme

The overlap action has an exact chiral symmetry, discovered by Luescher, which, combined with the choice of the improved current, or equivalently, of improved Green functions, should lead to $Z_A^{\mathrm{MOM}}/Z_V^{\mathrm{MOM}} = 1$ without any artifact. However, let us note that this is true only if one assumes the symmetry of the vacuum state, and therefore the absence of spontaneous symmetry breaking. In the presence of spontaneous symmetry breaking, one expects (1) a physical effect from the Goldstone boson, (2) discretization artifacts, both vanishing with inverse powers of momentum. Therefore this result only holds at large momenta.

We find indeed $Z_A^{\mathrm{MOM}}/Z_V^{\mathrm{MOM}} = 1$ to a high accuracy, Fig. 5, above 2.4 GeV, for the four β's. On the other hand, we see that for lower momentum: (i) the ratio differs from 1, and (ii) it depends on β. Observation (i) has also been made for domain wall fermions at $\beta = 6.0$ [40].

At $\beta = 6.0$, there is a rather abrupt change of regime slightly below $p = 2$ GeV. At higher $\beta = 6.4$, 6.6, the values below $p = 2$ GeV differ both from $\beta = 6.0$ and, slightly, from 1. Unhappily, we have no point in this region at $\beta = 6.8$. Therefore, at the two or three lowest β, we have a combination of discretization artifacts and perhaps a not well-determined physical effect; on the other hand, at $\beta = 6.8$, all the known points being at large p, it is impossible to conclude whether the flat result 1 is due to restoration of chiral symmetry at small volume or to extinction of spontaneous symmetry breaking effects at large p; working with larger volumes would allow the question to be settled.

2. Z_A from a hadronic WT identity against Z_V^{MOM}

In principle, once artifacts and power corrections have been eliminated, the various ways of defining the renormalization constants can be related by perturbation theory. In the case of Z_V or similar cases, for the same action, implying identical finite parts, they should even be equal.

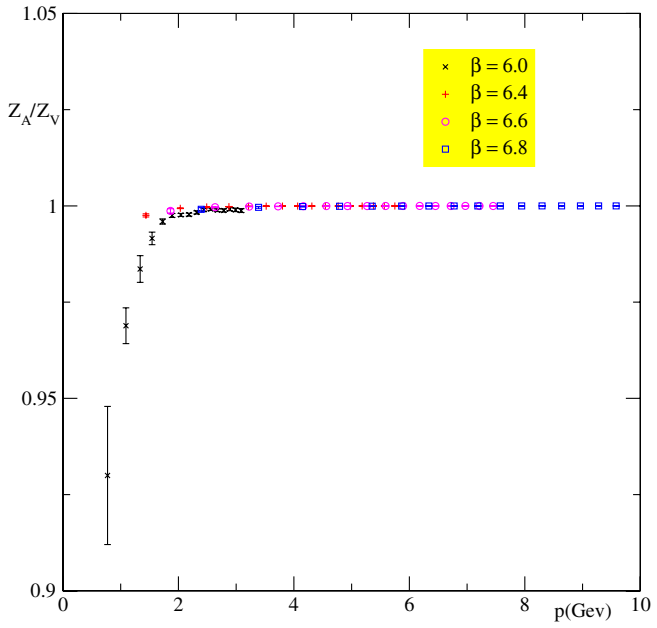

FIG. 5 (color online). $Z_A^{\mathrm{MOM}}/Z_V^{\mathrm{MOM}}$ for overlap fermions and small quark mass at the four $\beta = 6.0, 6.4, 6.6, 6.8$. The ratio is 1 above 2 GeV, but differs from 1 and depends on β below 2 GeV.

Moreover, Z_A should be equal to Z_V from the chiral symmetry of the overlap action.

We verify this statement, as a highly nontrivial consistency check of our treatment, by measuring Z_A from a standard hadronic Ward-Takayashi identity $Z_A^{\mathrm{WI}} = m_q/\rho$, where:

$$\rho = 1/2\langle \partial_0 A_0, P_5\rangle/\langle P_5, P_5\rangle. \tag{36}$$

In contrast to $Z_V^{\mathrm{MOM,\ large\ } p}$, Z_A^{WI} presents a very strong variation with β: at 6.0, the situation seems hopeless, with $Z_A^{\mathrm{WI}} \sim 3$, but the decrease is rapid:

$$\begin{aligned} Z_A^{\mathrm{WI}}(6.0) &= 3.03, \qquad Z_A^{\mathrm{WI}}(6.4) = 2.046, \\ Z_A^{\mathrm{WI}}(6.6) &= 1.90, \qquad Z_A^{\mathrm{WI}}(6.8) = 1.78 \end{aligned} \tag{37}$$

and one reaches finally a level close to $Z_V^{\mathrm{MOM,\ large\ } p}$. Moreover, $Z_A^{\mathrm{WI}}(a)$ is linear in a^2 to a very good precision:

$$Z_A^{\mathrm{WI}}(a) = 1.65 + 5.52a^2, \tag{38}$$

which shows that the difference $Z_A^{\mathrm{WI}} - Z_V^{\mathrm{MOM,\ large\ } p}$ is indeed a discretization artifact, as it should be, and one of the most canonical species, since we expect precisely chiral-symmetry breaking to be at most $\mathcal{O}(a^2)$. Indeed, on hadron states, the Ward identity is valid up to $\mathcal{O}(a^2)$, and Green functions at large p, where we measure $Z_V^{\mathrm{MOM,\ large\ } p}$, we have also only $\mathcal{O}(a^2)$ artifacts from chiral-symmetry arguments, in the chiral limit.

B. Check through the clover Z_ψ

In view of the rather paradoxical situation which allows us to determine $\langle A^2 \rangle$ from the $s = 0$ overlap action, it is important to reascertain this analysis by a study of the standard clover action, to check whether we can obtain a similar *continuum* result. Let us stress indeed that from renormalizability, the whole continuum function must be the same up to an overall constant for all the versions of the action. *A priori*, as we have said, the clover case may seem hopeless because there are large $O(4)$ artifacts even in Z_ψ; and even at large p, Z_V is not so flat.

However, one can benefit from the knowledge gained in the overlap case. In the end, the situation appears in fact somewhat similar for the clover action as concerns $O(4)$ symmetric artifacts: the main burden of noncanonical artifacts appears to concern g_1, and we can obtain a good description of Z_ψ with a minimal a^2p^2 *canonical* artifact, and just one a/p^2 term. Let us insist that one should consider this structure as a very empirical fitting form. One should not take literally the power exponents in p and a, since the artifacts are very large. The important point is that one has both canonical increasing artifacts and noncanonical decreasing artifacts. We obtain:

$$\langle A^2 \rangle = (2.4 \pm 0.3)\ \mathrm{GeV}^2 \tag{39}$$

with artifacts

$$-(0.005 \pm 0.0015)a^2p^2 - (1.9 \pm 0.4)\ \mathrm{GeV}^3 a/p^2. \tag{40}$$

In the present case, the role of these artifacts is crucial to obtain the condensate. In fact, before extraction of these artifacts, the clover data show a rather flat behavior at large p, or even a small increase at 6.8. The very large term a/p^2 considerably improves the fit, with a χ^2 divided by 3, and brings in a value of the condensate quite close to the one obtained with overlap fermions. These results seem to be a signal that we have obtained a rather accurate treatment of artifacts. Of course, one may be worried of having introduced a noncanonical artifact, which was not necessary in the overlap case; yet, we must remember that there is no logical reason why such terms should be absent (in the overlap case, they are present anyway in g_1).

With a subleading term

$$\propto 1/p^4 \tag{41}$$

we obtain a still somewhat better agreement with the overlap condensate,

$$\langle A^2 \rangle = (2.83 \pm 0.35)\ \mathrm{GeV}^2. \tag{42}$$

The subleading term is also of the same sign and comparable magnitude as for overlap action,

$$(-1.85 \pm 0.85)\ \mathrm{GeV}^4/p^4. \tag{43}$$

Note that the consistency of the clover 24^4 results with the overlap analysis is comforting the idea that the volume effects are not too important in the overlap case, although certainly they are less compelling because of the need to introduce the a/p^2 artifact.

C. Consistency with the gluon analysis

To stress the overall consistency of the analysis, one has also to consider the agreement with the previous gluon analysis. We quote the result of a combined fit of $\langle A^2\rangle$ and $\Lambda_{\rm QCD}$ to the gluon propagator and the symmetric three-gluon vertex [21]:

$$\begin{aligned}&\langle A^2\rangle = (3.6 \pm 1.2)\ \text{GeV}^2(\alpha_{\rm MOM}),\\ &(2.4 \pm 0.6)\ \text{GeV}^2(\text{gluon propagator})\ldots\end{aligned} \tag{44}$$

with $\Lambda_{\overline{\rm MS}} = 0.233 \pm 0.028$ GeV, and three-loop anomalous dimensions (the use of our present $\Lambda_{\overline{\rm MS}} = 0.237$ GeV would lower somewhat the condensate values). The fit has been done with the same convention as for quarks, that the Wilson coefficient (calculated only to one loop) is expressed in terms of $\alpha_s^{\rm MOM}$. (We could get free from this convention by comparing the magnitude of power corrections themselves as in Eq. (30)). The coincidence of the values of $\langle A^2\rangle$ with the quark value is highly significant since it concerns the continuum function, extracted by a series of various, independent, manipulations committed on the gluon and quark Green functions, and since these continuum functions are related only thanks to the OPE.

The quark measurement has much smaller statistical errors. This is simply due to the fact that in the gluon case, we have left free the value of $\Lambda_{\rm QCD}$, and moreover, the value of the condensate depends strongly on this value, whence the large errors in the gluon case. On the contrary, in the quark case, we have chosen a fixed $\Lambda_{\overline{\rm MS}} = 0.237$ GeV. Indeed, in this case, the dependence on $\Lambda_{\rm QCD}$ is rather weak. Then it is useless to try to determine $\Lambda_{\rm QCD}$ from the fit, and on the other hand the value obtained for $\langle A^2\rangle$ remains well determined if we allow some variation in $\Lambda_{\rm QCD}$. Another advantage of quarks is that we can reach the accuracy of four loops in the theoretical expression for the purely perturbative part. In the gluon case, such an accuracy is only possible for the asymmetric vertex, but in that case, the leading term in OPE is not given by $\langle A^2\rangle$ [22].

IX. CONCLUSIONS AND DISCUSSIONS

A. Physical results and evidences of various artifacts

Let us summarize our results in the following few points:

(i) First of all, and this is the main point, let us stress that we have finally obtained a rather nontrivial confirmation of the validity of the OPE in the nongauge-invariant sector of *lattice* QCD (treated numerically). The virtue of OPE is that one can describe the departure of all the various Green functions from the perturbative approximation at large momenta with the same set of expectation values. We have now obtained consistency not only between gluon Green functions, but also with the quark sector. This is highly nontrivial. Let us recall that there is often some doubt raised about OPE itself, and the possibility is sometimes considered that power corrections could be present not corresponding to an operator vacuum expectation value (v.e.v.). The final consistency of the determination of the condensate from three distinct types of Green functions strongly suggests that this is not the case, at least for leading power $1/p^2$ corrections. We do identify an OPE power correction, consistently related to A^2. $\langle A^2\rangle$ is found consistently large and positive. The precise magnitude of $\langle A^2\rangle$ is affected by an important uncertainty, due to the low accuracy of the theoretical calculation of the Wilson coefficient. But the power correction (i.e. the product of the coefficient and the condensate) is well determined by the lattice analysis, and the ratio of the power corrections in the various Green functions is actually as expected from lowest order OPE.

(ii) It turns out that the lattice discretization artifacts are unusually sizable in the quark propagator Z_ψ, but a clear-cut distinction must be made between hypercubic artifacts, which are gigantic, but can be efficiently eliminated, and the $O(4)$ symmetric ones, which are not so catastrophic, but that we have not been able to handle systematically.

(iii) We believe that we have been really efficient in getting rid of the hypercubic artifacts thanks to our (improved) method of "restoration of $O(4)$ symmetry."

(iv) Once these artifacts have been subtracted, the overlap $Z_V(p^2) \equiv Z_\psi(p^2)/g_1(p^2)$, which should be independent of p except for artifacts, is very close to a constant at large $p > 5$ GeV. This is far from trivial and supports the statement that we have no remaining $O(4)$ symmetric artifacts in this specific region. This is directly supported by the near constancy of the quantities as a function of β.

Moreover, also in the overlap case, but for $Z_\psi(p^2)$ only, the same statement of constancy with β extends down to the lowest momenta; this leads one to suspect that $O(4)$ symmetric artifacts are small in this case over the whole range of p. We are unable to explain these two special situations. Let us recall that a certain flatness $Z_V(p^2)$ at large p is also observed with the clover quark action, although it is not so good. Let us recall also that this region is the basis for the standard determi-

nations of MOM renormalization constants, usually with the clover quark action (see for example [41], which presumes that discretization artifacts are not large there. Let us then recall that we have no theoretical argument supporting the statement that they are not large. Quite the contrary. If any, the theoretical arguments would suggest for them to increase with p. The support is purely empirical. This is embarrassing if we aim at precision determinations.

(v) Z_V of the overlap action at $s = 0$ is large, around 1.8, in qualitative agreement with BPT perturbation theory which finds large self-energy contributions in $Z_\psi(p)$. But it is still much larger than the expectation, and the lack of dependence on β is not understood from perturbation theory.

(vi) Considering the cases where the $O(4)$ symmetric artifact-free results are suspected to be small, we try to fit them by OPE, i.e. by the four-loop perturbative contribution plus the $\langle A^2 \rangle$ condensate contribution computed to leading logarithm. The overlap Z_ψ and g_1 at large $p > 5$ GeV allows for a good fit for $\beta = 6.4, 6.6, 6.8$, leading to a consistent $\langle A^2 \rangle$ not far from 3 GeV^2. The overlap Z_ψ also allows for a good fit for the whole range $p > 2.6$ GeV, including in addition $\beta = 6.0$. The $\langle A^2 \rangle$ condensate is consistent with the former value. A very small $1/p^4$ term still improves the consistency.

(vii) In the other cases, namely, in g_1 and Z_V for p lower than $p \sim 5$ GeV, the $O(4)$ symmetric artifacts become large, especially at small p, and in fact, they increase regularly from large to small p. This trend, which is also contrary to the expectation of lattice perturbation theory, clearly indicates a nonperturbative origin. In fact, one important conclusion of our study is the existence of these very large nonperturbative artifacts at small p due to discretization. These are very embarrassing for any analysis of the Green functions, as we comment below.

Let us repeat our strong conviction that these low p effects (under 2 GeV) are indeed discretization artifacts and not volume artifacts. They behave quite counter to volume effects, as we have extensively argued.

(viii) A short study of $Z_A^{\mathrm{MOM}}/Z_V^{\mathrm{MOM}}$ shows that similar artifacts are still present in a ratio where they would be expected to cancel if one applies naively the exact chiral symmetry of the lattice action. They are clearly seen as being discretization artifacts because they are reduced at larger β. They seem to be present on top of an actual continuum effect, small but visible, which could be due to continuum chiral symmetry spontaneous breaking. One can suspect that the artifacts themselves are connected with the spontaneous breaking of chiral symmetry.

It seems logical that such chiral-symmetry violating artifacts, as well as the continuum effect, be only forbidden at large p, if they are connected with spontaneous breaking effects. Indeed, it is there and only there that such effects fade away. Therefore, it is consistent with this interpretation that we find $Z_A^{\mathrm{MOM}}/Z_V^{\mathrm{MOM}} = 1$ at large p to a high precision, without β dependence.

(ix) Of course, the clover action has the great advantage that it does not present the same very constraining limit in volume as the overlap one; however, it leads to less compelling results than the overlap one, because we have not found here the same particular situations where $O(4)$ artifacts can be neglected; the small p artifacts, which render so difficult the determination of power corrections, are present in Z_ψ and not only in g_1. As explained above, by including more and more terms to describe these artifacts, we destabilize the numerical value of $\langle A^2 \rangle$. Stopping with the first term a/p^2, we obtain consistency with the overlap results.

(x) In comparison with other works on the quark propagator, the question of the presence and magnitude of power corrections is a crucial test of the precision obtained in the treatment of Green functions: the condensate value $\langle A^2 \rangle$ should be independent of the action and of β with due renormalization. In our opinion, safe and accurate extraction of power corrections requires a very large range of momenta, and therefore a large range of β[7]; indeed, to use a rather large range of momenta at a fixed β would be dangerous because of the periodicity of the lattice; the large p behavior would then be highly dependent on empirical redefinitions of the momenta, aiming to remove empirically lattice artifacts at very large $p \simeq 10$ GeV. Of course, a crucial question is whether one can work with a β as large as 6.8 with our lattice size 16^4—the possible size is indeed strongly restricted for the overlap action. It is our conviction, for reasons which have been explained in detail. Another concern is that, according to our experience, to extract the real power corrections, one needs a particularly careful elimination of hypercubic artifacts.

(xi) The resulting value of $\langle A^2 \rangle$ from the OPE analysis of lattice QCD data should be compared to tenta-

[7]The lattice group in Adelaide has performed extensive studies of the quark propagator (see references above). Recently, the same group has extended its analysis to overlap action [11] and a special gauge action, with several β's, corresponding to $a^{-1} \sim 1$ GeV^{-1}–2 GeV^{-1}, yet notably lower than our largest cutoffs.

tive estimates made by various authors within analytical approaches. One will find abundant references in the paper of Dudal *et al.*, Ref. [42]. It is clear that this comparison must take care of the precise definition of the condensate, as regarding instance renormalization.

B. Systematic errors on $\langle A^2 \rangle$

Of course, we are making many assumptions which introduce uncertainty in the value of $\langle A^2 \rangle$. Recall that we do not claim to determinate it only from the present study, since we have several previous determinations from the gluonic sector. So we can also appreciate systematic errors from the consistency we obtain with these previous estimates (see above, Sec. VIII C). In fact, as we have observed in Sec. VIII C, the gluon propagator determination and the one from α_s are notably different, but the values are affected by very large errors, and are compatible with the present ones. From all the determinations, we could conclude that the systematic errors do no seem to exceed 1 GeV^2. But there is in fact an important source of the systematic error, which is explained below, and which cannot be estimated by comparison with gluons because it is present in both: it is the fact that the Wilson coefficient is calculated only at low order. Then, it remains useful to discuss the sources of errors inside the quark sector itself, for which anyway the conditions are intrinsically very favorable.

Since we do not claim to do phenomenology, but rather an exercise in quenched QCD, we have not bothered with the quenched approximation, which is also supposed for gluons. Chiral limit is assumed on the theoretical side, for instance to calculate the Wilson coefficients. Now, of course, we do not work at zero quark mass on the lattice. We have not tried to do a systematic chiral extrapolation on the lattice data, which would only lead to increase the statistical errors. We observe that Z_ψ seems very weakly dependent on our set of masses, which means that this limit is not *a priori* a problem at the smallest mass (at which we have made all our OPE analysis). Anyway, we can discard any catastrophic effect at very low quark masses through the consistency with the quenched gluon data.

Let us now pass to more relevant effects.

Some come from the treatment of artifacts, for which we lack theoretical basis, some are relative to our description of the continuum, which, although based on a much stronger theoretical basis, involves necessarily approximations.

We do not return to finite volume artifacts, since we have nothing quantitative to say about their magnitude in the overlap case. For what concerns hypercubic artifacts, we may have an idea on the error remaining in their treatment by the variation observed with two variants, with a slightly different description of the continuum limit:

$$\langle A^2 \rangle = (3.23 - 3.0)\ \text{GeV}^2. \tag{45}$$

Whenever large $O(4)$ symmetric artifacts are present, we are compelled, as we have seen, to rather arbitrary assumptions on their structure, since we have concluded that we cannot rely on lattice perturbation theory. The correlated errors in our determination of the ''nonperturbative'' artifacts and of the condensate seem very large, as judged from the range of values obtained in various fits, to such a point that we have renounced to extract any number in this case. Therefore, we consider only the case where we have strong hints that the artifacts are small, in which case we have mainly to consider uncertainties in the *continuum* description. They are themselves of two origins: the perturbative calculation of Wilson coefficients; the nonperturbative aspect, i.e. the relevance of the OPE expansion, the enumeration of operators, etc. As to uncertainties in perturbative calculations:

(i) We have checked that computing the perturbative contribution to third or fourth order in perturbation does not change significantly the estimated condensate (only 7% of change). Another test is to reexpress the series in terms of $\alpha_{\overline{\text{MS}}}$ instead of α_{MOM}. Z_ψ changes by less than 1% with various prescriptions. We can thus assume that the perturbative contribution has been expanded far enough.

(ii) As we have explained in Subsection II C, the problem is much more important for the Wilson coefficient of the A^2 operator which has, on the contrary, only been computed to leading logarithm. A sign of this problem is seen by changing α_{MOM} into $\alpha_{\overline{\text{MS}}}$. A change of α_{MOM} into $\alpha_{\overline{\text{MS}}}$ reduces it by 40% $\alpha_s(10\ \text{GeV})$, and more for the smaller momenta; whence a reduction of the Wilson coefficient by 50% in average. Through a conspiracy with the smaller change in the perturbative contribution, this change amounts to an increase of the resulting condensate by 70%. Of course, similar effects are present for gluons, and, as we have explained, the ratio of condensates obtained from quarks and gluons will remain the same. More importantly, one must be aware that the power correction by itself remains well determined; what is not well determined is the translation of the power correction into a $\langle A^2 \rangle$ condensate value. Indeed, this translation depends on the theoretical evaluation of the Wilson coefficient, which is not accurate at present.

As to the properly nonperturbative aspect, we may think of two sources of uncertainties in determining $\langle A^2 \rangle$. The one stems from other operators which could enter with the same power in the OPE expansion. However, we do not find any such operator contribution in Z_ψ in the chiral limit. We could have some contamination since we are not exactly in the chiral limit, but it must be small, since we observe a very weak quark mass dependence. The other could be the possibility that p is not sufficiently large for

the leading correction $1/p^2$ to completely dominate over the next ones. This possibility is represented by the $1/p^4$ term and the fact that it is small but nonzero in the fits shows indeed that we are not completely asymptotic at such large momenta; although it is very small, around 1% at 3 GeV, it leads to a change of around 15% in $\langle A^2 \rangle$. On the other hand, with this term included, we find a very good stability of $\langle A^2 \rangle$ over the large range $2.6\ \text{GeV} < p < 10\ \text{GeV}$, when varying the fitting window, which suggests that we have correctly accounted for the small subasymptotic effects. Actually, this term could also mimic a neglected logarithmic dependence; indeed, passing from α^{MOM} to $\alpha^{\overline{\text{MS}}}$ as expansion parameter as explained above, the $1/p^4$ term passes from $-1.1/p^4\ \text{GeV}^4$ to $-(0.65 \pm 0.34)\ \text{GeV}^4/p^4$, therefore there is an appreciable variation, although the sign and order of magnitude are encouragingly stable.

On the whole, the dependence of the Wilson coefficient on the scheme for α_s seems to be the most worrying source of uncertainty, yet it can be solved soon.

Another concern is the value we use for Λ_{QCD}, conventionally converted to the $\overline{\text{MS}}$ scheme. Let us vary by $\pm 10\%$ our $\Lambda_{\overline{\text{MS}}} = 0.237$ GeV. We find:

$$\Lambda_{\overline{\text{MS}}} = 0.215\text{–}0.260\ \text{GeV} \rightarrow \langle A^2 \rangle = 3.45\text{–}3.02\ \text{GeV}^2, \tag{46}$$

a quite moderate change indeed. There is naturally an increase for decreasing Λ_{QCD} because the larger power correction compensates for the slower falloff of the perturbative part.

C. General consequences for the nonperturbative MOM renormalization approach

True, the direct object of this study has been to verify the consistency of our OPE analysis of lattice data by extending it to the quark sector, and then comparing with the previous analysis of the gluon sector; and thereby to assess the soundness of our statement of large power corrections in ''elementary'' Green functions as well as of their interpretation in terms of the nongauge-invariant condensate $\langle A^2 \rangle$.

Now, one must also be aware of the strong consequences of this study, as well as of the preceding ones, on other problems, especially in precision studies of physical quantities. These consequences should be then reexamined in unquenched calculations.

A first example is the determination of Λ_{QCD} in the gluonic sector; a purely perturbative fit leads to a first determination of Λ_{QCD}; the necessary inclusion of a $1/p^2$ power correction induces a striking modification in the value of Λ_{QCD}, as was found some years ago [19]. In general, important consequences are expected on quantities requiring a renormalization procedure. Very commonly, the renormalization is performed in the MOM method, which uses elementary Green functions in momentum space to calculate the renormalization factors. The naive MOM method would just start from raw lattice data, with minimal improvements such as a democratic selection and choice of an optimal momentum window. Then, our study makes clear the difficulties of this naive method, which render impossible to get accurate results at the precision of a few percents. The present study allows some improvements, but fails in extracting the $O(4)$ artifacts. Let us enumerate the obstacles in the naive method and those who remain in spite of our attempts to improve it:

(1) *Power corrections.*—First, we have a problem independent of the discretization of the action: the presence of the power corrections has the effect of modifying the lattice estimate of the renormalization constants defined in the *perturbative regime*. Indeed, only perturbative renormalization schemes can be connected between one another by analytical calculations (e.g. the $\overline{\text{MS}}$ quark mass), and also connected to Wilson coefficients in order to produce physical quantities (e.g. weak four-fermion interaction). Now, the lattice measurement can be performed only at rather small momenta, where power corrections are present. These corrections must be subtracted, especially if one requires a good precision. This extraction is not performed in the naive method.

Such power corrections affect Z_ψ and various vertex functions. From the Ward identity, it happens that for quantities like Z_V, the A^2 power corrections cancel between the vertex function and Z_ψ. But this is not true in general: in Z_S and Z_P for instance, this cancellation does not occur. In such cases, the power corrections must be subtracted, in principle. Of course, one may wonder whether this is practically important. It depends on the accuracy we want to obtain. If we aim at a precision of a few percent, certainly we do require to take them into account, since they reach several percent around 4 GeV, 5% on Z_ψ in the clover case. Now, one often claims to go below 10% with dynamical quarks; then, such effects are deserving of consideration.

What simplifies somewhat the problem raised by power corrections is that not only are they independent of the chosen discretization of the action, but they are often related to the same $\langle A^2 \rangle$ condensate, at least as regards the dominant power. Once $\langle A^2 \rangle$ has been confidently determined by one analysis, it can be used for other Green functions.

(2) the discretization artifacts that we have found set a more general difficulty, and one which is more embarrassing, in view of our lack of theoretical control. We have no reason to suppose that the special case we have studied is especially cata-

strophic. Yet, it shows already an embarrassing situation.

α—The first step is to have control on hypercubic artifacts to a good accuracy. It is deserving of mention that this accuracy cannot be obtained by the naive method of selecting democratic points; since the determination of renormalization constants relies on measures at large p, the error is particularly large. Moreover, the simplest version of our alternative method of ''restoration of $O(4)$ symmetry,'' the old ''$p^{[4]}$'' method, has not allowed one to get a good accuracy. We have had to go further with the $p^{[2n]}$ method. This requires already a good deal of work. But, finally, it seems that we are able to produce a *systematic* and *accurate* procedure.

β—On the other hand, we have a clear proof from Z_V that there are large $O(4)$ invariant artifacts at small p. This type of large artifacts seems to be a problem specific to the quark sector: we had not found similar evidence for gluonic Green functions. We do not have any theoretical mastering about the magnitude of these artifacts. It is also surprising that Z_V becomes relatively flat at large p. This suggests, but does not prove, that these artifacts are small at large p. All this is counter to the expectation of perturbation theory which would predict artifacts increasing with p. This lack of theoretical understanding will remain a problem for all the MOM determinations of renormalization constants.

Indeed, we have not obtained on the other hand an empirical method which would be safe and systematic, to eliminate this new type of artifacts. This is disastrous. It also renders difficult the extraction of the power corrections, as we have observed in the clover case. The choice of the *overlap action* may help somewhat, but does not offer a general solution. Indeed, for Z_ψ, the overlap action seems to present small $O(4)$ invariant artifacts. It introduces larger hypercubic artifacts, but this is not a decisive obstacle, as we have explained how to eliminate them accurately. Then one would conclude that all in all, the overlap action has an important advantage. Yet, it is weakened by our lack of theoretical understanding of the underlying reasons why it is so, which precludes any estimation of accuracy (calculations with other values of $\rho = 1 + s$ may reveal instructive in this respect). Moreover, the problem remains entire for Z_V and other vertex renormalization constants.

Taking into account the large uncertainties coming from $O(4)$ invariant artifacts, the MOM method for renormalization, even with the improvements we have proposed, may reveal not very practicable for precision calculations, although it is appealing by its simplicity, and quite efficient for ordinary purposes. Of course, at this point, one may think of the method of the ALPHA collaboration as a complementary one, technically difficult, but which allows a very clean treatment of discretization artifacts by using on shell quantities, and also allows to work at a very high energy scale, therefore eliminating power corrections (see, for example, for Z_V, [43]).[8]

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APPENDIX A: PERTURBATIVE EXPANSION TO FOUR LOOPS IN THE FULL MOM SCHEME

Our aim is to express $Z_\psi(p^2)$, in terms of α_s^{MOM}, defined by the triple gluon vertex at symmetric momenta. More precisely, we are looking for the renormalization group improved expression, which resums the large logs $\log(p^2/\mu^2)$ at large p. The expression then takes the form of a series in $\alpha_s^{\mathrm{MOM}}(p)$ with pure number coefficients. This series can be obtained from the knowledge of the anomalous dimension of Z_ψ in terms of α_s^{MOM}, and of the β function of the MOM scheme. This is possible in the Landau gauge to the order of four loops included, thanks to the papers of Chetyrkin and collaborators, Refs. [26,27].

First, one has in Sec. IV B of the first paper the expansion of $\frac{\partial \ln Z_\psi(\mu^2)}{\partial \ln \mu^2}$ as a series in α_s of the $\overline{\mathrm{MS}}$ scheme to four loops (in fact the authors consider the inverse $Z_2(\mu^2) = (Z_\psi(\mu^2))^{-1}$, so we have to invert their formula). Since there is no α_s term in the Landau gauge, to reexpress the series in terms of the α_s of the MOM scheme at four loops requires the expansion of $\alpha_s^{\overline{\mathrm{MS}}}$ in terms of α_s^{MOM} only to order three included. This expansion is provided by Sec. V of the second paper, by inverting the first equation of this section. On the other hand, we need the MOM β function. It is given in the second paper, Sec. VI, at three loops, which is also sufficient to calculate the renormalization improved series for $Z_\psi(p^2)$ at four loops included; indeed, the four-loop coefficient of the β function enters only as a factor of the one-loop anomalous dimension of the fermion γ_0, which is zero in the Landau gauge. We then obtain the following expansions:

[8]After having issued the arXiv version, we have been advised of the paper [44] which has studied previously the same problem of extracting the A^2 condensate from the quark propagator.

Z_ψ is expressed in general as:

$$Z_\psi^{\rm pert} = \alpha^{\gamma_0/\beta_0}\Big(1 + \alpha\frac{\beta_0\gamma_1 - 2\beta_1\gamma_0}{4\pi\beta_0^2} + 0.5\alpha^2\Big(\frac{(\beta_0\gamma_1 - 2\beta_1\gamma_0)^2}{16\beta_0^4\pi^2} + \frac{8\beta_1^2\gamma_0 - \beta_0\beta_2\gamma_0 - 4\beta_0\beta_1\gamma_1 + 2\beta_0^2\gamma_2}{32\beta_0^3\pi^2}\Big) + \frac{\alpha^3}{768\beta_0^6\pi^3}(-16\beta_1^3\gamma_0^3 + 24\beta_0\beta_1^2\gamma_0^2(-2\beta_1 + \gamma_1) + 2\beta_0^2\beta_1\gamma_0(-16\beta_1^2 + 3\beta_2\gamma_0 + 24\beta_1\gamma_1 - 6\gamma_1^2) - 2\beta_0^4(2\beta_3\gamma_0 + \beta_2\gamma_1 + 4\beta_1\gamma_2 - 3\gamma_1\gamma_2) + \beta_0^3(16\beta_1^2\gamma_1 - 3\beta_2\gamma_0\gamma_1 + 2\gamma_1^3 + 4\beta_1(2\beta_2\gamma_0 - 3(\gamma_1^2 + \gamma_0\gamma_2))) + 4\beta_0^5\gamma_3)\Big). \tag{A1}$$

For $n_f = 0$ in the Landau gauge, we get:

$$\gamma_0 = 0.0 \qquad \gamma_1 = \tfrac{67}{3}, \qquad \gamma_2 = -94.7943 \qquad \gamma_3 = 14\,503.7. \tag{A2}$$

α, which represents here $\alpha_{\rm MOM}(p)$ (defined through the three-gluon vertex at symmetric momenta), is given, under the same conditions, by:

$$\alpha(p) = \frac{4\pi}{\beta_0 t} - 8\pi\frac{\beta_1}{\beta_0}\frac{\ln t}{(\beta_0 t)^2} + \frac{1}{(\beta_0 t)^3}\times\Big(2\pi\frac{\beta_2}{\beta_0} + 16\pi\frac{\beta_1^2}{\beta_0^2}((\ln t)^2 - \ln t - 1)\Big) + \frac{1}{(\beta_0 t)^4}\Big(-32\pi\frac{\beta_1^3}{\beta_0^3}(\ln t)^3 + 80\pi\frac{\beta_1^3}{\beta_0^3}(\ln t)^2 - \frac{12\pi\beta_0\beta_1\beta_2 - 64\pi\beta_1^3}{\beta_0^3}\ln t + \frac{2\pi\beta_0^2\beta_3 - 16\pi\beta_1^3}{\beta_0^3}\Big), \tag{A3}$$

where $t = \ln(p^2/\Lambda^2_{\rm MOM})$ (in the MOM case, one stops at the $1/t^3$ terms, and where

$$\beta_0 = 11, \qquad \beta_1 = 51, \qquad \beta_2 = 3072\,(10\%). \tag{A4}$$

APPENDIX B: A^2 WILSON COEFFICIENT IN THE QUARK PROPAGATOR

In order to renormalize the bare quark propagator in Eq. (15) we will define the following two renormalization constants, both in the momentum-subtraction (MOM) general scheme:

$$\begin{pmatrix} Z_\psi(\mu^2) \\ Z_\psi^{\rm pert}(\mu^2)\end{pmatrix}\delta_{a,b} = \frac{-i\not{p} + m(p^2)}{p^2 + m(p^2)}\begin{pmatrix} S^{-1}(p) \\ S^{-1}_{\rm pert}(p)\end{pmatrix}_{p^2=\mu^2}. \tag{B1}$$

The constant in the top of the left-hand side of Eq. (B1) includes the nonperturbative contributions to the quark propagator to let it take the tree-level value all over the energy range (not only in the perturbative regime). We renormalize Eq. (15) by multiplying by that of the bottom (this purely perturbative MOM renormalization constant, computed to four loops in Ref. [26], is presented in Appendix A); as to the Wilson coefficient of A^2, we calculate it at leading RG order.

$$\begin{aligned}(Z_\psi^{\rm pert}(\mu^2))^{-1}S^{-1}(p) &= i\not{p}\delta_{a,b}\frac{Z_\psi(p^2)}{Z_\psi^{\rm pert}(\mu^2)} + \cdots \\ &= (Z_\psi^{\rm pert}(\mu^2))^{-1}S^{-1}_{\rm pert}(p) \\ &\quad + i\not{p}\frac{d(\frac{p^2}{\mu^2},\alpha(\mu))}{p^2}\frac{\langle(A^2)_R(\mu)\rangle}{4(N_C^2-1)}\delta_{a,b} \\ &\quad + \cdots \\ &= i\not{p}\delta_{a,b}\Big(\frac{Z_\psi^{\rm pert}(p^2)}{Z_\psi^{\rm pert}(\mu^2)} + \frac{d(\frac{p^2}{\mu^2},\alpha(\mu))}{p^2} \\ &\quad \times\frac{\langle(A^2)_R(\mu)\rangle}{4(N_C^2-1)}\Big) + \cdots,\end{aligned} \tag{B2}$$

where $(A^2)_R(\mu^2) = Z^{-1}_{A^2}(\mu^2) : A^2_{\rm bare} :$,[9] and where we only write the leading terms in $\not{p}$. Concerning $d(\frac{p^2}{\mu^2},\alpha(\mu))$, the same procedure used for gluon Green functions in Refs. [20–22] is in order here. Then, from Eq. (B2), multiplying by $Z_\psi^{\rm pert}(\mu^2)$, and taking logarithm derivatives on μ in both sides, we obtain the following RG equation:

$$\Big\{-\gamma_0\frac{\alpha(\mu)}{4\pi} + \gamma^{(0)}_{A^2}\frac{\alpha(\mu)}{4\pi} + \frac{d}{d\ln\mu^2}\Big\}d\Big(\frac{p^2}{\mu^2},\alpha(\mu)\Big) = 0 \tag{B3}$$

that is identically satisfied with

$$d\Big(\frac{p^2}{\mu^2},\alpha(\mu)\Big) = d(1,\alpha(p))\Big(\frac{\alpha(\mu)}{\alpha(p)}\Big)^{(-\gamma_0+\gamma^{(0)}_{A^2})/\beta_0}, \tag{B4}$$

where $d((1,\alpha(p)))$ is in fact beginning with $\alpha(p)$ and $\gamma^{(0)}_{A^2}$ is defined through

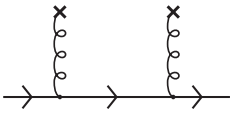
$$\frac{d}{d\ln\mu^2}\ln Z_{A^2} = -\gamma^{(0)}_{A^2}\frac{\alpha(\mu)}{4\pi} + \mathcal{O}(\alpha^2); \tag{B5}$$

[9] the: $\cdots$ fixes the normal order subtracting the additive divergencies in $A^2_{\rm bare}$ (see discussion in Ref. [45]).

and where all the involved one-loop coefficients are well known,

$$\gamma_{A^2}^{(0)} = \frac{35N_c}{12}, \qquad \gamma_0 = 0, \qquad \beta_0 = 11. \tag{B6}$$

On the other hand, $d(1, \alpha(p))$ can be obtained by computing the only diagram involved in the "OPE business" for this case,



$$\rightarrow \; d(1, \alpha(p)) = \frac{(N_C^2 - 1)}{3} g^2(p). \tag{B7}$$

Then, by applying the results from Eqs. (B4)–(B6) to Eq. (B2), we will obtain the final result Eq. (16) to be used for the fits.

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