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Towards the anomalous dimension to $\mathcal{O}(\Lambda_{\rm QCD}/m_b)$ for phase space restricted $\bar{B} \to X_u \ell \bar{\nu}$ and $\bar{B} \to X_s \gamma$

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We examine the anomalous dimension matrix appropriate for the phase space restricted $\bar{B} \to X_u \ell \bar{\nu}$ and $\bar{B} \to X_s \gamma$ decay spectra to subleading nonperturbative order. The time ordered products of the HQET Lagrangian with the leading order shape function operator are calculated, as are the anomalous dimensions of subleading operators. We establish the renormalizability and closure of a subset of the nonlocal operator basis, a requirement for the establishment of factorization theorems at this order. Operator mixing is found between the operators which occur to subleading order, requiring the subleading operator basis be extended. We comment on the requirement for new shape functions to be introduced to characterize the matrix elements of these new operators, and the phenomenological consequences for extractions of $|V_{ub}|$.

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I. INTRODUCTION

Extracting the CKM parameter $|V_{ub}|$ is an important step in testing of the CKM description of *CP* violation in the *B* meson system. Currently, the theoretically cleanest determinations of $|V_{ub}|$ come from inclusive semileptonic decays which are not sensitive to the details of hadronization; although recently an approach of extracting $|V_{ub}|$, utilizing $B \to \pi\pi$, has been advanced with a competitive error to inclusive methods. [1]

In inclusive extractions of $|V_{ub}|$, experimental cuts to exclude the charm background of $\bar{B} \to X_c \ell \bar{\nu}$ are imposed. This restricts the decay products to hadronic final states that have large energy $E_X \sim m_B$ and low invariant mass $M_X \sim \sqrt{m_B \Lambda_{\rm QCD}}$. With these phase space restrictions the local OPE expansion [2] appropriate for sufficiently inclusive decays used to extract $|V_{cb}|$ [3], typically breaks down [4].

As the local OPE and the clean separation of scales that the local OPE represented in the analysis of $\bar{B} \to X_c \ell \bar{\nu}$ is no longer valid, a more involved theoretical approach is required to separate the scales relevant to these decays. Decay rates are expressed as convolutions of hard (H), jet (J) and soft physics (S) associated with the scales $m_b \gg \sqrt{\Lambda_{\rm QCD} m_b} \gg \Lambda_{\rm QCD}$, in the following way,

$$d\Gamma = H\left(\frac{m_b}{\mu}, \alpha_s(\mu)\right) \int d\omega J\left(\frac{\sqrt{m_b \Lambda_{\rm QCD}}}{\mu}, \alpha_s(\mu), \omega\right) S(\omega). \tag{1}$$

This factorization theorem has been proven diagrammatically [5] at leading order in $1/m_b$, however it is not known to hold to all orders in the nonperturbative expansion. The form of corrections present in the expansion of the soft sector, which begins at $\mathcal{O}(1/\Lambda_{\rm QCD})$ to subleading order ($\mathcal{O}(1/m_b)$) has only recently been examined [6].

The systematic treatment of the nonperturbative corrections involves a two step matching procedure. One matches QCD onto the effective field theory of the intermediate scale, describing quarks and gluons with large energy and small offshellness, known as SCET [7–9], and uses the renormalization group evolution to run down to the soft scale. One then matches SCET onto the lightcone wavefunction of the *B* meson, expressed in terms of HQET fields. One can also match directly from QCD onto HQET, a much simpler procedure at the cost of not summing the logarithms of the ratio of scales $\log(\sqrt{m_b \Lambda_{\rm QCD}}/m_b)$ via SCET. In either case, the soft sector of the theory is expanded in terms of nonlocal operators. The leading order term in the $\Lambda_{\rm QCD}/m_b$ expansion of the lightcone distribution function of the *B* meson [10,11] is know as the shape function.

At subleading order in the nonperturbative expansion, four additional nonlocal operators have been determined to be present [12–20], the matrix elements of which are referred to as subleading shape functions.

It is of some intrinsic interest to examine the renormalization of these nonlocal operators, as they are nonlocal and their renormalizability is not know *a priori*. It is also important to know if this set of operators is complete for both determining their associated error in extractions of $|V_{ub}|$ and in considering the above factorization theorem beyond leading order. Examining the perturbative behavior of these subleading operators is also a necessary step in the one loop matching calculations onto the soft sector. For these reasons we have examined the anomalous dimension to subleading nonperturbative order.

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We have determined the contributions of the time ordered products of the subleading $\mathcal{L}_{\rm HQET}$ with the leading order shape function, and examined the anomalous dimensions of the subleading nonperturbative operators. We establish the renormalizability and closure of a subset of the subleading nonlocal operators. We find that the known operator basis mixes with new operators, requiring that the subleading operator basis be extended. We also comment on the phenomenological consequences of these results.

II. ANOMALOUS DIMENSION TO SUBLEADING ORDER

A. Notation

We introduce two lightlike vectors n^μ and $\bar{n}^\mu$ related to the velocity of the heavy quark by $v = \frac{1}{2}(n + \bar{n})$, and satisfying

$$n^2 = \bar{n}^2 = 0, \qquad v \cdot n = v \cdot \bar{n} = 1, \qquad n \cdot \bar{n} = 2. \quad (2)$$

In the frame in which the b quark is at rest, these vectors are given by $n^\mu = (1, 0, 0, 1)$, $\bar{n}^\mu = (1, 0, 0, -1)$ and $v^\mu = (1, 0, 0, 0)$. The projection of an arbitrary four-vector a^α onto the directions which are perpendicular to the lightcone is given by $a_\perp^\alpha = g_\perp^{\alpha\beta} a_\beta$, where

$$g_\perp^{\mu\nu} \equiv g^{\mu\nu} - \frac{1}{2}(n^\mu \bar{n}^\nu + n^\nu \bar{n}^\mu). \quad (3)$$

We also define a perpendicular Levi-Civita tensor

$$\epsilon_\perp^{\alpha\beta} = \epsilon^{\alpha\beta\sigma\rho} v_\sigma n_\rho. \quad (4)$$

We also use the projector $P_+ = 1/2(1 + \not{v})$ as well as the Dirac structure $s^\eta = P_+ \gamma^\eta \gamma_5 P_+$, so that $v \cdot s = 0$.

Distribution notation

Rather than the usual definitions of the star distribution as given in Neubert and deFazio [21],

$$\left(\frac{1}{x}\right)_* = \lim_{\beta \to 0}\left[\frac{\theta(x-\beta)}{x} + \delta(x-\beta)\log(x)\right]$$
$$\left(\frac{\log(x)}{x}\right)_* = \lim_{\beta \to 0}\left[\frac{\theta(x-\beta)}{x}\log(x) + \frac{1}{2}\delta(x-\beta)\log^2(x)\right], \quad (5)$$

we utilize the alternate notation, equivalent to the μ-distribution's in [22]

$$\phi_n(x) \equiv \lim_{\beta \to 0}\left[\frac{1}{n+1}\theta(x-\beta)\log^{n+1}(x)\right]. \quad (6)$$

This notation has a fairly easy correspondence to the usual star distribution notation

$$\phi_0'(x) = \left(\frac{1}{x}\right)_*, \qquad \phi_1'(x) = \left(\frac{\log(x)}{x}\right)_*. \quad (7)$$

A useful identity given by analytic continuation is

$$\frac{\theta(x)}{x^{1+\epsilon}} = -\frac{1}{\epsilon}\delta(x) + \phi_0'(x) - \epsilon\phi_1'(x) + \mathcal{O}(\epsilon^2) \quad (8)$$

This relationship is valid when integrated against arbitrary functions $f(x)$, where f is not singular at the origin. In general we can write the recursion relation

$$\frac{\theta(x)}{x^{n+\epsilon}} = \frac{-1}{n-1+\epsilon}\frac{d}{dx}\left[\frac{\theta(x)}{x^{n-1+\epsilon}}\right] \quad \text{for } n \geq 2. \quad (9)$$

Several other useful properties of this function are (for some positive constant a):

$$x\phi_0'(x) = \theta(x) \qquad x\phi_0''(x) = \delta(x) - \phi_0'(x)$$
$$a\phi_0'(ax) = \phi_0'(x) + \delta(x)\log(a), \quad (10)$$

as well as

$$\int_{-\infty}^a dx f(x)\phi_0'(x) = \int_0^a dx\left(\frac{\theta(x)}{x}\right)_+ f(x) + f(0)\log(a). \quad (11)$$

B. Operators to subleading order

At leading order a single nonlocal operator characterizes the nonperturbative physics,

$$Q_0(\omega, \Gamma) = \bar{h}_v \delta(\omega + in \cdot D)\Gamma h_v, \quad (12)$$

where the covariant derivative is $D_\mu = \partial_\mu + igA_\mu$.

The order $\Lambda_{\rm QCD}/m_b$ corrections to the $\bar{B} \to X_u \ell \bar{\nu}$ and $\bar{B} \to X_s \gamma$ decay spectra require the introduction of four additional nonlocal operators [12–16],

$$m_b Q_1^\mu(\omega, \Gamma) = \bar{h}_v\{iD_\perp^\mu, \delta(\omega + in \cdot D)\}\Gamma h_v,$$
$$m_b Q_2^\mu(\omega, \Gamma) = \bar{h}_v[iD_\perp^\mu, \delta(\omega + in \cdot D)]\Gamma h_v,$$
$$m_b Q_3(\omega, \Gamma) = \int d\omega_1 d\omega_2 \delta(\omega_1, \omega_2; \omega)\bar{h}_v \delta(\omega_2 + in \cdot D) \times g_\perp^{\mu\nu}\{iD_\perp^\mu, iD_\perp^\nu\}\delta(\omega_1 + in \cdot D)\Gamma h_v,$$
$$m_b Q_4(\omega, \Gamma) = -\int d\omega_1 d\omega_2 \delta(\omega_1, \omega_2; \omega)\bar{h}_v \times \delta(\omega_2 + in \cdot D) i\epsilon_\perp^{\mu\nu}[iD_\perp^\mu, iD_\perp^\nu] \times \delta(\omega_1 + in \cdot D)\Gamma h_v, \quad (13)$$

where

$$\delta(\omega_1, \omega_2; \omega) = \frac{\delta(\omega - \omega_1) - \delta(\omega - \omega_2)}{\omega_1 - \omega_2}. \quad (14)$$

We define these operators rescaled by m_b for later convenience in the anomalous dimension. This rescaling should be noted when comparing to other work dealing with subleading shape functions. We also use the convention of labeling operators as Q_i operators when the Dirac structure is general and refer to them as $O_i(\omega) = Q_i(\omega, 1)$ and $P_i^\eta = Q_i(\omega, s^\eta)$ for particular Dirac structures.

We find that the operator basis must be extended beyond tree level to include, at least the following operator

$$m_b \bar{Q}_1^\mu(\omega, \Gamma) = -2 \int d\omega_1 d\omega_2 \theta(\omega_1, \omega_2; \omega) K_2^\mu(\omega_1, \omega_2; \Gamma), \tag{15}$$

where we have defined the following kernel and coefficient functions

$$K_2^\mu(\omega_1, \omega_2; \Gamma) = \bar{h}_v \delta(\omega_1 + in \cdot D) i D_\perp^\mu \delta(\omega_2 + in \cdot D) \Gamma h_v, \tag{16}$$

$$\theta(\omega_1, \omega_2; \omega) = \frac{\theta(\omega - \omega_1) - \theta(\omega - \omega_2)}{\omega_1 - \omega_2}. \tag{17}$$

The operator $\bar{Q}_1$ was originally defined in [12] by Mannel and Tackmann [23,24] based on symmetry arguments and examining the endpoint of $\bar{B} \to X_c \ell \bar{\nu}$ and taking the massless limit. We find that the operator is unambiguously required beyond tree level due to the mixing experienced with the original set of operators.

C. Operator Feynman rules

We use Feynman gauge to calculate the anomalous dimension to subleading order as the usual choice of light-cone gauge introduces nonphysical poles in the calculation; for a review of the relevant issues see [25].

Below, we present the required zero and one gluon Feynman rules (see Fig. 1). The two gluon Feynman rules

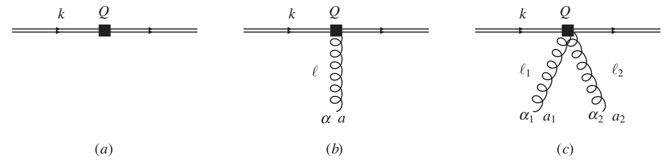

FIG. 1. Gluon labels and momentum routing for the zero, one and two gluon Feynman rules.

are also required but are too lengthy to include here.[1] The nonvanishing zero gluon Feynman rules in Feynman gauge are:

$$\begin{aligned} Q_0(\omega, \Gamma)[0 - \text{gluon}] &= \delta(\omega + n \cdot k)\Gamma, \\ Q_1^\mu(\omega, \Gamma))[0 - \text{gluon}] &= 2\frac{k_\perp^\mu}{m_b}\delta(\omega + n \cdot k)\Gamma, \\ \bar{Q}_1^\mu(\omega, \Gamma))[0 - \text{gluon}] &= 2\frac{k_\perp^\mu}{m_b}\delta(\omega + n \cdot k)\Gamma, \\ Q_3(\omega, \Gamma))[0 - \text{gluon}] &= -2\frac{k_\perp^2}{m_b}\delta'(\omega + n \cdot k)\Gamma. \end{aligned} \tag{18}$$

The one gluon Feynman rule for the leading order operator is

$$Q_0(\omega, \Gamma)[1 - \text{gluon}] = -gT_a n^\alpha \left(\frac{\delta_-(n \cdot \ell)}{n \cdot \ell}\right)\Gamma. \tag{19}$$

The one gluon Feynman rules for single covariant derivative operators are:

$$\begin{aligned} Q_1^\mu(\omega, \Gamma)[1 - \text{gluon}] &= -gT_a g_\perp^{\mu\alpha} \delta_+(n \cdot \ell)\frac{\Gamma}{m_b} - gT_a n^\alpha (2k + \ell)_\perp^\mu \left(\frac{\delta_-(n \cdot \ell)}{n \cdot \ell}\right)\frac{\Gamma}{m_b}, \\ \bar{Q}_1^\mu(\omega, \Gamma)[1 - \text{gluon}] &= -2gT_a g_\perp^{\mu\alpha}\left(\frac{\theta_-(n \cdot \ell)}{n \cdot \ell}\right)\frac{\Gamma}{m_b} + 2gT_a n^\alpha \ell_\perp^\mu \left(\frac{\theta_-(n \cdot \ell)}{(n \cdot \ell)^2}\right)\frac{\Gamma}{m_b} - 2gT_a n^\alpha k_\perp^\mu \left(\frac{\delta_-(n \cdot \ell)}{n \cdot \ell}\right)\frac{\Gamma}{m_b} \\ &\quad - 2gT_a n^\alpha \ell_\perp^\mu \left(\frac{\delta(\omega + n \cdot k + n \cdot \ell)}{n \cdot \ell}\right)\frac{\Gamma}{m_b}, \\ Q_2^\mu(\omega, \Gamma)[1 - \text{gluon}] &= gT_a g_\perp^{\mu\alpha}\delta_-(n \cdot \ell)\frac{\Gamma}{m_b} - gT_a n^\alpha \ell_\perp^\mu \left(\frac{\delta_-(n \cdot \ell)}{n \cdot \ell}\right)\frac{\Gamma}{m_b}. \end{aligned} \tag{20}$$

Finally, the one gluon Feynman rules for two covariant derivative operators are as follows:

$$\begin{aligned} Q_3(\omega, \Gamma)[1 - \text{gluon}] &= 2gT_a\left((2k + \ell)_\perp^\alpha\left(\frac{\delta_-(n \cdot \ell)}{n \cdot \ell}\right) - n^\alpha k_\perp^2\left(\frac{\delta'(\omega + n \cdot k)}{n \cdot \ell}\right)\right)\frac{\Gamma}{m_b} \\ &\quad + 2gT_a\left(n^\alpha(k + \ell)_\perp^2\left(\frac{\delta'(\omega + n \cdot k + n \cdot \ell)}{n \cdot \ell}\right) - n^\alpha(2k_\perp \cdot \ell_\perp + \ell_\perp^2)\left(\frac{\delta_-(n \cdot \ell)}{(n \cdot \ell)^2}\right)\right)\frac{\Gamma}{m_b}, \\ Q_4(\omega, \Gamma)[1 - \text{gluon}] &= 2gT_a i\epsilon_\perp^{\alpha\beta}\ell_{\perp\beta}\left(\frac{\delta_-(n \cdot \ell)}{n \cdot \ell}\right)\frac{\Gamma}{m_b}. \end{aligned} \tag{21}$$

[1]All the Feynman rules have been collected in a *Mathematica* file that can be supplied if requested.

where ℓ is the gluon momentum flowing into the vertex, and the gluon carries Lorentz index α and color index a. We have also made the convenient definitions

$$\theta_{\pm}(x) = \theta(\omega + n \cdot k + x) \pm \theta(\omega + n \cdot k) \tag{22}$$

$$\delta_{\pm}(x) = \delta(\omega + n \cdot k + x) \pm \delta(\omega + n \cdot k). \tag{23}$$

D. The anomalous dimension matrix

The renormalization of the operators $Q_i(\omega, \Gamma)$ is performed in the usual fashion,

$$Q_i(\omega, \Gamma)_{\text{bare}} = \int d\omega' Z_{ij}(\omega', \omega, \tilde{\mu}) Q_i(\omega', \tilde{\mu}, \Gamma)_{\text{ren}}, \tag{24}$$

where $Z_{ij}(\omega', \omega, \tilde{\mu})$ is a matrix of renormalization constants. The values of the elements of Z_{ij} can be found by taking arbitrary partonic matrix elements of both sides, which at leading order gives $Z_{ij}^{(0)}(\omega', \omega, \tilde{\mu}) = \delta_{ij}\delta(\omega - \omega')$.

To subleading order in α_s we have

$$\begin{aligned} &\langle Q_i(\omega, \Gamma)\rangle^{(0)}_{\text{bare}} + \alpha_s \langle Q_i(\omega, \Gamma)\rangle^{(1)}_{\text{bare}} \\ &\quad = \int d\omega' [Z_{ij}^{(0)}(\omega', \omega, \tilde{\mu}) + \alpha_s Z_{ij}^{(1)}(\omega', \omega, \tilde{\mu})] \\ &\qquad \times [\langle Q_j(\omega', \tilde{\mu}, \Gamma)\rangle^{(0)}_{\text{ren}} + \alpha_s \langle Q_j(\omega', \tilde{\mu}, \Gamma)\rangle^{(1)}_{\text{ren}}], \end{aligned} \tag{25}$$

from which one obtains

$$\begin{aligned} &\int d\omega' Z_{ij}^{(1)}(\omega', \omega, \tilde{\mu}) \langle Q_j(\omega', \tilde{\mu}, \Gamma)\rangle^{(0)} \\ &\quad = \langle Q_i(\omega, \Gamma)\rangle^{(1)}_{\text{bare}} - \langle Q_i(\omega, \Gamma)\rangle^{(1)}_{\text{ren}} \\ &\quad = (\langle Q_i(\omega, \Gamma)\rangle^{(1)}_{\text{bare}})_{\text{div}} \end{aligned} \tag{26}$$

where by $(\langle Q_i(\omega, \Gamma)\rangle^{(1)}_{\text{bare}})_{\text{div}}$, we refer to the UV divergent part of $\langle Q_i(\omega, \Gamma)\rangle^{(1)}_{\text{bare}}$. Because there are operators such as Q_2 and Q_4 which do not have a zero gluon form, we must consider matrix elements of Eq. (24) with at least one external gluon. These will be sufficient to identify the mixing of the various operators into Q_2 and Q_4. It should be noted that matrix elements with zero and one external gluon states are not sufficient in principle to determine the anomalous dimension matrix to subleading order. The operator

$$\begin{aligned} Q^{\mu\nu}(\omega_1, \omega_2, \Gamma) = \bar{h}_v [iD_\perp^\mu, \delta(\omega_2 + in \cdot D)] \\ \times [iD_\perp^\nu, \delta(\omega_1 + in \cdot D)] \Gamma h_v, \end{aligned} \tag{27}$$

does not have a zero gluon or one gluon Feynman rule. Its first nonvanishing Feynman rule contains two gluons. In this paper, we will not be calculating the two external gluon diagrams necessary to find mixing into this operator, if any exists. We extract the anomalous dimension matrix of the subleading operators by examining matrix elements containing one perpendicularly polarized external gluon:

$$\begin{aligned} &\int d\omega' Z_{ij}^{(1)}(\omega', \omega, \tilde{\mu}) \langle h_v A_\perp | Q_i(\omega', \Gamma) | h_v\rangle^{(0)}_{\text{ren}} \\ &\quad = (\langle h_v A_\perp | Q_i(\omega, \Gamma) | h_v\rangle^{(1)}_{\text{bare}})_{\text{div}}. \end{aligned} \tag{28}$$

The nonperpendicular components of the gluon field were also examined but found to induce no further mixing.

The mixing of Q_0 into the other operators is determined by calculating matrix elements of this operator with insertions of the subleading HQET Lagrangian. Zero gluon matrix elements are sufficient to find the mixing into Q_0, while one gluon matrix elements are required for mixing into the remaining operators. Because of the spin symmetry violating effects of the subleading HQET Lagrangian, the anomalous dimension of the P_i operators can differ from that of the O_i operators.

The wavefunction renormalization of the bare operators expressed in terms of renormalized fields is $Q_i(\omega, \Gamma)_{\text{bare}} = Z_h Z_3^{n/2} Q_i(\omega, \Gamma)$ where n is the number of gluons in the operator, and $Q_i(\omega, \Gamma)$ is written in terms of renormalized fields. For diagrams with an external state gluon we use the background field method to treat the external gluon as a background field [26].

E. Diagram calculations

1. One gluon matrix elements

The one gluon matrix elements are determined by calculating the diagrams shown in Fig. 2 for each operator. The external gluon in each of these diagrams is a background field gluon chosen to have perpendicular polarization. We utilize dimensional regularization and the $\overline{\text{MS}}$ scheme to regulate our divergences. To isolate and remove the IR divergences in the calculation we keep *all* the particles off shell by retaining factors of $v \cdot k$, $v \cdot \ell$ and ℓ^2, where ℓ is the external gluon momentum.

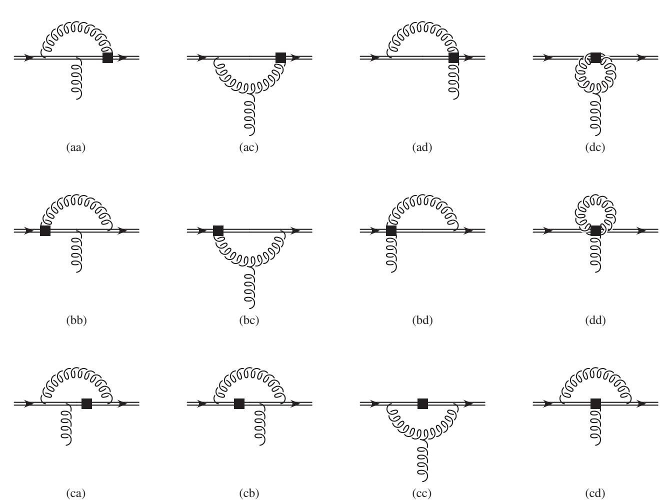

FIG. 2. The one gluon diagrams which must be calculated for each operator.

To clearly illustrate the need to extend the operator basis we present the results for Q_1 diagram by diagram. In general, for perpendicular polarized external gluons, only diagrams (ac), (ad), (bc), (bd) and (dc) contribute to the amplitude. For diagram (dc), the loop integrals to perform are as follows, with $c_1 \equiv (\alpha_s g T_a g_\perp^{\mu\nu})/(4\pi)$,

$$\langle i\mathcal{A}_{\rm dc}\rangle^{(1)} = -\frac{c_1}{m_b} C_A (n\cdot\ell)^2 \mu^{4-d} \int \frac{d^d q}{(2\pi)^d} \frac{\delta_+(n\cdot\ell)}{n\cdot q(n\cdot\ell + n\cdot q)(q^2+i\epsilon)((q+\ell)^2+i\epsilon)} + \frac{c_1}{m_b} C_A (n\cdot\ell)^2 \mu^{4-d} \times \int \frac{d^d q}{(2\pi)^d} \frac{\delta(\omega + n\cdot k - n\cdot q) + \delta(\omega + n\cdot k + n\cdot\ell + n\cdot q)}{n\cdot q(n\cdot\ell + n\cdot q)(q^2+i\epsilon)((q+\ell)^2+i\epsilon)}. \tag{29}$$

The integrals are perfomed via the standard techniques of dimensional regularization with $d = 4 - 2\epsilon$, the $\overline{\rm MS}$ renormalization scale $\tilde{\mu}^2 = 4\pi\mu^2 e^{-\gamma_E}$, and the utilization of Eq. (8) and we have suppressed the Lorentz and color indicies. The UV poles obtained from this diagram for Q_1 after the integrals are performed and consideration of the symmetry factor are

$$\langle i\mathcal{A}_{\rm dc}\rangle^{(1)}_{\rm div} = \frac{c_1 C_A}{m_b\epsilon}\left(\frac{n\cdot\ell}{\tilde{\mu}(\omega + n\cdot k)}\phi_0'\left(\frac{\omega + n\cdot k + n\cdot\ell}{\tilde{\mu}}\right) - \frac{n\cdot\ell}{\tilde{\mu}(\omega + n\cdot k + n\cdot\ell)}\phi_0'\left(\frac{\omega + n\cdot k}{\tilde{\mu}}\right)\right) + \frac{c_1 C_A}{m_b\epsilon}\left(\delta_+(n\cdot\ell)\log\left(\frac{n\cdot\ell}{\tilde{\mu}}\right)\right).$$

The results for diagrams ac and bc when inserting Q_1 are

$$\langle i\mathcal{A}_{\rm ac}\rangle^{(1)}_{\rm div} = \frac{c_1 C_A}{m_b\epsilon}\left(\frac{\delta(\omega + n\cdot k + n\cdot\ell)}{2\epsilon} + \delta(\omega + n\cdot k + n\cdot\ell)\right) - \frac{c_1 C_A}{m_b\epsilon}\left(\log\left(\frac{n\cdot\ell}{\tilde{\mu}}\right)\delta(\omega + n\cdot k + n\cdot\ell)\right) + \frac{c_1 C_A}{m_b n\cdot\ell\epsilon}\left(\frac{(\omega + n\cdot k)}{(\omega + n\cdot k + n\cdot\ell)}\theta\left(\frac{\omega + n\cdot k}{\tilde{\mu}}\right) - \theta\left(\frac{\omega + n\cdot k + n\cdot\ell}{\tilde{\mu}}\right)\right), \tag{30}$$

$$\langle i\mathcal{A}_{\rm bc}\rangle^{(1)}_{\rm div} = \frac{c_1 C_A}{m_b\epsilon}\left(\frac{\delta(\omega + n\cdot k)}{2\epsilon} + \delta(\omega + n\cdot k) - \log\left(\frac{n\cdot\ell}{\tilde{\mu}}\right)\delta(\omega + n\cdot k)\right) + \frac{c_1 C_A}{m_b n\cdot\ell\epsilon}\left(-\frac{(\omega + n\cdot k + n\cdot\ell)}{(\omega + n\cdot k)}\theta\left(\frac{\omega + n\cdot k + n\cdot\ell}{\tilde{\mu}}\right) + \theta\left(\frac{\omega + n\cdot k}{\tilde{\mu}}\right)\right). \tag{31}$$

Finally, the results for diagrams ad and bd for Q_1 insertions are

$$\begin{aligned}\langle i\mathcal{A}_{\rm ad}\rangle^{(1)}_{\rm div} &= \frac{c_1 C_F}{m_b\epsilon}\left(\frac{\delta_+(n\cdot\ell)}{\epsilon} - \frac{2}{\tilde{\mu}}\phi_0'\left(\frac{\omega + n\cdot k + n\cdot\ell}{\tilde{\mu}}\right) - \frac{2}{\tilde{\mu}}\phi_0'\left(\frac{\omega + n\cdot k}{\tilde{\mu}}\right)\right) + \frac{c_1 C_A}{m_b\epsilon}\left(-\frac{\delta(\omega + n\cdot k + n\cdot\ell)}{2\epsilon} + \frac{1}{\tilde{\mu}}\phi_0'\left(\frac{\omega + n\cdot k + n\cdot\ell}{\tilde{\mu}}\right)\right),\\ \langle i\mathcal{A}_{\rm bd}\rangle^{(1)}_{\rm div} &= \frac{c_1 C_F}{m_b\epsilon}\left(\frac{\delta_+(n\cdot\ell)}{\epsilon} - \frac{2}{\tilde{\mu}}\phi_0'\left(\frac{\omega + n\cdot k + n\cdot\ell}{\tilde{\mu}}\right) - \frac{2}{\tilde{\mu}}\phi_0'\left(\frac{\omega + n\cdot k}{\tilde{\mu}}\right)\right) + \frac{c_1 C_A}{m_b\epsilon}\left(-\frac{\delta(\omega + n\cdot k)}{2\epsilon} + \frac{1}{\tilde{\mu}}\phi_0'\left(\frac{\omega + n\cdot k}{\tilde{\mu}}\right)\right).\end{aligned} \tag{32}$$

The amplitudes combine to give the following UV poles

$$\langle i\mathcal{A}_{Q_1}\rangle^{(1)}_{\rm div} = \frac{c_1 C_F}{m_b\epsilon}\left(\frac{2\delta_+(n\cdot\ell)}{\epsilon} - \frac{4}{\tilde{\mu}}\phi_{0+}'\left(\frac{n\cdot\ell}{\tilde{\mu}}\right)\right) + \frac{c_1 C_A}{m_b\epsilon}\left(\delta_+(n\cdot\ell) - \frac{2\theta_-(n\cdot\ell)}{n\cdot\ell}\right). \tag{33}$$

Once the wavefunction renormalization terms are multiplicatively combined with the result, we express the amplitude in terms of renormalization matrix elements d_i and the one gluon Feynman rules for the operators Q_1^μ and $\bar{Q}_1^\mu$ as follows

$$\langle i\mathcal{A}_{Q_1}\rangle^{(1)}_{\rm div} = \frac{\alpha_s}{4\pi}\int d\omega' d_1(\omega, \omega', \tilde{\mu})\langle Q_1^\mu(\omega')\rangle^{(0)} + \frac{\alpha_s}{4\pi}\int d\omega' d_4(\omega, \omega')(\langle \bar{Q}_1^\mu(\omega')\rangle^{(0)} - \langle Q_1^\mu(\omega')\rangle^{(0)}), \tag{34}$$

where

$$d_1(\omega, \omega', \tilde{\mu}) = -\frac{2C_F}{\epsilon^2}\delta(\omega - \omega') + \frac{2C_F}{\epsilon}\delta(\omega - \omega') + \frac{4C_F}{\tilde{\mu}\epsilon}\phi_0'\left(\frac{\omega - \omega'}{\tilde{\mu}}\right), \quad (35)$$
$$d_5(\omega, \omega') = \frac{C_A}{\epsilon}\delta(\omega - \omega').$$

The form of the mixing between Q_1^μ and $\bar{Q}_1^\mu$ deserves some comment. At zero gluon the matrix elements of these operators are identical causing this mixing to be undetermined for zero gluon external state diagrams, even though the zero gluon matrix elements of both operators are nonzero, contrary to naive expectations. The contribution of the operator $\bar{Q}_1^\mu$ to the renormalization matrix was also determined. We find that this operator mixes with itself contributing a d_1 form to the matrix $Z_{\rm SL}$. The antisymmetric operators Q_2 and Q_4 mix only with themselves and contribute diagonal factors of d_1 to the matrix of renormalization constants.

The operator Q_3 is still under investigation and left for a future work. Because of this complication in determining the full anomalous dimension matrix and the need for a two gluon calculation to determine the possible mixing with $Q^{\mu\nu}(\omega_1, \omega_2, \Gamma)$ we present the results of our initial study of the anomalous dimension to subleading order in this paper and comment on the phenomenological consequences of the presented results. We collect our results in Sec. III.

2. The T products of $\mathcal{L}_{\rm HQET}$ with O_0 and P_0

To find the mixing of the operators

$$O_0 = \bar{h}_v\delta(\omega + in\cdot D)h_v, \qquad P_0^\mu = \bar{h}_v\delta(\omega + in\cdot D)\gamma^\mu\gamma_5 h_v \quad (36)$$

into the subleading operators, we must evaluate the time ordered products of the operators with the the subleading terms of the HQET Lagrangian ($\mathcal{L}_1$)

$$T_O(\omega) = \int d^4x T[i\mathcal{L}_1(x), O_0(0,\omega)], \qquad T_P^\mu(\omega) = \int d^4x T[i\mathcal{L}_1(x), P_0^\mu(0,\omega)]. \quad (37)$$

We now explicitly refer to the Dirac structure of the operators. This is necessary due to the Dirac structure of the operators in the subleading Lagrangian. We treat the subleading Lagrangian as a single operator insertion for the purposes of our calculation. The different renormalization of the kinetic and chromomagnetic terms is accommodated by breaking the T products up in to $T_{(O_0,O_k)}$, $T_{(O_0,O_m)}$, $T_{(P_0,O_k)}$, $T_{(P_0,O_m)}$ after the diagram calculations, where O_k, O_m refer to the kinetic and chromomagnetic operators of the subleading Lagrangian.

We start with the zero gluon diagrams. They are illustrated in Fig. 3. The crosses in the diagrams represent the possible locations where one inserts the subleading HQET Lagrangian, given by

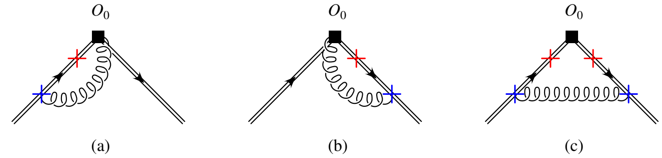

FIG. 3 (color online). The diagrams which must be calculated for Q_0. The crosses represent possible locations to insert the subleading HQET Lagrangian.

$$\mathcal{L}_1 = \bar{h}_v\frac{(iD_\perp)^2}{2m_b}h_v - a(\tilde{\mu})\bar{h}_v\frac{(g\sigma_{\alpha\beta}G^{\alpha\beta})}{4m_b}h_v. \quad (38)$$

The zero, one and two gluon Feynman rules for this Lagrangian, where we suppress the renormalization scale dependence of the O_m operator, are

$$i\mathcal{L}_1[0-\text{gluon}] = i\frac{k_\perp^2}{2m_b}P_+$$
$$i\mathcal{L}_1[1-\text{gluon}] = -igT_a\frac{(2k_\perp + \ell_\perp)^\alpha}{2m_b}P_+ + igT_a\frac{i\epsilon^{\alpha\mu\rho\eta}l_\mu v_\rho}{2m_b}s_\eta \quad (39)$$
$$i\mathcal{L}_1[2-\text{gluon}] = ig^2\{T^{a_1},T^{a_2}\}\frac{g_\perp^{\alpha_1\alpha_2}}{2m_b}P_+ + ig^2[T^{a_1},T^{a_2}]\frac{i\epsilon^{\alpha_1\alpha_2\rho\eta}v_\rho}{2m_b}s_\eta.$$

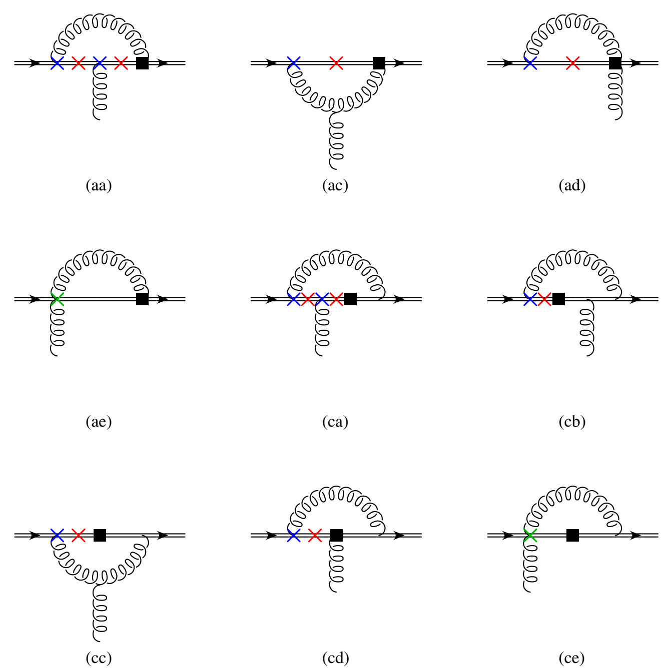

FIG. 4 (color online). The one gluon diagrams which must be calculated for O_0 and P_0. The crosses represent insertions of the subleading HQET Lagrangian. The mirror diagram corresponding to each of the above diagrams is not shown.

The UV divergent part of the sum of the subleading $\mathcal{L}_{\rm HQET}$ zero gluon results for both O_0 and P_0 (up to Dirac structure) is the same. Our result for the general Dirac structure operator Q_0, with indices suppressed, is

$$\begin{aligned}\langle h_v(k)|T_{(Q_0,O_k)}(\omega,\Gamma)|h_v(k)\rangle^{(1)}_{\rm div} &= \frac{\alpha_s}{4\pi}C_F\int d\omega'\frac{4\omega'\delta(\omega-\omega')}{m_b\epsilon}\langle h_v(k)|Q_0(\omega',\Gamma)|h_v(k)\rangle^{(0)} \\ &+\frac{\alpha_s}{4\pi}C_F\int d\omega'\frac{3\delta(\omega-\omega')}{\epsilon} \\ &\times\langle h_v(k)|v\cdot Q_1(\omega',\Gamma)|h_v(k)\rangle^{(0)}, \end{aligned}\tag{40}$$

where we have introduced the operator $v\cdot Q_1$ defined by,

$$m_b v\cdot Q_1(\omega,\Gamma)=\bar{h}_v\{iv\cdot D,\delta(\omega+in\cdot D)\}\Gamma h_v. \tag{41}$$

This operator vanishes by the leading order equations of motion and will not contribute to decay spectra. We do not include this operator in our operator basis for consistency in our stated results.

There are many more one gluon diagrams than zero gluon diagrams, as illustrated in Fig. 4. The diagrams explicitly given constitute half of the total number of diagrams that must be calculated for each of O_0 and P_0. The other diagrams can be looked upon as either the mirror diagrams of those given, or the transposed diagrams which have the $\mathcal{L}_1$ operator and lightcone operator interchanged.

The Dirac structure of the subleading Lagrangian force us to consider the Dirac structure of these diagrams. Let us consider the one gluon diagrams of Fig. 4, where the lightcone operator is Q_0. We will denote by $\langle ..|\mathcal{A}_R|..\rangle$ and $\langle ..|\mathcal{A}_L|..\rangle$ the amplitude of these diagrams and the amplitude of the mirror diagrams, respectively. Because of the simple relationship between O_0 and P_0^σ, the corresponding amplitudes for P_0^σ are $\langle ..|s^\sigma\mathcal{A}_R|..\rangle$ and $\langle ..|\mathcal{A}_L s^\sigma|..\rangle$.

Each of $\langle\mathcal{A}_R\rangle$ and $\langle\mathcal{A}_L\rangle$ can be decomposed into the two Dirac structures P_+ and s^η, for example, with a heavy quark target, with one gluon in the final external state:

$$\begin{aligned}\langle h_vA^\nu_a|\mathcal{A}_R|h_v\rangle &= \langle h_vA^\nu_a|\mathcal{A}_{R+}P_+|h_v\rangle \\ &+\langle h_vA^\nu_a|\mathcal{A}^\eta_{Rs}s_\eta|h_v\rangle, \\ \langle h_vA^\nu_a|\mathcal{A}_L|h_v\rangle &= \langle h_vA^\nu_a|\mathcal{A}_{L+}P_+|h_v\rangle \\ &+\langle h_vA^\nu_a|\mathcal{A}^\eta_{Ls}s_\eta|h_v\rangle.\end{aligned}\tag{42}$$

Thus for operator O_0 we can write the total amplitude proportional to each of the Dirac structures after the calculations of the 40 diagrams required. The results for insertions of the O_k are:

$$\begin{aligned}\langle h_vA^\nu_a|T_{(O_0,O_k)}|h_v\rangle^{(1)}_{\rm div} &= \langle h_vA^\nu_a|(\mathcal{A}_{R+}+\mathcal{A}_{L+})P_+|h_v\rangle^{(1)}_{\rm div}, \\ &= -\frac{C_F\omega\alpha_s g_s}{\pi m_b\epsilon}n^\nu T_a\left(\frac{\delta_-(n\cdot\ell)}{n\cdot\ell}\right)P_+ - \frac{3C_F\alpha_s g_s}{4\pi m_b\epsilon}T_a\left(v^\nu\delta_+(n\cdot\ell)-n^\nu(2k\cdot v+\ell\cdot v)\left(\frac{\delta_+(n\cdot\ell)}{n\cdot\ell}\right)\right)P_+.\end{aligned}\tag{43}$$

This result is easily matched, it is identical to the mixing form found in the zero gluon result with $\Gamma=P_+$, so that the mixing occurs with the operators O_0 and O_1:

$$\langle h_vA^\nu_a|T_{(O_0,O_k)}|h_v\rangle^{(1)}_{\rm div}=\frac{\alpha_s}{4\pi}C_F\int d\omega'\frac{4\omega'\delta(\omega-\omega')}{\epsilon m_b}\langle h_vA^\nu_a|O_0(\omega')|h_v\rangle^{(0)}+\frac{\alpha_s}{4\pi}C_F\int d\omega'\frac{3\delta(\omega-\omega')}{\epsilon}\langle h_vA^\nu_a|v\cdot O_1(\omega')|h_v\rangle^{(0)}. \tag{44}$$

The result of the T product with O_m is

$$\langle h_vA^\nu_a|T_{(O_0,O_m)}|h_v\rangle^{(1)}_{\rm div}=\langle h_vA^\nu_a|[\mathcal{A}^\eta_{Rs}+\mathcal{A}^\eta_{Ls}]s_\eta|h_v\rangle^{(1)}_{\rm div}, = -\frac{C_Ag_s\alpha T^a}{8m_b\epsilon\pi}\left(i\epsilon^{\nu s}_\perp\delta_-(n\cdot\ell)-i\epsilon^{\ell s}_\perp n^\nu\frac{\delta_-(n\cdot\ell)}{n\cdot\ell}\right). \tag{45}$$

This result matches onto the one gluon rule for P_2, as expected by the symmetry of the single derivative operators:

$$\langle h_vA^\nu_a|T_{(O_0,O_m)}|h_v\rangle^{(1)}_{\rm div}=-\frac{\alpha_s}{4\pi}C_A\int d\omega'\frac{\delta(\omega-\omega')}{2\epsilon}i\epsilon^{\mu\sigma}_\perp\langle h_vA^\nu_a|P_{2\mu\sigma}(\omega')|h_v\rangle^{(0)}.$$

The total $T^\sigma_{(P_0,O_k)}$ and $T^\sigma_{(P_0,O_m)}$ amplitudes can be written as

$$\begin{aligned}\langle h_vA^\nu_a|T^\sigma_{(P_0,O_k)}|h_v\rangle &= \langle h_vA^\nu_a|s^\sigma\mathcal{A}_{R+}|h_v\rangle+\langle h_vA^\nu_a|s^\sigma\mathcal{A}_{L+}|h_v\rangle, \\ \langle h_vA^\nu_a|T^\sigma_{(P_0,O_m)}|h_v\rangle &= \langle h_vA^\nu_a|s^\sigma s^\eta(A_{Rs})_\eta|h_v\rangle+\langle h_vA^\nu_a|s^\eta s^\sigma(A_{Ls})_\eta|h_v\rangle.\end{aligned}\tag{46}$$

Using the decomposition

$$s^\sigma s^\rho = i\epsilon^{\sigma\rho\eta\phi} s_\eta v_\phi - (g^{\sigma\rho} - v^\sigma v^\rho)P_+, \tag{47}$$

we can decompose in terms of the pieces proportional to P_+ and s^η for these amplitudes. For $T^\sigma_{(P_0,O_k)}$ and $T^\sigma_{(P_0,O_m)}$ we find the following mixing

$$\begin{aligned}\langle h_v (A^\nu_a | T^\sigma_{(P_0,O_k)} | h_v\rangle^{(1)}_{\rm div} &= \langle h_v A^\nu_a | s^\sigma (\mathcal{A}_{R+} + \mathcal{A}_{L+}) | h_v\rangle,\\ &= \frac{\alpha_s}{4\pi} C_F \int d\omega' \frac{4\omega' \delta(\omega - \omega')}{\epsilon m_b} \langle h_v A^\nu_a | P^\sigma_0(\omega') | h_v\rangle^{(0)}\\ &\quad + \frac{\alpha_s}{4\pi} C_F \int d\omega' \frac{3\delta(\omega - \omega')}{\epsilon} \langle h_v A^\nu_a | v\cdot P^\sigma_1(\omega') | h_v\rangle^{(0)},\end{aligned}$$

$$\begin{aligned}\langle h_v A^\nu_a | T^\sigma_{(P_0,O_m)} | h_v\rangle^{(1)}_{\rm div} &= (v^\sigma v^\mu - g^{\sigma\mu})\langle h_v A^\nu_a | ((A_{Rs})_\mu + (\mathcal{A}_{Ls})_\mu) P_+ | h_v\rangle + \langle h_v A^\nu_a | [i\epsilon^{\sigma\eta\phi\mu} v_\mu ((A_{Rs})_\eta - (A_{Ls})_\eta) s_\phi | h_v\rangle,\\ &= \frac{\alpha_s}{4\pi} C_A \int d\omega' \frac{\delta(\omega - \omega')}{2\epsilon} i\epsilon^{\mu\sigma}_\perp \langle h_v A^\nu_a | O_{2\mu}(\omega') | h_v\rangle - \frac{\alpha_s}{4\pi} C_A \int d\omega' \frac{\delta(\omega - \omega')}{2\epsilon} [(v^\sigma - n^\sigma) g^{\mu\eta}_\perp\\ &\quad + n^\eta g^{\sigma\mu}_\perp] (\langle h_v A^\nu_a | P^{\mu\eta}_1(\omega') | h_v\rangle^{(0)} - \langle h_v A^\nu_a | \bar{P}^{\mu\eta}_1(\omega') | h_v\rangle^{(0)})\\ &\quad + \frac{\alpha_s}{4\pi} C_A \int d\omega' \frac{\delta(\omega - \omega')}{\epsilon} n^\eta v^\sigma \langle h_v A^\nu_a | v\cdot P^\eta_1(\omega') - v\cdot \bar{P}^\eta_1(\omega') | h_v\rangle^{(0)}.\end{aligned}$$

where in analogy to $v\cdot Q_1$ we define

$$m_b v\cdot \bar{Q}_1(\omega, \Gamma) = -2\int d\omega_1 d\omega_2 \theta(\omega_1, \omega_2; \omega) \bar{h}_v \delta(\omega_1 + in\cdot D) iv\cdot D \delta(\omega_2 + in\cdot D)\Gamma h_v. \tag{48}$$

For consistency, we do not consider the mixing into this operator in Sec. III.

III. RESULTS

A. Leading nonperturbative order

The order α_s perturbative and leading order nonperturbative anomalous dimension matrix has been calculated by a variety of authors [22,27]. Our results agree with theirs, and in the basis

$$\{O_0, P_0\} \tag{49}$$

the pertrubative expansion is given by

$$Z^{(0)}(\omega, \omega') = \begin{bmatrix} \delta(\omega - \omega') & 0 \\ 0 & \delta(\omega - \omega') \end{bmatrix}, \tag{50}$$

$$Z^{(1)}(\omega, \omega', \tilde{\mu}) = \frac{\alpha_s(\tilde{\mu})}{4\pi} \begin{bmatrix} d_1(\omega, \omega', \tilde{\mu}) & 0 \\ 0 & d_1(\omega, \omega', \tilde{\mu}) \end{bmatrix}. \tag{51}$$

The distribution $d_1(\omega, \omega', \tilde{\mu})$ is the combination of the operator and wavefunction renormalization counter terms, given by

$$d_1(\omega, \omega', \tilde{\mu}) = -\frac{2C_F}{\epsilon^2}\delta(\omega - \omega') + \frac{2C_F}{\epsilon}\delta(\omega - \omega') + \frac{4C_F}{\tilde{\mu}\epsilon}\phi_0'\left(\frac{\omega - \omega'}{\tilde{\mu}}\right). \tag{52}$$

Recall, our initial expression related the bare and renormalized operators,

$$Q_0(\omega, \Gamma)_{\rm bare} = \int d\omega' Z(\omega', \omega, \tilde{\mu}) Q_0(\omega', \tilde{\mu}, \Gamma)_{\rm ren}. \tag{53}$$

We differentiate this equation with respect to $\tilde{\mu}$ to obtain our renormalization group equation

$$\tilde{\mu}\frac{d}{d\tilde{\mu}} Q_0(\omega, \tilde{\mu}, \Gamma)_{\rm ren} = -\int d\omega' \gamma(\omega', \omega, \tilde{\mu}) Q_0(\omega', \tilde{\mu}, \Gamma)_{\rm ren}. \tag{54}$$

The anomalous dimension matrix is determined using the useful result for $\overline{\rm MS}$ [28]

$$\gamma(g_s) = -2\alpha_s \frac{dZ_1(\alpha_s)}{d\alpha_s}, \tag{55}$$

where Z_1 is the coefficient of the $1/\epsilon$ poles. We find

$$\gamma(\omega, \omega', \tilde{\mu}) = \begin{bmatrix} \gamma_1(\omega, \omega', \tilde{\mu}) & 0 \\ 0 & \gamma_1(\omega, \omega', \tilde{\mu}) \end{bmatrix}, \tag{56}$$

with,

$$\gamma_1(\omega, \omega', \tilde{\mu}) = -\frac{\alpha_s(\tilde{\mu})}{\pi} C_F\left[\delta(\omega - \omega') + \frac{2}{\tilde{\mu}}\phi_0'\left(\frac{\omega - \omega'}{\tilde{\mu}}\right)\right]. \tag{57}$$

B. Subleading nonperturbative order

We have determined the matrix of renormalization constants at subleading nonperturbative order $Z_{\rm SL}$, excluding operators of class Q_3. If we order our Q_i operators as

$$\begin{aligned} \mathcal{O}_i &= \{O_0, T_{(O_0,O_k)}, T_{(O_0,O_m)}, O_1^\mu, \bar{O}_1^\mu, O_2^\mu, O_4\}, \\ \mathcal{P}_i^\sigma &= \{P_0^\sigma, T^\sigma_{(P_0,O_k)}, T^\sigma_{(P_0,O_m)}, P_1^{\sigma\mu}, \bar{P}_1^{\sigma\mu}, P_2^{\sigma\mu}, P_4^\sigma\}, \end{aligned} \tag{58}$$

the leading order term in the perturbative expansion in the basis $\{\mathcal{O}_i, \mathcal{P}_i\}$ is given in block form as

$$Z^{(0)}_{\rm SL}(\omega, \omega') = \begin{pmatrix} \Gamma^0_{\mathcal{O}_i,\mathcal{O}_j}(\omega,\omega') & \Gamma^0_{\mathcal{O}_i,\mathcal{P}_j}(\omega,\omega') \\ \Gamma^0_{\mathcal{P}_i,\mathcal{O}_j}(\omega,\omega') & \Gamma^0_{\mathcal{P}_i,\mathcal{P}_j}(\omega,\omega') \end{pmatrix}. \tag{59}$$

Where the entries in the matrices in the above expression with $i, j = 0, 1 \ldots 6$ are given by

$$\begin{aligned} \Gamma^0_{\mathcal{O}_i,\mathcal{O}_j}(\omega,\omega') &= (\delta_{i,j} - \delta_{0,j}\delta_{i,0})\delta(\omega-\omega'), \\ \Gamma^0_{\mathcal{P}_i,\mathcal{P}_j}(\omega,\omega') &= (\delta_{i,j} - \delta_{0,j}\delta_{i,0})\delta(\omega-\omega'), \\ \Gamma^0_{\mathcal{O}_i,\mathcal{P}_j}(\omega,\omega') &= 0, \qquad \Gamma^0_{\mathcal{P}_i,\mathcal{O}_j}(\omega,\omega') = 0. \end{aligned} \tag{60}$$

While the $\mathcal{O}(\alpha_s)$ term in the expansion is

$$Z^{(1)}_{\rm SL}(\omega, \omega', \tilde{\mu}) = \frac{\alpha_s(\tilde{\mu})}{4\pi} \begin{pmatrix} \Gamma^1_{\mathcal{O}_i,\mathcal{O}_j}(\omega,\omega') & \Gamma^1_{\mathcal{O}_i,\mathcal{P}_j}(\omega,\omega') \\ \Gamma^1_{\mathcal{P}_i,\mathcal{O}_j}(\omega,\omega') & \Gamma^1_{\mathcal{P}_i,\mathcal{P}_j}(\omega,\omega') \end{pmatrix}. \tag{61}$$

with the diagonal block matrices

$$\Gamma^1_{\mathcal{O}_i,\mathcal{O}_j}(\omega,\omega') = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ d_2(\omega,\omega',\tilde{\mu}) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & d_4(\omega,\omega',\tilde{\mu}) & d_5(\omega,\omega') & 0 & 0 \\ 0 & 0 & 0 & 0 & d_1(\omega,\omega',\tilde{\mu}) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & d_1(\omega,\omega',\tilde{\mu}) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & d_1(\omega,\omega',\tilde{\mu}) \end{bmatrix},$$

$$\Gamma^1_{\mathcal{P}_i,\mathcal{P}_j}(\omega,\omega') = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ d_2(\omega,\omega',\tilde{\mu}) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -d_7^{\mu\sigma\eta}(\omega,\omega') & d_7^{\mu\sigma\eta}(\omega,\omega') & 0 & 0 \\ 0 & 0 & 0 & d_4(\omega,\omega',\tilde{\mu}) & d_5(\omega,\omega') & 0 & 0 \\ 0 & 0 & 0 & 0 & d_1(\omega,\omega',\tilde{\mu}) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & d_1(\omega,\omega',\tilde{\mu}) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & d_1(\omega,\omega',\tilde{\mu}) \end{bmatrix}.$$

The off diagonal block matrices are as follows

$$\Gamma^1_{\mathcal{O}_i,\mathcal{P}_j}(\omega,\omega') = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -d_6^{\mu\sigma}(\omega,\omega') & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\Gamma^1_{\mathcal{P}_i,\mathcal{O}_j}(\omega,\omega') = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & d_6^{\mu\sigma}(\omega,\omega') & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The $d_i(\omega, \omega', \tilde{\mu})$ distributions are given by

$$d_2(\omega, \omega', \tilde{\mu}) = \frac{4C_F}{\epsilon}\frac{\omega'}{m_b(\tilde{\mu})}\delta(\omega - \omega'), \qquad d_4(\omega, \omega', \tilde{\mu}) = d_1(\omega, \omega', \tilde{\mu}) - \frac{C_A}{\epsilon}\delta(\omega - \omega'),$$
$$d_5(\omega, \omega') = \frac{C_A}{\epsilon}\delta(\omega - \omega'), \qquad d_6^{\mu\sigma}(\omega, \omega') = -\frac{C_A}{2\epsilon}(i\epsilon_\perp^{\mu\sigma})\delta(\omega - \omega'), \tag{62}$$
$$d_7^{\mu\sigma\eta}(\omega, \omega') = \frac{C_A}{2\epsilon}((v^\sigma - n^\sigma)g_\perp^{\mu\eta} + n^\eta g_\perp^{\sigma\mu})\delta(\omega - \omega').$$

We directly determine the anomalous dimension matrix to subleading order using Eq. (55) to be the following

$$\gamma_{\rm SL}(\omega, \omega', \tilde{\mu}) = \begin{pmatrix} \gamma_{\mathcal{O}_i,\mathcal{O}_j}(\omega, \omega', \tilde{\mu}) & \gamma_{\mathcal{O}_i,\mathcal{P}_j}(\omega, \omega', \tilde{\mu}) \\ \gamma_{\mathcal{P}_i,\mathcal{O}_j}(\omega, \omega', \tilde{\mu}) & \gamma_{\mathcal{P}_i,\mathcal{P}_j}(\omega, \omega', \tilde{\mu}) \end{pmatrix}. \tag{63}$$

The diagonal entries of the anomalous dimension matrix are

$$\gamma_{\mathcal{O}_i,\mathcal{O}_j}(\omega, \omega', \tilde{\mu}) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \gamma_2(\omega, \omega', \tilde{\mu}) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma_4(\omega, \omega', \tilde{\mu}) & \gamma_5(\omega, \omega', \tilde{\mu}) & 0 & 0 \\ 0 & 0 & 0 & 0 & \gamma_1(\omega, \omega', \tilde{\mu}) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \gamma_1(\omega, \omega', \tilde{\mu}) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \gamma_1(\omega, \omega', \tilde{\mu}) \end{bmatrix},$$

$$\gamma_{\mathcal{P}_i,\mathcal{P}_j}(\omega, \omega', \tilde{\mu}) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \gamma_2(\omega, \omega', \tilde{\mu}) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\gamma_7^{\mu\sigma\eta}(\omega, \omega', \tilde{\mu}) & \gamma_7^{\mu\sigma\eta}(\omega, \omega', \tilde{\mu}) & 0 & 0 \\ 0 & 0 & 0 & \gamma_4(\omega, \omega', \tilde{\mu}) & \gamma_5(\omega, \omega', \tilde{\mu}) & 0 & 0 \\ 0 & 0 & 0 & 0 & \gamma_1(\omega, \omega', \tilde{\mu}) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \gamma_1(\omega, \omega', \tilde{\mu}) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \gamma_1(\omega, \omega', \tilde{\mu}) \end{bmatrix}.$$

The off diagonal entries are as follows

$$\gamma_{\mathcal{O}_i,\mathcal{P}_j}(\omega, \omega', \tilde{\mu}) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\gamma_6^{\mu\sigma}(\omega, \omega') & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\gamma_{\mathcal{P}_i,\mathcal{O}_j}(\omega, \omega', \tilde{\mu}) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \gamma_6^{\mu\sigma}(\omega, \omega') & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

With the following entries in the anomalous dimension matrix,

$$\gamma_1(\omega, \omega', \tilde{\mu}) = -\frac{\alpha_s(\tilde{\mu})}{\pi} C_F\Big[\delta(\omega - \omega') + \frac{2}{\tilde{\mu}}\phi_0'\Big(\frac{\omega - \omega'}{\tilde{\mu}}\Big)\Big], \qquad \gamma_2(\omega, \omega', \tilde{\mu}) = -2\alpha_s(\tilde{\mu})\frac{\omega' C_F}{\pi m_b(\tilde{\mu})}\delta(\omega - \omega'),$$
$$\gamma_4(\omega, \omega', \tilde{\mu}) = \gamma_1(\omega, \omega', \tilde{\mu}) + \frac{\alpha_s(\tilde{\mu})C_A}{2\pi}\delta(\omega - \omega'), \qquad \gamma_5(\omega, \omega', \tilde{\mu}) = -\frac{\alpha_s(\tilde{\mu})C_A}{2\pi}\delta(\omega - \omega'), \tag{64}$$
$$\gamma_6^{\mu\sigma}(\omega, \omega', \tilde{\mu}) = \frac{\alpha_s(\tilde{\mu})C_A}{4\pi}(i\epsilon_\perp^{\mu\sigma})\delta(\omega - \omega'), \qquad \gamma_7^{\mu\sigma\eta}(\omega, \omega', \tilde{\mu}) = -\frac{\alpha_s(\tilde{\mu})C_A}{4\pi}((v^\sigma - n^\sigma)g_\perp^{\mu\eta} + n^\eta g_\perp^{\sigma\mu})\delta(\omega - \omega').$$

IV. CONCLUSIONS

We have examined the anomalous dimension matrix appropriate for the phase space restricted $\bar{B} \to X_u \ell \bar{\nu}$ and $\bar{B} \to X_s \gamma$ decay spectra to subleading nonperturbative order. The effects of the time ordered products of the HQET Lagrangian with the leading order shape function operator were determined and the renormalizability and closure of a subset of the nonlocal operator basis used to describe these spectra, to subleading order, was established.

Operator mixing was found between the operators which occur to subleading order, requiring that the subleading operator basis be extended to include the operator $\bar{Q}_1$. This requires the introduction of new shape functions to characterize the decay spectra of $\bar{B} \to X_u \ell \bar{\nu}$ and $\bar{B} \to X_s \gamma$ beyond tree level. The mixing determined between the operators Q_1 and $\bar{Q}_1$ is of the pernicious form that required a one gluon external state calculation to determine, despite the nonvanishing zero gluon Feynman rules of the operators. We have also demonstrated that the possible mixing with the operator $Q^{\mu\nu}(\omega_1, \omega_2, \Gamma)$ (see (27)) in a similar manner; with vanishing Feynman rules for zero and one gluon, requires a two gluon external state calculation to completely determine the anomalous dimension at subleading nonperturbative order.

Mixing was also determined between the T product $T_{(O_0, O_k)}$ and the leading order shape function, and the T product $T_{(O_0, O_m)}$ was shown to lead to mixing between the P_i and O_i operators at this order.

The anomalous dimension and running of the $\bar{Q}_1^\mu$, Q_2^μ and Q_4 operators was shown to be identical to the leading order shape function Q_0.

This work can be built upon in a number of ways. The anomalous dimension of the operator Q_3 is under investigation by the authors to establish the closure at one loop of the set of subleading operators discussed in this paper. The anomalous dimension of the subleading four quark operators should be investigated to determine the full anomalous dimension matrix at subleading order. Once the full anomalous dimension is determined, Sudakov logarithms in the perturbative corrections to the subleading operators can be resummed, so that renormalization group improved calculations can be undertaken for the $\bar{B} \to X_u \ell \bar{\nu}$ and $\bar{B} \to X_s \gamma$ decay spectra to subleading nonperturbative order. As the Sudakov resummation that can be accomplished with these results are resummations of perturbative corrections to order α_s/m_b we expect the numerical size of these corrections on the extraction of $|V_{ub}|$ to be small. The fact that this work establishes the renormalizability of a subset of the soft sector nonperturbative expansion beyond leading order is more significant. This is a necessary step in extending QCD factorization theorems beyond leading nonperturbative order, validating the factorization based approach used for the phase space restricted $\bar{B} \to X_u \ell \bar{\nu}_\ell$ and $\bar{B} \to X_s \gamma$ decay spectra beyond leading nonperturbative order.

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