Heavy-quark potentials and AdS/QCD

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We give an example of modeling phenomenological heavy-quark potentials in a five-dimensional framework nowadays known as AdS/QCD. In particular we emphasize the absence of infrared renormalons.

I. INTRODUCTION

Heavy-quark potential is one of the basic observables relevant to confinement. It has been measured in great detail in lattice simulations,[1] and the results reveal a remarkable agreement with the so called Cornell potential [2]

$$V(r) = -\frac{\kappa}{r} + \frac{r}{a^2} + C, \tag{1}$$

where the coefficients are adjusted to fit the charmonium spectrum
$$\kappa \approx 0.48, \qquad a \approx 2.34 \ \text{GeV}^{-1}, \qquad C = -0.25 \ \text{GeV}. \tag{2}$$

As follows from above, κ and $1/a^2$ can be interpreted as $4/3\alpha_s$ and the string tension, respectively.

One of the implications of the AdS/CFT correspondence [3] is that it resumed interest in finding a string description of strong interactions. For the case of interest, let us briefly mention a couple of results.

First, in the approach called usually gauge/string duality one tries to keep the underlying string structure. As a consequence, the theory is ten dimensional and its reductions to five dimensions in general contain additional higher derivative terms (stringy α' corrections).[2] According to [4], the expectation value of the Wilson loop is given by

$$\langle W(\mathcal{C}) \rangle \sim e^{-S}, \tag{3}$$

where S is an area of a string world-sheet bounded by a curve $\mathcal{C}$ at the boundary of AdS space.[3]

Second, in more phenomenological approach called AdS/QCD one starts from a five-dimensional effective field theory somehow motivated by string theory and tries to fit it to QCD as much as possible. It was recently pointed out [6,7] that asymptotic linearity of Regge trajectories arises for some backgrounds. Such backgrounds reduce to the standard AdS background in the UV but differ from it in the IR. The latter turns out to be crucial for linearity. In this case, it would be natural to expect that the interquark interaction would include the dominant Coulomb term at short distances as well as the dominant linear term at large distances.

In this note we explore this expectation in the context of (3). For simplicity we will concentrate here on the case of [7]. However, it should not be hard to adapt the arguments to [6]. It is worth noting a recent attempt to derive the Cornell type potential within the model based on a truncated AdS space [8].[4]

Before proceeding to the detailed analysis, let us set the basic framework. We consider the following Euclidean background metric

$$ds^2 = G_{nm}dX^n dX^m = R^2 \frac{h}{z^2}(dx^i dx^i + dz^2),$$
$$h = e^{(1/2)cz^2}, \tag{4}$$

where $i = 0, \ldots, 3$. In the region of small z the metric behaves asymptotically as Euclidean AdS$_5$, as expected. We also take a constant dilaton. Note that the use of the Euclidean signature for the background metric slightly modifies h. So, the exponent has an opposite sign to that of [7].

II. CALCULATING THE POTENTIAL

Given the background metric, we can attempt to calculate the corresponding potential. In doing so, we adapt the conjecture (3) to AdS/QCD.

We consider a rectangular Wilson loop $\mathcal{C}$ living on the boundary ($z = 0$) of five-dimensional space as shown in

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[1]For a review, see [1].

[2]These corrections are of order $1/\sqrt{N_c}$. Thus, they might be relevant at $N_c = 3$.

[3]The literature on the Wilson loops within the AdS/CFT correspondence is very vast. For a discussion of this issue, see, e.g., [5] and references therein.
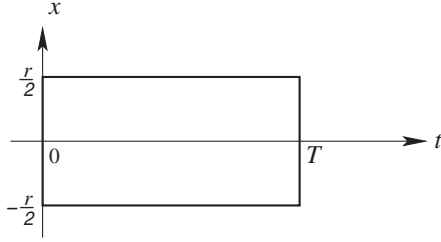
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[4]In this case there is a subtle point. The use of two different solutions leads to a discontinuity in the interquark force.

FIG. 1.   A loop $\mathcal{C}$.

Fig. 1. The quark and antiquark are set at $x = \frac{r}{2}$ and $x = -\frac{r}{2}$, respectively. As known, taking the limit $T \to \infty$ allows one to read off the energy of such a pair from the expectation value of the Wilson loop, namely, $\langle W(\mathcal{C}) \rangle \sim e^{-TE(r)}$.

Now we are ready to evaluate the expectation value of the loop. To this end, we make use of the Nambu-Goto action equip with the background metric (4)

$$S = \frac{1}{2\pi\alpha'} \int d^2\xi \sqrt{\det G_{nm} \partial_\alpha X^n \partial_\beta X^m}. \quad (5)$$

Next, we choose $\xi^1 = t$ and $\xi^2 = x$. This yields

$$S = \frac{\mathfrak{g}}{2\pi} T \int_{-(r/2)}^{r/2} dx \frac{h}{z^2} \sqrt{1 + (z')^2}, \quad (6)$$

where $\mathfrak{g} = R^2/\alpha'$. A prime denotes a derivative with respect to x.

Now it is easy to find the equation of motion for z

$$zz'' + (2 - cz^2)(1 + (z')^2) = 0 \quad (7)$$

as well as the first integral

$$\frac{h}{z^2\sqrt{1 + (z')^2}} = \text{const.} \quad (8)$$

The integration constant can be expressed via the maximum value of z. On symmetry grounds, z reaches it at $x = 0$. By virtue of (8), the integral over $[-\frac{r}{2}, \frac{r}{2}]$ of dx is equal to

$$r = 2\sqrt{\frac{\lambda}{c}} \int_0^1 dv\, v^2 e^{(1/2)\lambda(1-v^2)} (1 - v^4 e^{\lambda(1-v^2)})^{-(1/2)}, \quad (9)$$

where $v = z/z_0$, $\lambda = cz_0^2$, and $z_0 = z|_{x=0}$.

At this point a comment is in order. A simple analysis shows that the integral (9) is real for $\lambda < 2$. It develops a logarithmic singularity at $\lambda = 2$ and becomes complex for larger λ. Hence, there exists the upper bound on the maximum value of z

$$z_0 < \sqrt{\frac{2}{c}}. \quad (10)$$

Note that in the limit as c goes to zero z_0 is not bounded, as should be for the AdS space. As function of z, the effective string tension reaches its minimum at $z = z_0$. Thus, there

exists a kind of horizon which is a generic feature of confining theories.

Now, as in [4], we will compute the energy of the configuration. First, we reduce the integral over x in Eq. (6) to that over z. This is done by using the first integral (8). Since the integral is divergent at $z = 0$ due to the factor z^{-2} in the metric (4), in the process we regularize it by imposing a cutoff ϵ. Then we replace z with v as in (9). Finally, the regularized expression takes the form

$$E_R = \frac{\mathfrak{g}}{\pi}\sqrt{\frac{c}{\lambda}} \int_{\epsilon/z_0}^1 dv\, v^{-2} e^{(1/2)\lambda v^2} (1 - v^4 e^{\lambda(1-v^2)})^{-(1/2)}. \quad (11)$$

Its ϵ-expansion is simply

$$E_R = \frac{\mathfrak{g}}{\pi\epsilon} + E + O(\epsilon), \quad (12)$$

where

$$E = \frac{\mathfrak{g}}{\pi}\sqrt{\frac{c}{\lambda}}\left(-1 + \int_0^1 dv\, v^{-2} \right. $$
$$\left. \times [e^{(1/2)\lambda v^2}(1 - v^4 e^{\lambda(1-v^2)})^{-(1/2)} - 1]\right). \quad (13)$$

Similarly as r, E is real only for $\lambda < 2$. Having observed that the energy acquires an imaginary part, it is tempting to interpret this as the string breaking. However, this occurs at complex r. Our model is therefore stable.

In contrast to the AdS case [4], the potential in question is written in parametric form given by Eqs. (9) and (13).[5] We can, however, gain some important insights into the problem by considering two limiting cases.

First, let us have a close look at Eq. (9). As noted earlier, the range of λ is $0 \leq \lambda < 2$. After a short inspection we find that r is a continuously growing function of λ. The asymptotic behavior near zero is given by[6]

$$r = \frac{1}{\rho}\sqrt{\frac{\lambda}{c}}\left(1 - \frac{1}{4}\lambda(1 - \pi\rho^2) + O(\lambda^2)\right), \quad (14)$$

where $\rho = \Gamma^2(\frac{1}{4})/(2\pi)^{3/2}$. From this it follows that small λ's correspond to small values of r.

The asymptotic behavior near 2 is given by[7]

$$r = -\sqrt{\frac{2}{c}}\ln(2 - \lambda) + O(1). \quad (15)$$

Thus, this region corresponds to large values of r.

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[5]It is unclear to us how to eliminate the parameter λ and find E as a function of r and c.

[6]All the integrals can be evaluated in terms of the beta functions as discussed in the Appendix A.

[7]The integral is dominated by $v \sim 1$, where it takes the following form $2\sqrt{\frac{\lambda}{c}} \int_0^1 dv/\sqrt{a(1 - v) + b(1 - v)^2}$, with $a = 2(2 - \lambda)$ and $b = \lambda - (2 - \lambda)(3 + 2\lambda)$. The remaining integral may be found in tables in [9].

Having understood the correspondence between λ and r, we can investigate the properties of the interquark interaction at short and long distances.

We begin with the case of small r. Expanding the right-hand side of Eq. (13) up to the quadratic terms in λ, we get

$$E = -\frac{\mathfrak{g}}{2\pi\rho}\sqrt{\frac{c}{\lambda}}\left(1 + \frac{1}{4}\lambda(1 - 3\pi\rho^2) + O(\lambda^2)\right). \quad (16)$$

Combining this with (14), we find the energy of the configuration as a function of r and c

$$E = \mathfrak{g}\left(-\frac{\kappa_0}{r} + \sigma_0 r + O(r^3)\right), \quad (17)$$

where $\kappa_0 = \frac{1}{2\pi}\rho^{-2}$ and $\sigma_0 = \frac{1}{4}c\rho^2$. Thus, we have the expected $1/r$ behavior at short distances.

In a similar spirit, we can explore the long distance behavior of E. It follows from (13) that in the neighbor of $\lambda = 2$ the energy behaves as

$$E = -\frac{e\mathfrak{g}}{2\pi}\sqrt{\frac{c}{2}}\ln(2 - \lambda) + O(1). \quad (18)$$

Along with the relation (15), this means that the interquark interaction at long distances is given by

$$E = \mathfrak{g}(\sigma r + O(1)) \quad (19)$$

that is nothing but the desired linear potential. Here we have set $\sigma = \frac{e}{4\pi}c$.

Having understood the two limiting cases, we can now make a couple of estimates relevant to phenomenology.

It is natural to fix the overall constant $\mathfrak{g}$ from the slope of the potential at large distances. Indeed, the stringy approach is to be most reliable at large distances. From (1) and (19) we have

$$\mathfrak{g} = \frac{4\pi}{e}(ca^2)^{-1} \approx 0.94, \quad (20)$$

where we have used $c \approx 0.9$ GeV^2 as it follows from the fits to the slope of the Regge trajectories [7].

Next, we can estimate the slope of the linear potential at short distances. According to the Cornell model (1), the slope is the same at all the distances while Eqs. (17) and (19) imply that the coefficients in front of the linear terms at large and small distances are different. However, a simple estimate of their ratio yields

$$\frac{\sigma}{\sigma_0} = 8\pi^2 e\Gamma^{-4}\left(\frac{1}{4}\right) \approx 1.24. \quad (21)$$

Clearly, the difference in the slopes is not significant for our phenomenological estimates and the agreement with the lattice data is very satisfactory at this point.

Finally, we can compare evaluate the $1/r$ term in the potential. Phenomenologically, a little algebra shows that in (1) the coefficients obey $1/\kappa a^2 \approx 0.38$ GeV^2. If we

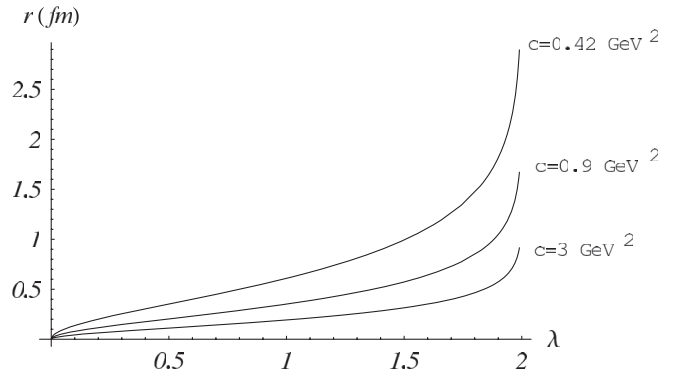FIG. 2. r as a function of λ at $c = 0.42$, 0.9, and 3 GeV^2.

truncate our model by keeping only the two terms as in the Cornell model, we find

$$\frac{\sigma}{\kappa_0} = \frac{1}{16\pi^3}e\Gamma^4\left(\frac{1}{4}\right)c \approx 0.85 \text{ GeV}^2. \quad (22)$$

The value is more than twice bigger than that of the Cornell model. So, this looks disappointing. However, the Coulomb-like potential at short distances is controlled by the running coupling $\alpha_s(r)$ and can hardly be predicted within the simplified stringy model we are considering. We will come back to discuss this point in the next section.

To complete the picture, let us present the results of numerical calculations. The parametric Eq. (9) predicts a characteristic form for r, as shown in Fig. 2. It has an interesting effect on the form of the interquark potential in the phenomenologically important interval 0.1 fm $\leq r \leq 1$ fm. It is clear that for quite small values of c this interval corresponds to small λ's. As a result, the approximate formula (17) is valid. So, if the desired linear behavior holds, it has the slope proportional to σ_0. On the other hand, larger values of c result in λ's ~ 2. In this case the linear behavior has the slope proportional to σ. The effect can be seen in Fig. 3 for the window 0.5 fm $\leq r \leq 1$ fm and the two values of c namely, $c = 0.42$ and 3 GeV^2.
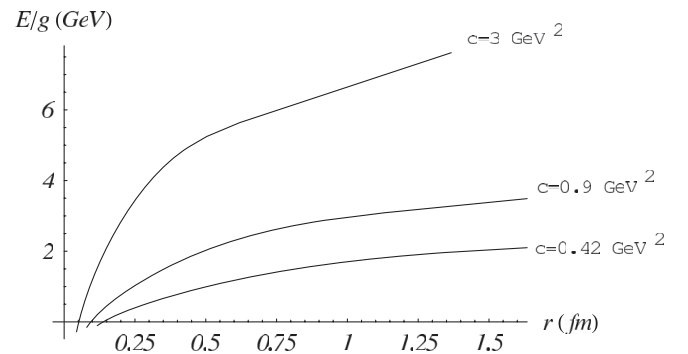

FIG. 3. $E/\mathfrak{g}$ as a function of r at $c = 0.42$, 0.9, and 3 GeV^2.

III. DISCUSSIONS

In this note we have evaluated the heavy-quark potential using the now standard ideas motivated by the gauge/string duality. A key point is the use of the background metric (4) which is singled out by the observation [7] that it provides, in terms of the same duality, linear Regge trajectories. The overall conclusion is that the same background metric results in a phenomenologically satisfactory description of the confining potential as well. However, there is a number of open problems which are mostly rooted in the heuristic nature of the gauge/string duality in the case of pure Yang-Mills theory. In conclusion, we list a few such problems and compare the results with theoretical expectations based on more traditional approaches.

(i) The potential of interest behaves as $1/r$ at short distances. This behavior is predicted, of course, by perturbative QCD as well. On the string theory side, one might fix the overall factor $\mathfrak{g}$ in the world-sheet action (6) by fitting the coefficients of the $1/r$ terms in the energy (16) and the perturbative calculations. If so, then $\mathfrak{g} \sim \alpha_s$. By contrast, the AdS/CFT correspondence requires $\mathfrak{g} \sim \sqrt{\alpha_s}$ [3]. Thus, this way of fixing the overall factor does not look satisfactory.[8]

On the other hand, in the case of pure Yang-Mills theory the status of the $1/r$ term in the potential (1) is also unclear. Its numerical value fits the theoretical prediction for the so called Lüscher term [11] which is derived at large distances. Phenomenologically the fit (1) works at small distances as well. Moreover, a straightforward application of the expansion in the running coupling $\alpha_s(r)$ results in a badly divergent series at presently available distances r.[9] Thus, it is not ruled out that a pure perturbative description sets in at much smaller distances. In view of all these theoretical uncertainties, the lack of agreement between the stringy potential (17) and the Cornell model at short distances might be not so significant.

(ii) On the lattice, the so called Casimir scaling has been observed [1]. The point is that if one measures the heavy-quark potential for various color representations of the quarks, at large distances the string tension turns out to be proportional to the coefficient of the Coulomb-like term. Reproducing the Casimir scaling theoretically is a strong challenge to the model building [12]. Within AdS/QCD, such a scaling is obvious. It is reproduced without fixing any parameter. Indeed, the fitting parameter is the overall factor $\mathfrak{g}$ in the expression for the energy, see the discussion above.

(iii) Power-like corrections to the heavy-quark potential at short distances were studied earlier mostly in terms of infrared renormalons.[10] Within this approach, the potential can be represented, in somewhat symbolical form, as

$$V(r) = (\text{perturbative series}) + \sum_{k=0} b_k r^{2k} \Lambda_{\mathrm{QCD}}^{2k+1},$$
(23)

where the coefficients b_k cannot be determined consistently within short-distance physics and correspond to the infrared renormalons. In particular, the leading nonperturbative correction to the potential is of order [14]

$$V_{\text{non-pert}} \sim r^2 \langle \alpha_s G^2 \rangle \rho_{\text{inst}},$$
(24)

where $\langle \alpha_s G^2 \rangle$ is the gluon condensate and ρ_{inst} is a typical instanton size. Note that the instanton size plays the role of an infrared cutoff. Without such a cutoff, the calculation is divergent.

In our example, the potential is calculable consistently at short distances. There are no uncertainties corresponding to the infrared renormalons. The only exception might be the leading renormalon corresponding to $k = 0$ in (23). Indeed, by the potential one can understand the difference between the total energy and self-energy of the quarks. In the bulk of the paper, we were calculating the total energy. Separation of the potential from the self-energy involves then the infrared uncertainty in the mass of the heavy quark which corresponds to the renormalon.[11]

We close the discussion of the infrared renormalons with a few short comments:

(1) There exists a simple picture of the absence of the renormalons. The key point is that at short distances a string does not go far away into the fifth dimension or, in other words, it does not reach a vicinity of the horizon that would correspond to the IR.

(2) We discuss a class of metric leading to the absence of the renormalons with $k > 0$ in the Appendix B.

(3) Phenomenologically, there is no significant r^2 term in the potential.[12] Thus, the absence of the infrared renormalons from the stringy potential can be considered as a success of the model.

(iv) The AdS/QCD approach provides a natural framework for appearance of a linear piece in the potential at short distances. Moreover, as follows from

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[8]This difficulty was pointed out in different contexts. See, e.g. [4,10].

[9]For a review, see [1].

[10]For further discussion, see [13].

[11]At technical level this means that the use of minimal subtraction in Eq. (12) is not appropriate.

[12]For further discussion, see [15].

Eq. (21), the coefficients of the linear terms in the short and large distance expansions turn out to be close to each other. Phenomenological arguments in favor of such a contribution can be found in [16].

ACKNOWLEDGMENTS

APPENDIX A

This appendix collects together some of the formulae that are used in Sec. II.

All the integrals can be expressed in terms of integrals of the form

$$I(a, b) = \int_0^1 dv v^a (1 - v^4)^b. \tag{A1}$$

Making the variable change $w = v^4$, we can easily evaluate the integral over w. The final result is

$$I(a, b) = \frac{1}{4} B\left(\frac{a + 1}{4}, b + 1\right), \tag{A2}$$

where B denotes the beta function.

In manipulating the beta functions, we use the relation to the gamma function and the formulae [9]

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}, \qquad \Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right) = \sqrt{2}\pi,$$

$$\Gamma(x + 1) = x\Gamma(x). \tag{A3}$$

APPENDIX B

In this appendix we will investigate the question of whether for a generic form of the warp factor $h(z)$ in the metric (4) the renormalons with $k > 0$ are missing as well.[13]

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[13]We assume $h(z) > 0$.

Since at $z = 0$ the metric reduces to that of AdS$_5$, we take

$$h(0) = 1. \tag{B1}$$

The Nambu-Goto action and the first integral of equation of motion are given by Eq. (6) and (8), respectively. From the first integral we get

$$r = 2z_0 \int_0^1 dv v^2 \frac{h_0}{h}\left(1 - v^4\left(\frac{h_0}{h}\right)^2\right)^{-(1/2)}, \tag{B2}$$

where z_0 is the maximum value of z, $v = z/z_0$, and $h_0 = h(z_0)$.

We compute the energy of the configuration as in Sec. III. In the process we regularize the integral over z by imposing a cutoff ϵ. Finally, we get

$$E_R = \frac{\mathfrak{g}}{\pi \epsilon} + E + O(\epsilon), \tag{B3}$$

where

$$E = \frac{\mathfrak{g}}{\pi z_0}\left(-1 + \int_0^1 dv v^{-2}\left[h\left(1 - v^4\left(\frac{h_0}{h}\right)^2\right)^{-(1/2)} - 1\right]\right). \tag{B4}$$

As a result, the potential is written in parametric form given by Eqs. (B2) and (B4).

Assume that we eliminated the parameter z_0 and found E as a function of r.[14] Now the question arises: when does this potential have no terms like r^{2k}? The answer to this is clear from the form of equations. If we transform $z_0 \rightarrow -z_0$, then $r \rightarrow -r$ for any even function h. Meanwhile, the energy transforms as $E \rightarrow -E$. Thus, E is an odd function of r if h is even. The latter means that in this case the problem of the renormalons with $k > 0$ is missing.

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[14]As in Sec. III, we should consider a range of z_0 corresponding to real r and E. For small r (equivalently, small z_0) this is indeed the case.

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