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Discrete flavor symmetry D_5

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We consider the standard model (SM) extended by the flavor symmetry D_5 and search for a minimal model leading to viable phenomenology. We find that it contains four Higgs fields apart from the three generations of fermions whose left-handed and left-handed conjugate parts do not transform in the same way under D_5. We provide two numerical fits for the case of Dirac and Majorana neutrinos to show the viability of our low energy model. The fits can accommodate all data with the neutrinos being normally ordered. For Majorana neutrinos two of the right-handed neutrinos are degenerate. Concerning the Higgs sector we find that all potentials constructed with three SM-like Higgs doublets transforming as $\underline{\mathbf{1}} + \underline{\mathbf{2}}$ under D_5 have a further unwanted global $U(1)$ symmetry. Therefore we consider the case of four Higgs fields forming two D_5 doublets and show that this potential leads to viable solutions in general, however it does not allow spontaneous CP violation (SCPV) for an arbitrary vacuum expectation value (VEV) configuration. Finally, we discuss extensions of our model to grand unified theories (GUTs) as well as embeddings of D_5 into the continuous flavor symmetries $SO(3)_f$ and $SU(3)_f$.

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I. INTRODUCTION

Gauge interactions and charge quantization of quarks and leptons can successfully be described by the mathematical concept of Lie groups, e.g. in the framework of grand unified theories (GUTs). Albeit the number of fermion generations, the diverse masses and mixing parameters of quarks and leptons remain free parameters. It is tempting to assume that these properties can also be explained by some (flavor) symmetry G_f. For several reasons G_f is chosen to be discrete and non-Abelian in many models. In the literature the permutation symmetries S_3 [1], A_4 [2], and S_4 [3], the single- and double-valued dihedral groups such as D_4 [4] and D_2' [5] and groups D_n, D_n' with larger index n [6–8] have been discussed. Furthermore the two-valued group T' [9] and subgroups of $SU(3)$, $\Delta(48)$, and $\Delta(75)$ [10], belonging to the series of $\Delta(3n^2)$ and $\Delta(6n^2)$ with $n \in \mathbb{N}$ have been studied. Most of these groups have been used to maintain a certain fermion mass texture. However, proceeding in this way does not answer the question which fundamental group structure of a discrete symmetry is favorable for describing nature and which is not.

In order to investigate the generic features of a certain group structure it is enough to discuss the smallest group which reveals this structure. Therefore we choose the flavor symmetry to be D_5 which is the smallest group with two irreducible (faithful) inequivalent two-dimensional representations. This group is used in [6] to produce certain mass textures for the lepton sector, but mass matrices for the quarks as well as the Higgs sector are not discussed. Apart from D_5 only the discussed groups D_n for $n \geq 6$, D_n' for $n > 2$, and T' have more than one irreducible two-dimensional representations. However, in general the groups differ in the product structure.

Our starting point is thus the SM gauge group extended by the flavor group D_5. Both groups are broken only spontaneously at the electroweak scale. We require a partial unification for left-handed and left-handed conjugate fields, i.e. both should transform as $\underline{\mathbf{1}} + \underline{\mathbf{2}}$ under D_5 where $\underline{\mathbf{1}}$ and $\underline{\mathbf{2}}$ do not need to be the same for both. Since we do not want to give up the idea of unified gauge groups we further require that our model is embeddable into the Pati-Salam group $SU(4)_C \times SU(2)_L \times SU(2)_R$, $SU(5)$, $SO(10)$, or $E(6)$. The resulting mass matrices should allow a viable fit of all data which will be demonstrated by numerical examples. For this and for the spontaneous breaking of D_5, we have to take at least three $SU(2)_L$ doublet Higgs fields which transform nontrivially under D_5. Since there exist strong bounds on flavor changing neutral currents (FCNCs), the number of Higgs fields should be as small as possible and they should be sufficiently heavy. Furthermore we discard the possible existence of $SU(2)_L$ triplet and standard model (SM) gauge singlet (scalar) fields. Taking all these constraints and the requirement that there are no leftover massless Goldstone bosons coming from accidental symmetries of the Higgs potential we will show that we need at least four Higgs fields. With these it turns out to be favorable to have different transformation properties of left-handed and left-handed conjugate fermions under D_5. The neutrinos can be either Dirac or Majorana particles. In the second case two of the right-handed neutrinos are degenerate, since there are no SM gauge singlets in the theory. We contrast this minimal D_5

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invariant model with the corresponding one invariant under the flavor symmetry D_3 which is isomorphic to S_3 and considered very often in the literature [1].

We also discuss the three Higgs potential in detail and show the existence of an accidental global $U(1)$ symmetry in the potential. Furthermore we study the phenomenology of the four Higgs sector analytically and numerically and demonstrate that the VEV configurations chosen in the numerical examples of the fermion mass matrices cannot be minima of the potential, if CP is only spontaneously violated. In the case of explicit CP violation a numerical analysis indicates the possibility that the chosen VEV configurations can be minima of the general D_5 invariant potential. The D_5 invariant three Higgs sector as well as the four Higgs sector are compared with the corresponding Higgs sectors invariant under the dihedral groups D_3, D_4, and D_6. Thereby we show the importance to classify the symmetries according to their product structure rather than to pick one freely.

Finally, we briefly mention the possible embeddings of our minimal model into GUT groups and continuous flavor groups.

The paper is organized as follows: Sec. II contains the group theory of the dihedral symmetries. Our minimal model is presented in Sec. III and the numerical analysis in Sec. IV. Section V is dedicated to the Higgs sectors of D_5 and the differences to D_3, D_4, and D_6. Section VI contains possible extensions of our model from a low to a high energy theory. Finally, we conclude in Sec. VII and comment on nontrivial subgroups of D_5. Clebsch Gordan coefficients and embeddings of D_5 are delegated to Appendix A. Appendix B lists the numerical solutions for the Yukawa couplings and Higgs VEVs and Appendix C contains the used experimental data.

II. GROUP STRUCTURE OF DIHEDRAL GROUPS

A. General properties of dihedral groups D_n

The groups D_n are well known in solid state and molecular physics. Their double-valued counterparts are the groups D'_n. Since $n \in \mathbb{N}$, there are infinitely many of them. Apart from the two trivial groups with $n = 1, 2$ all groups D_n are non-Abelian. They only contain real one- and two-dimensional irreducible representations. If its index n is even, the group D_n has four one- and $\frac{n}{2} - 1$ two-dimensional representations and for n being odd D_n has two one- and $\frac{n-1}{2}$ two-dimensional representations. The order of the group D_n is $2n$. The four smallest non-Abelian discrete groups can be found among the family of the dihedral symmetries: D_3, D_4, D'_2, and D_5. Generators of the two-dimensional representations can be given for all n [11]:

$$\mathrm{A} = \begin{pmatrix} e^{[(2\pi i)/n]j} & 0 \\ 0 & e^{-[(2\pi i)/n]j} \end{pmatrix}, \qquad \mathrm{B} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (1)$$

with $j = 1, \ldots, \frac{n}{2} - 1$ for n even and $j = 1, \ldots, \frac{n-1}{2}$ for n odd. They fulfill the relations:

$$\mathrm{A}^n = \mathbb{1}, \qquad \mathrm{B}^2 = \mathbb{1}, \qquad \mathrm{ABA} = \mathrm{B}. \quad (2)$$

The corresponding character tables can also be found in [11]. Note that we have chosen complex generators for the two-dimensional representations. Since these are real, there exists a unitary matrix U which links their generators to its complex conjugates:

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

For any

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \sim \underline{\mathbf{2}}$$

the combination

$$U\begin{pmatrix} a_1^\star \\ a_2^\star \end{pmatrix} = \begin{pmatrix} a_2^\star \\ a_1^\star \end{pmatrix}$$

transforms as $\underline{\mathbf{2}}$ instead of

$$\begin{pmatrix} a_1^\star \\ a_2^\star \end{pmatrix},$$

as it would be the case for real generators A and B.

B. The group D_5

D_5 is of order ten and has two one- and two two-dimensional irreducible representations, since its index is odd. They are denoted as $\underline{\mathbf{1}}_1$, $\underline{\mathbf{1}}_2$, $\underline{\mathbf{2}}_1$, and $\underline{\mathbf{2}}_2$. Both two-dimensional representations are faithful. Their characters χ, i.e. the traces of their representation matrices, are given in the character table, shown in Table I. There we use the following notations: $\mathcal{C}_i$ with $i = 1, \ldots, 4$ are the four classes of the group, ${}^\circ\mathcal{C}_i$ is the order of the ith class, i.e. the number of distinct elements contained in this class, ${}^\circ\mathrm{h}_{\mathcal{C}_i}$ is the order of the elements R in the class $\mathcal{C}_i$, i.e. the

TABLE I. Character table of the group D_5. α and β are given as $\alpha = \frac{1}{2}(-1 + \sqrt{5}) = 2\cos(\frac{2\pi}{5})$ and $\beta = \frac{1}{2}(-1 - \sqrt{5}) = 2\cos(\frac{4\pi}{5})$ and therefore $\alpha + \beta = -1$. For further explanations see text.

	Classes			
	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$
G	$\mathbb{1}$	B	A	A^2
${}^\circ\mathcal{C}_i$	1	5	2	2
${}^\circ\mathrm{h}_{\mathcal{C}_i}$	1	2	5	5
$\underline{\mathbf{1}}_1$	1	1	1	1
$\underline{\mathbf{1}}_2$	1	-1	1	1
$\underline{\mathbf{2}}_1$	2	0	α	β
$\underline{\mathbf{2}}_2$	2	0	β	α

smallest integer (>0) for which the equation $R^{\circ h_{C_i}} = \mathbb{1}$ holds. Furthermore the table contains one representative for each class $\mathcal{C}_i$ given as product of the generators A and B of the group. The elements belonging to the classes $\mathcal{C}_i$ are: $\mathcal{C}_1 = \{\mathbb{1}\}$, $\mathcal{C}_2 = \{B, BA, BA^2, BA^3, BA^4\}$, $\mathcal{C}_3 = \{A, A^4\}$, and $\mathcal{C}_4 = \{A^2, A^3\}$. With the help of the character table the Kronecker products can be calculated. They are $\underline{\mathbf{1}}_\mathbf{i} \times \underline{\mathbf{1}}_\mathbf{j} = \underline{\mathbf{1}}_{(\mathrm{i+j})\mathrm{mod}2+1}$, $\underline{\mathbf{1}}_\mathbf{i} \times \underline{\mathbf{2}}_\mathbf{j} = \underline{\mathbf{2}}_\mathrm{j}$ for $\{\mathrm{i,j}\} \in \{1,2\}$ and $\underline{\mathbf{2}}_\mathbf{1} \times \underline{\mathbf{2}}_\mathbf{2} = \underline{\mathbf{2}}_\mathbf{1} + \underline{\mathbf{2}}_\mathbf{2}$, and $[\underline{\mathbf{2}}_\mathbf{i} \times \underline{\mathbf{2}}_\mathbf{i}] = \underline{\mathbf{1}}_\mathbf{1} + \underline{\mathbf{2}}_\mathbf{j}$, $\{\underline{\mathbf{2}}_\mathbf{i} \times \underline{\mathbf{2}}_\mathbf{i}\} = \underline{\mathbf{1}}_\mathbf{2}$ for $\mathrm{i} \neq \mathrm{j}$ where $[\mu \times \mu]$ is the symmetric part of the product $\mu \times \mu$ and $\{\mu \times \mu\}$ the antisymmetric one. Note further that $\mu \times \nu = \nu \times \mu$ for all representations μ and ν. Taking $n = 5$ in Eqs. (1) and (2) gives the generators A and B and their relations for D_5. They are required for the calculation of the Clebsch Gordan coefficients being shown in Appendix A. They actually coincide with the matrices chosen in [6]. The embedding of D_5 into continuous groups is very interesting with respect to grand unified model building. Therefore we show how D_5 can be embedded into $SO(3)$ and $SU(3)$ in Appendix A. We will discuss this in more detail in Sec. VI.

III. MINIMAL MODEL

Here we present a minimal model which leads to viable mass spectra and mixing parameters for quarks as well as leptons with the Higgs potential being free from accidental symmetries (see Sec. V). We assign the left-handed quarks $Q_i = (u_i, d_i)^T_L$ and their conjugates u^c_{Li}, d^c_{Li} of the ith generation to:

$$Q_1 \sim \underline{\mathbf{1}}_\mathbf{1}; \qquad \begin{pmatrix} Q_2 \\ Q_3 \end{pmatrix} \sim \underline{\mathbf{2}}_\mathbf{2};$$

$$u^c_{L1}, d^c_{L1} \sim \underline{\mathbf{1}}_\mathbf{1}, \quad \text{and} \quad \begin{pmatrix} u^c_{L2} \\ u^c_{L3} \end{pmatrix}, \begin{pmatrix} d^c_{L2} \\ d^c_{L3} \end{pmatrix} \sim \underline{\mathbf{2}}_\mathbf{1}.$$

The left-handed lepton doublets $L_i = (\nu_i, e_i)^T_L$ and their conjugates e^c_{Li} and ν^c_{Li} transform in a similar way, i.e.:

$$L_1 \sim \underline{\mathbf{1}}_\mathbf{1}; \qquad \begin{pmatrix} L_2 \\ L_3 \end{pmatrix} \sim \underline{\mathbf{2}}_\mathbf{2};$$

$$e^c_{L1}, \nu^c_{L1} \sim \underline{\mathbf{1}}_\mathbf{1}, \quad \text{and} \quad \begin{pmatrix} e^c_{L2} \\ e^c_{L3} \end{pmatrix}, \begin{pmatrix} \nu^c_{L2} \\ \nu^c_{L3} \end{pmatrix} \sim \underline{\mathbf{2}}_\mathbf{1}.$$

The four Higgs fields χ_i and ψ_i which are $SU(2)_L$ doublets with hypercharge $Y = -1$ (like the Higgs field in the SM) transform as

$$\begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} \sim \underline{\mathbf{2}}_\mathbf{1}$$

and

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \sim \underline{\mathbf{2}}_\mathbf{2}$$

under D_5.

The fermion mass matrices arise from the coupling $y_{ij} \mathrm{L}^T_i \epsilon \xi \mathrm{L}^c_j$ for down-type quarks ($\mathrm{L}_i = Q_i$, $\mathrm{L}^c_i = d^c_{Li}$) and charged leptons ($\mathrm{L}_i = L_i$, $\mathrm{L}^c_i = e^c_{Li}$) and $y_{ij} \mathrm{L}^T_i \epsilon \tilde{\xi} \mathrm{L}^c_j$ for up-type quarks ($\mathrm{L}_i = Q_i$, $\mathrm{L}^c_i = u^c_{Li}$) and neutrinos ($\mathrm{L}_i = L_i$, $\mathrm{L}^c_i = \nu^c_{Li}$). Thereby, the Higgs field ξ is $\xi^T = (\xi^0, \xi^-)^T$ and its complex conjugate $\tilde{\xi}$ is $\tilde{\xi} = \epsilon \xi^\star$ with ϵ being the antisymmetric 2-by-2 matrix in $SU(2)_L$ space and the star $\star$ denotes the complex conjugation.

The resulting Dirac mass matrices are:

$$\begin{aligned} \mathcal{M}_{u,\nu} &= \begin{pmatrix} 0 & \alpha_2^{u,\nu} \langle \chi_1 \rangle^\star & \alpha_2^{u,\nu} \langle \chi_2 \rangle^\star \\ \alpha_3^{u,\nu} \langle \psi_1 \rangle^\star & \alpha_1^{u,\nu} \langle \psi_2 \rangle^\star & \alpha_0^{u,\nu} \langle \chi_1 \rangle^\star \\ \alpha_3^{u,\nu} \langle \psi_2 \rangle^\star & \alpha_0^{u,\nu} \langle \chi_2 \rangle^\star & \alpha_1^{u,\nu} \langle \psi_1 \rangle^\star \end{pmatrix}, \\ \mathcal{M}_{d,l} &= \begin{pmatrix} 0 & \alpha_2^{d,l} \langle \chi_2 \rangle & \alpha_2^{d,l} \langle \chi_1 \rangle \\ \alpha_3^{d,l} \langle \psi_2 \rangle & \alpha_1^{d,l} \langle \psi_1 \rangle & \alpha_0^{d,l} \langle \chi_2 \rangle \\ \alpha_3^{d,l} \langle \psi_1 \rangle & \alpha_0^{d,l} \langle \chi_1 \rangle & \alpha_1^{d,l} \langle \psi_2 \rangle \end{pmatrix}, \end{aligned} \tag{3}$$

where $\langle \xi \rangle$ denotes the VEV of the field $\xi = \psi_i, \chi_i$. The VEVs and the Yukawa couplings $\alpha_j^{u,d,l,\nu}$ are in general complex. The (1, 1) element of the mass matrices is zero, since there is no Higgs field transforming trivially under D_5. Even though there are more parameters in our model than observables to fit, this is a rather nontrivial task, since apart from the number of free parameters also the structure of the mass matrices plays an important role in fitting the observables.

The number of parameters could obviously be reduced, if some of the Yukawa couplings were assumed to be equal. Since our flavor symmetry D_5 cannot explain this, we do not use such assumptions. Another way to reduce the number of parameters could be to set some of the VEVs to be equal or zero. For two VEVs being zero we either have two massless quarks or cannot generate CP violation, since $\mathcal{J}_{CP} \propto \det([\mathcal{M}_u \mathcal{M}_u^\dagger, \mathcal{M}_d \mathcal{M}_d^\dagger])$ [12] vanishes. Furthermore some of these configurations lead to the appearance of accidental symmetries in the Higgs potential (see Sec. V B). For one VEV being zero or two VEVs being equal we cannot find an obvious reason to exclude these assumptions, but one does not gain much in doing so, since most of the free parameters in our model come from the (in total) 16 Yukawa couplings which have to be compared with the 20 (22) observable masses and mixing parameters in the quark and lepton sector for Dirac (Majorana) neutrinos. Therefore we do not make such assumptions in the following numerical study.

We have chosen a structure which is similar to a mass texture which has already been discussed in the literature [13]. It actually arises from our mass matrix for real parameters and in the limit that all VEVs are equal in Eq. (3) together with $\alpha_2^i = \alpha_3^i$ for $i = u, d, l, \nu$ or for $\langle \chi_1 \rangle = \langle \chi_2 \rangle$, $\langle \psi_1 \rangle = \langle \psi_2 \rangle$ and $\alpha_3^{u,\nu} = \frac{\langle \chi_2 \rangle^\star}{\langle \psi_2 \rangle^\star} \alpha_2^{u,\nu}$, $\alpha_3^{d,l} = \frac{\langle \chi_2 \rangle}{\langle \psi_2 \rangle} \alpha_2^{d,l}$. Then all mass matrices are invariant under the interchange of the second and third generation which always leads to mixing angles $\theta_{13} = 0$ and $\theta_{23} = \frac{\pi}{4}$ with unconstrained θ_{12}. In the leptonic sector this is called $\mu - \tau$ interchange symmetry [14].

Here we assume that the first generation transforms as $\underline{\mathbf{1}}_{\mathbf{1}}$ and the second and third one as $\underline{\mathbf{2}}_{\mathbf{i}}$ under D_5. This choice is inspired by the observation that the masses of the particles which belong to the first generation are much smaller than the masses of the ones of the second and third one and by the fact that the mixing in the 2-3 sector of the leptons is large, possibly maximal. In general there are six possibilities to assign the three generations $\{1, 2, 3\}$ to $\underline{\mathbf{1}}_{\mathbf{i}} + \underline{\mathbf{2}}_{\mathbf{j}}$: $\{[1], [2, 3]\}$, $\{[1], [3, 2]\}$, $\{[2], [1, 3]\}$, $\{[2], [3, 1]\}$, $\{[3], [1, 2]\}$, and $\{[3], [2, 1]\}$ where [.] forms the one-dimensional representation and [. , .] the two-dimensional one under D_5. If the left-handed fields are permuted by P: $P\{[1], [2, 3]\} = \{[P_1 \cdot (1, 2, 3)^T], [P_2 \cdot (1, 2, 3)^T, P_3 \cdot (1, 2, 3)^T]\}$ with P_i being the ith row of the matrix P and the left-handed conjugate fields by Q, the mass matrix $\mathcal{M}$ changes to $\widetilde{\mathcal{M}} = P\mathcal{M}Q^T$, since all permutations are orthogonal. As one can see these permutations change neither the eigenvalues of the mass matrix, i.e. $\det(\mathcal{M}\mathcal{M}^\dagger - \Lambda\,\mathrm{diag}(1, 1, 1)) = 0$ remains invariant, nor the mixing matrices V_{CKM} and U_{MNS}. If the mass matrix of the up-type quarks $\mathcal{M}_u$ is diagonalized by U_u fulfilling $U_u^\dagger \mathcal{M}_u \mathcal{M}_u^\dagger U_u = \mathrm{diag}(m_u^2, m_c^2, m_t^2)$ and the same holds for $\widetilde{\mathcal{M}}_u$ and $\tilde{U}_u$, then U_u and $\tilde{U}_u$ are connected by $\tilde{U}_u = PU_u$. Similarly one gets for the down-type quarks $\tilde{U}_d = PU_d$[1] and therefore, for example, $V_{\mathrm{CKM}} = \tilde{U}_u^T \tilde{U}_d^\star = U_u^T P^T P^\star U_d^\star = U_u^T U_d^\star$, such that V_{CKM} is not affected by this permutation. For the mass matrix texture it seems to be most convenient to have a vanishing (1, 1) element instead of, for example, a (2, 3) or (3, 3) one.

Apart from permuting the three generations among each other one can interchange the transformation properties of the left-handed and left-handed conjugate fields. This leads to matrices which are transposed to the ones shown in Eq. (3). Furthermore one can ask whether there is a considerable change, if the first generation is not assigned to $\underline{\mathbf{1}}_{\mathbf{1}}$, but to $\underline{\mathbf{1}}_{\mathbf{2}}$. The answer is no, since it only introduces a relative sign between the (1, 2) and (1, 3) and (2, 1) and (3, 1) elements of the mass matrix.

Our choice for the assignment of fermion generations allows an embedding into the Pati-Salam gauge group, where all left-handed fields are unified into one representation as well as all left-handed conjugate fields into the conjugated one. One can also attempt to embed the model into $SO(10)$, but then all fermions have to transform in the same way under D_5. In doing so one arrives at mass matrices which have two additional texture zeros in the (2, 3) and (3, 2) element (and one Yukawa coupling less than the matrices shown above). In the case of Hermitian matrices such a texture is excluded for quarks [15]. This does not strictly apply in our case, because our matrices are in general not Hermitian, but we believe that this does not change the result of [15]. For an embedding into $SU(5)$, the generations Q_i and u^c_{Li} would have to transform in the same way under D_5, since these fields are unified into the 10-plet of $SU(5)$. Then again the mass matrix for the up-type quarks has to have three texture zeros in the positions (1, 1), (2, 3), and (3, 2) combined with a mass matrix for the down-type quarks with one zero in the (1, 1) element, since Q_i and d^c_{Li} do not belong to the same $SU(5)$ representation and therefore can transform differently under D_5. Generally, such a structure is not excluded, but taking into account the various relations among the nonvanishing matrix elements, it seems to be unfavorable. Therefore we do not discuss this possibility here. Concerning the number of possible different assignments for quarks and leptons the Pati-Salam group has an advantage over $SU(5)$, since the 16 fermions of one generation (i.e. the right-handed neutrino is always included in our considerations) are unified into two and not into three representations of the gauge group. Its disadvantage is the fact that the three SM gauge factors are not unified into a single group, but rather in a product one.

If neutrinos are Majorana particles, the Majorana mass matrix for the right-handed neutrinos looks very simple, since our model does not include SM gauge singlets transforming nontrivially under D_5:

$$\mathcal{M}_{RR} = \begin{pmatrix} M_1 & 0 & 0 \\ 0 & 0 & M_2 \\ 0 & M_2 & 0 \end{pmatrix}. \tag{4}$$

The resulting mass matrix for the light neutrinos is then given through the type I seesaw [16] formula

$$M_\nu = (-)\mathcal{M}_\nu \mathcal{M}_{RR}^{-1} \mathcal{M}_\nu^T. \tag{5}$$

As one can see, two of the right-handed neutrinos are degenerate at tree level. This can be used for resonant leptogenesis [17].

An important aspect of our symmetry driven discussion is that different from the usual assumption in papers treating a certain texture of the mass matrices (like [13]) the Majorana mass matrix for the right-handed neutrinos strongly differs from the structure of the Dirac masses. Therefore also the effective mass matrix for the light neutrinos is in general distinct from the (Dirac) mass matrices of the other fermions. The reason for this simply lies in the fact that Majorana and Dirac masses do arise from completely different mechanisms with different symmetry aspects: first the Dirac masses connect different fields whereas Majorana masses connect the same field with itself and second Dirac masses arise through the coupling of $SU(2)_L$ doublet Higgs fields with hypercharge $Y = \pm 1$ unlike Majorana masses which are direct mass terms for right-handed neutrinos and are mediated by $SU(2)_L$ Higgs triplets for left-handed ones.

[1] The permutations have to be both P, since u_L and d_L transform in the same representation of the SM.

D_5 has two distinct two-dimensional representations instead of only one like D_3. The differences in the mass matrices which follow from this fact will be studied next. In [18] the authors assigned the fermion generations and three Higgs fields to $\underline{\mathbf{1}}_{\mathbf{1}} + \underline{\mathbf{2}}$ under D_3. We observe that we cannot use the same representation structure in the Higgs sector in our D_5 model for a realistic theory due to an accidental $U(1)$ symmetry in the potential (see Sec. VA). If we do so anyway, we can distinguish two cases in D_5: both fermion generations and Higgs fields transform as $\underline{\mathbf{1}}_{\mathbf{1}} + \underline{\mathbf{2}}_{\mathbf{i}}$ under D_5 or the fermions are in $\underline{\mathbf{1}}_{\mathbf{1}} + \underline{\mathbf{2}}_{\mathbf{i}}$ and the Higgs fields are in $\underline{\mathbf{1}}_{\mathbf{1}} + \underline{\mathbf{2}}_{\mathbf{j}}$ with i $\neq$ j. In the first case the D_5 invariance leads to a mass matrix with two zeros on its diagonal, i.e. the (2, 2) and (3, 3) element vanish, since $\underline{\mathbf{2}}_{\mathbf{i}} \times \underline{\mathbf{2}}_{\mathbf{i}}$ does not contain $\underline{\mathbf{2}}_{\mathbf{i}}$ for i $= 1, 2$ in contrast to $\underline{\mathbf{2}} \times \underline{\mathbf{2}} \ni \underline{\mathbf{2}}$ in D_3. In the latter case the first generation transforming trivially under D_5 is decoupled from the two other ones forming a two-dimensional representation, since $\underline{\mathbf{1}}_{\mathbf{1}} \times \underline{\mathbf{2}}_{\mathbf{i}} = \underline{\mathbf{2}}_{\mathbf{i}}$ for i $=1, 2$.

Thus the existence of two two-dimensional representations in the flavor group has two main consequences on the structure of the mass matrices: on the one hand it tends to reduce the number of allowed Yukawa couplings and so maintaining texture zeros becomes easier, on the other hand it leaves the freedom of assigning the three generations of fermions to different two-dimensional representations (as done here).

IV. PHENOMENOLOGICAL ANALYSIS

One appropriate example for a starting point of our numerical analysis is given by

$$\mathcal{M}_{\text{start}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & a & b \\ 0 & b & a \end{pmatrix}. \tag{6}$$

With this matrix one can already fit the masses of the second and third generation fermions by fixing a and b. The eigenvalues of $\mathcal{M}_{\text{start}}$ are $(0, a - b, a + b)$. The mass of the third generation can be taken to be $a - b$ and the one of the second one $a + b$. It is clear then that $\text{sign}(a) = -\text{sign}(b)$. The mass of the third generation determines the absolute values of a and b and the second generation the difference of $|a|$ and $|b|$. The vanishing eigenvalue of $\mathcal{M}_{\text{start}}$ also explains the smallness of the first generation compared to the two other ones. Such a matrix is closely connected to the mass matrix of the light neutrinos for $b \neq 0$ [14,19] where it leads to maximal atmospheric mixing. Although it contains this large mixing angle we can use it for the description of quarks, because taking this form for up-type as well as down-type quark mass matrices makes the two large mixing angles cancel such that the angle θ_{23} can be arbitrarily small in this sector.

The matrix in Eq. (6) arises from Eq. (3) for $\langle\chi_1\rangle = \langle\chi_2\rangle$, $\langle\psi_1\rangle = \langle\psi_2\rangle$, and $\alpha^i_{2,3} = 0$ for $i = u, d, l, \nu$. As argued in Sec. V B one can arrange the Higgs potential to have an extremum for VEVs being pairwise equal. Since the difference of $|a|$ and $|b|$ is determined by the mass of the second generation, $|a| \approx |b|$ holds. This can be maintained if all VEVs are nearly equal and $|\alpha^i_0| \approx |\alpha^i_1|$. As shown in Sec. V B also this is allowed by the minimization conditions. Note that D_5 does not restrict the Yukawa couplings α^j_i. Therefore $|\alpha^i_0| \approx |\alpha^i_1|$ is not favored by the flavor symmetry. Also our assumption $|\alpha^i_{2,3}| \ll |\alpha^i_{0,1}|$ is not guaranteed by any symmetry of the model. In order to achieve this, one could, for example, introduce a $U(1)_{\text{FN}}$ factor acting nontrivially in flavor space to implement the Froggatt Nielsen (FN) mechanism [20]. We could assign a nonvanishing charge $+q$ to the first generation and let the second and third generation be neutral under this $U(1)_{\text{FN}}$. We then gain a suppression factor of ϵ^q with $\epsilon \equiv \frac{\langle\theta\rangle}{M}$ for the matrix elements of the first row and column compared to the others. $\langle\theta\rangle$ is the VEV of the scalar SM gauge singlet θ having charge -1 under $U(1)_{\text{FN}}$ and M is the mass of some vectorlike fermions. These fields are assumed to be very heavy and therefore actually decouple from our low energy theory. Note that the second and third generation of fermions have to transform in the same way under the $U(1)_{\text{FN}}$, since otherwise the $U(1)_{\text{FN}}$ would not commute with our flavor symmetry D_5. Note further that the zero in the (1, 1) element is independent of the FN mechanism, since it comes from our assignment of fermions and Higgs fields under D_5.

Next we present our numerical examples for Dirac and Majorana neutrinos. As already stated above, the mass matrices contain in general too many parameters to make predictions. In order to reduce the number of free parameters we restrict ourselves to real Yukawa couplings and allow the VEVs $\langle\chi_{1,2}\rangle$ and $\langle\psi_1\rangle$ to have nonvanishing (complex) phases. We show the numerical values of the Yukawa couplings and VEVs in Appendix B. With these the best fit values of the measured quantities shown in Appendix C can be accommodated within the given error bars. Interestingly, all phases of the VEVs turn out to be small. Although the Yukawa couplings are chosen to be real, spontaneous CP violation (SCPV) is excluded, since the parameters in the Higgs sector have to be complex in order to allow the shown VEV configurations to be minima of the potential. This fact will be explained in detail in Sec. V B. The mass ordering of the (light) neutrinos is normal in both examples. This is not a general feature of our model, but rather chosen by us for simplicity. As we can fit all measured quantities, we only discuss the results for the unmeasured ones.

In the case of Dirac neutrinos the sum of the neutrino masses is 0.2255 eV. This is below the current bound obtained from cosmology, even if the Lyman α data are included [21]. However, it will be measurable in the next five to ten years [22]. s^2_{13} is about 0.012 and hence a factor of 3 below the current CHOOZ bound, but detectable quite soon in the next generation of reactor experiments [23]. The Dirac phase δ is $\sim$3.6 radian. The quantity m_β measured in beta decay experiments is 0.07 eV. This is below the current limit of 2.2 eV [24] and also a factor of 3 below the one of the planned KATRIN experiment [25].

Before presenting the corresponding results in the case of Majorana neutrinos, we comment on the generic problem of Dirac neutrinos. As one can see in Appendix B the Yukawa couplings of the neutrinos α_i^ν have to be suppressed by nine to 12 orders of magnitude compared to the other fermions to ensure that the neutrinos have masses of the order 1 eV. Clearly, our flavor symmetry D_5 does not explain this, but an additional $U(1)_{\rm FN}$ family symmetry can do so. If the right-handed neutrinos have a charge $\sim q_f + 10$ under $U(1)_{\rm FN}$ where q_f is the charge of any other fermion under $U(1)_{\rm FN}$, the neutrino couplings can be suppressed by an additional factor ϵ^{10}. For $\epsilon \sim 0.1$ this gives the right order of magnitude for the neutrino masses. However, then the model cannot be embedded into the Pati-Salam group.

Next we consider the neutrinos to be Majorana particles. In this case the type I seesaw [16] explains the smallness of the neutrino masses without an extra suppression of their Yukawa couplings α_i^ν. The masses for the light neutrinos are (0.1146, 0.1149, 0.1242) eV. The sum of their masses is therefore still below the current bound, but could be measured by the planned experiments. The scale of the right-handed neutrino masses is about 10^{14} GeV, but it can be rescaled by proper redefinition of the neutrino Yukawa couplings α_i^ν. Interestingly, s_{13}^2 is around the 2σ limit of the CHOOZ experiment. The CP phases which are not constrained by experiments are $(\delta, \varphi_1, \varphi_2) \sim (3.9, 0.74, 0.33)$ radian. m_β is—similar to the Dirac case—a factor of 2 smaller than the bound which can be obtained by the KATRIN experiment. $|m_{ee}|$ which is measured in neutrinoless double beta decay is about 0.1 eV. This is an order of magnitude below the upper bound [26], but can be measured in the next five to ten years [27].

The smallness of m_β and $|m_{ee}|$ is due to the normal ordering of the (light) neutrinos. Finally, we summarize all mentioned quantities in Table II.

TABLE II. Numerical values for the unmeasured quantities of the leptonic sector. The Majorana phases $\varphi_{1,2}$ are given by the convention: $U_{\rm MNS} = \tilde{V}_{\rm CKM} \cdot ({\rm e}^{i\varphi_1}, {\rm e}^{i\varphi_2}, 1)$ with $0 \le \varphi_{1,2} \le \pi$.

Quantity	Dirac neutrinos	Majorana neutrinos
m_1 [eV]	0.0701	0.1146
m_2 [eV]	0.0706	0.1149
m_3 [eV]	0.0848	0.1242
$\sum_i m_i$ [eV]	0.2255	0.3537
M_{R1} [GeV]	0	1.878×10^{14}
$M_{R2,3}$ [GeV]	0	2.011×10^{14}
s_{13}^2	0.0119	0.0303
δ [rad.]	3.5775	3.8619
φ_1 [rad.]	0	0.7396
φ_2 [rad.]	0	0.3312
m_β [eV]	0.0704	0.1150
$\lvert m_{ee}\rvert$ [eV]	0	0.1002

V. MINIMAL HIGGS POTENTIALS IN D_5

A. Three Higgs potential

In this subsection we discuss the potential arising from the three Higgs fields ϕ, ψ_1, and ψ_2 where ϕ transforms as any one-dimensional representation and

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

forms any doublet under D_5. The potential reads:

$$\begin{aligned} V_3(\phi, \psi_i) = & -\mu_1^2 \phi^\dagger \phi - \mu_2^2 \sum_{i=1}^2 \psi_i^\dagger \psi_i + \lambda_s (\phi^\dagger \phi)^2 \\ & + \lambda_1 \left(\sum_{i=1}^2 \psi_i^\dagger \psi_i \right)^2 + \lambda_2 (\psi_1^\dagger \psi_1 - \psi_2^\dagger \psi_2)^2 \\ & + \lambda_3 |\psi_1^\dagger \psi_2|^2 + \sigma_1 (\phi^\dagger \phi) \left(\sum_{i=1}^2 \psi_i^\dagger \psi_i \right) \\ & + \{ \sigma_2 (\phi^\dagger \psi_1)(\phi^\dagger \psi_2) + {\rm H.c.} \} \\ & + \sigma_3 \sum_{i=1}^2 |\phi^\dagger \psi_i|^2, \end{aligned} \tag{7}$$

where only σ_2 is complex. It can be made real by appropriate redefinition of the field ϕ, for example. We want to show that there exists an accidental $U(1)$ symmetry in this potential apart from the gauge symmetry $U(1)_Y$. In order to see this let the Higgs fields ϕ and ψ_i transform as

$$\phi \to {\rm e}^{i\alpha} \phi, \qquad \psi_1 \to {\rm e}^{i\beta} \psi_1, \qquad \psi_2 \to {\rm e}^{i\gamma} \psi_2. \tag{8}$$

The only nontrivial condition for the phases α, β, and γ arises from the term σ_2:

$$2\alpha - \beta - \gamma = 0, \tag{9}$$

i.e. α can be expressed as $\frac{1}{2}(\beta + \gamma)$ while β and γ can have any value. Consequently, there exist two $U(1)$ symmetries, called $U(1)_\beta$ and $U(1)_\gamma$, under which the three fields have the charges: $Q(\phi; \beta) = Q(\phi; \gamma) = \frac{1}{2}$, $Q(\psi_1; \beta) = 1$, $Q(\psi_1; \gamma) = 0$ and vice versa for ψ_2: $Q(\psi_2; \beta) = 0$, $Q(\psi_2; \gamma) = 1$. Taking the two linear independent combinations of the charges $Q(\chi; Y) = -[Q(\chi; \beta) + Q(\chi; \gamma)]$ and $Q(\chi; X) = Q(\chi; \beta) - Q(\chi; \gamma)$ for $\chi = \phi, \psi_1, \psi_2$ one recovers the $U(1)_Y$ and a further $U(1)_X$ under which the two fields ψ_i transform with opposite charges and ϕ remains invariant. Alternatively, the $U(1)_X$ could be defined such that $Q(\phi; X) = -Q(\psi_i; X)$ and $Q(\psi_j; X) = 0$ with $i \neq j$. Taking the first definition of the $U(1)_X$ charges one sees that any nonvanishing VEV for a field ψ_i leads to the spontaneous breaking of the $U(1)_X$ and therefore to the appearance of a massless Goldstone boson which is phenomenologically unacceptable. There are two ways to circumvent this: first introduce terms in the potential which explicitly break $U(1)_X$, but also the D_5 symmetry[2] or second leave $U(1)_X$ unbroken. The first possibility increases the number of parameters by at least four and is not explained in terms of any (further) symmetry while the second one cannot be realized, if our model should accommodate the fermion masses at tree level without further fields. Hence we abandon this three Higgs potential which actually contains the minimal set of Higgs fields needed for the construction of viable mass matrices.

The accidental $U(1)$ symmetry found here becomes obvious in the basis where the generators A and B of D_5 are taken to be the ones shown in Eq. (1). If one chooses, for example, real representation matrices (found in [11]), the resulting potential still contains the extra $U(1)$, but it is rather nontrivial to show this.

If one sets $\sigma_2 = 0$, the symmetry of the potential is further increased to $U(1)^3$, since then the condition Eq. (9) is no longer valid. The $U(1)^2$ which then exists in the (ψ_1, ψ_2) space can be enhanced to a $SU(2)$ by setting $\lambda_3 = 4\lambda_2$. Then the terms λ_2 and λ_3 can be written as $\sum_a (\Psi^\dagger \tau_a \Psi)^2$ with $\Psi = (\psi_1, \psi_2)^T$ and τ_a are the Pauli matrices, i.e. it equals the invariant arising from $(\underline{\mathbf{2}} \times \underline{\mathbf{2}})^2 \ni \underline{\mathbf{3}} \times \underline{\mathbf{3}} \ni \underline{\mathbf{1}}$ in $SU(2)$. $\sigma_2 = 0$ and/or $\lambda_3 = 4\lambda_2$ can be enforced by the VEV conditions. One example for this is given by the configuration where the VEVs of ϕ and ψ_1 are unequal to zero and $\langle\psi_2\rangle = 0$.

Let us comment on the origin of this accidental $U(1)$. For this we compare our D_5 invariant Higgs potential to one being invariant under D_3 and D_4, respectively. The D_3 invariant version of our potential has already been discussed in the literature [28]. Apart from the terms contained in the D_5 invariant potential it allows a further term, namely: $\{\tau[(\phi^\dagger\psi_1)(\psi_2^\dagger\psi_1) \pm (\phi^\dagger\psi_2)(\psi_1^\dagger\psi_2)] + \text{H.c.}\}$ with $+$ for $\phi \sim \underline{\mathbf{1}}_{\mathbf{1}}$ and $-$ for $\phi \sim \underline{\mathbf{1}}_{\mathbf{2}}$ (under D_3). This term is D_3 invariant, since the product $\underline{\mathbf{2}} \times \underline{\mathbf{2}}$ contains the representation $\underline{\mathbf{2}}$ itself and therefore $\underline{\mathbf{2}} \times \underline{\mathbf{2}} \times \underline{\mathbf{2}} \times \underline{\mathbf{1}}_{\mathbf{i}} \ni \underline{\mathbf{1}}_{\mathbf{1}}$ for i = 1, 2. In D_5 the corresponding coupling is of the form $\underline{\mathbf{2}}_{\mathbf{i}} \times \underline{\mathbf{2}}_{\mathbf{i}} \not\ni \underline{\mathbf{2}}_{\mathbf{i}}$ for both i = 1, 2. Clearly, the τ term does not allow for a further $U(1)$ symmetry, since it enforces the relations $2\beta - \alpha - \gamma = 0$ and $2\gamma - \alpha - \beta = 0$ for the phases α, β, and γ. This term has to vanish, if the potential should be invariant under the reflection symmetry $\phi \to -\phi$ and $\psi_{1,2} \to \psi_{1,2}$ as mentioned in [29]. Then there exists an accidental $U(1)$ which was already realized in [30].

To compare our potential to the one being invariant under D_4 one has to notice that the product $\underline{\mathbf{2}} \times \underline{\mathbf{2}}$ decomposes into $\sum_{\mathrm{i}=1}^4 \underline{\mathbf{1}}_{\mathbf{i}}$ there. Hence the quartic coupling λ_3 has to be replaced by

$$\lambda_3(\psi_1^\dagger\psi_2 - \psi_2^\dagger\psi_1)^2 + \tilde{\lambda}_3(\psi_1^\dagger\psi_2 + \psi_2^\dagger\psi_1)^2. \qquad (10)$$

The rest of the potential remains the same. Thereby the field ϕ can transform as any one-dimensional representation of D_4. λ_3 and $\tilde{\lambda}_3$ lead to $\beta = \gamma$ such that $\alpha = \beta = \gamma$ is enforced. The accidental $U(1)$ can be restored, if $\tilde{\lambda}_3 = -\lambda_3$ is chosen, since then Eq. (10) simplifies to $-4\lambda_3|\psi_1^\dagger\psi_2|^2$.

Since D_6 has also been mentioned as flavor symmetry in the literature and is the next smallest D_n symmetry after D_5, we briefly comment on D_6 invariant three Higgs potentials. If the three fields transform as faithful two-dimensional and as trivial representation, their potential incorporates an accidental $U(1)$ symmetry. However, using instead one of the two further one-dimensional representations of D_6 which are not present in D_5 one can get rid of this $U(1)$. Products of the faithful representation with these have the structure $\underline{\mathbf{1}} \times \underline{\mathbf{2}} = \underline{\mathbf{2}}'$ and therefore lead together with $\underline{\mathbf{2}} \times \underline{\mathbf{2}} = \underline{\mathbf{1}}_{\mathbf{1}} + \underline{\mathbf{1}}' + \underline{\mathbf{2}}'$ to a potential which coincides with the one obtained from D_3.

This demonstrates that a thorough discussion of the Higgs potential is always necessary to ensure the validity of the model as a whole. A more complete discussion about the possible potentials arising from D_n flavor symmetries and also D_n' symmetries will be given elsewhere [31].

B. Four Higgs potential

In this subsection we consider a potential containing four Higgs fields. There exist two possible choices. First we can augment our three Higgs potential with a further Higgs field χ transforming as a one-dimensional representation. If $\phi \sim \underline{\mathbf{1}}_{\mathbf{i}}$ then χ should transform as $\underline{\mathbf{1}}_{\mathbf{j}}$ with i $\neq$ j. Writing down all possible D_5 invariant couplings shows that they cannot break the $U(1)_X$ symmetry. Therefore we will consider a four Higgs potential with fields χ_i and ψ_i, $i = 1, 2$. Each pair forms a doublet under D_5, without loss of generality:

$$\begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} \sim \underline{\mathbf{2}}_{\mathbf{1}}$$

and

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \sim \underline{\mathbf{2}}_{\mathbf{2}}.$$

The potential then has the following form:

[2] This is similar to the soft breaking terms invoked in the minimal supersymmetric standard model (MSSM).

$$\begin{aligned}V_4(\chi_i,\psi_i) = &-\mu_1^2\sum_{i=1}^{2}\chi_i^\dagger\chi_i - \mu_2^2\sum_{i=1}^{2}\psi_i^\dagger\psi_i + \lambda_1\Big(\sum_{i=1}^{2}\chi_i^\dagger\chi_i\Big)^2 + \tilde{\lambda}_1\Big(\sum_{i=1}^{2}\psi_i^\dagger\psi_i\Big)^2 + \lambda_2(\chi_1^\dagger\chi_1 - \chi_2^\dagger\chi_2)^2 + \lambda_3|\chi_1^\dagger\chi_2|^2\\ &+ \tilde{\lambda}_2(\psi_1^\dagger\psi_1 - \psi_2^\dagger\psi_2)^2 + \tilde{\lambda}_3|\psi_1^\dagger\psi_2|^2 + \sigma_1\Big(\sum_{i=1}^{2}\chi_i^\dagger\chi_i\Big)\Big(\sum_{j=1}^{2}\psi_j^\dagger\psi_j\Big) + \sigma_2(\chi_1^\dagger\chi_1 - \chi_2^\dagger\chi_2)(\psi_1^\dagger\psi_1 - \psi_2^\dagger\psi_2)\\ &+ \{\tau_1(\chi_1^\dagger\psi_1)(\chi_2^\dagger\psi_2) + \mathrm{H.c.}\} + \{\tau_2(\chi_1^\dagger\psi_2)(\chi_2^\dagger\psi_1) + \mathrm{H.c.}\} + \{\kappa_1[(\chi_1^\dagger\chi_2)(\chi_1^\dagger\psi_2) + (\chi_2^\dagger\chi_1)(\chi_2^\dagger\psi_1)] + \mathrm{H.c.}\}\\ &+ \{\kappa_2[(\psi_1^\dagger\psi_2)(\chi_2^\dagger\psi_2) + (\psi_2^\dagger\psi_1)(\chi_1^\dagger\psi_1)] + \mathrm{H.c.}\} + \kappa_3[|\chi_1^\dagger\psi_1|^2 + |\chi_2^\dagger\psi_2|^2] + \kappa_4[|\chi_1^\dagger\psi_2|^2 + |\chi_2^\dagger\psi_1|^2],\end{aligned} \tag{11}$$

where the couplings $\tau_{1,2}$ and $\kappa_{1,2}$ are in general complex. We checked that this potential does not have any accidental (global) symmetries. Assuming that the fields $\chi_{1,2}$, $\psi_{1,2}$ transform in the following way:

$$\chi_1 \to \chi_1 e^{i\alpha}, \qquad \chi_2 \to \chi_2 e^{i\beta},$$
$$\psi_1 \to \psi_1 e^{i\gamma}, \qquad \psi_2 \to \psi_2 e^{i\delta},$$

one finds that the couplings $\mu_{1,2}$, $\lambda_{1,2,3}$, $\tilde{\lambda}_{1,2,3}$, $\sigma_{1,2}$, $\kappa_{3,4}$ leave the full $U(1)^4$ invariant, $\tau_{1,2}$ breaks it down to $U(1)^3$, and $\kappa_{1,2}$ down to $U(1)^2$, i.e. none of the couplings themselves is only invariant under $U(1)_Y$. $\tau_{1,2}$ leave the same $U(1)^3$ invariant with the condition $\alpha = \gamma + \delta - \beta$. The $U(1)^2$ symmetries which are preserved by $\kappa_{1,2}$ are constrained by the conditions $2\alpha = \beta + \delta$, $2\beta = \alpha + \gamma$ and $2\delta = \beta + \gamma$, $2\gamma = \alpha + \delta$, respectively. As one can see only $\kappa_1 \neq 0$ and $\kappa_2 \neq 0$ can reduce $U(1)^4$ to $U(1)_Y$, i.e. taking the $\tau_{1,2}$ terms with only the κ_1 term still leaves the potential invariant under $U(1)^2$. Consequently, none of the VEV conditions should enforce κ_1 or κ_2 to vanish. A simple example for this is the configuration $\langle\chi_1\rangle \neq 0$, $\langle\psi_2\rangle \neq 0$, and $\langle\chi_2\rangle = \langle\psi_1\rangle = 0$ with all VEVs being real. It leads to $\kappa_1 = 0$. However, it cannot produce phenomenological viable mass matrices anyway as discussed above.

In the following we show that the VEV configuration which is used in the zeroth order approximation in our numerical study represents one possible minimum of the Higgs potential V_4. As one can see, the equivalence of all four VEVs is not obligatory, since for example μ_1 and μ_2 and λ_1 and $\tilde{\lambda}_1$ are not restricted to have the same value, respectively. Therefore we search for a symmetry which can maintain these restrictions such that the equivalence of all four VEVs becomes more natural. The simplest choice is to first interchange the fields χ_i with ψ_i in order to enforce, for example, the equivalence of μ_1 and μ_2 and to further exchange the fields χ_1 and χ_2 preventing the couplings $\kappa_{1,2}$ from being set to zero.[3] This symmetry will be called T in the following. It restricts the parameters as follows:

$$\begin{aligned}&\mu_1 = \mu_2, \qquad \lambda_i = \tilde{\lambda}_i, \qquad \sigma_2 = 0,\\ &\tau_1 = \tau_2^\star, \qquad \kappa_1 = \kappa_2^\star, \qquad \kappa_3 = \kappa_4.\end{aligned} \tag{12}$$

Note that setting σ_2 to zero does not lead to an accidental continuous symmetry. Especially, we do not enforce $\kappa_{1,2}$ to vanish. Note also that changing the order of the actions $\chi_i \leftrightarrow \psi_i$ and $\chi_1 \leftrightarrow \chi_2$ does not change the result.

Next we analyze the potential invariant under $D_5 \times T$ for real VEVs $\langle\chi_1\rangle = \frac{v}{\sqrt{2}}\cos(\alpha)$, $\langle\chi_2\rangle = \frac{v}{\sqrt{2}}\sin(\alpha)$, $\langle\psi_1\rangle = \frac{u}{\sqrt{2}}\cos(\beta)$, and $\langle\psi_2\rangle = \frac{u}{\sqrt{2}}\sin(\beta)$. The form of the potential at the extremum is:

$$\begin{aligned}V_{4T\,\mathrm{min}} = &-\tfrac{1}{2}\mu_1^2(u^2+v^2) + \tfrac{1}{32}(u^4+v^4)(8\lambda_1+4\lambda_2+\lambda_3)\\ &+ \tfrac{1}{4}u^2v^2(\sigma_1+\kappa_3) + \tfrac{1}{32}(v^4\cos(4\alpha)\\ &+ u^4\cos(4\beta))(4\lambda_2-\lambda_3) + \tfrac{1}{4}uv[u^2\cos(\alpha-\beta)\\ &\times\sin(2\beta) + v^2\sin(2\alpha)\sin(\alpha+\beta)]\mathrm{Re}(\kappa_1)\\ &+ \tfrac{1}{4}u^2v^2\sin(2\alpha)\sin(2\beta)\mathrm{Re}(\tau_1).\end{aligned} \tag{13}$$

The minimization conditions which can be deduced from $V_{4T\,\mathrm{min}}$ are:

$$\begin{aligned}\frac{\partial V_{4T\,\mathrm{min}}}{\partial\alpha} = &-\frac{1}{8}v^4\sin(4\alpha)y + \frac{1}{2}u^2v^2\cos(2\alpha)\sin(2\beta)\mathrm{Re}(\tau_1)\\ &+\frac{1}{4}uv[v^2(\cos(2\alpha)\sin(\alpha+\beta)+\sin(3\alpha+\beta))\\ &- u^2\sin(\alpha-\beta)\sin(2\beta)]\mathrm{Re}(\kappa_1),\end{aligned} \tag{14a}$$

$$\begin{aligned}\frac{\partial V_{4T\,\mathrm{min}}}{\partial\beta} = &-\frac{1}{8}u^4\sin(4\beta)y + \frac{1}{2}u^2v^2\sin(2\alpha)\cos(2\beta)\mathrm{Re}(\tau_1)\\ &+\frac{1}{4}uv[u^2(\cos(\alpha-\beta)\cos(2\beta)+\cos(\alpha-3\beta))\\ &+ v^2\sin(2\alpha)\cos(\alpha+\beta)]\mathrm{Re}(\kappa_1),\end{aligned} \tag{14b}$$

where $y = 4\lambda_2 - \lambda_3$. Equations (14a) and (14b) are fulfilled for $\alpha = \frac{\pi}{4}$ and $\beta = \frac{\pi}{4}$. Then each of the terms vanishes separately, there is especially no constraint on $\mathrm{Re}(\tau_1)$, $\mathrm{Re}(\kappa_1)$, or $4\lambda_2 - \lambda_3$. This is important, since constraining these parameters to be zero could lead to accidental symmetries. For $\alpha = \frac{\pi}{4}$ and $\beta = \frac{\pi}{4}$ there is also one solution with $u = v$. Therefore the equivalence of all (real) VEVs is a natural result of the potential.

[3] The exchange of the fields ψ_1 and ψ_2 gives the same result.

Apart from this zeroth order solution it is important to check whether phenomenological viable VEV configurations can be a minimum of the potential for an appropriate choice of parameters. As we tried to restrict ourselves above to SCPV, it is especially necessary to find out whether this is possible for the chosen VEVs in our numerical examples. Unfortunately, it turns out to be impossible for the potential invariant under $D_5 \times T$. For VEVs parametrized as $\langle\chi_1\rangle = \frac{v_1}{\sqrt{2}}e^{i\alpha}$, $\langle\chi_2\rangle = \frac{v_2}{\sqrt{2}}e^{i\beta}$, $\langle\psi_1\rangle = \frac{v_3}{\sqrt{2}}e^{i\gamma}$, and $\langle\psi_2\rangle = \frac{v_4}{\sqrt{2}}$ one can deduce, for example, the following equations from the minimization conditions for $v_i \neq 0$, $\alpha \neq 0$, $\beta \neq 0$, $\gamma \neq 0$:

$$5v_3v_4\,\mathrm{Re}(\kappa_1)(v_1v_3\sin(\alpha-2\gamma) - v_2v_4\sin(\beta+\gamma)) = 0, \tag{15a}$$

$$5v_1v_2\,\mathrm{Re}(\kappa_1)(v_1v_4\sin(2\alpha-\beta) + v_2v_3\sin(\alpha-2\beta+\gamma)) = 0. \tag{15b}$$

These directly lead to the conclusion that $\mathrm{Re}(\kappa_1) = 0$. As we consider SCPV, also $\mathrm{Im}(\kappa_1) = 0$ and therefore the coupling κ_1 vanishes.[4] This increases the symmetry of the potential, as explained above. With $v_i \neq 0$, $\alpha \neq 0$, $\beta \neq 0$, $\gamma \neq 0$ it is then clear that this additional symmetry will be broken and hence further massless Goldstone bosons will appear which are phenomenologically unacceptable. In this case we have not gained anything by discussing the four Higgs potential compared to the three Higgs one. Abandoning the T symmetry and only requiring that the potential is invariant under D_5 does not change the situation, since then one can deduce the equations:

$$5v_3v_4\,\mathrm{Re}(\kappa_2)(v_1v_3\sin(\alpha-2\gamma) - v_2v_4\sin(\beta+\gamma)) = 0, \tag{16a}$$

$$5v_1v_2\,\mathrm{Re}(\kappa_1)(v_1v_4\sin(2\alpha-\beta) + v_2v_3\sin(\alpha-2\beta+\gamma)) = 0. \tag{16b}$$

These enforce the vanishing of $\mathrm{Re}(\kappa_1)$ and $\mathrm{Re}(\kappa_2)$ for general VEV configurations. Again $\mathrm{Im}(\kappa_{1,2})$ are already set to zero, since we want to study the case of SCPV. In the end, the constraints $\kappa_1 = 0$ and $\kappa_2 = 0$ lead to an increase of the symmetry of the potential. Similar to the case above this further symmetry is broken by arbitrary VEV configurations resulting in extra Goldstone bosons. This proves that SCPV can only exist for special VEV configurations, but not in general.

For a general D_5 invariant four Higgs potential with complex parameters one can successfully solve all minimization conditions without the necessity to set parameters to zero. Furthermore one is able to maintain that all masses of the Higgs fields at this extremum are positive, i.e. this extremum can be a minimum of the potential. As all relevant equations are invariant under $v_i \to -v_i$, the VEV configurations $\langle\chi_1\rangle = \frac{v_1}{\sqrt{2}}e^{i\alpha}$, $\langle\chi_2\rangle = \frac{v_2}{\sqrt{2}}e^{i\beta}$, $\langle\psi_1\rangle = \frac{v_3}{\sqrt{2}}e^{i\gamma}$, $\langle\psi_2\rangle = \frac{v_4}{\sqrt{2}}$ and $\langle\chi_1\rangle = \frac{-v_1}{\sqrt{2}}e^{i\alpha}$, $\langle\chi_2\rangle = \frac{-v_2}{\sqrt{2}}e^{i\beta}$, $\langle\psi_1\rangle = \frac{-v_3}{\sqrt{2}}e^{i\gamma}$, $\langle\psi_2\rangle = \frac{-v_4}{\sqrt{2}}$ are degenerate. Finally, one can check numerically whether the potential is stable as a whole. This clearly is not a proof of the stability of the potential, but is enough for our considerations.

All this has been done for the two VEV configurations used in the numerical examples. The parameters of the potential can be chosen in such a way that all constraints are fulfilled. The mass of the lightest Higgs field is usually smaller (~ 40 GeV) than the experimental bounds (≤ 114.4 GeV) [32], if the mass parameters μ_i are of the order of the electroweak scale (100–200 GeV) and the quartic couplings are in the perturbative range. This problem can be cured by simply assuming that the μ_is are larger than $\mathcal{O}(100\text{ GeV})$ or adding some other mass dimension two terms which break D_5. In order to pass not only the direct Higgs mass bounds, but also the stringent bounds on FCNCs, the Higgs masses should be even larger than a few TeV. The mechanism of adding D_5 breaking terms is unmotivated from the theoretical point of view, but seems to be necessary for a phenomenological viable model in this context.

One could ask whether it is also possible to achieve that arbitrary VEV configurations can be minima of the potential, if this is invariant under $D_5 \times T$. The answer is no, since one can deduce three linear independent equations containing $\mathrm{Re}(\kappa_1)$, $\mathrm{Im}(\kappa_1)$, and $\mathrm{Re}(\tau_1)$ which are in general only solved, if $\mathrm{Re}(\kappa_1) = 0$, $\mathrm{Im}(\kappa_1) = 0$, and $\mathrm{Re}(\tau_1) = 0$. Again, the minimization conditions enforce a parameter setup which leads to an additional global symmetry in the Higgs potential.

Finally, we compare the D_5 invariant potential of four Higgs fields to the equivalent one in D_6. Similar to D_5 also D_6 has two inequivalent two-dimensional representations (one faithful and one unfaithful one). However, in contrast to D_5 the D_6 invariant four Higgs potential contains a further $U(1)$ symmetry. The reason for this is the D_6 product structure $\underline{\mathbf{2}}_\mathbf{i} \times \underline{\mathbf{2}}_\mathbf{i} = \underline{\mathbf{1}}_\mathbf{1} + \underline{\mathbf{1}}_\mathbf{4} + \underline{\mathbf{2}}_\mathbf{2}$ for i = 1, 2 and $\underline{\mathbf{2}}_\mathbf{1} \times \underline{\mathbf{2}}_\mathbf{2} = \underline{\mathbf{1}}_\mathbf{2} + \underline{\mathbf{1}}_\mathbf{3} + \underline{\mathbf{2}}_\mathbf{1}$ which does not allow for invariant couplings of the form $\underline{\mathbf{2}}_\mathbf{i}^3\underline{\mathbf{2}}_\mathbf{j}$ with i ≠ j. Precisely, these couplings, κ_1 and κ_2, exist in D_5 and therefore prevent the potential from having an accidental $U(1)$.

VI. EXTENSIONS OF THE MODEL

Finally, we would like to comment on how the model has to be changed in order to be embedded into an $SO(10)$ GUT and—maybe simultaneously—into a continuous flavor symmetry, like $SO(3)_f$ or $SU(3)_f$. This is desirable, since GUTs turned out to be very successful in unifying the SM gauge interactions and fermions of one generation and in explaining, for example, charge quantization. These features should not be given up when flavored models are considered. Second, the embedding of a discrete flavor

[4]Actually in general even more parameters of the potential are constrained to be zero or have to fulfill certain relations.

symmetry into a continuous group G_f allows one to unify it with the GUT group being also continuous into one group containing gauge and flavor symmetries. Attempts to find such a group can be found in the literature [33]. Albeit these have not been very successful, the idea is still appealing. Furthermore gauged symmetries are the only ones which remain unbroken in the presence of quantum gravitational corrections [34] which suggests that any flavor symmetry should also be gauged. However, gauging a discrete symmetry can be performed in the easiest way, if it is embedded into a continuous one which is then gauged. Nevertheless, in the context of string theory discrete flavor symmetries could also arise without such an embedding.

Since all fermions of one generation reside in the $\underline{\mathbf{16}}$ of $SO(10)$ they need to transform in the same way under D_5, for example, as $\underline{\mathbf{1}}_\mathbf{1} + \underline{\mathbf{2}}_\mathbf{1}$. In our minimal model with just the four Higgs fields χ_i and ψ_i the resulting mass matrices do hardly lead to phenomenological viable masses at tree level and at low energies. Therefore we have to extend the Higgs sector by at least one Higgs field ϕ transforming trivially under D_5. The mass matrices are then of the form

$$\mathcal{M}_{u,\nu} = \begin{pmatrix} \alpha_0^{u,\nu}\langle\phi\rangle^\star & \alpha_1^{u,\nu}\langle\chi_1\rangle^\star & \alpha_1^{u,\nu}\langle\chi_2\rangle^\star \\ \alpha_2^{u,\nu}\langle\chi_1\rangle^\star & \alpha_4^{u,\nu}\langle\psi_1\rangle^\star & \alpha_3^{u,\nu}\langle\phi\rangle^\star \\ \alpha_2^{u,\nu}\langle\chi_2\rangle^\star & \alpha_3^{u,\nu}\langle\phi\rangle^\star & \alpha_4^{u,\nu}\langle\psi_2\rangle^\star \end{pmatrix},$$
$$\mathcal{M}_{d,l} = \begin{pmatrix} \alpha_0^{d,l}\langle\phi\rangle & \alpha_1^{d,l}\langle\chi_2\rangle & \alpha_1^{d,l}\langle\chi_1\rangle \\ \alpha_2^{d,l}\langle\chi_2\rangle & \alpha_4^{d,l}\langle\psi_2\rangle & \alpha_3^{d,l}\langle\phi\rangle \\ \alpha_2^{d,l}\langle\chi_1\rangle & \alpha_3^{d,l}\langle\phi\rangle & \alpha_4^{d,l}\langle\psi_1\rangle \end{pmatrix}, \tag{17}$$

i.e. the Higgs field ϕ fills the zeros in the (1, 1), (2, 3), and (3, 2) elements. Note that the form of the right-handed Majorana mass terms does not change. In a complete $SO(10)$ model the Higgs doublet fields have to be embedded into the representations $\underline{\mathbf{10}}$, $\underline{\mathbf{120}}$, and $\overline{\underline{\mathbf{126}}}$, since these do couple to $\underline{\mathbf{16}} \times \underline{\mathbf{16}}$. Still this setup has to be embedded into the continuous flavor group G_f. For G_f being $SO(3)_f$ this is not possible, since we cannot identify $\underline{\mathbf{1}}_\mathbf{1} + \underline{\mathbf{2}}_\mathbf{i}$ with the fundamental representation of $SO(3)_f$. The same holds for $SU(3)_f$. In order to do so, the first generation has to transform as $\underline{\mathbf{1}}_\mathbf{2}$ rather than $\underline{\mathbf{1}}_\mathbf{1}$. This leads to a sign in the (1, 3) and (3, 1) elements of the mass matrices in Eq. (15), but does not alter the discussion. The five Higgs fields χ_i, ψ_i and $\phi \sim \underline{\mathbf{1}}_\mathbf{1} + \underline{\mathbf{2}}_\mathbf{1} + \underline{\mathbf{2}}_\mathbf{2}$ can be identified with the $\underline{\mathbf{5}}$ of $SO(3)_f$ and together with an additional field $\phi' \sim \underline{\mathbf{1}}_\mathbf{1}$ also with the six-dimensional representation of $SU(3)_f$.

A more minimal choice for an embedding into $SO(10) \times G_f$ would be given by the three generations transforming as $\underline{\mathbf{1}}_\mathbf{2} + \underline{\mathbf{2}}_\mathbf{1}$ and three Higgs fields doing the same. Unfortunately, this leads to traceless mass matrices for the fermions which seem to be highly disfavored by the observed mass hierarchies among the generations. This problem can be cured by adding another Higgs field transforming trivially under D_5. Furthermore this increases the number of allowed Yukawa couplings by two. Since the added Higgs field transforms as $\underline{\mathbf{1}}_\mathbf{1}$, the model can still be embedded into the continuous flavor symmetries $SO(3)_f$ and $SU(3)_f$ with this field being identified with the singlet of $SO(3)_f$ or $SU(3)_f$. Although we showed that the Higgs sector is not phenomenological viable in this case (see Sec. V), we cannot exclude it as a GUT model, because the Higgs couplings might change through the embedding of the $SU(2)_L$ Higgs doublet fields into $SO(10)$ representations.

VII. CONCLUSIONS AND OUTLOOK

In this paper, we constructed a minimal model with the SM gauge group enlarged by the flavor symmetry D_5. Both are broken only spontaneously at the electroweak scale. We chose D_5, since it is the smallest discrete group with two inequivalent irreducible two-dimensional representations. We demanded the left-handed and left-handed conjugate fields of the three generations to unify partially, i.e. transform as $\underline{\mathbf{1}} + \underline{\mathbf{2}}$ under D_5, combined with the requirement that our model should be embeddable at least into the Pati-Salam gauge group. Furthermore we have chosen the minimal possible number of Higgs doublets with a potential free of accidental symmetries and did not include scalar fields transforming as $SU(2)_L$ triplets or gauge singlets. We showed that under these constraints a minimal model can be built in which the left-handed fields transform as $\underline{\mathbf{1}}_\mathbf{1} + \underline{\mathbf{2}}_\mathbf{2}$ under D_5, the left-handed conjugate ones as $\underline{\mathbf{1}}_\mathbf{1} + \underline{\mathbf{2}}_\mathbf{1}$ and the four Higgses χ_i and $\psi_i (i = 1, 2)$ as $\underline{\mathbf{2}}_\mathbf{1} + \underline{\mathbf{2}}_\mathbf{2}$. By a numerical study we showed that all fermion masses and mixing parameters can be accommodated at tree level. We considered the case of Majorana as well as Dirac neutrinos and we discussed the results of the unmeasured leptonic quantities. By our choice the spectrum of the light neutrinos is always normally ordered. The structure of the right-handed neutrino mass matrix is (almost) trivial, since we did not include SM gauge singlets. As a consequence two of the right-handed neutrinos are degenerate at tree level. We compared the structure of the D_5 invariant mass matrices with those of D_3 invariant ones which are often discussed in the literature. The main difference is the tendency to get more texture zeros for a similar assignment of fermions and Higgs fields arising from the existence of the two inequivalent two-dimensional representations in D_5. We then turned to a discussion of the Higgs sector and found that all potentials with three Higgs fields transforming as $\underline{\mathbf{1}} + \underline{\mathbf{2}}$ are not only D_5 invariant, but also incorporate an accidental $U(1)$ symmetry which is broken by any VEV configuration leading to phenomenological viable mass matrices for the fermions at tree level. To find the group theoretical reason for this accidental $U(1)$ we considered similar potentials invariant under D_3 and D_4, respectively, and found that they do not have an accidental $U(1)$ symmetry. The difference lies in the D_5 product structure $\underline{\mathbf{1}}_\mathbf{1,2} \times \underline{\mathbf{2}} = \underline{\mathbf{2}}$ and $\underline{\mathbf{2}} \times \underline{\mathbf{2}} = \underline{\mathbf{1}}_\mathbf{1} + \underline{\mathbf{1}}_\mathbf{2} + \underline{\mathbf{2}}'$ such that the coupling $\underline{\mathbf{2}} \times \underline{\mathbf{2}} \times \underline{\mathbf{2}} \times \underline{\mathbf{1}}_\mathbf{1,2}$ is not invariant under D_5. Therefore we had to extend the Higgs sector to four fields χ_i and ψ_i transforming as the doublets of D_5. We explicitly showed that this potential is free of accidental symmetries and analyzed its VEV configurations. For a zeroth order solution we imposed a further discrete symmetry—called T—on the potential in order to maintain the configuration that all (real) VEVs are equal as natural outcome of the minimization conditions. In a second step we proved that SCPV is not possible for the VEV configurations used in our numerical examples of the fermion mass matrices. Nevertheless these configurations can be minima of the D_5 invariant four Higgs potential, if its parameters are complex. Furthermore we calculated the masses for the Higgs fields. We found that they are naturally of the order $\mathcal{O}(100\text{ GeV})$ up to $\mathcal{O}(1\text{ TeV})$ with the smallest mass below the LEP bound of 114.4 GeV [32], if the mass parameters of the potential are of the order of the electroweak scale and the quartic couplings are in the perturbative regime. Therefore FCNCs might be a problem which can probably be cured by adding large mass dimension two terms which break D_5. In our numerical examples the FCNCs involving the first generation get additionally suppressed, since the relevant Yukawa couplings are at most 10^{-4}. Finally, we considered extensions of our low energy model and showed the necessary changes in the particle assignment and content to achieve the embedding into $SO(10) \times G_f$ where G_f can be either $SO(3)_f$ or $SU(3)_f$.

In our numerical examples the VEVs of the fields χ_i and ψ_i break D_5 completely. However, D_5 has two nontrivial Abelian subgroups Z_2 and Z_5 which can be generated by the generator B and the generator A alone, respectively. As one can see, Z_5 is always broken by a nonvanishing VEV of χ_i and ψ_i. In contrast to this a residual Z_2 is preserved in the Lagrangian, if $\langle\chi_1\rangle = \langle\chi_2\rangle$ and $\langle\psi_1\rangle = \langle\psi_2\rangle$. Interestingly, the resulting mass matrices $\mathcal{M}_{u,\nu}$ and $\mathcal{M}_{d,l}$ are then invariant under the interchange of the second and third generation and therefore produce a maximal mixing in the 2-3 sector, a vanishing mixing in the 1-3 sector, and leave the mixing angle θ_{12} undetermined. For an exact Z_2 thus the mixing matrices $V_{\rm CKM}$ and $U_{\rm MNS}$ have two vanishing mixing angles θ_{13} and θ_{23}, since the maximal mixing angles in the 2-3 sectors of the up-type quarks (neutrinos) and the down-type quarks (charged leptons) cancel each other. This leads to the conclusion that this residual Z_2 is only weakly broken in the quark sector, but strongly broken in the lepton/neutrino sector. Actually the equalities $\langle\chi_1\rangle = \langle\chi_2\rangle$ and $\langle\psi_1\rangle = \langle\psi_2\rangle$ have also been employed when we searched for an appropriate zeroth order structure of the fermion mass matrices in our phenomenological analysis (see Sec. IV). By considering the nontrivial subgroups of D_5 this choice gains further significance.

This discussion can be compared with the studies of the nontrivial subgroups of A_4 [35]. A_4 can be broken to either Z_2 or Z_3 by different VEV configurations of Higgs fields forming a triplet under A_4. It turns out that preserving the subgroup Z_3 for charged fermions leads to $V_{\rm CKM} = 1$ whereas Z_2 is preserved in the neutrino sector leading to tribimaximal mixing. Concerning the quark sector our flavor symmetry D_5 has the advantage that its breaking to Z_2 leads to vanishing 1-3 and 2-3 mixing, but does not constrain the Cabibbo angle. In this way we can explain why the Cabibbo angle is about 1 order of magnitude larger than the two other mixing angles whereas models using A_4 might have problems to generate a 1-2 mixing angle being large enough to accommodate the data. On the other hand in our minimal model shown here the residual subgroups of D_5 can hardly give reason for the tribimaximal or bimaximal mixing pattern observed in the leptonic sector which can nicely be explained by the residual Z_2 symmetry of A_4 in the neutrino sector.

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APPENDIX A: DETAILS OF GROUP THEORY

Here, we show the Clebsch Gordan coefficients for all Kronecker products in case that none of the representations is complex conjugated. This choice corresponds to the Yukawa couplings for the down-type quarks and charged leptons (see Sec. III). All other Clebsch Gordan coefficients needed, for example, for the quartic couplings in the Higgs sector and the Yukawas for the up-type quarks which involve at least one complex conjugated representation can be generated from the given Clebsch Gordan coefficients taking into account the similarity transformation between the representation matrices and its complex conjugates as shown in Sec. II A.

For $A \sim \underline{\mathbf{1}}_{\mathbf{i}}$ and $B \sim \underline{\mathbf{1}}_{\mathbf{j}}$ the product is $AB \sim \underline{\mathbf{1}}_{(\mathrm{i+j})\mathrm{mod}2+1}$. Combining the two-dimensional representation

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \sim \underline{\mathbf{2}}_{\mathbf{i}}$$

with the trivial singlet $A \sim \underline{\mathbf{1}}_{\mathbf{1}}$ leads to

$$\begin{pmatrix} Aa_1 \\ Aa_2 \end{pmatrix} \sim \underline{\mathbf{2}}_{\mathbf{i}}.$$

Similarly for the nontrivial singlet $B \sim \underline{\mathbf{1}}_{\mathbf{2}}$ one finds

$$\begin{pmatrix} Ba_1 \\ -Ba_2 \end{pmatrix} \sim \underline{\mathbf{2}}_{\mathbf{i}}.$$

The D_5 covariant combinations of $\underline{\mathbf{2}}_{\mathbf{1}} \times \underline{\mathbf{2}}_{\mathbf{1}}$ for

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \begin{pmatrix} a'_1 \\ a'_2 \end{pmatrix} \sim \underline{\mathbf{2}}_{\mathbf{1}}$$

are $a_1 a'_2 + a_2 a'_1 \sim \underline{\mathbf{1}}_{\mathbf{1}}$, $a_1 a'_2 - a_2 a'_1 \sim \underline{\mathbf{1}}_{\mathbf{2}}$, and

$$\begin{pmatrix} a_1 a_1' \\ a_2 a_2' \end{pmatrix} \sim \underline{\mathbf{2}}_{\mathbf{2}}$$

and for the product $\underline{\mathbf{2}}_{\mathbf{2}} \times \underline{\mathbf{2}}_{\mathbf{2}}$ they read $b_1 b_2' + b_2 b_1' \sim \underline{\mathbf{1}}_{\mathbf{1}}$, $b_1 b_2' - b_2 b_1' \sim \underline{\mathbf{1}}_{\mathbf{2}}$, and

$$\begin{pmatrix} b_2 b_2' \\ b_1 b_1' \end{pmatrix} \sim \underline{\mathbf{2}}_{\mathbf{1}}$$

with

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \begin{pmatrix} b_1' \\ b_2' \end{pmatrix} \sim \underline{\mathbf{2}}_{\mathbf{2}}.$$

For the mixed product $\underline{\mathbf{2}}_{\mathbf{1}} \times \underline{\mathbf{2}}_{\mathbf{2}}$ we find

$$\begin{pmatrix} a_2 b_1 \\ a_1 b_2 \end{pmatrix} \sim \underline{\mathbf{2}}_{\mathbf{1}}$$

and

$$\begin{pmatrix} a_2 b_2 \\ a_1 b_1 \end{pmatrix} \sim \underline{\mathbf{2}}_{\mathbf{2}}$$

with a_i being the upper and lower components of $\underline{\mathbf{2}}_{\mathbf{1}}$ and b_i of $\underline{\mathbf{2}}_{\mathbf{2}}$, respectively.

The Clebsch Gordan coefficients for the product $\nu \times \mu$ can be constructed from the ones given for $\mu \times \nu$ by simply taking the transpose of these. Therefore the shown Clebsch Gordan coefficients are sufficient for the calculation of all Yukawa and Higgs couplings.

Finally, we display the resolution of the smallest representations of $SO(3)$ ($SU(3)$) into irreducible ones of D_5.

$SO(3) \to \mathrm{D}_5$	$SU(3) \to \mathrm{D}_5$
$\underline{\mathbf{1}} \to \underline{\mathbf{1}}_{\mathbf{1}}$	$\underline{\mathbf{1}} \to \underline{\mathbf{1}}_{\mathbf{1}}$
$\underline{\mathbf{3}} \to \underline{\mathbf{1}}_{\mathbf{2}} + \underline{\mathbf{2}}_{\mathbf{1}}$	$\underline{\mathbf{3}} \to \underline{\mathbf{1}}_{\mathbf{2}} + \underline{\mathbf{2}}_{\mathbf{1}}$
$\underline{\mathbf{5}} \to \underline{\mathbf{1}}_{\mathbf{1}} + \underline{\mathbf{2}}_{\mathbf{1}} + \underline{\mathbf{2}}_{\mathbf{2}}$	$\underline{\mathbf{6}} \to 2\underline{\mathbf{1}}_{\mathbf{1}} + \underline{\mathbf{2}}_{\mathbf{1}} + \underline{\mathbf{2}}_{\mathbf{2}}$
$\underline{\mathbf{7}} \to \underline{\mathbf{1}}_{\mathbf{2}} + \underline{\mathbf{2}}_{\mathbf{1}} + 2\underline{\mathbf{2}}_{\mathbf{2}}$	$\underline{\mathbf{8}} \to \underline{\mathbf{1}}_{\mathbf{1}} + \underline{\mathbf{1}}_{\mathbf{2}} + 2\underline{\mathbf{2}}_{\mathbf{1}} + \underline{\mathbf{2}}_{\mathbf{2}}$
$\underline{\mathbf{9}} \to \underline{\mathbf{1}}_{\mathbf{1}} + 2\underline{\mathbf{2}}_{\mathbf{1}} + 2\underline{\mathbf{2}}_{\mathbf{2}}$	$\underline{\mathbf{10}} \to 2\underline{\mathbf{1}}_{\mathbf{2}} + 2\underline{\mathbf{2}}_{\mathbf{1}} + 2\underline{\mathbf{2}}_{\mathbf{2}}$

One can interchange $\underline{\mathbf{2}}_{\mathbf{1}}$ with $\underline{\mathbf{2}}_{\mathbf{2}}$ to get an alternative possible embedding. These breaking sequences can be calculated with the methods shown in [36].

APPENDIX B: TABLES OF NUMERICAL EXAMPLES

Tables III and IV contain solutions for Dirac and Majorana neutrinos.

APPENDIX C: EXPERIMENTAL DATA

The masses for the quarks and charged leptons at $\mu = M_Z$ are [37,38]:

$$\begin{aligned}
m_u(M_Z) &= (1.7 \pm 0.4)\ \mathrm{MeV}, \\
m_c(M_Z) &= (0.62 \pm 0.03)\ \mathrm{GeV}, \\
m_t(M_Z) &= (171 \pm 3)\ \mathrm{GeV}, \\
m_d(M_Z) &= (3.0 \pm 0.6)\ \mathrm{MeV}, \\
m_s(M_Z) &= (54 \pm 11)\ \mathrm{MeV}, \\
m_b(M_Z) &= (2.87 \pm 0.03)\ \mathrm{GeV}, \\
m_e(M_Z) &= (0.486\,847\,27 \pm 0.000\,000\,14)\ \mathrm{MeV}, \\
m_\mu(M_Z) &= (102.751\,38 \pm 0.000\,33)\ \mathrm{MeV}, \\
m_\tau(M_Z) &= 1.746\,69^{+0.000\,30}_{-0.000\,27}\ \mathrm{GeV}.
\end{aligned}$$

The Cabibbo-Kobayashi-Maskawa (CKM) mixing angles hardly depend on the scale μ at low energies. Therefore we take the values found in [39] which are measured in tree-level processes only:

$$\begin{aligned}
\sin(\theta_{12}) &\equiv s_{12} = 0.2243 \pm 0.0016, \\
\sin(\theta_{23}) &\equiv s_{23} = 0.0413 \pm 0.0015, \\
\sin(\theta_{13}) &\equiv s_{13} = 0.0037 \pm 0.0005, \\
\delta &= 1.05 \pm 0.24, \quad \text{and} \quad \mathcal{J}_{CP} = (2.88 \pm 0.33) \times 10^{-5}.
\end{aligned}$$

In the neutrino sector only the two mass squared differences measured in atmospheric and solar neutrino experiments are known [40]:

$$\begin{aligned}
\Delta m_{21}^2 &= m_2^2 - m_1^2 = (7.9^{+0.6}_{-0.6}) \times 10^{-5}\ \mathrm{eV}^2, \\
|\Delta m_{31}^2| &= |m_3^2 - m_1^2| = (2.2^{+0.7}_{-0.5}) \times 10^{-3}\ \mathrm{eV}^2.
\end{aligned}$$

The leptonic mixing angles are constrained: $s_{13}^2 \leq 0.031$,

TABLE III. Numerical solution for Dirac neutrinos. The Yukawa couplings of the neutrinos have to be multiplied by $10^{-12.35}$ and $\sum_{i=1}^{2}(|\langle\chi_i\rangle|^2 + |\langle\psi_i\rangle|^2) = (172.02\ \mathrm{GeV})^2$.

Yukawas	α_0^i	α_1^i	α_2^i	α_3^i
$i = u$	$-0.993\,413$	$0.994\,834$	$0.000\,478\,57$	$0.000\,151\,179$
$i = d$	$-0.016\,992\,5$	$0.016\,399\,6$	$0.000\,087\,483\,3$	$0.000\,127\,194$
$i = l$	$-0.010\,791\,2$	$0.009\,540\,78$	$-0.000\,060\,939$	$0.000\,056\,597$
$i = \nu$	$-1.006\,16$	$0.979\,207$	1.3039	$1.453\,44$
VEVs	$\langle\chi_1\rangle$	$\langle\chi_2\rangle$	$\langle\psi_1\rangle$	$\langle\psi_2\rangle$
abs. [GeV]	97.3856	71.386	101.872	68.0594
phase [rad.]	$-0.007\,651\,5$	$0.007\,171\,1$	$0.014\,899$	0

TABLE IV. Numerical solution for Majorana neutrinos. The sum of the squares of the absolute values of the VEVs is $\approx (174.65\text{ GeV})^2$.

Yukawas	α_0^i	α_1^i	α_2^i	α_3^i
$i = u$	$-0.972\,907$	$0.985\,35$	$0.000\,447\,89$	$0.000\,161\,069$
$i = d$	$-0.016\,679\,9$	$0.016\,192$	$0.000\,083\,112\,3$	$0.000\,133\,83$
$i = l$	$-0.010\,641\,8$	$0.009\,378\,77$	$-0.000\,163\,66$	$0.000\,021\,183\,6$
$i = \nu$	$0.837\,83$	$-0.983\,826$	$1.224\,05$	$1.222\,14$
VEVs	$\langle\chi_1\rangle$	$\langle\chi_2\rangle$	$\langle\psi_1\rangle$	$\langle\psi_2\rangle$
abs. [GeV]	60.385	106.489	56.6084	110.954
phase [rad.]	$-0.031\,356\,9$	$-0.035\,866\,5$	$-0.050\,002\,6$	0
M_{RR}	$M_1 = 1.878 \times 10^{14}$ GeV		$M_2 = 2.011 \times 10^{14}$ GeV	

$s_{12}^2 = 0.3^{+0.04}_{-0.05}$, and $s_{23}^2 = 0.5^{+0.14}_{-0.12}$. All values observed in neutrino oscillations are given at 2σ level. Three further quantities connected to the neutrinos are measurable: the sum of the neutrino masses from cosmology, m_β in beta decay experiments, and $|m_{ee}|$ in neutrinoless double beta decay. The experimental bounds on these quantities are:

$$\sum_{i=1}^{3} m_i \leq (0.42\ldots1.8)\text{ eV}\,[21],$$

$$m_\beta = \left(\sum_{i=1}^{3} |U_{\rm MNS}^{ei}|^2 m_i^2\right)^{1/2} \leq 2.2\text{ eV}\,[24],$$

$$\text{and}\quad |m_{ee}| = \left|\sum_{i=1}^{3} (U_{\rm MNS}^{ei})^2 m_i\right| \leq 0.9\text{ eV}\,[26].$$

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