Maxwell stress in Rindler space and the Archimedes Law

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We construct a theory of Maxwell stresses in Rindler space, presenting both a noncovariant and a covariant formulation. The theory shows how the Maxwell stresses are modified by a nonvanishing acceleration of gravity and that the Maxwell stresses of an electromagnetic field produce a volume stress force which vanishes in an inertial frame. In the noncovariant formulation we deduce the Maxwellian force due to the bending of the field lines by the acceleration of gravity. In the covariant formulation we show that Archimedes' law is valid for stationary electromagnetic fields in Rindler space: The Maxwellian surface forces acting from the outside field upon a closed surface produce a buoyancy equal to the weight of the electromagnetic field enclosed by the surface. Generally the mechanism is different from that in a fluid in which the buoyancy is due to a pressure which increases with depth. In a vertical electrical field the buoyancy is due to a tension that increases with height, but in a horizontal field it is due to a Maxwellian pressure which increases with the depth.

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I. INTRODUCTION

In a usual laboratory at the surface of the Earth we experience a uniform gravitational field. According to Einstein's principle of equivalence we may explore the physics in such a reference frame by considering a uniformly accelerated reference frame. This is usually called a Rindler frame [1], and the 3-space of the Minkowski spacetime as referred to such a frame is known as the Rindler space.

In this article we shall consider some effects upon the electromagnetic field of a point charge at rest in the Rindler frame, of the gravity experienced in this frame. In particular we shall investigate the field from the point of view of the Maxwell stresses in order to understand the origin of the field force acting upon a charge at rest in a gravitational field. The existence of this force has earlier been found in another way by Harpaz [2] and interpreted as a radiation reaction force. Our analysis leads to a different interpretation of this force.

Calculating the Maxwellian stress force on a closed surface in a stationary electromagnetic field and a volume force being the weight of the field, we find that Archimedes' law is valid for stationary electromagnetic fields in Rindler space. The mechanism of the buoyancy is exhibited for the simple cases of vertical and horizontal fields.

II. MAXWELL'S EQUATIONS IN RINDLER SPACE

In this section we shall establish Maxwell's equations for arbitrary electromagnetic fields in Rindler space.

The line element of spacetime in the Rindler frame may be written as

$$ds^2 = -{g_0}^2 x^2 dt^2 + dx^2 + dy^2 + dz^2, \tag{2.1}$$

where the constant g_0 is an acceleration, and x is the distance from the horizon in the Rindler frame at $x = 0$. The kinematics of this frame has been thoroughly discussed in Ref. [1]. At the point x the acceleration of gravity is $g(x)$, which is the acceleration of the point in the local system of inertia

$$g(x) = 1/x. \tag{2.2}$$

Thus g_0 is the acceleration of gravity at the point $x = 1/g_0$.

In the following $F^{\mu\nu}$ is the field tensor in the Rindler frame. The components of the field tensor in a field of orthonormal basis vectors in the Rindler frame are denoted by $F^{\hat{\mu}\hat{\nu}}$ and are related to the coordinate components by

$$\begin{aligned} F^{\hat{0}\hat{i}} &= \sqrt{|g_{00}|g_{ii}}F^{0i} = g_0 x F^{0i}, \\ F^{\hat{i}\hat{j}} &= \sqrt{g_{ii}g_{jj}}F^{ij} = F^{ij}. \end{aligned} \tag{2.3}$$

The electric field is defined as the electric force per unit charge upon a charge at rest. The covariant expression of this force is

$$f^\mu = qF^\mu{}_\nu \frac{dx^\nu}{ds}. \tag{2.4}$$

Hence the electric field $\vec{E} = E^i \vec{e}_i$ is given by

$$f^i = qE^i = qF^i{}_0 \frac{dx^0}{ds} = q\frac{F^i{}_0}{g_0 x} = -qg_0 x F^{i0} = -qF^{\hat{i}\hat{0}} = qF^{\hat{0}\hat{i}} \tag{2.5}$$

which leads to

$$E^i = F^{\hat{0}\hat{i}} = g_0 x F^{0i}. \tag{2.6}$$

In the same way the magnetic flux density is given by

$$\begin{aligned} &\vec{B} = B^i \vec{e}_i, \\ &B^1 = F^{\hat{2}\hat{3}} = F^{23} \quad \text{and cyclic permutations.} \end{aligned} \tag{2.7}$$

Note that the field strengths are the same in an accelerated orthonormal basis comoving with the Rindler frame as in a local instantaneous inertial rest system of the uniformly accelerated reference frame. The components of a tensor with reference to a field of orthonormal basis vector has been called the physical components of the tensor [3], because one has then become rid of coordinate effects present in the expressions of the components when the tensor is decomposed in an arbitrary coordinate basis. Although there is Minkowski metric in the orthonormal basis field comoving with the Rindler frame, this basis field is different from the orthonormal basis of an instantaneous inertial rest frame of the Rindler frame. The timelike basis vector of the orthonormal basis field comoving with a frame is equal to the four-velocity field of the reference particles of the frame. Mathematically the difference between the two orthonormal basis fields appear in the connection coefficients. The connection coefficients of the inertial basis field vanish, but the orthonormal basis field comoving with the Rindler frame has nonvanishing connection coefficients. In particular $\Gamma^{\hat{x}}{}_{\hat{t}\hat{t}} = 0$ in the inertial basis field and $\Gamma^{\hat{x}}{}_{\hat{t}\hat{t}} = 1/x$ in the basis field comoving with the Rindler frame. From the geodesic equation follows that a free particle instantaneously at rest has an acceleration $-\Gamma^{\hat{x}}{}_{\hat{t}\hat{t}}$. This is the acceleration of gravity in the frame.

Maxwell's equations in the Rindler frame are obtained from the covariant form of the equations

$$\frac{1}{\sqrt{|\det g_{\alpha\beta}|}}\frac{\partial}{\partial x^\mu}\left(\sqrt{|\det g_{\alpha\beta}|}F^{\mu\nu}\right) = -4\pi s^\nu \tag{2.8}$$

$$\frac{\partial F_{\mu\nu}}{\partial x^\sigma} + \frac{\partial F_{\nu\sigma}}{\partial x^\mu} + \frac{\partial F_{\sigma\mu}}{\partial x^\nu} = 0, \tag{2.9}$$

where s^ν is the four-current density,

$$s^\nu = \frac{1}{\sqrt{-g_{00}}}(\rho, \vec{j}). \tag{2.10}$$

ρ is the current density.

In the Rindler frame $\det g_{\alpha\beta} = -g_{00} = g_0^2 x^2$. Hence, in terms of the fields $\vec{E}$ and $\vec{B}$ defined by Eqs. (2.6), (2.7), (2.8), and (2.9) lead to

$$\nabla \cdot \vec{E} = 4\pi\rho \qquad \frac{\partial \vec{E}}{\partial t} - \nabla \times (g_0 x \vec{B}) = -4\pi g_0 x \vec{j}$$
$$\nabla \cdot \vec{B} = 0 \qquad \nabla \times (g_0 x \vec{E}) + \frac{\partial \vec{B}}{\partial t} = 0, \tag{2.11}$$

where ρ and $\vec{j}$ are the charge density and current density, respectively. In the stationary case Maxwell's equations in the Rindler frame reduce to

$$\nabla \cdot \vec{E} = 4\pi\rho(\vec{r}), \qquad \nabla \times (x\vec{E}) = 0,$$
$$\text{i.e.} \quad \nabla \times \vec{E} = \frac{1}{x}\vec{E} \times \vec{e}_x \tag{2.12a}$$
$$\nabla \cdot \vec{B} = 0, \qquad \nabla \times (x\vec{B}) = 4\pi x \vec{j}(\vec{r}),$$
$$\text{i.e.} \quad \nabla \times \vec{B} = 4\pi\vec{j}(\vec{r}) + \frac{1}{x}\vec{B} \times \vec{e}_x, \tag{2.12b}$$

where $\vec{E}$ and $\vec{B}$ are the fields strength which where defined in local orthonormal basis in Eqs. (2.6) and (2.7), $\rho(\vec{r})$ is the charge density and $\vec{j}(\vec{r})$ the current density.

III. MAXWELL STRESS IN THE FIELD OF A POINT CHARGE IN THE RINDLER SPACE

Electrodynamics in the Rindler space has earlier been investigated by I. Brevik [4]. He deduces expressions for the fundamental electromagnetic modes of a pure radiation field in Rindler space. However, he does not consider Maxwell stress, which is the topic of the present article. We shall first examine the Maxwell stress force in a region of space from a very simple point of view. In Sec. V the result obtained in this section will be applied to the field of a point charge. For the present purpose a noncovariant approach, not taking into account the weight of the electric field, is most suitable. A covariant description is presented in Sec. VI.

We now consider an electrostatic field and an element of a field line tube of infinitesimal cross section in a region free of charge. The product of E and the area A of the cross section is constant along the tube due to the continuity equation $\nabla \cdot \vec{E} = 0$. The element is of infinitesimal length ds such that to lowest order its volume is Ads. In order to find the Maxwell stress force which acts on the infinitesimal volume element, we shall apply well known properties of the electromagnetic field [5]: The electric field transmits a tension $E^2/8\pi$ parallel to the direction of the field and a pressure of magnitude $E^2/8\pi$ transverse to the direction of the field as measured in an instantaneous inertial rest system of the Rindler frame. The forces acting upon the volume element are tensions $\vec{T}_1$ and $\vec{T}_2$ at the boundary surfaces at the ends of the tube and pressure forces acting upon the side.

The total surface force $\hat{\vec{S}}$, on the element, as measured in local inertial frames at the surface points of the element, is

$$\hat{\vec{S}} = -\int_{\sigma'} \frac{E^2}{8\pi}\vec{n}\,d\sigma + \vec{T}_1 + \vec{T}_2$$
$$= -\int_{\sigma} \frac{E^2}{8\pi}\vec{n}\,d\sigma + 2(\vec{T}_1 + \vec{T}_2), \tag{3.1}$$

where $\vec{n}$ is the outwards pointing surface normal, and $d\sigma$ a surface element. In the surface σ' the end surfaces of the tube are not included, but they are included in the surface σ. The equality of the two expressions are due to the fact that tension and pressure are opposite in direction and of the same value, $E^2/8\pi$. Since σ is a closed surface, the first term of the last expression can be transformed into a volume integral,

$$\int_\sigma \frac{E^2}{8\pi}\vec{n}d\sigma = \int_V \frac{1}{8\pi}\nabla(E^2)dV = \frac{1}{8\pi}\nabla(E^2)Ads, \quad (3.2)$$

where A is the cross section area and ds the length of the field line tube. Let Δ denote increment over the length of the tube, and $\vec{e}_t$ a unit tangential vector. Then

$$\vec{T}_1 + \vec{T}_2 = \Delta\left(A\frac{E^2}{8\pi}\vec{e}_t\right) = \frac{AE}{8\pi}\Delta(E\vec{e}_t) = \frac{AE}{8\pi}\Delta\vec{E} = \frac{AEds}{8\pi}\frac{d\vec{E}}{ds} \quad (3.3)$$

From Eqs. (3.1), (3.2), and (3.3) we find that the total surface force is

$$\vec{S} = \left(-\frac{1}{8\pi}\nabla(E^2) + \frac{E}{4\pi}\frac{d\vec{E}}{ds}\right)Ads. \quad (3.4)$$

Making use of the identity

$$\nabla(E^2) = 2(\vec{E}\cdot\nabla)\vec{E} + 2\vec{E}\times(\nabla\times\vec{E}) = 2E\frac{d\vec{E}}{ds} + 2\vec{E}\times(\nabla\times\vec{E}) \quad (3.5)$$

we obtain

$$\vec{S} = -\frac{1}{4\pi}\vec{E}\times(\nabla\times\vec{E})Ads. \quad (3.6)$$

We introduce the force $\hat{\vec{f}}$ per unit volume,

$$\hat{\vec{f}} = \frac{1}{4\pi}\vec{E}\times(\nabla\times\vec{E}) \quad (3.7)$$

and write Eq. (3.6) in the form

$$\vec{S} + \hat{\vec{f}}Ads = 0. \quad (3.8)$$

This equation may be interpreted as an equation of equilibrium for the volume element acted upon by a surface force $\vec{S}$ and a volume force $\hat{\vec{f}}Ads$.

At this point one may ask which properties of space and field have been assumed to obtain the result (3.7). The assumptions are that 3-space is Euclidean and the field is static. Thus the result is valid for any electrostatic situation in inertial frames and in the Rindler frame at points free of charge.

In a global inertial frame $\nabla\times\vec{E} = 0$ which shows that $\hat{\vec{S}} = 0$ and $\hat{\vec{f}} = 0$, i.e. in an inertial system the total Maxwell stress force on the surface of a volume free of charge is zero and so is the corresponding force density.

We shall now consider an electrostatic field in the Rindler frame. We insert $\nabla\times\vec{E} = \vec{E}\times\vec{e}_x/x$ from Eq. (2.12a) into Eq. (3.7), and get

$$\hat{\vec{f}} = \frac{1}{4\pi}\vec{E}\times\left(\vec{E}\times\frac{\vec{e}_x}{x}\right) = \frac{1}{4\pi x}(E_x\vec{E} - E^2\vec{e}_x). \quad (3.9)$$

This equation will now be adjusted to the field from a charge Q on the x-axis at the position x_Q. The electric field lines from the charge Q are conveniently described in cylindrical coordinates (x, ρ, φ), where $\rho = (y^2 + z^2)^{1/2}$ is the distance from the x-axis, and φ is the angle round this axis. The component $E_\varphi = 0$, so the field lines are in vertical planes φ = constant. The components E_x and E_ρ are given in Eq. (5.3). The field lines leave the charge Q in different directions. They are bending downwards in the gravitational field, each line forming an arc of a circle and reaching the horizon, $x = 0$ at the border of the Rindler space in vertical direction. The center of such a circle is consequently in the horizon, i.e. in the plane $x = 0$.

Putting $\vec{E} = \vec{E}_x + \vec{E}_\rho$ in Eq. (3.9) we get

$$\hat{\vec{f}} = \frac{1}{4\pi x}(E_x\vec{E}_\rho - E_\rho^2\vec{e}_x). \quad (3.10)$$

Introducing (see Fig. 1)

$$x = R\sin\theta, \qquad E_x = E\cos\theta, \qquad E_\rho = E\sin\theta, \quad (3.11)$$

we get

$$\hat{\vec{f}} = \frac{E^2}{4\pi R}\vec{e}_n \quad (3.12)$$

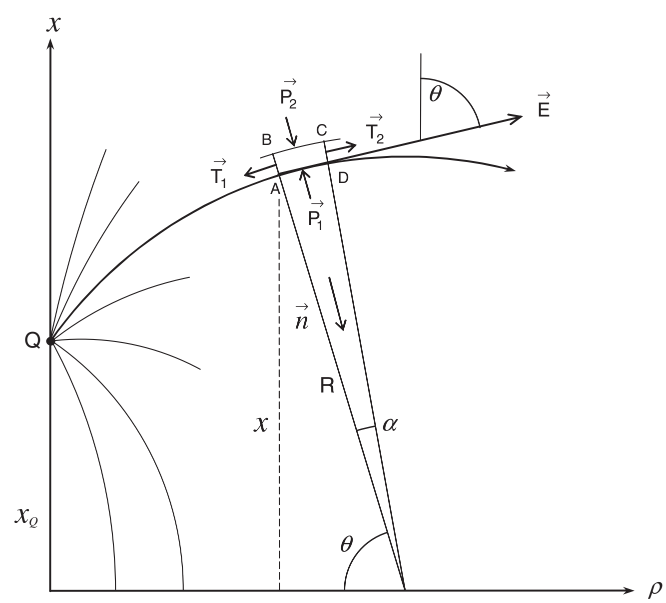

FIG. 1. The figure shows an electric field line from a charge Q in Rindler space and an element of a field line tube. The field line is an arc of a circle. Rotating the pictured plane around the x-axis one obtains the whole field of the charge. The field lines are bent downwards due to the acceleration of gravity like the paths of droplets in a water fountain. The Maxwellian stress produces tensions T_1 and T_2 in the directions of the field line, and a pressure normal to the field line.

which agrees with eq. (4) in Ref. [2]. The unit vector $\vec{e}_n = -\vec{e}_x \sin\theta + \vec{e}_\rho \cos\theta$ is the principal normal vector of the field line pointing in the direction of the curvature center, R is the radius of curvature, and $\vec{e}_\rho$ is the basis vector of the ρ-coordinate.

A transition to an inertial frame is most easily obtained by introducing the Møller coordinate $\bar{x} = x - 1/g_0$ where g_0 is the acceleration of gravity at $\bar{x} = 0$. Then the line element of the uniformly accelerated reference frame takes the form [6]

$$ds^2 = -(1 + g_0\bar{x})^2 dt^2 + d\bar{x}^2 + dy^2 + dz^2. \tag{3.13}$$

Since

$$\frac{1}{x} = \frac{g_0}{1 + g_0\bar{x}} \tag{3.14}$$

we get from Eq. (3.11)

$$R = \frac{1 + g_0\bar{x}}{g_0 \sin\theta}. \tag{3.15}$$

In the inertial frame limit $g_0 \to 0$, we get $\lim_{g_0\to 0} R = \infty$ and the force $\hat{\vec{f}}$ due to the electric stresses vanishes, as expected for the field of a point charge with straight field lines in an inertial frame.

IV. ON THE ELECTRIC STRESS TENSOR IN THE RINDLER FRAME

We proceed with a general treatment of electrostatics in the Euclidean 3-space of the Rindler frame. The charge distribution is given as $\rho = \rho(x, y, z)$. Maxwell's equations, Eqs. (2.12a), are now written in component form as follows

$$\frac{\partial E_j}{\partial x_j} = 4\pi\rho, \qquad \frac{\partial E_i}{\partial x_j} - \frac{\partial E_j}{\partial x_i} = \frac{1}{x}(\delta_{1i}E_j - \delta_{1j}E_i). \tag{4.1}$$

The electric force density is

$$F_i = \rho E_i. \tag{4.2}$$

In textbooks on electricity it is shown how F_i in inertial frames may be expressed as the divergence of the electric stress tensor [5]. We shall here show how this is modified in the Rindler space.

By means of Eqs. (4.1) and (4.2) we get

$$\begin{aligned} F_i = \rho E_i &= \frac{1}{4\pi}\frac{\partial E_j}{\partial x_j}E_i = \frac{1}{4\pi}\frac{\partial}{\partial x_j}(E_jE_i) - \frac{1}{4\pi}E_j\frac{\partial E_i}{\partial x_j} \\ &= \frac{1}{4\pi}\frac{\partial}{\partial x_j}(E_jE_i) - \frac{1}{4\pi}E_j\left[\frac{\partial E_j}{\partial x_i} + \frac{1}{x}(\delta_{1i}E_j - \delta_{1j}E_i)\right] \\ &= \frac{1}{4\pi}\frac{\partial}{\partial x_j}(E_jE_i) - \frac{1}{8\pi}\delta_{ij}\frac{\partial E^2}{\partial x_j} + \frac{1}{4\pi x}(E_1E_i - \delta_{1i}E^2) \end{aligned} \tag{4.3}$$

which may be written as

$$F_i = \frac{\partial t_{\hat{i}\hat{j}}}{\partial x_j} + \hat{f}_i, \tag{4.4}$$

where $t_{\hat{i}\hat{j}}$ are the components of the Maxwell stress tensor as referred to an instantaneous inertial rest system of the Rindler frame,

$$t_{\hat{i}\hat{j}} = \frac{1}{4\pi}\left(E_iE_j - \frac{1}{2}\delta_{ij}E^2\right) \tag{4.5}$$

and

$$\begin{aligned} \hat{f}_i &= \frac{1}{4\pi x}(E_1E_i - \delta_{1i}E^2) \\ &= \frac{1}{4\pi x}(-E_2^2 - E_3^2, E_1E_2, E_1E_3) \end{aligned} \tag{4.6}$$

or

$$\hat{\vec{f}} = \frac{1}{4\pi x}(E_1\vec{E} - E^2\vec{e}_1) = \frac{1}{4\pi}\frac{g_0}{1 + g_0\bar{x}}(E_1\vec{E} - E^2\vec{e}_1) \tag{4.7}$$

in accordance with Eq. (3.9). Here $t_{\hat{i}\hat{j}}$ is the electric stress tensor, which has the same form as in inertial frames, and $\hat{\vec{f}}$ is the force due to electric stress per unit volume, which vanishes in inertial frames where $g_0 = 0$.

Equation (4.4) may be interpreted physically by integrating it over a volume V of Rindler space,

$$\int F_i dV = \int \frac{\partial t_{\hat{i}\hat{j}}}{\partial x_j} dV + \int \hat{f}_i dV. \tag{4.8}$$

Here $\int F_i dV = \int \rho E_i dV$ is the force on the charge in V. The integral $\int \hat{f}_i dV$ expresses the force due to electric stresses in V. We express the first integral on the right-hand side of Eq. (4.8) as a surface integral by means of Gauss's theorem. Let $\vec{n}$ be the unit outward normal and $d\sigma$ an area element. Then

$$\int \frac{\partial t_{\hat{i}\hat{j}}}{\partial x_j} dV = \int t_{\hat{i}\hat{j}} n_j d\sigma \tag{4.9}$$

and Eq. (4.8) may be written

$$\int F_i dV = \int t_{\hat{i}\hat{j}} n_{\hat{j}} d\sigma + \int \hat{f}_i dV. \tag{4.10}$$

Here $t_{\hat{i}\hat{j}} n_j d\sigma$ is the i-component of the force acting on the area $d\sigma$ from outside. By means of Eq. (4.5) we get

$$t_{\hat{i}\hat{j}} n_j = \frac{1}{4\pi}\left(E_iE_j - \frac{1}{2}\delta_{ij}E^2\right)n_j = \frac{1}{4\pi}\left(E_i\vec{E}\cdot\vec{n} - \frac{1}{2}n_iE^2\right). \tag{4.11}$$

That is, the force acting upon the area from the outside may be written

$$\hat{\vec{t}}d\sigma = -\frac{1}{4\pi}\left((\vec{E}\cdot\vec{n})\vec{E} - \frac{1}{2}E^2\vec{n}\right)d\sigma. \quad (4.12)$$

In general, to account for the electric stress on an element, we may introduce + and − for the two sides of the element. Let $\vec{n}$ be the unit normal on the positive side (pointing outwards from the element), and $\vec{n}' = -\vec{n}$ the normal on the opposite side. The force of electric stress acting on the element from the positive side is $\hat{\vec{t}}d\sigma$, and the force from the negative side is $\hat{\vec{t}}'d\sigma$, where

$$\hat{\vec{t}}' = \frac{1}{4\pi}\left((\vec{E}\cdot\vec{n}')\vec{E} - \frac{1}{2}E^2\vec{n}'\right) = -\hat{\vec{t}} \quad (4.13)$$

i.e. there is an opposite force acting on the opposite side. Putting $\vec{E} = \pm E\vec{n}$ in Eq. (4.12) we get for the force per unit area

$$\hat{\vec{t}} = \frac{E^2}{8\pi}\vec{n} \quad (4.14)$$

demonstrating that there is a tension parallel to the direction of the field. If $\vec{E}\perp\vec{n}$ Eq. (4.12) gives $\hat{\vec{t}} = -(1/8\pi)E^2\vec{n}$, showing that there is a pressure transverse to the direction of the field.

For arbitrary directions of $\vec{n}$ relative to $\vec{E}$ the area will be exposed to a shear stress in addition to the tension or pressure. It is easily seen from Eq. (4.12) that the absolute value of $\hat{\vec{t}}$ is independent of the angle between $\vec{E}$ and $\vec{n}$, and equal to the energy density $E^2/8\pi$. Further, the direction of $\hat{\vec{t}}$ is such that $\vec{E}$ bisects the angle between $\vec{n}$ and $\hat{\vec{t}}$.

V. APPLICATION TO THE FIELD OF A POINT CHARGE

Suppose a point charge Q is situated on the x-axis at the height $x = x_Q$. The electrical field is rotationally symmetric about the x-axis. The field lines define vertical planes, and are circles with centers on the horizontal plane $x = 0$, as shown in Fig. 1. (Note that in this construction the centers of the circles are not restricted to that part of the horizontal line in the paper plane which is covered by the ρ-axis, but fill the whole of this line. The circles with centers to the left of the x-axis give the field lines starting with a downwards component at the charge, and the centers to the right of the x-axis give the field lines with an upwards component at the charge.)

The electrical field, found by solving Eqs. (2.12), may be expressed as

$$\vec{E} = -\frac{1}{x}\nabla\Psi, \quad (5.1)$$

where the potential Ψ is given in eq. (4.41) of Ref. [7], and may be expressed as

$$\Psi = \frac{Q}{2}\left(\frac{r}{r'} + \frac{r'}{r}\right). \quad (5.2)$$

Here r is the distance from the point charge, and r' is the distance from a point which is symmetrically placed on the other side of the $x = 0$ plane. The components of the electrical field strength are,

$$E_x = 4Qx_Q^2(x^2 - x_Q^2 - \rho^2)/\xi_R^3, \qquad E_\rho = 8Qx_Q^2 x\rho/\xi_R^3,$$
$$\xi_R = rr', \qquad r = [(x - x_Q)^2 + \rho^2]^{1/2},$$
$$r' = [(x + x_Q)^2 + \rho^2]^{1/2} \quad (5.3)$$

in accordance with eq. (16) and (19) in [7].

We now apply Eq. (4.10) to a region which does not include the charge Q. Then $\vec{F} = 0$ and the equation may be written

$$\int t_{\hat{i}\hat{j}}n_j d\sigma + \int \hat{f}_i dV = 0 \quad (5.4)$$

which says that the sum of the force from the outside on the surface and the volume force is equal to zero.

Let us now apply Eq. (5.4) to the region between two spherical surfaces which are concentric about the charge point, i.e. $r_1 < r < r_2$ where $0 < r_1 < r_2 < x_Q$. Because of the symmetry of the field and the region, the total force in the volume and on the surfaces points in the x-direction. We therefore restrict ourselves to the component of forces in that direction. Expressing $\vec{E}$ by polar coordinates (r, ϕ), given by,

$$x = x_Q + r\cos\phi, \qquad \rho = r\sin\phi \quad (5.5)$$

we have

$$E_r = \frac{Q}{r^2}\frac{1 + b\cos\phi}{[1 + 2b\cos\phi + b^2]^{3/2}},$$
$$E_\phi = \frac{Q}{r^2}\frac{b\sin\phi}{[1 + 2b\cos\phi + b^2]^{3/2}}, \qquad b = \frac{r}{2x_Q} \quad (5.6)$$

and the magnitude of the field strength is

$$E = \frac{Q}{r^2}\frac{1}{1 + 2b\cos\phi + b^2}. \quad (5.7)$$

Let $\hat{S}_1(r)$ denote the x-component of the electric stress force acting on the convex side of a spherical surface of radius r ($< x_Q$). We introduce the unit normal vector $\vec{n} = \vec{e}_r = \vec{e}_x\cos\phi + \vec{e}_\rho\sin\phi$, which together with Eq. (4.11) gives,

$$t_{\hat{x}\hat{j}}n_j = \frac{1}{8\pi}[(E_x^2 - E_\rho^2)\cos\phi + 2E_xE_\rho\sin\phi]. \quad (5.8)$$

Hence, by integration over angles we obtain

$$\hat{S}_{\hat{1}}(r) = \int_r t_{\hat{1}\hat{j}}n_j d\sigma$$
$$= \frac{Q^2}{16x_Q^2}\left(\frac{1}{b^3} - \frac{2}{b(1 - b^2)} - \frac{1 + 3b^2}{2b^4}\ln\frac{1 + b}{1 - b}\right). \quad (5.9)$$

Expanding this expression in powers of r/x_Q leads to

$$\hat{S}_1(r) = \frac{Q^2}{x_Q^2}\left(-\frac{2}{3}\frac{x_Q}{r} - \frac{1}{10}\frac{r}{x_Q} - \frac{3}{140}\frac{r^3}{x_Q^3} + \ldots\right). \quad (5.10)$$

The force acting on the convex side of the outer surface is $\hat{S}_1(r_2)$, and the force acting on the concave side of the inner surface is $-S_1(r_1)$. Thus, the force acting on the surface of the considered region is $-\hat{S}_1(r_1) + \hat{S}_1(r_2)$. According to Eq. (5.4) this force is equal and opposite of the volume force,

$$\int_{r_1<r<r_2} \hat{f}_1 dV = \hat{S}_1(r_1) - \hat{S}_1(r_2) \quad (5.11)$$

where, from Eq. (4.6), $\hat{f}_1 = -E_\rho^2/4\pi x$, and $S_1(r)$ is given by Eq. (5.9). If $\hat{S}_1(r_2)$ may be neglected compared to $\hat{S}_1(r_1)$, Eq. (5.11) explains how the force on the inner sphere may be found by the volume force, as was done by Harpaz [2].

To lowest order in r_1 and r_2 we get from Eqs. (5.10) and (5.11)

$$\int \hat{f}_1 dV = \frac{2}{3}\frac{Q^2}{x_Q}\left(\frac{1}{r_2} - \frac{1}{r_1}\right) = \frac{2}{3}\frac{Q^2}{x_Q r_1}\left(\frac{r_1}{r_2} - 1\right) \quad (5.12)$$

i.e.

$$\int \hat{f}_1 dV \approx -\frac{2}{3}\frac{Q^2}{x_Q r_1}, \quad \text{where} \quad r_1 \ll r_2. \quad (5.13)$$

Introducing the acceleration of gravity, $g_Q = 1/x_Q$, at the point $x = x_Q$, we find that the force on the concave side of the inner sphere is

$$-S_1(r_1) = -\int \hat{f}_1 dV = \frac{2}{3}Q^2\frac{g_Q}{r_1} \quad (5.14)$$

which is the approximation found by Harpaz [2] (in our notation).

It may be noted that the procedure employed in this deduction cannot be used in an inertial frame, for example, in connection with the field of an electric dipole. The reason is that in an inertial system $\hat{f}_1 = 0$ even in the case of an electric field with curved field lines, and $\hat{S}_1(r_1) = \hat{S}_1(r_2)$ according to Eq. (5.11).

It may be useful, here, to put in mind a result given on p. 162 of Ref. [7]. It concerns an extended spherical charge with an equipotential volume. Utilizing the fact that the field outside the charge may be described as the field due to a fictitious charge, at the point $x = \sqrt{x_Q^2 - r_1^2}$ on the x-axis we found the field at the surface. Considering the Maxwell stress, we found in Ref. [7] the following exact expression for the self force $\vec{F}_s$ on an extended charge with radius r_1,

$$\vec{F}_s = -\frac{2}{3}Q^2\frac{g_Q}{r_1}\frac{\vec{e}_x}{1 - g_Q^2 r_1^2}. \quad (5.15)$$

The dominating term of this expression agrees with Eq. (5.14) when $g_Q r_1 \ll 1$.

We now proceed with our analysis of the field due to a point charge. Let us consider the total 3-space, V_R, of the Rindler frame, $x > 0$, excluding a sphere around the point charge. The total 3-space is considered as the limit of a cylindrical region coaxial with the x-axis. Let $\vec{N} = -\vec{e}_x$ be the outward normal at the plane $x = 0$. Then, since the surface integrals at infinity have zero limits, we have from Eq. (5.4) the equation

$$\int_{x=0} t_{\hat{1}\hat{i}} N_{\hat{i}} d\sigma - \hat{S}_1(r_1) = -\int_{V_R} \hat{f}_1 dV \quad (5.16)$$

where the left-hand side is the force acting on the surface of the 3-space considered. The first term on the left-hand side is the Maxwellian stress force acting on the plane $x = 0$, i.e. on the horizon of the Rindler space. It points in the negative x-direction. The force $-\hat{S}_1(r_1)$ which is acting on the concave side of the sphere, is directed upwards and is given by Eqs. (5.9) and (5.10). Its magnitude is approximately $2Q^2/3x_Q r_1$.

From Eq. (5.3) we get, in accordance with eq. 20 in [8],

$$\vec{E}(\text{at } x = 0) = -\frac{4Qx_Q^2}{(x_Q^2 + \rho^2)^2}\vec{e}_x \quad (5.17)$$

and from Eq. (4.5)

$$t_{\hat{1}\hat{1}} = \frac{E^2}{8\pi}. \quad (5.18)$$

Hence,

$$\int_{x=0} t_{\hat{1}\hat{i}} N_{\hat{i}} d\sigma = -\int_{x=0} t_{\hat{1}\hat{1}} d\sigma = -\frac{1}{8\pi}\int E^2 d\sigma = -\frac{2}{3}\frac{Q^2}{x_Q^2}. \quad (5.19)$$

The volume force is now given by Eq. (5.16),

$$\int_{V_R} \hat{f}_1 dV = \frac{2}{3}\frac{Q^2}{x_Q^2} + S_1(r_1) \approx \frac{2}{3}\frac{Q^2}{x_Q^2} - \frac{2}{3}\frac{Q^2}{x_Q r_1} = -\frac{2}{3}\frac{Q^2}{x_Q r_1}\left(1 - \frac{r_1}{x_Q}\right), \quad (5.20)$$

where $S_1(r_1)$ is given by Eq. (5.9).

To sum up, there are two forces acting downwards on the field in V_R, the volume force $\int_{V_R} \hat{f}_1 dV$ and the tension $(2/3)(Q^2/x_Q^2)$ at the horizon, $x = 0$. The equilibrium is maintained by a force $(-S_1)$ which is acting upwards on the inner surface of the sphere of radius r_1 surrounding the charged particle.

Equation (5.19) is an expression of the tension at the plane $x = 0$, i.e. at the horizon in the Rindler space. We introduce the acceleration of gravity $g_Q = 1/x_Q$ and get

$$\int_{x=0} t_{\hat{1}\hat{i}} N_i d\sigma = -\frac{2}{3} Q^2 g_Q^2. \tag{5.21}$$

This is a force. In units with $c = 1$ force and power have the same dimension. However physically it is remarkable that the expression (5.21) is identical to Larmor's formula for radiated power. The charge is permanently at rest in the Rindler space, however. The situation is static and there is no magnetic field at any time or any place in the Rindler space, so as observed in this space the charge does not radiate [7]. As observed in an inertial frame, on the other hand, the charge does indeed radiate with the power (5.21).

VI. COVARIANT THEORY AND ARCHIMEDES'S LAW FOR AN ELECTROMAGNETIC FIELD IN RINDLER SPACE

The electromagnetic energy-momentum tensor in an arbitrary frame of reference is given by

$$T_\alpha^\beta = \frac{1}{4\pi}\left(F_{\alpha\mu} F^{\beta\mu} - \frac{1}{4}\delta_\alpha^\beta F_{\mu\nu} F^{\mu\nu}\right), \tag{6.1}$$

where $F^{\mu\nu}$ is the field tensor. The tensor T_α^β is constructed such that the four-force density $\kappa_\alpha = F_{\alpha\sigma} s^\sigma$, by means of Maxwell's equations, may be expressed as the covariant divergence of T_α^β,

$$-\kappa_\alpha = T_{\alpha;\beta}^\beta. \tag{6.2}$$

This divergence may be expressed as [6]

$$T_{\alpha;\beta}^\beta = \frac{1}{\sqrt{|\det g_{\mu\nu}|}} \frac{\partial}{\partial x^\beta}\left(\sqrt{|\det g_{\mu\nu}|} T_\alpha^\beta\right) - \frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial x^\alpha} T^{\mu\nu}. \tag{6.3}$$

In the Rindler frame with metric $(-{g_0}^2 x^2, 1, 1, 1)$ this expression takes the form

$$-\kappa_\alpha = T_{\alpha;\beta}^\beta = \frac{1}{g_0 x}\frac{\partial}{\partial x^\beta}(g_0 x T_\alpha^\beta) - \frac{1}{x}\frac{\partial x}{\partial x^\alpha} T_0^0, \tag{6.4}$$

where T_α^β is given by Eq. (6.1) with $F^{\mu\nu}$ as the field tensor in the Rindler frame.

At a point where the charge density $\rho = 0$, Eq. (6.4) becomes

$$\frac{\partial}{\partial x^\beta}(g_0 x T_\alpha^\beta) = g_0 T_0^0 \frac{\partial x}{\partial x^\alpha}. \tag{6.5}$$

Here we may put $\alpha = 0$ and get

$$\frac{\partial}{\partial x^\beta}(g_0 x T_0^\beta) = 0 \tag{6.6}$$

which expresses the energy conservation law,

$$\frac{\partial h}{\partial t} + \nabla \cdot \vec{S} = 0, \tag{6.7}$$

where

$$h = -g_0 x T_0^0 = g_0 x \frac{1}{8\pi}(\vec{E}^2 + \vec{B}^2) \tag{6.8}$$

represents energy density, and

$$S^i = -g_0 x T_0^i = {g_0}^2 x^2 \frac{1}{4\pi}(\vec{E} \times \vec{B})^i \tag{6.9}$$

represents energy flux.

With the Møller coordinate $\bar{x} = x - 1/g_0$ we have $g_0 x = 1 + g_0 \bar{x}$. In the inertial limit, where $g_0 \to 0$ and $g_0 x \to 1$, the Rindler quantities h and $\vec{S}$ get their inertial values.

Next we put $\alpha = i = 1, 2, 3$ in Eq. (6.5), and get

$$\frac{\partial}{\partial x^\beta}(g_0 x T^\beta{}_i) = g_0 T^0{}_0 \delta_{i1}. \tag{6.10}$$

By means of Eq. (6.1) and the definitions of $\vec{E}$ and $\vec{B}$ in Eqs. (2.6) and (2.7) we find

$$\frac{1}{4\pi}\frac{\partial}{\partial t}(\vec{E} \times \vec{B})_i - \frac{\partial}{\partial x^j} t_{ij} = -\delta_{i1} g(x) h, \tag{6.11}$$

where

$$t_{ij} = \frac{g_0 x}{4\pi}\left[E_i E_j + B_i B_j - \frac{1}{2}\delta_{ij}(\vec{E}^2 + \vec{B}^2)\right]. \tag{6.12}$$

Here $g(x) = 1/x$ is the acceleration of gravity, and h is the energy density of the field as given in Eq. (6.8).

We integrate Eq. (6.11) over a constant volume V and obtain by means of Gauss's law,

$$\int_\sigma t_{ij} n_j d\sigma - \delta_{i1}\int_V g(x) h dV = \frac{1}{4\pi}\frac{d}{dt}\int_V (\vec{E} \times \vec{B})_i dV \tag{6.13}$$

which suggests the following interpretation. The left-hand side is the force acting on the system, i.e. a force on the surface of the system due to the Maxwell stress, and a volume force which is the weight of the field. At the right-hand side $\vec{E} \times \vec{B}/4\pi$ is the momentum density of the field. Hence, the equation says that the sum of the Maxwellian forces acting upon the system is equal to the time derivative of the momentum of the system.

We note that according to Eq. (6.9) the momentum density of an electromagnetic field in the Rindler space is

$$\frac{1}{4\pi}\vec{E} \times \vec{B} = \frac{1}{g_0^2 x^2}\vec{S}, \tag{6.14}$$

where $\vec{S}$ is the energy flux density (Poynting's vector). In the following we let the fields be stationary, i.e. $\partial\vec{E}/\partial t = \partial\vec{B}/\partial t = 0$. Then the right-hand side of Eq. (6.13) is zero, and we get

$$\int_\sigma t_{ij} n_j d\sigma = \delta_{i1}\int_V g(x) h dV, \tag{6.15}$$

where the left-hand side is the i-component of the force acting on the surface. The equation tells that this force has no horizontal component, and that the vertical component ($i = 1$) is pointing upwards. Thus, the surface force acts like a buoyancy, and this buoyancy is equal to $\int_V g(x)hdV$ which is the weight of the field in the volume. This is the law of Archimedes of Syracuse (287–212 B.C.) as applied to a stationary electromagnetic field in a gravitational field.

In vector notation Eq. (6.15) reads

$$\vec{S} + \vec{W} = 0, \tag{6.16}$$

where $\vec{S}$ is the buoyancy and $\vec{W}$ the weight,

$$\vec{S} = \vec{e}_1 \int_\sigma t_{1j} n_j d\sigma \tag{6.17}$$

$$\vec{W} = \int_V \vec{f} dV. \tag{6.18a}$$

The volume force $\vec{f}$ is the weight per unit volume,

$$\vec{f} = -\vec{e}_1 g(x) h = -\vec{e}_1 g_0 \frac{1}{8\pi}(\vec{E}^2 + \vec{B}^2). \tag{6.18b}$$

We shall apply Eq. (6.15) to the electrostatic field $\vec{E}$ in Eq. (5.3), which is produced by a point charge Q situated at rest on the x-axis in the point $x = x_Q$. Putting $\vec{B} = 0$ in Eqs. (6.8) and (6.12) we get

$$h = g_0 x \frac{\vec{E}^2}{8\pi} \tag{6.19}$$

$$t_{ij} = \frac{g_0 x}{4\pi}\left(E_i E_j - \frac{1}{2}\delta_{ij}\vec{E}^2\right). \tag{6.20}$$

We let the volume V be the total 3-dimensional volume V_R above the horizon of the Rindler frame (i.e. $x > 0$), except a small sphere of radius r_1 with center at the point charge. Because of the factor x in the expression (6.21) for t_{ij} there is no contribution to the surface integral from the plane $x = 0$. Neither is there any contribution from infinitely far. So, from Archimedes' law, we may conclude: On the electric field in the volume V_R there acts a surface force upwards on the inside of the sphere. This force is equal and opposite to the weight of the field in V_R, keeping the field in static equilibrium. (A corresponding situation is that of a garment hanging on a peg. The peg corresponds to the force that keeps the charge at rest, and the garment corresponds to the electric field. The garment acts upon the peg with a force equal to its weight.)

We shall finally calculate the surface force acting on the inner surface of the sphere of radius r_1. For this purpose we utilize the fact that this force is the buoyancy $\vec{S}$ acting on the field which keeps it at rest. Without this force the field would be freely falling. For the sake of comparing with the corresponding expression (5.14), which was found without taking the weight of the field into account, it is convenient to choose $g_0 = g_Q = 1/x_Q$. According to Eqs. (6.16), (6.18), and (6.19),

$$\vec{S} = \vec{e}_x g_Q \int_{V_R} \frac{E^2}{8\pi} dV. \tag{6.21}$$

By integration we find

$$U \equiv \int \frac{E^2}{8\pi} dV = \frac{Q^2}{8r_1}\left(6 - \frac{2}{1-b^2} - 3b \ln\frac{1+b}{1-b}\right), \tag{6.22}$$

where $b = r_1/2x_Q$. Expanding this in powers of r_1/x_Q leads to

$$U = \frac{Q^2}{2r_1}\left(1 - 2\left(\frac{r_1}{x_Q}\right)^2 - \frac{1}{16}\left(\frac{r_1}{x_Q}\right)^4 + \cdots\right) \tag{6.23}$$

which gives $U \approx Q^2/2r_1$ when $r_1 \ll x_Q$. From Eq. (6.21) we now get the buoyancy, i.e. force from the stresses in the electric field, upon the inside surface of the sphere,

$$S = g_Q U \approx \frac{g_Q Q^2}{2r_1} \tag{6.24}$$

which is different from the expression (5.14) found by Harpaz [2].

The Maxwellian stress force acting on the outside of the sphere is opposite to the force S. Thus, it is acting downwards and is equal to the weight of the electric field in V_R.

VII. THE MECHANISM OF THE BUOYANCY IN A STATIONARY ELECTROMAGNETIC FIELD

We consider a region of Rindler space which is free from charge and current. Under stationary conditions Eq. (2.12) give the following equations for $\vec{E}$ and $\vec{B}$,

$$\nabla \cdot \vec{E} = 0, \qquad \nabla \times \vec{E} = \frac{1}{x}\vec{E} \times \vec{e}_x, \tag{7.1a}$$

$$\nabla \cdot \vec{B} = 0, \qquad \nabla \times \vec{B} = \frac{1}{x}\vec{B} \times \vec{e}_x. \tag{7.1b}$$

The equations are identical in form. Further the expressions (6.12) and (6.18b) are symmetric in $\vec{E}$ and $\vec{B}$, so the field $\vec{E}$ in the examples below may everywhere be replaced by $\vec{B}$.

Let $\vec{E}$ be vertical field,

$$\vec{E} = E\vec{e}_x. \tag{7.2}$$

Then Eq. (7.1a) is satisfied when E is constant. We now consider a volume in the form of a cube with side length s and situated as shown in Fig. 2. According to Eq. (6.12) there is a tension $g_0 x E^2/8\pi$ parallel to the direction of the field, and a pressure of the same magnitude normal to the field. Thus, on the vertical sides of the cube there are pressure forces acting horizontally. Their sum is zero. On the bottom plane there is a drag $T_1 = g_0 x_1 (E^2/8\pi)s^2$ acting downwards, and on the upper plane there is a drag $T_2 = g_0(x_1 + s)(E^2/8\pi)s^2$ acting upwards. The net Maxwell stress forces acting on the cube (the buoyancy) is directed

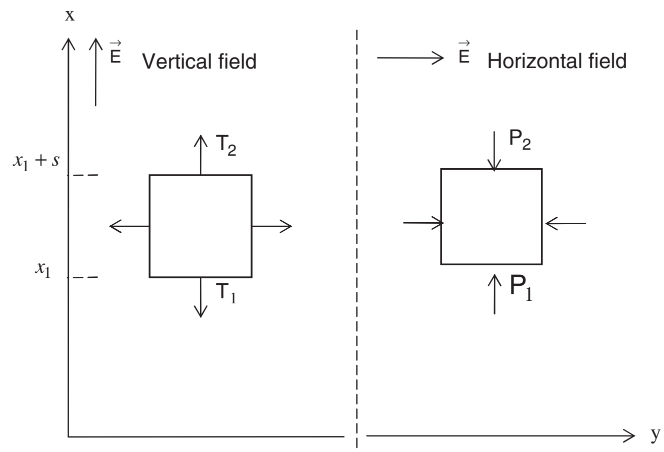

FIG. 2. The figure shows the buoyancy in the Rindler frame due to Maxwell stress in an electrostatic field. In a vertical field the buoyancy is due to a tension that increases with height. In a horizontal field the mechanism is a pressure that increases with depth.

upwards and given by

$$T_2 - T_1 = g_0 \frac{E^2}{8\pi} s^3 \tag{7.3}$$

which is just the weight of the field in the volume, according to Eq. (6.8).

Next we let $\vec{E}$ be in the y-direction,

$$\vec{E} = E_y \vec{e}_y. \tag{7.4}$$

Then Eq. (7.1a) requires that $E_y = k/x$ where k is a constant. Also in this case the forces on the vertical sides are horizontal. On two of them there are opposite drags of the same magnitude, as shown in Fig. 2. On the other two vertical sides there are pressure forces of equal magnitude in opposite z-directions. The sum of the horizontal forces is zero.

On the bottom plane there is a pressure force acting upwards,

$$P_1 = g_0 x_1 \frac{E^2}{8\pi} s^2 = \frac{g_0}{8\pi} \frac{k^2}{x_1} s^2 \tag{7.5}$$

and on the top plane there is a pressure force acting downwards

$$P_2 = g_0 (x_1 + s) \frac{E^2}{8\pi} s^2 = \frac{g_0}{8\pi} \frac{k^2}{x_1 + s} s^2. \tag{7.6}$$

Thus there is a buoyancy on the cube given by

$$P_1 - P_2 = \frac{g_0}{8\pi} k^2 \left(\frac{1}{x_1} - \frac{1}{x_1 + s} \right) s^2 \tag{7.7}$$

which according to Eq. (6.18) is the weight W of the field in the cube

$$W = \int_V g_0 \frac{1}{8\pi} \vec{E}^2 dV = \frac{g_0}{8\pi} \int_V \frac{k^2}{x^2} dV$$
$$= \frac{g_0}{8\pi} \int_{x_1}^{x_1+s} \frac{k^2 s^2}{x^2} dx = \frac{g_0}{8\pi} k^2 \left(\frac{1}{x_1} - \frac{1}{x_1 + s} \right) s^2. \tag{7.8}$$

These examples show that in a vertical field the buoyancy is due to a tension that increases with height. In a horizontal field the mechanism is like that in a fluid in which the buoyancy is due to a pressure which increases with depth.

It should be noted that the buoyancy is extremely small in an electrical field due to its small mass density compared to that of, for example, air, which is 1.3 kg/m^3 at standard temperature and pressure. Consider an electrical field equal to the ionization field of the Earth's atmosphere, $E = 2.4 \times 10^6$ V/m. The energy density of this field is about 25 J/m^3. However, the mass density is only 2.8×10^{-16} kg/m^3. The weight of one cubic meter of this field at the surface of the Earth is approximately 2.7×10^{-15} N. At the surface of a neutron star the acceleration of gravity is 3×10^{12} m/s^2. Even in such a strong gravitational field the weight of the above electrical field is only about a hundredth of a Newton, i.e. equal to the weight of a milligram mass at the surface of the Earth.

VIII. CONCLUSION

A laboratory at the surface of the Earth is not freely falling. It is not an inertial frame. Relative to an inertial frame it accelerates upwards with the acceleration of gravity. It accelerates uniformly. Hence one may obtain knowledge about the physics experienced in a laboratory at the surface of the Earth by studying corresponding phenomena in a uniformly accelerated reference frame. Flat spacetime as experienced in such a frame is called Rindler space. In this space one experiences an acceleration of gravity.

In the present work we have studied modifications of Maxwell stresses in electromagnetic fields due to gravity. In the electric field of a point charge at rest in the Rindler space, the field lines are bent downwards and shaped like a fountain. In this field there is a force density due to the Maxwell stresses, as given in Eq. (3.12). It is inversely proportional to the curvature radius of the field lines.

We have calculated the exact expression for the Maxwellian stress force acting upwards on the inner spherical surface around a point charge. An approximate form of this expression, valid for a small sphere, is given in Eq. (5.14). A related expression is Eq. (5.21) which gives the Maxwell stress force acting upon the plane $x = 0$, representing the horizon plane of the Rindler space. Multiplying this force by the velocity of light we obtain Larmor's formula for the power radiated by an accelerated charge. This does not mean, however, the presence of any electromagnetic radiation in the Rindler space. The situation we have considered has been static.

From a covariant theory of electromagnetic stresses in Rindler space we have formulated Archimedes' law for stationary electromagnetic fields in Rindler space. In this case the buoyancy is due to Maxwell stresses. In a vertical electric (or magnetic) field they are tensions acting upwards on the upper horizontal surface of a volume element in the field and downwards on the bottom part. The tension increases with height, producing a surface force—a buoyancy—equal to the weight of the electric field in the volume element. In a horizontal field the Maxwell stresses produce a vertical pressure which increases with depth just as in a fluid. For an arbitrary electromagnetic field these two mechanisms act together to give a buoyancy on a surface enclosing a region of the field equal to the weight of the enclosed field. Hence we arrive at the Archimedian law for an electromagnetic field. There is a buoyancy in the field equal to the weight of the displaced field.

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