Gravitons enhance fermions during inflation

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We solve the effective Dirac equation for massless fermions during inflation in the simplest gauge, including all one loop corrections from quantum gravity. At late times the result for a spatial plane wave behaves as if the classical solution were subjected to a time-dependent field strength renormalization of $Z_2(t) = 1 - \frac{17}{4\pi} G H^2 \ln(a) + O(G^2)$. We show that this also follows from making the Hartree approximation, although the numerical coefficients differ.

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I. INTRODUCTION

Gravitons and massless, minimally coupled scalars can mediate vastly enhanced quantum effects during inflation because they are simultaneously massless and not conformally invariant [1]. One naturally wonders how interactions with these quanta affect themselves and other particles. The first step in answering this question on the linearized level is to compute the one particle irreducible (1PI) 2-point function for the field whose behavior is in question. This has been done at one loop order for gravitons in pure quantum gravity [2], for photons [3,4] and charged scalars [5] in scalar quantum electrodynamics (SQED), for fermions [6,7] and Yukawa scalars [8] in Yukawa theory, for fermions in Dirac + Einstein [9] and, at two loop order, for scalars in ϕ^4 theory [10]. The next step is using the 1PI 2-point function to correct the linearized equation of motion for the field in question. That is what we shall do here for the fermions of massless Dirac + Einstein.

It is worth reviewing the conventions used in computing the fermion self-energy [9]. We worked on de Sitter background in conformal coordinates,

$$ds^2 = a^2(\eta)(-d\eta^2 + d\vec{x}\cdot d\vec{x}) \quad \text{where } a(\eta) = -\frac{1}{H\eta} = e^{Ht}. \tag{1}$$

We used dimensional regularization and obtained the self-energy for the conformally rescaled fermion field,

$$\Psi(x) \equiv a^{[(D-1)/2]}\psi(x). \tag{2}$$

The local Lorentz gauge was fixed to allow an algebraic expression for the vierbein in terms of the metric [11]. The general coordinate gauge was fixed to make the tensor structure of the graviton propagator decouple from its spacetime dependence [12,13]. The result we obtained is

$$[\Sigma^{\rm ren}](x;x') = \frac{i\kappa^2 H^2}{2^6\pi^2}\left\{\frac{\ln(aa')}{H^2 aa'}\not{\partial}\partial^2 + \frac{15}{2}\ln(aa')\not{\partial} - 7\ln(aa')\bar{\not{\partial}}\right\}\delta^4(x-x') + \frac{\kappa^2}{2^8\pi^4}(aa')^{-1}\not{\partial}\partial^4\left[\frac{\ln(\mu^2\Delta x^2)}{\Delta x^2}\right] + \frac{\kappa^2 H^2}{2^8\pi^4}\left\{\left(\frac{15}{2}\not{\partial}\partial^2 - \bar{\not{\partial}}\partial^2\right)\left[\frac{\ln(\mu^2\Delta x^2)}{\Delta x^2}\right] + (-8\bar{\not{\partial}}\partial^2 + 4\not{\partial}\nabla^2)\left[\frac{\ln(\frac{1}{4}H^2\Delta x^2)}{\Delta x^2}\right] + 7\not{\partial}\nabla^2\left[\frac{1}{\Delta x^2}\right]\right\} + O(\kappa^4), \tag{3}$$

where $\kappa^2 \equiv 16\pi G$ is the loop counting parameter of quantum gravity. The various differential and spinor-differential operators are

$$\partial^2 \equiv \eta^{\mu\nu}\partial_\mu\partial_\nu, \qquad \nabla^2 \equiv \partial_i\partial_i, \qquad \not{\partial} \equiv \gamma^\mu\partial_\mu \quad \text{and} \quad \bar{\not{\partial}} \equiv \gamma^i\partial_i, \tag{4}$$

where $\eta^{\mu\nu}$ is the Lorentz metric and γ^μ are the gamma matrices. The conformal coordinate interval is basically $\Delta x^2 \equiv (x-x')^\mu(x-x')^\nu\eta_{\mu\nu}$, up to a subtlety about the imaginary part which will be explained shortly.

The linearized, effective Dirac equation we will solve is

$$i\not{\partial}_{ij}\Psi_j(x) - \int d^4x'[{}_i\Sigma_j](x;x')\Psi_j(x') = 0. \tag{5}$$

In judging the validity of this exercise it is important to answer five questions:

(1) What is the relation between the $\mathbb{C}$-number, effective field Eq. (5) and the Heisenberg operator equations of Dirac + Einstein?
(2) How do solutions to (5) change when different gauges are used?
(3) How do solutions to (5) depend upon the finite parts of counterterms?
(4) What is the imaginary part of Δx^2?
(5) What can we do without the higher loop contributions to the fermion self-energy?

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Issues (1) and (2) are closely related, and require a lengthy digression that we have consigned to Sec. II of this paper. In this Introduction we will comment on issues (3)–(5).

Dirac + Einstein is not perturbatively renormalizable [14], so we could only obtain a finite result by absorbing divergences in the sense of Bogoliubov, Parasiuk, Hepp and Zimmermann (BPHZ) [15–18] using three higher derivative counterterms,

$$-\kappa^2 H^2 \left\{ \frac{\alpha_1}{H^2 a a'} \not{\partial} \partial^2 + \alpha_2 D(D-1) \not{\partial} + \alpha_3 \bar{\not{\partial}} \right\} \delta^D(x - x'). \tag{6}$$

No physical principle seems to fix the finite parts of these counterterms so any result which derives from their values is arbitrary. We chose to null local terms at the beginning of inflation ($a = 1$), but any other choice could have been made and would have affected the solution to (5). Hence, there is no point in solving the equation exactly. However, each of the three counterterms is related to a term in (3) which carries a factor of $\ln(aa')$,

$$\frac{\alpha_1}{H^2 a a'} \not{\partial} \partial^2 \Leftrightarrow \frac{\ln(aa')}{H^2 a a'} \not{\partial} \partial^2, \tag{7}$$

$$\alpha_2 D(D-1) \not{\partial} \Leftrightarrow \frac{15}{2} \ln(aa') \not{\partial}, \tag{8}$$

$$\alpha_3 \bar{\not{\partial}} \Leftrightarrow -7 \ln(aa') \bar{\not{\partial}}. \tag{9}$$

Unlike the α_i's, the numerical coefficients of the right-hand terms are uniquely fixed and completely independent of renormalization. The factors of $\ln(aa')$ on these right-hand terms mean that they dominate over any finite change in the α_i's at late times. It is in this late time regime that we can make reliable predictions about the effect of quantum gravitational corrections.

The analysis we have just made is a standard feature of low energy effective field theory, and has many distinguished antecedents [19–33]. Loops of massless particles make finite, nonanalytic contributions which cannot be changed by counterterms and which dominate the far infrared. Further, these effects must occur as well, with precisely the same numerical values, in whatever fundamental theory ultimately resolves the ultraviolet problems of quantum gravity.

We must also clarify what is meant by the conformal coordinate interval $\Delta x^2(x; x')$ which appears in (3). The in-out effective field equations correspond to the replacement,

$$\Delta x^2(x;x') \to \Delta x^2_{++}(x;x') \equiv \|\vec{x} - \vec{x}'\|^2 - (|\eta - \eta'| - i\delta)^2. \tag{10}$$

These equations govern the evolution of quantum fields under the assumption that the universe begins in free vacuum at asymptotically early times and ends up the same way at asymptotically late times. This is valid for scattering in flat space but not for cosmological settings in which particle production prevents the in vacuum from evolving to the out vacuum. Persisting with the in-out effective field equations would result in quantum correction terms which are dominated by events from the infinite future. This is the correct answer to the question being asked, which is, "what must the field be in order to make the universe to evolve from in vacuum to out vacuum?" However, that question is not very relevant to any observation we can make.

A more realistic question is, "what happens when the universe is released from a prepared state at some finite time and allowed to evolve as it will?" This sort of question can be answered using the Schwinger-Keldysh formalism [34–40]. For a recent derivation in the position-space formalism we are using, see [41]. We confine ourselves here to noting four simple rules:

(i) The end points of lines in the Schwinger-Keldysh formalism carry a $\pm$ polarity, so every n-point 1PI function of the in-out formalism gives rise to 2^n 1PI functions in the Schwinger-Keldysh formalism.

(ii) The linearized effective Dirac equation of the Schwinger-Keldysh formalism takes the form (5) with the replacement,

$$[{}_i\Sigma_j](x;x') \to [{}_i\Sigma_j]_{++}(x;x') + [{}_i\Sigma_j]_{+-}(x;x'). \tag{11}$$

(iii) The $_{++}$ fermion self-energy is (3) with the replacement (10).

(iv) The $_{+-}$ fermion self-energy is,

$$\begin{aligned} &-\frac{\kappa^2}{2^8 \pi^4 a a'} \not{\partial} \partial^4 \left[\frac{\ln(\mu^2 \Delta x^2)}{\Delta x^2} \right] \\ &\quad - \frac{\kappa^2 H^2}{2^8 \pi^4} \left\{ \left(\frac{15}{2} \not{\partial} \partial^2 - \bar{\not{\partial}} \partial^2 \right) \left[\frac{\ln(\mu^2 \Delta x^2)}{\Delta x^2} \right] \right. \\ &\quad + (-8 \bar{\not{\partial}} \partial^2 + 4 \not{\partial} \nabla^2) \left[\frac{\ln(\frac{1}{4} H^2 \Delta x^2)}{\Delta x^2} \right] \\ &\quad \left. + 7 \not{\partial} \nabla^2 \left[\frac{1}{\Delta x^2} \right] \right\} + O(\kappa^4), \end{aligned} \tag{12}$$

with the replacement,

$$\begin{aligned} \Delta x^2(x;x') &\to \Delta x^2_{+-}(x;x') \\ &\equiv \|\vec{x} - \vec{x}'\|^2 - (\eta - \eta' + i\delta)^2. \end{aligned} \tag{13}$$

The difference of the $_{++}$ and $_{+-}$ terms leads to zero contribution in (5) unless the point x'^μ lies on or within the past light-cone of x^μ.

We can only solve for the one loop corrections to the field because we lack the higher loop contributions to the self-energy. The general perturbative expansion takes the form

$$\Psi(x) = \sum_{\ell=0}^{\infty} \kappa^{2\ell} \Psi^\ell(x) \quad \text{and}$$

$$[\Sigma](x;x') = \sum_{\ell=1}^{\infty} \kappa^{2\ell} [\Sigma^\ell](x;x'). \tag{14}$$

One substitutes these expansions into the effective Dirac equation (5) and then segregates powers of κ^2,

$$i\not{\partial}\Psi^0(x) = 0,$$

$$i\not{\partial}\Psi^1(x) = \int d^4x'[\Sigma^1](x;x')\Psi^0(x') \qquad \text{et cetera.} \tag{15}$$

We shall work out the late time limit of the one loop correction $\Psi_i^1(\eta,\vec{x};\vec{k},s)$ for a spatial plane wave of helicity s,

$$\Psi_i^0(\eta,\vec{x};\vec{k},s) = \frac{e^{-ik\eta}}{\sqrt{2k}}u_i(\vec{k},s)e^{i\vec{k}\cdot\vec{x}}$$
$$\text{where } k^\ell\gamma_{ij}^\ell u_j(\vec{k},s) = k\gamma_{ij}^0 u_j(\vec{k},s). \tag{16}$$

In the next section we derive the effective field equation. In Sec. III we derive some key simplifications. In Sec. IV we solve for the late time limit of $\Psi_i^1(\eta,\vec{x};\vec{k},s)$. The result takes the surprising form of a time-dependent field strength renormalization of the tree order solution. In Sec. V we show that this can be understood qualitatively using mean-field theory. Our results are summarized and discussed in Sec. VI.

II. THE EFFECTIVE FIELD EQUATIONS

The purpose of this section is to elucidate the relation between the Heisenberg operators of Dirac + Einstein—$\bar{\psi}_i(x)$, $\psi_i(x)$ and $h_{\mu\nu}(x)$—and the $\mathbb{C}$-number plane wave mode solutions $\Psi_i(x;\vec{k},s)$ of the linearized, effective Dirac equation (5). After explaining the relation we work out an example, at one loop order, in a simple scalar analogue model. Finally, we return to Dirac + Einstein to explain how $\Psi_i(x;\vec{k},s)$ changes with variations of the gauge.

A. Heisenberg operators and effective field equations

The invariant Lagrangian of Dirac + Einstein in D spacetime dimensions is

$$\mathcal{L} = \frac{1}{16\pi G}(R-(D-1)(D-2)H^2)\sqrt{-g} + \bar{\psi}e^\mu{}_b\gamma^b\Big(i\partial_\mu - \frac{1}{2}A_{\mu cd}J^{cd}\Big)\psi\sqrt{-g}. \tag{17}$$

Here $e_{\mu b}$ is the vierbein field and $g_{\mu\nu} \equiv e_{\mu b}e_{\nu c}\eta^{bc}$ is the metric. The metric and vierbein-compatible connections are

$$\Gamma^\rho{}_{\mu\nu} \equiv \tfrac{1}{2}g^{\rho\sigma}(g_{\sigma\mu,\nu} + g_{\nu\sigma,\mu} - g_{\mu\nu,\sigma})$$
$$\text{and} \quad A_{\mu cd} \equiv e^\nu{}_c(e_{\nu d,\mu} - \Gamma^\rho{}_{\mu\nu}e_{\rho d}). \tag{18}$$

The Ricci scalar is

$$R \equiv g^{\mu\nu}(\Gamma^\rho{}_{\nu\mu,\rho} - \Gamma^\rho{}_{\rho\mu,\nu} + \Gamma^\rho{}_{\rho\sigma}\Gamma^\sigma{}_{\nu\mu} - \Gamma^\rho{}_{\nu\sigma}\Gamma^\sigma{}_{\rho\mu}). \tag{19}$$

The gamma matrices γ_{ij}^b have spinor indices $i, j \in \{1,2,3,4\}$ and obey the usual anticommutation relations,

$$\{\gamma^b,\gamma^c\} = -2\eta^{bc}I. \tag{20}$$

The Lorentz generators of the bispinor representation are

$$J^{bc} \equiv \frac{i}{4}[\gamma^b,\gamma^c]. \tag{21}$$

We employ the Lorentz symmetric gauge, $e_{\mu b} = e_{b\mu}$, which permits one to perturbatively determine the vierbein in terms of the metric and their respective backgrounds (denoted with overlines) [11],

$$e_{\mu b}[g] = (\sqrt{g\bar{g}_0^{-1}})_\mu{}^\nu\bar{e}_{\nu b}. \tag{22}$$

We define the graviton field $h_{\mu\nu}$ in de Sitter conformal coordinates as follows:

$$g_{\mu\nu}(x) \equiv a^2(\eta_{\mu\nu} + \kappa h_{\mu\nu}(x)) \quad \text{where } a = -\frac{1}{H\eta}. \tag{23}$$

By convention the indices of $h_{\mu\nu}$ are raised and lowered with the Lorentz metric. We fix the general coordinate freedom by adding the gauge fixing term,

$$\mathcal{L}_{\rm GF} = -\tfrac{1}{2}a^{D-2}\eta^{\mu\nu}F_\mu F_\nu$$
$$\text{where } F_\mu = \eta^{\rho\sigma}(h_{\mu\rho,\sigma} - \tfrac{1}{2}h_{\rho\sigma,\mu} + (D-2)Hah_{\mu\rho}\delta_\sigma^0). \tag{24}$$

One solves the gauge-fixed Heisenberg operator equations perturbatively,

$$h_{\mu\nu}(x) = h_{\mu\nu}^0(x) + \kappa h_{\mu\nu}^1(x) + \kappa^2 h_{\mu\nu}^2(x) + \cdots, \tag{25}$$

$$\psi_i(x) = \psi_i^0(x) + \kappa\psi_i^1(x) + \kappa^2\psi_i^2(x) + \cdots. \tag{26}$$

Because our state is released in free vacuum at $t = 0$ ($\eta = -1/H$), it makes sense to express the operator as a functional of the creation and annihilation operators of this free state. So our initial conditions are that $h_{\mu\nu}$ and its first time derivative coincide with those of $h_{\mu\nu}^0(x)$ at $t = 0$, and also that $\psi_i(x)$ coincides with $\psi_i^0(x)$. The zeroth order solutions to the Heisenberg field equations take the form

$$h_{\mu\nu}^0(x) = \int\frac{d^{D-1}k}{(2\pi)^{D-1}}\sum_\lambda\{\epsilon_{\mu\nu}(\eta;\vec{k},\lambda)e^{i\vec{k}\cdot\vec{x}}\alpha(\vec{k},\lambda) + \epsilon_{\mu\nu}^*(\eta;\vec{k},\lambda)e^{-i\vec{k}\cdot\vec{x}}\alpha^\dagger(\vec{k},\lambda)\}, \tag{27}$$

$$\psi_i^0(x) = a^{-[(D-1)/2]}\int\frac{d^{D-1}k}{(2\pi)^{D-1}}\sum_s\Big\{\frac{e^{-ik\eta}}{\sqrt{2k}}u_i(\vec{k},s)e^{i\vec{k}\cdot\vec{x}}b(\vec{k},s) + \frac{e^{ik\eta}}{\sqrt{2k}}v_i(\vec{k},\lambda)e^{-i\vec{k}\cdot\vec{x}}c^\dagger(\vec{k},s)\Big\}. \tag{28}$$

The graviton mode functions are proportional to Hankel functions whose precise specification we do not require. The Dirac mode functions $u_i(\vec{k},s)$ and $v_i(\vec{k},s)$ are precisely those of flat space by virtue of the conformal invariance of massless fermions. The canonically normalized creation and annihilation operators obey

$$[\alpha(\vec{k},\lambda),\alpha^\dagger(\vec{k}',\lambda')] = \delta_{\lambda\lambda'}(2\pi)^{D-1}\delta^{D-1}(\vec{k}-\vec{k}'), \quad (29)$$

$$\{b(\vec{k},s),b^\dagger(\vec{k}',s')\} = \delta_{ss'}(2\pi)^{D-1}\delta^{D-1}(\vec{k}-\vec{k}') = \{c(\vec{k},s),c^\dagger(\vec{k}',s')\}. \quad (30)$$

The zeroth order Fermi field $\psi_i^0(x)$ is an anticommuting operator whereas the mode function $\Psi^0(x;\vec{k},s)$ is a $\mathbb{C}$-number. The latter can be obtained from the former by anticommuting with the fermion creation operator,

$$\Psi_i^0(x;\vec{k},s) = a^{(D-1)/2}\{\psi_i^0(x),b^\dagger(\vec{k},s)\} = \frac{e^{-ik\eta}}{\sqrt{2k}}u_i(\vec{k},s)e^{i\vec{k}\cdot\vec{x}}. \quad (31)$$

The higher order contributions to $\psi_i(x)$ are no longer linear in the creation and annihilation operators, so anticommuting the full solution $\psi_i(x)$ with $b^\dagger(\vec{k},s)$ produces an operator. The quantum-corrected fermion mode function we obtain by solving (5) is the expectation value of this operator in the presence of the state which is free vacuum at $t=0$,

$$\Psi_i(x;\vec{k},s) = a^{(D-1)/2}\langle\Omega|\{\psi_i(x),b^\dagger(\vec{k},s)\}|\Omega\rangle. \quad (32)$$

This is what the Schwinger-Keldysh field equations give. The more familiar, in-out effective field equations obey a similar relation except that one defines the free fields to agree with the full ones in the asymptotic past, and one takes the in-out matrix element after anticommuting.

B. A worked-out example

It is perhaps worth seeing a worked-out example, at one loop order, of the relation (32) between the Heisenberg operators and the Schwinger-Keldysh field equations. To simplify the analysis we will work with a model of two scalars in flat space,

$$\mathcal{L} = -\partial_\mu\varphi^*\partial^\mu\varphi - m^2\varphi^*\varphi - \lambda\chi{:}\varphi^*\varphi{:} - \tfrac{1}{2}\partial_\mu\chi\partial^\mu\chi. \quad (33)$$

In this model φ plays the role of our fermion ψ_i, and χ plays the role of the graviton $h_{\mu\nu}$. Note that we have normal-ordered the interaction term to avoid the harmless but time-consuming digression that would be required to deal with χ developing a nonzero expectation value. We shall also omit discussion of counterterms.

The Heisenberg field equations for (33) are

$$\partial^2\chi - \lambda{:}\varphi^*\varphi{:} = 0, \quad (34)$$

$$(\partial^2 - m^2)\varphi - \lambda\chi\varphi = 0. \quad (35)$$

As with Dirac + Einstein, we solve these equations perturbatively,

$$\chi(x) = \chi^0(x) + \lambda\chi^1(x) + \lambda^2\chi^2(x) + \cdots, \quad (36)$$

$$\varphi(x) = \varphi^0(x) + \lambda\varphi^1(x) + \lambda^2\varphi^2(x) + \cdots. \quad (37)$$

The zeroth order solutions are

$$\chi^0(x) = \int\frac{d^{D-1}k}{(2\pi)^{D-1}}\left\{\frac{e^{-ikt}}{\sqrt{2k}}e^{i\vec{k}\cdot\vec{x}}\alpha(\vec{k}) + \frac{e^{ikt}}{\sqrt{2k}}e^{-i\vec{k}\cdot\vec{x}}\alpha^\dagger(\vec{k})\right\}, \quad (38)$$

$$\varphi^0(x) = \int\frac{d^{D-1}k}{(2\pi)^{D-1}}\left\{\frac{e^{-i\omega t}}{\sqrt{2\omega}}e^{i\vec{k}\cdot\vec{x}}b(\vec{k}) + \frac{e^{i\omega t}}{\sqrt{2\omega}}e^{-i\vec{k}\cdot\vec{x}}c^\dagger(\vec{k})\right\}. \quad (39)$$

Here $k \equiv \|\vec{k}\|$ and $\omega \equiv \sqrt{k^2+m^2}$. The creation and annihilation operators are canonically normalized,

$$[\alpha(\vec{k}),\alpha^\dagger(\vec{k}')] = [b(\vec{k}),b^\dagger(\vec{k}')] = [c(\vec{k}),c^\dagger(\vec{k}')] = (2\pi)^{D-1}\delta^{D-1}(\vec{k}-\vec{k}'). \quad (40)$$

We choose to develop perturbation theory so that all the operators and their first time derivatives agree with the zeroth order solutions at $t=0$. The first few higher order terms are

$$\chi^1(x) = \int_0^t dt'\int d^{D-1}x'\left\langle x\left|\frac{1}{\partial^2}\right|x'\right\rangle_{\rm ret}{:}\varphi^{0*}(x')\varphi^0(x'){:}, \quad (41)$$

$$\varphi^1(x) = \int_0^t dt'\int d^{D-1}x'\left\langle x\left|\frac{1}{\partial^2-m^2}\right|x'\right\rangle_{\rm ret}\times\chi^0(x')\varphi^0(x'), \quad (42)$$

$$\varphi^2(x) = \int_0^t dt'\int d^{D-1}x'\left\langle x\left|\frac{1}{\partial^2-m^2}\right|x'\right\rangle_{\rm ret}\times\{\chi^1(x')\varphi^0(x') + \chi^0(x')\varphi^1(x')\}. \quad (43)$$

The commutator of $\varphi^0(x)$ with $b^\dagger(\vec{k})$ is a $\mathbb{C}$-number,

$$[\varphi^0(x),b^\dagger(\vec{k})] = \frac{e^{-i\omega t}}{\sqrt{2\omega}}e^{i\vec{k}\cdot\vec{x}} \equiv \Phi^0(x;\vec{k}). \quad (44)$$

However, commuting the full solution with $b^\dagger(\vec{k})$ leaves operators,

$$[\varphi(x),b^\dagger(\vec{k})] = \Phi^0(x;\vec{k}) + \lambda\int_0^t dt'\int d^{D-1}x'\left\langle x\left|\frac{1}{\partial^2-m^2}\right|x'\right\rangle_{\rm ret}\chi^0(x')\Phi^0(x';\vec{k}) + \lambda^2\int_0^t dt'\int d^{D-1}x'\left\langle x\left|\frac{1}{\partial^2-m^2}\right|x'\right\rangle_{\rm ret}\times\{[\chi^1(x'),b^\dagger(\vec{k})]\varphi^0(x') + \chi^1(x')\Phi^0(x';\vec{k}) + \chi^0(x')[\varphi^1(x'),b^\dagger(\vec{k})]\} + O(\lambda^3). \quad (45)$$

The commutators in (45) are easily evaluated,

$$[\chi^1(x'), b^\dagger(\vec{k})]\varphi^0(x') = \int_0^{t'} dt'' \int d^{D-1}x'' \Big\langle x' \Big| \frac{1}{\partial^2} \Big| x'' \Big\rangle_{\rm ret} \varphi^{0*}(x'')\varphi^0(x')\Phi^0(x''; \vec{k}), \tag{46}$$

$$\chi^0(x')[\varphi^1(x'), b^\dagger(\vec{k})] = \int_0^{t'} dt'' \int d^{D-1}x'' \Big\langle x' \Big| \frac{1}{\partial^2 - m^2} \Big| x'' \Big\rangle_{\rm ret} \chi^0(x')\chi^0(x'')\Phi^0(x''; \vec{k}). \tag{47}$$

Hence, the expectation value of (45) gives

$$\langle\Omega|[\varphi(x), b^\dagger(\vec{k})]|\Omega\rangle = \Phi^0(x;\vec{k}) + \lambda^2 \int_0^t dt' \int d^{D-1}x' \Big\langle x \Big| \frac{1}{\partial^2 - m^2} \Big| x' \Big\rangle_{\rm ret} \int_0^{t'} dt'' \int d^{D-1}x'' \Big\{ \Big\langle x' \Big| \frac{1}{\partial^2} \Big| x'' \Big\rangle_{\rm ret} \times \langle\Omega|\varphi^{0*}(x'')\varphi^0(x')|\Omega\rangle + \Big\langle x' \Big| \frac{1}{\partial^2 - m^2} \Big| x'' \Big\rangle_{\rm ret} \langle\Omega|\chi^0(x')\chi^0(x'')|\Omega\rangle \Big\} \Phi^0(x''; \vec{k}) + O(\lambda^4). \tag{48}$$

To make contact with the effective field equations we must first recognize that the retarded Green's functions can be written in terms of expectation values of the free fields,

$$\Big\langle x' \Big| \frac{1}{\partial^2} \Big| x'' \Big\rangle_{\rm ret} = -i\theta(t' - t'')[\chi^0(x'), \chi^0(x'')] \tag{49}$$

$$= -i\theta(t' - t'')\{\langle\Omega|\chi^0(x')\chi^0(x'')|\Omega\rangle - \langle\Omega|\chi^0(x'')\chi^0(x')|\Omega\rangle\}, \tag{50}$$

$$\Big\langle x' \Big| \frac{1}{\partial^2 - m^2} \Big| x'' \Big\rangle_{\rm ret} = -i\theta(t' - t'')[\varphi^0(x'), \varphi^{0*}(x'')] \tag{51}$$

$$= -i\theta(t' - t'')\{\langle\Omega|\varphi^0(x')\varphi^{0*}(x'')|\Omega\rangle - \langle\Omega|\varphi^{*0}(x'')\varphi^0(x')|\Omega\rangle\}. \tag{52}$$

Substituting these relations into (48) and canceling some terms gives the expression we have been seeking:

$$\langle\Omega|[\varphi(x), b^\dagger(\vec{k})]|\Omega\rangle = \Phi^0(x;\vec{k}) - i\lambda^2 \int_0^t dt' \int d^{D-1}x' \Big\langle x \Big| \frac{1}{\partial^2 - m^2} \Big| x' \Big\rangle_{\rm ret} \int_0^{t'} dt'' \int d^{D-1}x'' \{\langle\Omega|\chi^0(x')\chi^0(x'')|\Omega\rangle \times \langle\Omega|\varphi^0(x')\varphi^{0*}(x'')|\Omega\rangle - \langle\Omega|\chi^0(x'')\chi^0(x')|\Omega\rangle\langle\Omega|\varphi^{0*}(x'')\varphi^0(x')|\Omega\rangle\}\Phi^0(x''; \vec{k}) + O(\lambda^4). \tag{53}$$

We turn now to the effective field equations of the Schwinger-Keldysh formalism. The $\mathbb{C}$-number field corresponding to $\varphi(x)$ at linearized order is $\Phi(x)$. If the state is released at $t = 0$ then the equation $\Phi(x)$ obeys is

$$(\partial^2 - m^2)\Phi(x) - \int_0^t dt' \int d^{D-1}x' \{M^2_{++}(x;x') + M^2_{+-}(x;x')\}\Phi(x') = 0. \tag{54}$$

The one loop diagram for the self-mass-squared of φ is depicted in Fig. 1.

Because the self-mass-squared has two external lines, there are $2^2 = 4$ polarities in the Schwinger-Keldysh formalism. The two we require are [8,41]

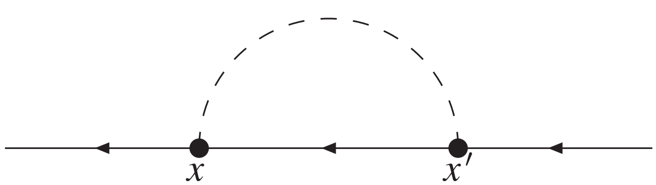

FIG. 1. Self-mass-squared for φ at one loop order. Solid lines stand for φ propagators while dashed lines represent χ propagators.

$$-iM^2_{++}(x;x') = (-i\lambda)^2 \Big\langle x \Big| \frac{i}{\partial^2} \Big| x' \Big\rangle_{++} \Big\langle x \Big| \frac{i}{\partial^2 - m^2} \Big| x' \Big\rangle_{++} + O(\lambda^4), \tag{55}$$

$$-iM^2_{+-}(x;x') = (-i\lambda)(+i\lambda) \Big\langle x \Big| \frac{i}{\partial^2} \Big| x' \Big\rangle_{+-} \times \Big\langle x \Big| \frac{i}{\partial^2 - m^2} \Big| x' \Big\rangle_{+-} + O(\lambda^4). \tag{56}$$

To recover (53) we must express the various Schwinger-Keldysh propagators in terms of expectation values of the free fields. The $_{++}$ polarity gives the usual Feynman propagator [41],

$$\Big\langle x \Big| \frac{i}{\partial^2} \Big| x' \Big\rangle_{++} = \theta(t - t')\langle\Omega|\chi^0(x)\chi^0(x')|\Omega\rangle + \theta(t' - t)\langle\Omega|\chi^0(x')\chi^0(x)|\Omega\rangle, \tag{57}$$

$$\Big\langle x \Big| \frac{i}{\partial^2 - m^2} \Big| x' \Big\rangle_{++} = \theta(t - t')\langle\Omega|\varphi^0(x)\varphi^{0*}(x')|\Omega\rangle + \theta(t' - t)\langle\Omega|\varphi^{0*}(x')\varphi^0(x)|\Omega\rangle. \tag{58}$$

The $_{+-}$ polarity propagators are [41]

$$\left\langle x \left| \frac{i}{\partial^2} \right| x' \right\rangle_{+-} = \langle \Omega | \chi^0(x') \chi^0(x) | \Omega \rangle, \tag{59}$$

$$\left\langle x \left| \frac{i}{\partial^2 - m^2} \right| x' \right\rangle_{+-} = \langle \Omega | \varphi^{0*}(x') \varphi^0(x) | \Omega \rangle. \tag{60}$$

Substituting these relations into (55) and (56) and making use of the identity $1 = \theta(t - t') + \theta(t' - t)$ gives

$$\begin{aligned} M^2_{++}(x;x') + M^2_{+-}(x;x') = {} & -i\lambda^2 \theta(t - t')\{\langle \Omega | \chi^0(x) \chi^0(x') | \Omega \rangle \\ & \times \langle \Omega | \varphi^0(x) \varphi^{0*}(x') | \Omega \rangle \\ & - \langle \Omega | \chi^0(x') \chi^0(x) | \Omega \rangle \\ & \times \langle \Omega | \varphi^{0*}(x') \varphi^0(x) | \Omega \rangle\} + O(\lambda^4). \end{aligned} \tag{61}$$

We now solve (54) perturbatively. The free plane wave mode function (44) is of course a solution at order λ^0. With (61) we easily recognize its perturbative development as

$$\begin{aligned} \Phi(x;\vec{k}) = {} & \Phi^0(x;\vec{k}) - i\lambda^2 \int_0^t dt' \int d^{D-1}x' \left\langle x \left| \frac{1}{\partial^2 - m^2} \right| x' \right\rangle_{\rm ret} \\ & \times \int_0^{t'} dt'' \int d^{D-1}x'' \{\langle \Omega | \chi^0(x') \chi^0(x'') | \Omega \rangle \\ & \times \langle \Omega | \varphi^0(x') \varphi^{0*}(x'') | \Omega \rangle - \langle \Omega | \chi^0(x'') \chi^0(x') | \Omega \rangle \\ & \times \langle \Omega | \varphi^{0*}(x'') \varphi^0(x') | \Omega \rangle\} \Phi^0(x'';\vec{k}) + O(\lambda^4). \end{aligned} \tag{62}$$

That agrees with (53), so we have established the desired connection,

$$\Phi(x;\vec{k}) = \langle \Omega | [\varphi(x), b^\dagger(\vec{k})] | \Omega \rangle, \tag{63}$$

at one loop order.

C. The gauge issue

The preceding discussion has made clear that we are working in a particular local Lorentz and general coordinate gauge. We are also doing perturbation theory. The function $\Psi^0_i(x; \vec{k}, s)$ describes how a free fermion of wave number $\vec{k}$ and helicity s propagates through classical de Sitter background in our gauge. What $\Psi^1_i(x; \vec{k}, s)$ gives is the first quantum correction to this mode function. It is natural to wonder how the effective field $\Psi_i(x; \vec{k}, s)$ changes if a different gauge is used.

The operators of the original, invariant Lagrangian transform as follows under diffeomorphisms ($x^\mu \longrightarrow x'^\mu$) and local Lorentz rotations (Λ_{ij}):[1]

$$\psi'_i(x) = \Lambda_{ij}(x'^{-1}(x)) \psi_j(x'^{-1}(x)), \tag{64}$$

$$e'_{\mu b}(x) = \frac{\partial x^\nu}{\partial x'^\mu} \Lambda_b{}^c(x'^{-1}(x)) e_{\nu c}(x'^{-1}(x)). \tag{65}$$

The invariance of the theory guarantees that the transformation of any solution is also a solution. Hence, the possibility of performing local transformations precludes the existence of a unique initial value solution. This is why no Hamiltonian formalism is possible until the gauge has been fixed sufficiently to eliminate transformations which leave the initial value surface unaffected.

Different gauges can be reached using field-dependent gauge transformations [42]. This has a relatively simple effect upon the Heisenberg operator $\psi_i(x)$, but a complicated one on the linearized effective field $\Psi_i(x; \vec{k}, s)$. Because local Lorentz and diffeomorphism gauge conditions are typically specified in terms of the gravitational fields, we assume x'^μ and Λ_{ij} depend upon the graviton field $h_{\mu\nu}$. Hence so too does the transformed field,

$$\psi'_i[h](x) = \Lambda_{ij}[h](x'^{-1}[h](x)) \psi_j(x'^{-1}[h](x)). \tag{66}$$

In the general case that the gauge changes even on the initial value surface, the creation and annihilation operators also transform,

$$b'[h](\vec{k}, s) = \frac{1}{\sqrt{2k}} u^*_i(\vec{k}, s) \int d^{D-1}x e^{-i\vec{k}\cdot\vec{x}} \psi'_i[h](\eta_i, \vec{x}), \tag{67}$$

where $\eta_i \equiv -1/H$ is the initial conformal time. Hence, the linearized effective field transforms to

$$\Psi'_i(x; \vec{k}, s) = a^{(D-1)/2} \langle \Omega | \{\psi'_i[h](x), b'^\dagger[h](\vec{k}, s)\} | \Omega \rangle. \tag{68}$$

This is quite a complicated relation. Note, in particular, that the $h_{\mu\nu}$ dependence of $x'^\mu[h]$ and $\Lambda_{ij}[h]$ means that $\Psi'_i(x; \vec{k}, s)$ is not simply a Lorentz transformation of the original function $\Psi_i(x; \vec{k}, s)$ evaluated at some transformed point.

III. SOME KEY REDUCTIONS

The purpose of this section is to derive three results that are used repeatedly in reducing the nonlocal contributions to the effective field equations. We observe that the nonlocal terms of (3) contain $1/\Delta x^2$. We can avoid denominators by extracting another derivative,

$$\frac{1}{\Delta x^2} = \frac{\partial^2}{4} \ln(\Delta x^2) \quad \text{and}$$
$$\frac{\ln(\Delta x^2)}{\Delta x^2} = \frac{\partial^2}{8}[\ln^2(\Delta x^2) - 2\ln(\Delta x^2)]. \tag{69}$$

The Schwinger-Keldysh field equations involve the difference of $_{++}$ and $_{+-}$ terms, for example,

[1]Of course the spinor and vector representations of the local Lorentz transformation are related as usual, with the same parameters $\omega_{cd}(x)$ contracted into the appropriate representation matrices,

$$\Lambda_{ij} \equiv \delta_{ij} - \frac{i}{2}\omega_{cd} J^{cd}{}_{ij} + \cdots \quad \text{and} \quad \Lambda_b{}^c = \delta_b{}^c - \omega_b{}^c + \cdots.$$

$$\frac{\ln(\mu^2\Delta x_{++}^2)}{\Delta x_{++}^2} - \frac{\ln(\mu^2\Delta x_{+-}^2)}{\Delta x_{+-}^2} = \frac{\partial^2}{8}\{\ln^2(\mu^2\Delta x_{++}^2) - 2\ln(\mu^2\Delta x_{++}^2) - \ln^2(\mu^2\Delta x_{+-}^2) + 2\ln(\mu^2\Delta x_{+-}^2)\}. \tag{70}$$

We now define the coordinate intervals $\Delta\eta \equiv \eta - \eta'$ and $\Delta x \equiv \|\vec{x} - \vec{x}'\|$ in terms of which the $_{++}$ and $_{+-}$ intervals are

$$\Delta x_{++}^2 = \Delta x^2 - (|\Delta\eta| - i\delta)^2 \quad \text{and} \quad \Delta x_{+-}^2 = \Delta x^2 - (\Delta\eta + i\delta)^2. \tag{71}$$

When $\eta' > \eta$ we have $\Delta x_{++}^2 = \Delta x_{+-}^2$, so the $_{++}$ and $_{+-}$ terms in (70) cancel. This means there is no contribution from the future. When $\eta' < \eta$ and $\Delta x > \Delta\eta$ (past spacelike separation) we can take $\delta = 0$,

$$\ln(\mu^2\Delta x_{++}^2) = \ln[\mu^2(\Delta x^2 - \Delta\eta^2)] = \ln(\mu^2\Delta x_{+-}^2) \quad (\Delta x > \Delta\eta > 0). \tag{72}$$

So the $_{++}$ and $_{+-}$ terms again cancel. Only for $\eta' < \eta$ and $\Delta x < \Delta\eta$ (past timelike separation) are the two logarithms different,

$$\ln(\mu^2\Delta x_{+\pm}^2) = \ln[\mu^2(\Delta\eta^2 - \Delta x^2)] \pm i\pi \quad (\Delta\eta > \Delta x > 0). \tag{73}$$

Hence, Eq. (70) can be written as

$$\frac{\ln(\mu^2\Delta x_{++}^2)}{\Delta x_{++}^2} - \frac{\ln(\mu^2\Delta x_{+-}^2)}{\Delta x_{+-}^2} = \frac{i\pi}{2}\partial^2\{\theta(\Delta\eta - \Delta x)[\ln(\mu^2(\Delta\eta^2 - \Delta x^2) - 1]\}. \tag{74}$$

This step shows that the Schwinger-Kledysh formalism is causal.

To integrate (74) up against the plane wave mode function (16) we first pull the x^μ derivatives outside the integration, then make the change of variables $\vec{x}' = \vec{x} + \vec{r}$ and perform the angular integrals,

$$\begin{aligned}\int d^4x'\Big\{\frac{\ln(\mu^2\Delta x_{++}^2)}{\Delta x_{++}^2} - \frac{\ln(\mu^2\Delta x_{+-}^2)}{\Delta x_{+-}^2}\Big\}\Psi_i^0(\eta', \vec{x}, \vec{k}, s) &= \frac{i2\pi^2}{k}u_i(\vec{k}, s)\partial^2 e^{i\vec{k}\cdot\vec{x}}\int_{\eta_i}^{\eta} d\eta'\frac{e^{-ik\eta'}}{\sqrt{2k}} \\ &\quad\times\int_0^{\Delta\eta} dr r\sin(kr)\{\ln[\mu^2(\Delta\eta^2 - r^2)] - 1\} \\ &= \frac{i2\pi^2}{k\sqrt{2k}}e^{i\vec{k}\cdot\vec{x}}u_i(\vec{k}, s)[-\partial_0^2 - k^2]\int_{\eta_i}^{\eta} d\eta'\Delta\eta^2 e^{-ik\eta'} \\ &\quad\times\int_0^1 dz z\sin(\alpha z)\Big\{\ln(1 - z^2) + 2\ln\Big(\frac{\mu\alpha}{k}\Big) - 1\Big\}.\end{aligned} \tag{75}$$

Here $\alpha \equiv k\Delta\eta$ and $\eta_i \equiv -1/H$ is the initial conformal time, corresponding to physical time $t = 0$. The integral over z is facilitated by the special function,

$$\begin{aligned}\xi(\alpha) &\equiv \int_0^1 dz z\sin(\alpha z)\ln(1 - z^2) \\ &= \frac{2}{\alpha^2}\sin(\alpha) - \frac{1}{\alpha^2}[\cos(\alpha) + \alpha\sin(\alpha)]\Big[\mathrm{si}(2\alpha) + \frac{\pi}{2}\Big] \\ &\quad + [\sin(\alpha) - \alpha\cos(\alpha)]\Big[\mathrm{ci}(2\alpha) - \gamma - \ln\Big(\frac{\alpha}{2}\Big)\Big].\end{aligned} \tag{76}$$

Here γ is the Euler-Mascheroni constant and the sine and cosine integrals are

$$\mathrm{si}(x) \equiv -\int_x^\infty dt\frac{\sin(t)}{t} = -\frac{\pi}{2} + \int_0^x dt\frac{\sin t}{t}, \tag{77}$$

$$\mathrm{ci}(x) \equiv -\int_x^\infty dt\frac{\cos t}{t} = \gamma + \ln(x) + \int_0^x dt\Big[\frac{\cos(t) - 1}{t}\Big]. \tag{78}$$

After substituting the ξ function and performing the elementary integrals, (75) becomes

$$\begin{aligned}&\int d^4x'\Big\{\frac{\ln(\mu^2\Delta x_{++}^2)}{\Delta x_{++}^2} - \frac{\ln(\mu^2\Delta x_{+-}^2)}{\Delta x_{+-}^2}\Big\}\Psi_i^0(\eta', \vec{x}, \vec{k}, s) \\ &= \frac{i2\pi^2}{k\sqrt{2k}}e^{i\vec{k}\cdot\vec{x}}u_i(\vec{k}, s)(\partial_{k\eta}^2 + 1)\int_{\eta_i}^{\eta} d\eta' e^{-ik\eta'}\Big\{\alpha^2\xi(\alpha) \\ &\quad + \Big[2\ln\Big(\frac{\mu\alpha}{k}\Big) - 1\Big][\sin(\alpha) - \alpha\cos(\alpha)]\Big\}.\end{aligned} \tag{79}$$

One can see that the integrand is of order $\alpha^3\ln(\alpha)$ for small α, which means we can pass the derivatives through the integral. After some rearrangements, the first key identity emerges,

$$\begin{aligned}&\int d^4x'\Big\{\frac{\ln(\mu^2\Delta x_{++}^2)}{\Delta x_{++}^2} - \frac{\ln(\mu^2\Delta x_{+-}^2)}{\Delta x_{+-}^2}\Big\}\Psi^0(\eta', \vec{x}'; \vec{k}, s) \\ &= -i4\pi^2k^{-1}\Psi^0(\eta, \vec{x}; \vec{k}, s) \\ &\quad\times\int_{\eta_i}^{\eta} d\eta' e^{ik\Delta\eta}\Big\{-\cos(k\Delta\eta)\int_0^{2k\Delta\eta} dt\frac{\sin(t)}{t} \\ &\quad + \sin(k\Delta\eta)\Big[\int_0^{2k\Delta\eta} dt\Big(\frac{\cos(t) - 1}{t}\Big) + 2\ln(2\mu\Delta\eta)\Big]\Big\}.\end{aligned} \tag{80}$$

Note that we have written $e^{-ik\eta'} = e^{-ik\eta} \times e^{+ik\Delta\eta}$ and extracted the first phase to reconstruct the full tree order solution $\Psi^0(\eta, \vec{x}; \vec{k}, s) = \frac{e^{-ik\eta}}{\sqrt{2k}} u_i(\vec{k}, s) e^{i\vec{k}\cdot\vec{x}}$.

The second identity derives from acting a d'Alembertian on (80). The d'Alembertian passes through the tree order solution to give

$$\partial^2 \Psi^0(\eta, \vec{x}; \vec{k}, s) = \Psi^0(\eta, \vec{x}; \vec{k}, s)\partial_\eta(\partial_\eta - 2ik). \quad (81)$$

Because the integrand goes like $\alpha \ln(\alpha)$ for small α, we can pass the first derivative through the integral to give

$$\partial^2 \int d^4x' \left\{\frac{\ln(\mu^2\Delta x_{++}^2)}{\Delta x_{++}^2} - \frac{\ln(\mu^2\Delta x_{+-}^2)}{\Delta x_{+-}^2}\right\}\Psi^0(\eta', \vec{x}'; \vec{k}, s) = i4\pi^2 \Psi^0(\eta, \vec{x}; \vec{k}, s)\partial_\eta \int_{\eta_i}^{\eta} d\eta' \left\{\int_0^{2\alpha} dt \left(\frac{e^{it} - 1}{t}\right) + 2\ln\left(\frac{2\mu\alpha}{k}\right)\right\}. \quad (82)$$

We can pass the final derivative through the first integral but, for the second, we must carry out the integration. The result is our second key identity,

$$\partial^2 \int d^4x' \left\{\frac{\ln(\mu^2\Delta x_{++}^2)}{\Delta x_{++}^2} - \frac{\ln(\mu^2\Delta x_{+-}^2)}{\Delta x_{+-}^2}\right\}\Psi^0(\eta', \vec{x}'; \vec{k}, s) = i4\pi^2 \Psi^0(\eta, \vec{x}; \vec{k}, s)\left\{2\ln\left[\frac{2\mu}{H}(1 + H\eta)\right] + \int_{\eta_i}^{\eta} d\eta' \left(\frac{e^{i2k\Delta\eta} - 1}{\Delta\eta}\right)\right\}. \quad (83)$$

The final key identity is derived through the same procedures. Because they should be familiar by now we simply give the result,

$$\int d^4x' \left\{\frac{1}{\Delta x_{++}^2} - \frac{1}{\Delta x_{+-}^2}\right\}\Psi^0(\eta', \vec{x}'; \vec{k}, s) = -i4\pi^2 k^{-1} \Psi^0(\eta, \vec{x}; \vec{k}, s) \int_{\eta_i}^{\eta} d\eta' e^{ik\Delta\eta} \sin(k\Delta\eta). \quad (84)$$

IV. SOLVING THE EFFECTIVE DIRAC EQUATION

In this section we first evaluate the various nonlocal contributions using the three identities of the previous section. Then we evaluate the vastly simpler and, as it turns out, more important, local contributions. Finally, we solve for $\Psi^1(\eta, \vec{x}; \vec{k}, s)$ at late times.

The various nonlocal contributions to (5) take the form

$$\int d^4x' \sum_{I=1}^{5} U_{ij}^I \left\{\frac{\ln(\alpha_I^2\Delta x_{++}^2)}{\Delta x_{++}^2} - \frac{\ln(\alpha_I^2\Delta x_{+-}^2)}{\Delta x_{+-}^2}\right\}\Psi_j^0(\eta', \vec{x}'; \vec{k}, s) + \int d^4x' U_{ij}^6 \left\{\frac{1}{\Delta x_{++}^2} - \frac{1}{\Delta x_{+-}^2}\right\}\Psi_j^0(\eta', \vec{x}'; \vec{k}, s). \quad (85)$$

The spinor-differential operators U_{ij}^I are listed in Table I. The constants α_I are μ for $I = 1, 2, 3$, and $\frac{1}{2}H$ for $I = 4, 5$.

As an example, consider the contribution from U_{ij}^2:

$$\frac{15}{2}\frac{\kappa^2H^2}{2^8\pi^4}\not{\partial}\partial^2 \int d^4x' \left\{\frac{\ln(\mu^2\Delta x_{++}^2)}{\Delta x_{++}^2} - \frac{\ln(\mu^2\Delta x_{+-}^2)}{\Delta x_{+-}^2}\right\}\Psi^0(\eta', \vec{x}'; \vec{k}, s) = \frac{15}{2}\frac{\kappa^2H^2}{2^8\pi^4}\not{\partial} \times i4\pi^2 \Psi^0(\eta, \vec{x}; \vec{k}, s) \times \left\{2\ln\left[\frac{2\mu}{H}(1 + H\eta)\right] + \int_{\eta_i}^{\eta} d\eta' \left(\frac{e^{2ik\Delta\eta} - 1}{\Delta\eta}\right)\right\}, \quad (86)$$

$$= \frac{\kappa^2H^2}{2^6\pi^2} iH\gamma^0 \Psi^0(\eta, \vec{x}; \vec{k}, s) \times \frac{15}{2}\frac{1}{1 + H\eta}\{e^{2i(k/H)(1+H\eta)} + 1\}. \quad (87)$$

In these reductions we have used $i\not{\partial}\Psi^0(\eta, \vec{x}; \vec{k}, s) = i\gamma^0 \Psi^0(\eta, \vec{x}; \vec{k}, s)\partial_\eta$ and (83). Recall from the Introduction that reliable predictions are only possible for late times, which corresponds to $\eta \to 0^-$. We therefore take this limit,

$$\frac{15}{2}\frac{\kappa^2H^2}{2^8\pi^4}\not{\partial}\partial^2 \int d^4x' \left\{\frac{\ln(\mu^2\Delta x_{++}^2)}{\Delta x_{++}^2} - \frac{\ln(\mu^2\Delta x_{+-}^2)}{\Delta x_{+-}^2}\right\}\Psi^0(\eta', \vec{x}'; \vec{k}, s) \to \frac{\kappa^2H^2}{2^6\pi^2} iH\gamma^0 \Psi^0(\eta, \vec{x}; \vec{k}, s) \times \frac{15}{2}\left\{\exp\left(2i\frac{k}{H}\right) + 1\right\}. \quad (88)$$

The other five nonlocal terms have very similar reductions. Each of them also goes to $\frac{\kappa^2H^2}{2^6\pi^2} \times iH\gamma^0\Psi^0(\eta, \vec{x}; \vec{k}, s)$ times a finite constant at late times. We summarize the results in Table II and relegate the details to an appendix.

The next step is to evaluate the local contributions. This is a straightforward exercise in calculus, using only the properties of the tree order solution (16) and the fact that $\partial_\mu a = Ha^2\delta_\mu^0$. The result is

TABLE I. Derivative operators U_{ij}^I: Their common prefactor is $\frac{\kappa^2H^2}{2^8\pi^4}$.

I	U_{ij}^I	I	U_{ij}^I
1	$(H^2aa')^{-1}\not{\partial}\partial^4$	4	$-8\bar{\not{\partial}}\partial^2$
2	$\frac{15}{2}\not{\partial}\partial^2$	5	$4\not{\partial}\nabla^2$
3	$-\bar{\not{\partial}}\partial^2$	6	$7\not{\partial}\nabla^2$

TABLE II. Nonlocal contributions to $\int d^4x'[\Sigma](x;x')\times \Psi^0(\eta',\vec{x}';\vec{k},s)$ at late times. Multiply each term by $\frac{\kappa^2H^2}{2^6\pi^2}\times iH\gamma^0\Psi^0(\eta,\vec{x};\vec{k},s)$.

I	Coefficient of the late time contribution from each U^I_{ij}
1	0
2	$\frac{15}{2}\{\exp(2i\frac{k}{H})+1\}$
3	$-i\frac{k}{H}\{2\ln(\frac{2\mu}{H})-\int_{\eta_i}^0 d\eta'(\frac{\exp(-2k\eta')-1}{\eta'})\}$
4	$8i\frac{k}{H}\int_{\eta_i}^0 d\eta'(\frac{\exp(-2k\eta')-1}{\eta'})$
5	$4\frac{k^2}{H}\int_{\eta_i}^0 d\eta' e^{-2ik\eta'}\{\int_0^{-2k\eta'}dt(\frac{\exp(-it)-1}{t})+2\ln(H\eta')\}$
6	$-\frac{7}{2}i\frac{k}{H}\{\exp(2i\frac{k}{H})-1\}$

$$\frac{i\kappa^2H^2}{2^6\pi^2}\int d^4x'\Big\{\frac{\ln(aa')}{H^2aa'}\bar{\partial\!\!\!/}\partial^2+\frac{15}{2}\ln(aa')\partial\!\!\!/ - 7\ln(aa')\bar{\partial\!\!\!/}\Big\}\delta^4(x-x')\Psi^0(\eta',\vec{x}';\vec{k},s)$$
$$=\frac{i\kappa^2H^2}{2^6\pi^2}\Big\{\frac{\ln(a)}{H^2a}\partial\!\!\!/\partial^2\Big(\frac{1}{a}\Psi^0(\eta,\vec{x};\vec{k},s)\Big)+\frac{1}{H^2a}\partial\!\!\!/\partial^2\Big(\frac{\ln(a)}{a}\Psi^0(\eta,\vec{x};\vec{k},s)\Big)+\frac{15}{2}(\ln(a)\partial\!\!\!/ + \partial\!\!\!/\ln(a))\Psi^0(\eta,\vec{x};\vec{k},s)-14\ln(a)\bar{\partial\!\!\!/}\Psi^0(\eta,\vec{x};\vec{k},s)\Big\}, \tag{89}$$

$$=\frac{\kappa^2H^2}{2^6\pi^2}iH\gamma^0\Psi^0(\eta,\vec{x};\vec{k},s)\Big\{\frac{17}{2}a-14i\frac{k}{H}\ln(a)-2i\frac{k}{H}\Big\}. \tag{90}$$

The local quantum corrections (90) are evidently much stronger than their nonlocal counterparts in Table II. Whereas the nonlocal terms approach a constant, the leading local contribution grows like the inflationary scale factor, $a=e^{Ht}$. Even factors of $\ln(a)$ are negligible by comparison. We can therefore write the late time limit of the one loop field equation as

$$i\partial\!\!\!/\kappa^2\Psi^1(\eta,\vec{x};\vec{k},s)\longrightarrow\frac{\kappa^2H^2}{2^6\pi^2}\frac{17}{2}iHa\gamma^0\Psi^0(\eta,\vec{x};\vec{k},s). \tag{91}$$

The only way for the left-hand side to reproduce such rapid growth is for the time derivative in $i\partial\!\!\!/$ to act on a factor of $\ln(a)$,

$$i\gamma^\mu\partial_\mu\ln(a)=i\gamma^\mu\frac{Ha^2}{a}\delta^0_\mu=iHa\gamma^0. \tag{92}$$

We can therefore write the late time limit of the tree plus one loop mode functions as

$$\Psi^0(\eta,\vec{x};\vec{k},s)+\kappa^2\Psi^1(\eta,\vec{x};\vec{k},s)\longrightarrow\Big\{1+\frac{\kappa^2H^2}{2^6\pi^2}\frac{17}{2}\ln(a)\Big\}\Psi^0(\eta,\vec{x};\vec{k},s). \tag{93}$$

All other corrections actually fall off at late times. For example, those from the $\ln(a)$ terms in (90) go like $\ln(a)/a$.

There is a clear physical interpretation for the sort of solution we see in (93). When the corrected field goes to the free field times a constant, that constant represents a field strength renormalization. When the quantum-corrected field goes to the free field times a function of time that is independent of the form of the free field solution, it is natural to think in terms of a *time-dependent field strength renormalization*,

$$\Psi(\eta,\vec{x};\vec{k},s)\longrightarrow\frac{\Psi^0(\eta,\vec{x};\vec{k},s)}{\sqrt{Z_2(t)}}\quad\text{where } Z_2(t)=1-\frac{17\kappa^2H^2}{2^6\pi^2}\ln(a)+O(\kappa^4). \tag{94}$$

Of course we only have the order κ^2 correction, so one does not know if this behavior persists at higher orders. If no higher loop correction supervenes, the field would switch from positive norm to negative norm at $\ln(a)=2^6\pi^2/17\kappa^2H^2$. In any case, it is safe to conclude that perturbation theory must break down near this time.

V. HARTREE APPROXIMATION

The appearance of a time-dependent field strength renormalization is such a surprising result that it is worth noting we can understand it on a simple, qualitative level using the Hartree, or mean-field, approximation. This technique has proved useful in a wide variety of problems from atomic physics [43] and statistical mechanics [44], to nuclear physics [45] and quantum field theory [46]. Of particular relevance to our work is the insight the Hartree approximation provides into the generation of photon mass by inflationary particle production in SQED [47–49].

The idea is that we can approximate the dynamics of Fermi fields interacting with the graviton field operator, $h_{\mu\nu}$, by taking the expectation value of the Dirac Lagrangian in the graviton vacuum. To the order we shall need it, the Dirac Lagrangian is [9]

$$\mathcal{L}_{\rm Dirac}=\bar{\Psi}i\partial\!\!\!/\Psi+\frac{\kappa}{2}\{h\bar{\Psi}i\partial\!\!\!/\Psi-h^{\mu\nu}\bar{\Psi}\gamma_\mu i\partial_\nu\Psi-h_{\mu\rho,\sigma}\bar{\Psi}\gamma^\mu J^{\rho\sigma}\Psi\}+\kappa^2\Big[\frac{1}{8}h^2-\frac{1}{4}h^{\rho\sigma}h_{\rho\sigma}\Big]\bar{\Psi}i\partial\!\!\!/\Psi+\kappa^2\Big[-\frac{1}{4}hh^{\mu\nu}+\frac{3}{8}h^{\mu\rho}h_\rho{}^\nu\Big]\bar{\Psi}\gamma_\mu i\partial_\nu\Psi+\kappa^2\Big[-\frac{1}{4}hh_{\mu\rho,\sigma}+\frac{1}{8}h^\nu{}_\rho h_{\nu\sigma,\mu}+\frac{1}{4}(h^\nu{}_\mu h_{\nu\rho})_{,\sigma}+\frac{1}{4}h^\nu{}_\sigma h_{\mu\rho,\nu}\Big]\bar{\Psi}\gamma^\mu J^{\rho\sigma}\Psi+O(\kappa^3). \tag{95}$$

Of course the expectation value of a single graviton field is zero, but the expectation value of the product of two fields is the graviton propagator [12,13],

$$\langle\Omega|T[h_{\mu\nu}(x)h_{\rho\sigma}(x')]|\Omega\rangle = i\Delta_A(x;x')[_{\mu\nu}T^A_{\rho\sigma}] + i\Delta_B(x;x')[_{\mu\nu}T^B_{\rho\sigma}] + i\Delta_C(x;x')[_{\mu\nu}T^C_{\rho\sigma}]. \quad (96)$$

The various tensor structures are

$$[_{\mu\nu}T^A_{\rho\sigma}] = 2\bar{\eta}_{\mu(\rho}\bar{\eta}_{\sigma)\nu} - \frac{2}{D-3}\bar{\eta}_{\mu\nu}\bar{\eta}_{\rho\sigma}, \quad [_{\mu\nu}T^B_{\rho\sigma}] = -4\delta^0_{\ (\mu}\bar{\eta}_{\nu)(\rho}\delta^0_{\ \sigma)}, \quad (97)$$

$$[_{\mu\nu}T^C_{\rho\sigma}] = \frac{2}{(D-2)(D-3)}[(D-3)\delta^0_\mu\delta^0_\nu + \bar{\eta}_{\mu\nu}] \times [(D-3)\delta^0_\rho\delta^0_\sigma + \bar{\eta}_{\rho\sigma}]. \quad (98)$$

Parenthesized indices are symmetrized and a bar over a common tensor such as the Kronecker delta function denotes that its temporal components have been nulled,

$$\bar{\delta}^\mu_\nu \equiv \delta^\mu_\nu - \delta^\mu_0\delta^0_\nu, \qquad \bar{\eta}_{\mu\nu} \equiv \eta_{\mu\nu} + \delta^0_\mu\delta^0_\nu. \quad (99)$$

The three scalar propagators that appear in (96) have complicated expressions which we omit in favor of simply giving their coincidence limits and the coincidence limits of their first derivatives [50],

$$\lim_{x'\to x} i\Delta_A(x;x') = \frac{H^{D-2}}{(4\pi)^{D/2}}\frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})}\left\{-\pi\cot\left(\frac{\pi}{2}D\right) + 2\ln(a)\right\}, \quad (100)$$

$$\lim_{x'\to x}\partial_\mu i\Delta_A(x;x') = \frac{H^{D-2}}{(4\pi)^{D/2}}\frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})}\times Ha\delta^0_\mu = \lim_{x'\to x}\partial'_\mu i\Delta_A(x;x'), \quad (101)$$

$$\lim_{x'\to x} i\Delta_B(x;x') = \frac{H^{D-2}}{(4\pi)^{D/2}}\frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})}\times -\frac{1}{D-2}, \quad (102)$$

$$\lim_{x'\to x}\partial_\mu i\Delta_B(x;x') = 0 = \lim_{x'\to x}\partial'_\mu i\Delta_B(x;x'), \quad (103)$$

$$\lim_{x'\to x} i\Delta_C(x;x') = \frac{H^{D-2}}{(4\pi)^{D/2}}\frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})}\times\frac{1}{(D-2)(D-3)}, \quad (104)$$

$$\lim_{x'\to x}\partial_\mu i\Delta_C(x;x') = 0 = \lim_{x'\to x}\partial'_\mu i\Delta_C(x;x'). \quad (105)$$

We are interested in terms which grow at late times. Because the B-type and C-type propagators go to constants, and their derivatives vanish, they can be neglected. The same is true for the divergent constant in the coincidence limit of the A-type propagator. In the full theory it would be absorbed into a constant counterterm. Because the remaining, time-dependent terms are finite, we may as well take $D = 4$. Our Hartree approximation therefore amounts to making the following replacements in (95):

$$h_{\mu\nu}h_{\rho\sigma} \to \frac{H^2}{4\pi^2}\ln(a)[\bar{\eta}_{\mu\rho}\bar{\eta}_{\nu\sigma} + \bar{\eta}_{\mu\sigma}\bar{\eta}_{\nu\rho} - 2\bar{\eta}_{\mu\nu}\bar{\eta}_{\rho\sigma}], \quad (106)$$

$$h_{\mu\nu}h_{\rho\sigma,\alpha} \to \frac{H^2}{8\pi^2}Ha\delta^0_\alpha[\bar{\eta}_{\mu\rho}\bar{\eta}_{\nu\sigma} + \bar{\eta}_{\mu\sigma}\bar{\eta}_{\nu\rho} - 2\bar{\eta}_{\mu\nu}\bar{\eta}_{\rho\sigma}]. \quad (107)$$

It is now just a matter of contracting (106) and (107) appropriately to produce each of the quadratic terms in (95). For example, the first term gives

$$\frac{\kappa^2}{8}h^2\bar{\Psi}i\not{\partial}\Psi \to \frac{\kappa^2H^2}{2^5\pi^2}\ln(a)[\eta^{\mu\nu}\eta^{\rho\sigma}][\bar{\eta}_{\mu\rho}\bar{\eta}_{\nu\sigma} + \bar{\eta}_{\mu\sigma}\bar{\eta}_{\nu\rho} - 2\bar{\eta}_{\mu\nu}\bar{\eta}_{\rho\sigma}]\bar{\Psi}i\not{\partial}\Psi, \quad (108)$$

$$= \frac{\kappa^2H^2}{2^5\pi^2}\ln(a)[3+3-18]\bar{\Psi}i\not{\partial}\Psi. \quad (109)$$

The second quadratic term gives a proportional result,

$$\frac{-\kappa^2}{4}h^{\rho\sigma}h_{\rho\sigma}\bar{\Psi}i\not{\partial}\Psi \to \frac{-\kappa^2H^2}{2^4\pi^2}\ln(a)[9+3-6]\bar{\Psi}i\not{\partial}\Psi. \quad (110)$$

The total for these first two terms is $\frac{-3\kappa^2H^2}{4\pi^2}\ln(a)\bar{\Psi}i\not{\partial}\Psi$.

The third and fourth of the quadratic terms in (95) result in only spatial derivatives,

$$\frac{-\kappa^2H^2}{4}hh^{\mu\nu}\bar{\Psi}\gamma_\mu i\partial_\nu\Psi \to \frac{-\kappa^2H^2}{2^4\pi^2}\ln(a)[1+1-6]\bar{\Psi}i\bar{\not{\partial}}\Psi, \quad (111)$$

$$\frac{3}{8}\kappa^2h^{\mu\rho}h^\nu_\rho\bar{\Psi}\gamma_\mu i\partial_\nu\Psi \to \frac{3\kappa^2H^2}{2^5\pi^2}\ln(a)[3+1-2]\bar{\Psi}i\bar{\not{\partial}}\Psi. \quad (112)$$

The total for this type of contribution is $\frac{7\kappa^2H^2}{2^4\pi^2}\ln(a)\bar{\Psi}i\bar{\not{\partial}}\Psi$.

The final four quadratic terms in (95) involve derivatives acting on at least one of the two graviton fields,

$$-\frac{\kappa^2}{4}hh_{\mu\rho,\sigma}\bar{\Psi}\gamma^\mu J^{\rho\sigma}\Psi \to \frac{-\kappa^2H^2}{2^5\pi^2}Ha[1+1-6]\bar{\eta}_{\mu\rho} \times \bar{\Psi}\gamma^\mu J^{\rho 0}\Psi, \quad (113)$$

$$\frac{\kappa^2}{8}h^\nu_\rho h_{\nu\sigma,\mu}\bar{\Psi}\gamma^\mu J^{\rho\sigma}\Psi \to \frac{\kappa^2H^2}{2^6\pi^2}Ha[3+1-2]\bar{\eta}_{\rho\sigma} \times \bar{\Psi}\gamma^0 J^{\rho\sigma}\Psi, \quad (114)$$

$$\frac{\kappa^2}{4}(h^\nu_\mu h_{\nu\rho})_{,\sigma}\bar{\Psi}\gamma^\mu J^{\rho\sigma}\Psi \to \frac{\kappa^2H^2}{2^4\pi^2}Ha[3+1-2]\eta_{\mu\rho} \times \bar{\Psi}\gamma^\mu J^{\rho 0}\Psi, \quad (115)$$

$$\frac{\kappa^2}{4} h^{\nu}_{\sigma} h_{\mu\rho,\nu} \bar{\Psi} \gamma^{\mu} J^{\rho\sigma} \Psi \rightarrow 0. \tag{116}$$

The second of these contributions vanishes owing to the antisymmetry of the Lorentz representation matrices, $J^{\mu\nu} \equiv \frac{i}{4}[\gamma^{\mu}, \gamma^{\nu}]$, whereas $\bar{\eta}_{\mu\rho} \gamma^{\mu} J^{\rho 0} = -\frac{3i}{2} \gamma^0$. Hence, the sum of all four terms is $\frac{-3\kappa^2 H^2}{8\pi^2} Ha \bar{\Psi} i \gamma^0 \Psi$.

Combining these results gives

$$\langle \mathcal{L}_{\rm Dirac} \rangle = \bar{\Psi} i \not{\partial} \Psi - \frac{3\kappa^2 H^2}{4\pi^2} \ln(a) \bar{\Psi} i \not{\partial} \Psi - \frac{3\kappa^2 H^2}{8\pi^2} Ha \bar{\Psi} i \gamma^0 \Psi + \frac{7\kappa^2 H^2}{16\pi^2} \ln(a) \bar{\Psi} i \bar{\not{\partial}} \Psi + O(\kappa^4), \tag{117}$$

$$= \bar{\Psi} \left[1 - \frac{3\kappa^2 H^2}{8\pi^2} \ln(a) \right] i \not{\partial} \left[1 - \frac{3\kappa^2 H^2}{8\pi^2} \ln(a) \right] \Psi + \frac{7\kappa^2 H^2}{16\pi^2} \ln(a) \bar{\Psi} i \bar{\not{\partial}} \Psi + O(\kappa^4). \tag{118}$$

If we express the equations associated with (118) according to the perturbative scheme of Sec. I, the first order equation is

$$i \not{\partial} \kappa^2 \Psi^1(\eta, \vec{x}; \vec{k}, s) = \frac{\kappa^2 H^2}{2^6 \pi^2} i H \gamma^0 \Psi^0(\eta, \vec{x}; \vec{k}, s) \times \left\{ 24a - 28i \frac{k}{H} \ln(a) \right\}. \tag{119}$$

This is similar, but not identical to, what we got in expression (90) from the delta function terms of the actual one loop self-energy (3). In particular, the exact calculation gives $\frac{17}{2} a - 14i \frac{k}{H} \ln(a)$, rather than the Hartree approximation of $24a - 28i \frac{k}{H} \ln(a)$. Of course the $\ln(a)$ terms make corrections to Ψ^1 which fall like $\ln(a)/a$, so the real disagreement between the two methods is limited to the differing factors of $\frac{17}{2}$ versus 24.

We are pleased that such a simple technique comes so close to recovering the result of a long and tedious calculation. The slight discrepancy is no doubt due to terms in the Dirac Lagrangian (95) which are linear in the graviton field operator. As described in relation (32) of Sec. II, the linearized effective field $\Psi_i(x; \vec{k}, s)$ represents $a^{(D-1)/2}$ times the expectation value of the anticommutator of the Heisenberg field operator $\psi_i(x)$ with the free fermion creation operator $b(\vec{k}, s)$. At the order we are working, quantum corrections to $\Psi_i(x; \vec{k}, s)$ derive from perturbative corrections to $\psi_i(x)$ which are quadratic in the free graviton creation and annihilation operators. Some of these corrections come from a single $hh\bar{\psi}\psi$ vertex, while others derive from two $h\bar{\psi}\psi$ vertices. The Hartree approximation recovers corrections of the first kind, but not the second, which is why we believe it fails to agree with the exact result. Yukawa theory presents a fully worked-out example [6,7,51] in which the *entire* lowest-order correction to the fermion mode functions derives from the product of two such linear terms, so the Hartree approximation fails completely in that case.

VI. DISCUSSION

We have used the Schwinger-Keldysh formalism to include one loop, quantum gravitational corrections to the Dirac equation, in the simplest local Lorentz and general coordinate gauge, in the locally de Sitter background which is a paradigm for inflation. Because Dirac + Einstein is not perturbatively renormalizable, it makes no sense to solve this equation generally. However, the equation should give reliable predictions at late times when the arbitrary finite parts of the BPHZ counterterms (6) are insignificant compared to the completely determined factors of $\ln(aa')$ on terms (7)–(9) which otherwise have the same structure. In this late time limit we find that the one loop corrected, spatial plane wave mode functions behave as if the tree order mode functions were simply subject to a time-dependent field strength renormalization,

$$Z_2(t) = 1 - \frac{17}{4\pi} G H^2 \ln(a) + O(G^2)$$

$$\text{where } G = 16\pi\kappa^2. \tag{120}$$

If unchecked by higher loop effects, this would vanish at $\ln(a) \simeq 1/GH^2$. What actually happens depends upon higher order corrections, but there is no way to avoid perturbation theory breaking down at this time, at least in this gauge.

Might this result be a gauge artifact? One reaches different gauges by making field-dependent transformations of the Heisenberg operators. We have worked out the change (68) this induces in the linearized effective field, but the result is not simple. Although the linearized effective field obviously changes when different gauge conditions are employed to compute it, we believe (but have not proven) that the late time factors of $\ln(a)$ do not change.

It is important to realize that the 1PI functions of a gauge theory in a fixed gauge are not devoid of physical content by virtue of depending upon the gauge. In fact, they encapsulate the physics of a quantum gauge field every bit as completely as they do when no gauge symmetry is present. One extracts this physics by forming the 1PI functions into gauge independent and physically meaningful combinations. The S-matrix accomplishes this in flat space quantum field theory. Unfortunately, the S-matrix fails to exist for Dirac + Einstein in de Sitter background, nor would it correspond to an experiment that could be performed if it did exist [52–54].

If it is conceded that we know what it means to release the universe in a free state then it would be simple enough—albeit tedious—to construct an analogue of $\psi_i(x)$ which is invariant under gauge transformations that do not affect the initial value surface. For example, one might extend to fermions the treatment given for pure gravity by [55]:

(i) Propagate an operator-valued geodesic a fixed invariant time from the initial value surface.
(ii) Use the spin connection $A_{\mu cd}J^{cd}$ to parallel transport along the geodesic.
(iii) Evaluate ψ at the operator-valued geodesic, in the Lorentz frame which is transported from the initial value surface.

This would make an invariant, as would any number of other constructions [56]. For that matter, the gauge-fixed 1PI functions also correspond to the expectation values of invariant operators [57]. Mere invariance does not guarantee physical significance, nor does gauge dependence preclude it.

What is needed is for the community to agree upon a relatively simple set of operators which stand for experiments that could be performed in de Sitter space. There is every reason to expect a successful outcome because the past few years have witnessed a resolution of the similar issue of how to measure quantum gravitational backreaction during inflation, driven either by a scalar inflaton [58–61] or by a bare cosmological constant [62]. That process has begun for quantum field theory in de Sitter space [53,54,56] and one must wait for it to run its course. In the meantime, it is safest to stick with what we have actually shown: perturbation theory must break down for Dirac + Einstein in the simplest gauge.

This is a surprising result but we were able to understand it qualitatively using the Hartree approximation in which one takes the expectation value of the Dirac Lagrangian in the graviton vacuum. The physical interpretation seems to be that fermions propagate through an effective geometry whose ever-increasing deviation from de Sitter is controlled by inflationary graviton production. At one loop order the fermions are passive spectators to this effective geometry.

It is significant that inflationary graviton production enhances fermion mode functions by a factor of $\ln(a)$ at one loop. Similar factors of $\ln(a)$ have been found in the graviton vacuum energy [63,64]. These infrared logarithms also occur in the vacuum energy of a massless, minimally coupled scalar with a quartic self-interaction [65,66], and in the VEV's of almost all operators in Yukawa theory [51] and SQED [67]. A recent all orders analysis was not even able to exclude the possibility that they might contaminate the power spectrum of primordial density fluctuations [68].

The fact that infrared logarithms grow without bound raises the exciting possibility that quantum gravitational corrections may be significant during inflation, in spite of the minuscule coupling constant of $GH^2 \lesssim 10^{-12}$. However, the only thing one can legitimately conclude from the perturbative analysis is that infrared logarithms cause perturbation theory to break down, in our gauge, if inflation lasts long enough. Inferring what happens after this breakdown requires a nonperturbative technique.

Starobinskiĭ has long advocated that a simple stochastic formulation of scalar potential models serves to reproduce the leading infrared logarithms of these models at each order in perturbation theory [69]. This fact has recently been proved to all orders [70,71]. When the scalar potential is bounded below it is even possible to sum the series of leading infrared logarithms and infer their net effect at asymptotically late times [72]. Applying Starobinskiĭ's technique to more complicated theories which also show infrared logarithms is a formidable problem, but solutions have recently been obtained for Yukawa theory [51] and for SQED [67]. It would be very interesting to see what this technique gives for the infrared logarithms we have exhibited, to lowest order, in Dirac + Einstein. And it should be noted that even the potentially complicated, invariant operators which might be required to settle the gauge issue would be straightforward to compute in such a stochastic formulation.

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APPENDIX: NONLOCAL TERMS FROM SEC. IV

It is important to establish that the nonlocal terms make no significant contribution at late times, so we will derive the results summarized in Table II. For simplicity we denote as $[U^I]$ the contribution from each operator U^I_{ij} in Table I. We also abbreviate $\Psi^0(\eta, \vec{x}; \vec{k}, s)$ as $\Psi^0(x)$.

Owing to the factor of $1/a'$ in U^1_{ij}, and to the larger number of derivatives, the reduction of $[U^1]$ is atypical,

$$[U^1] \equiv \frac{\kappa^2}{2^8 \pi^4} \frac{1}{a} \bar{\partial} \partial^4 \int d^4x' \frac{1}{a'} \Big\{ \frac{\ln(\mu^2 \Delta x^2_{++})}{\Delta x^2_{++}} - \frac{\ln(\mu^2 \Delta x^2_{+-})}{\Delta x^2_{+-}} \Big\} \Psi^0(x'), \tag{A1}$$

$$= \frac{-i\kappa^2}{2^6 \pi^2 a} \gamma^0 \Psi^0(x) [-2ik\partial_\eta + \partial^2_\eta] \Big\{ \partial_\eta \int_{\eta_i}^{\eta} d\eta' (-H\eta') \times \Big(\frac{e^{2ik\Delta\eta} - 1}{\Delta\eta} \Big) + \partial^2_\eta \int_{\eta_i}^{\eta} d\eta' (-2H\eta') \ln(2\mu\Delta\eta) \Big\}, \tag{A2}$$

$$= \frac{-i\kappa^2}{2^6 \pi^2 a} \gamma^0 \Psi^0 (-2ik + \partial_\eta) \Big\{ -\frac{e^{2ik(\eta + (1/H))} - 1}{(\eta + \frac{1}{H})^2} + \frac{(2ik - H) e^{2ik(\eta + (1/H))}}{\eta + \frac{1}{H}} - \frac{3H^2}{(1 + H\eta)} + \frac{2H^3 \eta}{(1 + H\eta)^2} \Big\}, \tag{A3}$$

$$= \frac{\kappa^2 H^2}{2^6 \pi^2}(H\eta) iH\gamma^0 \Psi \Big\{ \frac{2[e^{(2ik/H)(1+H\eta)} - 1 - 2H\eta]}{(1+H\eta)^3} + \frac{(1 - \frac{2ik}{H})e^{(2ik/H)(1+H\eta)}}{(1+H\eta)^2} + \frac{5 - 4ik\eta - \frac{2ik}{H}}{(1+H\eta)^2} + \frac{\frac{6ik}{H}}{1+H\eta}\Big\}. \tag{A4}$$

This expression actually vanishes in the late time limit of $\eta \to 0^-$.

$[U^2]$ was reduced in Sec. IV so we continue with $[U^3]$,

$$[U^3] \equiv -\frac{\kappa^2 H^2}{2^8 \pi^4} \bar{\partial\!\!\!/} \partial^2 \int d^4x' \Big\{ \frac{\ln(\mu^2 \Delta x^2_{++})}{\Delta x^2_{++}} - \frac{\ln(\mu^2 \Delta x^2_{+-})}{\Delta x^2_{+-}} \Big\} \Psi^0(x'), \tag{A5}$$

$$= -\frac{\kappa^2 H^2}{2^8 \pi^4} \bar{\partial\!\!\!/} i 4\pi^2 \Psi^0(x) \Big\{ 2\ln\Big[\frac{2\mu}{H}(1+H\eta)\Big] + \int_{\eta_i}^{\eta} d\eta' \Big(\frac{e^{2ik\Delta\eta} - 1}{\Delta\eta}\Big)\Big\}, \tag{A6}$$

$$= \frac{\kappa^2 H^2}{2^6 \pi^2} k\gamma^0 \Psi^0(x) \Big\{ 2\ln\Big[\frac{2\mu}{H}(1+H\eta)\Big] + \int_{\eta_i}^{\eta} d\eta' \Big(\frac{e^{2ik\Delta\eta} - 1}{\Delta\eta}\Big)\Big\}, \tag{A7}$$

$$\to \frac{\kappa^2 H^2}{2^6 \pi^2} iH\gamma^0 \Psi^0(x) \times -\frac{ik}{H} \Big\{ 2\ln\Big(\frac{2\mu}{H}\Big) - \int_{\eta_i}^{0} d\eta' \Big(\frac{e^{-2ik\eta'} - 1}{\eta'}\Big)\Big\}. \tag{A8}$$

U^4_{ij} has the same derivative structure as U^3_{ij}, so $[U^4]$ follows from (A8),

$$[U^4] \equiv -\frac{\kappa^2 H^2}{2^8 \pi^4} \times 8 \bar{\partial\!\!\!/} \partial^2 \int d^4x' \Big\{ \frac{\ln(\frac{1}{4}H^2 \Delta x^2_{++})}{\Delta x^2_{++}} - \frac{\ln(\frac{1}{4}H^2 \Delta x^2_{+-})}{\Delta x^2_{+-}} \Big\} \Psi^0(x'), \tag{A9}$$

$$= \frac{\kappa^2 H^2}{2^6 \pi^2} 8k\gamma^0 \Psi^0(x) \Big\{ 2\ln[(1+H\eta)] + \int_{\eta_i}^{\eta} d\eta' \Big(\frac{e^{2ik\Delta\eta} - 1}{\Delta\eta}\Big)\Big\}, \tag{A10}$$

$$\to \frac{\kappa^2 H^2}{2^6 \pi^2} iH\gamma^0 \Psi^0(x) \times 8i\frac{k}{H} \int_{\eta_i}^{0} d\eta' \Big(\frac{e^{-2ik\eta'} - 1}{\eta'}\Big). \tag{A11}$$

U^5_{ij} has a Laplacian rather than a d'Alembertian so we use identity (80) for $[U^5]$. We also employ the abbreviation $k\Delta\eta = \alpha$,

$$[U^5] \equiv 4\frac{\kappa^2 H^2}{2^8 \pi^4} \partial\!\!\!/ \nabla^2 \int d^4x' \Big\{ \frac{\ln(\mu^2 \Delta x^2_{++})}{\Delta x^2_{++}} - \frac{\ln(\mu^2 \Delta x^2_{+-})}{\Delta x^2_{+-}} \Big\} \Psi^0(x'), \tag{A12}$$

$$= 4\frac{\kappa^2 H^2}{2^8 \pi^4} \partial\!\!\!/ \nabla^2 \Big(\frac{-4i\pi^2}{k}\Big) \Psi^0(x) \int_{\eta_i}^{\eta} d\eta' e^{i\alpha} \Big\{ -\cos(\alpha) \times \int_0^{2\alpha} dt \frac{\sin(t)}{t} + \sin(\alpha)\Big[\int_0^{2\alpha} dt \Big(\frac{\cos(t) - 1}{t}\Big) + 2\ln\Big(\frac{H\alpha}{k}\Big)\Big]\Big\}, \tag{A13}$$

$$= \frac{\kappa^2 H^2}{2^6 \pi^2} iH\gamma^0 \Psi^0(x) \times 4\frac{k^2}{H} \times \int_{\eta_i}^{\eta} d\eta' e^{2i\alpha} \Big[\int_0^{2\alpha} dt \Big(\frac{e^{-it} - 1}{t}\Big) + \ln(H\Delta\eta)^2\Big], \tag{A14}$$

$$\to \frac{\kappa^2 H^2}{2^6 \pi^2} iH\gamma^0 \Psi^0(x) \times 4\frac{k^2}{H} \times \int_{\eta_i}^{0} d\eta' e^{2i\alpha} \Big[\int_0^{2\alpha} dt \Big(\frac{e^{-it} - 1}{t}\Big) + \ln(H\eta')^2\Big]. \tag{A15}$$

U^6_{ij} has the same derivative structure as U^5_{ij} but it acts on a different integrand. We therefore apply identity (84) for $[U^6]$,

$$[U^6] \equiv 7\frac{\kappa^2 H^2}{2^8 \pi^4} \partial\!\!\!/ \nabla^2 \int d^4x' \Big\{ \frac{1}{\Delta x^2_{++}} - \frac{1}{\Delta x^2_{+-}} \Big\} \Psi^0(x'), \tag{A16}$$

$$= 7\frac{\kappa^2 H^2}{2^8 \pi^4} \partial\!\!\!/ \nabla^2 \times (-i4\pi^2) k^{-1} \Psi^0(x) \times \int_{\eta_i}^{\eta} d\eta' e^{ik\Delta\eta} \sin(k\Delta\eta), \tag{A17}$$

$$= \frac{\kappa^2 H^2}{2^6 \pi^2} iH\gamma^0 \Psi^0(x) \times -\frac{7}{2}\frac{ik}{H}[e^{(2ik/H)(1+H\eta)} - 1], \tag{A18}$$

$$\to \frac{\kappa^2 H^2}{2^6 \pi^2} iH\gamma^0 \Psi^0(x) \times -\frac{7}{2}\frac{ik}{H}[e^{2ik/H} - 1]. \tag{A19}$$

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