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Holography with a gravitational Chern-Simons term

Sergey N. Solodukhin*

School of Engineering and Science, International University Bremen, P.O. Box 750561, Bremen 28759, Germany

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The holographic description in the presence of the gravitational Chern-Simons term is studied. The modified gravitational equations are integrated by using the Fefferman-Graham expansion, and the holographic stress-energy tensor is identified. The stress-energy tensor has both conformal anomaly and gravitational or, if reformulated in terms of the *zweibein*, Lorentz anomaly. We comment on the structure of anomalies in two dimensions and show that the two-dimensional stress-energy tensor can be reproduced by integrating the conformal and gravitational anomalies. We study the black hole entropy in theories with a gravitational Chern-Simons term and find that the usual Bekenstein-Hawking entropy is modified. For the Banados-Teitelboim-Zanelli (BTZ) black hole, the modification is determined by the area of the inner horizon. We show that the total entropy of the BTZ black hole is precisely reproduced in a boundary conformal field theory calculation using the Cardy formula.

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I. INTRODUCTION

It is amazing how much of physics is encoded in geometry of asymptotically anti-de Sitter (AdS) space-time. This includes the information on the ultraviolet divergences, quantum effective action, and the conformal anomaly. The latter is an important element in the holographic description and is due to the peculiar nature of the asymptotic diffeomorphisms that generate conformal symmetry [1,2]. The gravitational Einstein-Hilbert action, in which the metric is fixed on the boundary, breaks the asymptotic conformal symmetry and is thus the source of the anomalies. This and other related issues were much studied in recent years [3–12].

In a recent interesting paper [13], Kraus and Larsen have modified the gravitational action in three dimensions by adding the gravitational Chern-Simons term. The resultant theory is known as the topologically massive gravity [14,15]. The appearance of the Chern-Simons terms is generically predicted in string theory. The gravitational Chern-Simons term, explicitly depending on connection, is gauge invariant only up to some boundary terms, so that the presence of the Chern-Simons term necessarily breaks the asymptotic coordinate invariance. The appearance of the gravitational anomaly[1] in the boundary theory should be expected in addition to the already existing conformal anomaly. Alternatively, if the Chern-Simons term is defined in terms of the Lorentz connection, the asymptotic local Lorentz symmetry is broken and the Lorentz anomaly should appear. All these expectations were explicitly confirmed in [13] by looking at how the gravitational action changes under the gauge transformations. On the boundary side, the anomaly arises due to different central charge in holomorphic and antiholomorphic sectors. Such theories were studied some time ago, see in particular [18,19].

In the present note, inspired by [13], we take a different route to anomalies and show that they follow directly from the bulk gravitational equations. The latter are integrated by expanding the bulk metric in powers of distance from the boundary. This is much in the spirit of [2,5,8]. The integration procedure involves the fixing of the boundary data. The data are the boundary metric and the holographic stress-energy tensor. This helps to determine explicitly the structure of the stress-energy tensor in terms of the coefficients in the expansion and completely fix the form of the anomalies. The gravitational anomaly we get agrees with the one obtained in [16,18]. In the dual picture, the holographic stress-energy tensor should be identified with the quasilocal stress-energy tensor which determines the values of mass and angular momentum.

The gravitational Chern-Simons term is an eligible gravitational action which produces covariant equations of motion that are, in particular, solved by the Banados-Teitelboim-Zanelli (BTZ) metric. It is therefore an interesting question whether the black hole entropy is modified when the Chern-Simons term is included. We study this question and obtain a contribution to the entropy due to the gravitational Chern-Simons term. This contribution depends on the value of the Lorentz connection at the horizon and is, nevertheless, gauge invariant. For the BTZ black hole the entropy due to the Chern-Simons term is proportional to the area of the inner horizon. This is surprising, considering that in any theory nonlinear in Riemann curvature the entropy of BTZ black hole is always, as we argue in the paper, determined by area of the outer horizon. In the theory at hand, the total entropy has dual meaning in terms of the boundary conformal field theory (CFT) and is precisely reproduced by means of the Cardy formula, as we show.

*Electronic address: s.solodukhin@iu-bremen.de

[1]For review on the gravitational anomalies, see the original works [16,17].

II. FEFFERMAN-GRAHAM EXPANSION AND GRAVITATIONAL ANOMALY

The gravitational theory on three-dimensional spacetime is given by the action

$$I_{\rm gr} = I_{\rm EH} + I_{\rm CS}, \tag{2.1}$$

which is the sum of ordinary Einstein-Hilbert action (with cosmological constant)

$$I_{\rm gr} = -\frac{1}{16\pi G_N}\left[\int_M (R[G] + 2/l^2) + \int_{\partial M} 2K\right], \tag{2.2}$$

where K is the trace of the second fundamental form of boundary ∂M, and the gravitational Chern-Simons term

$$I_{\rm CS} = \frac{\beta}{64\pi G_N}\int_M dx^3 \epsilon^{\mu\nu\alpha}\Big[R_{ab\mu\nu}\omega^{ab}{}_{,\alpha} + \frac{2}{3}\omega^a{}_{b,\mu}\omega^b{}_{c,\nu}\omega^c{}_{a,\alpha}\Big]. \tag{2.3}$$

Parameter l in (2.2) sets the AdS scale. To simplify things, on an intermediate stage of calculation, we take liberty to use units $l = 1$ restoring l explicitly in the final expressions. Parameter β has dimension of length. We use the following definition for the curvature $R^a{}_{b\mu\nu} \equiv \partial_\mu \omega^a{}_{b,\nu} + \omega^a{}_{c,\mu}\omega^c{}_{b,\nu} - (\mu \leftrightarrow \nu)$. The torsion-free Lorentz connection $\omega^a{}_b = \omega^a{}_{b,\mu} dx^\mu$ is determined as usual by the equation

$$de^a + \omega^a{}_b \wedge e^b = 0, \tag{2.4}$$

where the orthonormal basis $e^a = h^a_\mu dx^\mu$, $a = 1, 2, 3$ is the "square root" of the metric, $G_{\mu\nu} = h^a_\mu h^b_\mu \eta_{ab}$. Equation (2.4) can be used to express components of the Lorentz connection in terms of h^a_μ and their derivatives[2]

$$\begin{aligned}\omega_{ab,\mu} &= \tfrac{1}{2}(C_{a\nu\mu}h^\nu_b + C_{b\mu\nu}h^\nu_a - C_{d\alpha\beta}h^\alpha_a h^\beta_b h^d_\mu),\\ C^a_{\mu\nu} &\equiv \partial_\mu h^a_\nu - \partial_\nu h^a_\mu.\end{aligned} \tag{2.5}$$

The Levi-Cività symbol is determined as $\epsilon^{\mu\nu\alpha} = h^\mu_a h^\nu_b h^\alpha_c \epsilon^{abc}$. To complete our brief diving into theory of gravity in the orthonormal basis, we remind the reader that h^a_μ is covariantly constant,

$$\nabla_\mu h^a_\nu = \partial_\mu h^a_\nu - \Gamma^\lambda_{\mu\nu} h^a_\lambda + \omega^a{}_{b,\mu} h^b_\nu = 0, \tag{2.6}$$

that is of course equivalent to Eq. (2.4). The latter property is useful in that we may freely manipulate with h^a_μ by pulling it inside the covariant derivative or taking it out. This property also means that the Levi-Cività symbol is covariantly constant, $\nabla_\sigma \epsilon^{\mu\nu\alpha} = 0$.

The theory described by the action (2.1) is quite well known and belongs to the class of theories with topological mass [14,15]. A remarkable property of this theory is that it describes a propagating degree of freedom although each term in (2.1) taken separately is topological and thus does not contain local degrees of freedom. Another interesting property of (2.1) is that the gravitational Chern-Simons term explicitly breaks the asymptotic coordinate invariance if expressed in terms of the metric connection $\Gamma^\alpha{}_{\beta\mu}$ or the asymptotic local Lorentz invariance if the term is written using the Lorentz connection as in (2.3). The violation of the gauge symmetry happens only asymptotically because the variation of the Chern-Simons term under local gauge symmetry generates terms on the boundary of the spacetime. This violation thus should be manifest in the boundary theory. Indeed, in [13] this was related to the appearance of the gravitational or Lorentz anomalies in the boundary theory. Such anomalies are natural when $c_L \neq c_R$ in the boundary conformal field theory. Such theories were studied some time ago, see for example [19]. In the present context, these anomalies are obtained holographically and are encoded in the dynamics of the gravitational field in the bulk. In [13] the anomalies were derived by looking at how the gravitational action (2.1) changes under the gauge symmetries. Here we look at the problem at a somewhat different angle. We demonstrate that anomalies show up in the process of the holographic reconstruction of the bulk metric from the boundary data. In the absence of the gravitational Chern-Simons term, the bulk metric is uniquely determined once the holographic boundary data, the boundary metric representing the conformal class, and the boundary stress tensor are specified. Usually the boundary stress tensor is not entirely arbitrary. It is covariantly conserved and its trace should reproduce the conformal anomaly. The anomaly itself is completely specified by the boundary metric. The details of this analysis can be found in [3,5,8]. The presence of the gravitational Chern-Simons term in the bulk action manifests itself in an interesting way: the boundary stress tensor is no more covariantly conserved. This is how the gravitational anomaly in the boundary theory shows up. In order to see this explicitly, we solve the gravitational bulk equations modified by the presence of the Chern-Simons term starting from the boundary and finding the bulk metric as an expansion, well known in the physics and mathematics literature as the Fefferman-Graham expansion [2].

The gravitational bulk equations obtained by varying the action (2.1) with respect to metric takes the form

$$R_{\mu\nu} - \tfrac{1}{2}G_{\mu\nu}R - G_{\mu\nu} + \beta C_{\mu\nu} = 0, \tag{2.7}$$

where all curvature tensors are determined with respect to the bulk metric $G_{\mu\nu}$. The tensor $C_{\mu\nu}$ is the result of the variation of the gravitational Chern-Simons term. It is known as the Cotton tensor and takes the form

$$C_{\mu\nu} = \epsilon_\mu{}^{\alpha\beta}\nabla_\alpha(R_{\beta\nu} - \tfrac{1}{4}G_{\beta\nu}R). \tag{2.8}$$

Although the Chern-Simons (2.3) is defined in terms of the

[2]We use the Latin letters (a, b, c, d) for the inner Lorentz indices and Greek letters $(\mu, \nu, \alpha, \beta, \ldots)$ for the coordinate indices.

Lorentz connection which is not a gauge invariant object, its variation is presented in the covariant and gauge invariant form (2.8). This is just a manifestation of the fact that the "noninvariance" of the Chern-Simons term resides on the boundary and does not appear in the bulk field equations. By virtue of the Bianchi identities this quantity is symmetric, manifestly traceless, and identically covariantly conserved,

$$C_{\mu\nu}G^{\mu\nu} = 0, \qquad \nabla_\mu C^\mu{}_\nu = 0, \qquad \epsilon_{\alpha\mu\nu}C^{\mu\nu} = 0.$$

Because of these properties, we find that solution to the Eq. (2.7) is space-time with constant Ricci scalar $R = -6$. This is exactly what we had when the Chern-Simons term was not included in the action. In that case we had moreover that $R_{\mu\nu} = -2G_{\mu\nu}$ and the solution was the constant curvature space. It is no longer the case in the presence of the Chern-Simons term and we have

$$R_{\mu\nu} = -2G_{\mu\nu} - \beta C_{\mu\nu}. \tag{2.9}$$

This is that equation which we are going to solve. We start with choosing the bulk metric in the form

$$ds^2 = G_{\mu\nu}dX^\mu dX^\nu = dr^2 + g_{ij}(r,x)dx^i dx^j \tag{2.10}$$

that always can be done by using appropriate normal coordinates. The quantity $g_{ij}(r,x)$ is the induced metric on the hypersurface of constant radial coordinate r. The following expansion,

$$g(r,x) = e^{2r}[g_{(0)} + g_{(2)}e^{-2r} + g_{(4)}e^{-4r} + \cdots], \tag{2.11}$$

is assumed so that the metric (2.10) describes asymptotically anti-de Sitter space-time with $g_{(0)}$ being the metric on its two-dimensional boundary. In the case of pure general relativity (GR), described by action (2.2), the solution to the gravitational equations contains [5] only these three terms in the expansion (2.11). This is no more true when the Chern-Simons term is turned on and the whole infinite series should be expected in (2.11). The presence of an infinite number of terms in the expansion (2.11) generically seems to be related to the presence of local propagating degrees of freedom in the theory.

Now the routine is to insert the expansion (2.11) into the gravitational equations (2.9) and equate the coefficients appearing in front of the same power of e^r on both sides of the equation. This gives certain constraints on the coefficients $g_{(2n)}$ appearing in the expansion (2.11) allowing express $g_{(2n)}$ in terms of the coefficients $g_{(2k)}$ with $k < n$. Generically, in odd dimension $(d+1)$ there may appear also a "logarithmic" term $h_{(d)}re^{-dr}$ in (2.11). $h_{(d)}$ is traceless and covariantly conserved and is the local function of boundary metric $g_{(0)}$. In $d = 2$ no such local function of two-dimensional metric exists so that $h_{(2)}$ identically vanishes (see [5,8]). When the gravitational Chern-Simons term is present the same arguments are valid so that no logarithmic term is likely to appear. In any event, it would not affect our calculation of the anomalies. Appendices A and B contain details of calculation of the expansion for the Ricci tensor and the Cotton tensor. A good starting point in the analysis is to look at the expansion for the Ricci scalar (A4). Since the Ricci scalar is constant for the solution of Eq. (2.9), the subleading terms in the expansion for R should vanish. In the first subleading order, as it is seen from (A4), this gives constraint

$$\mathrm{Tr}\, g_{(2)} = -\tfrac{1}{2}R(g_{(0)}). \tag{2.12}$$

Note that hereafter we define trace with the help of metric $g_{(0)}$. Now looking at the Eq. (2.9) for components $(\mu\nu) = (ij)$, we find that the leading term vanishes identically and the first subleading term vanishes provided constraint (2.12) is taken into account. Thus, no new constraint on $g_{(2)}$ appears. Further order terms in the expansion give constraint on higher order terms $g_{(2k)}$, $k > 2$. At present we are not interested in those terms. A simple relation of this sort comes from the component $(\mu\nu) = (rr)$ of Eq. (2.9),

$$\mathrm{Tr}\, g_{(4)} = \tfrac{1}{4}\mathrm{Tr} g_{(2)}^2 - \beta\epsilon^{ij}\nabla_i\nabla_k g^k_{(2)j}, \tag{2.13}$$

and indicates that (2.11) is not a "total square" as it happened to be in the case of pure GR [5]. The most important constraint comes from components $(\mu\nu) = (r,i)$ of the equation (2.9). As it follows from (A4) and (B3), we have that

$$-\nabla_j g^j_{(2)i} + \partial_i \mathrm{Tr} g_{(2)} + \beta[\epsilon_i{}^j(-\nabla_k g^k_{(2)j} + \partial_j \mathrm{Tr} g_{(2)})] = 0. \tag{2.14}$$

This can be represented in the form

$$\nabla_j t^j{}_i = \frac{\beta}{2}\epsilon_i{}^j\partial_j \mathrm{Tr} g_{(2)}, \tag{2.15}$$

where we have introduced symmetric tensor

$$t_{ij} = g^{(2)}_{ij} - g^{(0)}_{ij}\mathrm{Tr} g_{(2)} + \frac{\beta}{2}(\epsilon_i{}^k g_{(2)kj} + \epsilon_j{}^k g_{(2)ki}). \tag{2.16}$$

Equations (2.12) and (2.14) are the only restrictions on coefficient $g^{(2)}_{ij}$. Obviously, we cannot redefine t_{ij}, provided it remains symmetric, to include the right-hand side of (2.15) so that t_{ij} would be covariantly conserved. That it is impossible means that in fact we deal with an anomaly. Indeed, the holographic boundary stress tensor defined as

$$T_{ij} = \frac{1}{8\pi G_N}t_{ij} \tag{2.17}$$

has both conformal and gravitational anomalies

$$\mathrm{Tr}\, T = \frac{l}{16\pi G_N}R, \qquad \nabla_j T^j{}_i = -\frac{\beta}{32\pi G_N}\epsilon_i{}^j\partial_j R. \tag{2.18}$$

When $\beta = 0$, the stress tensor defined as in (2.17) agrees with the stress tensor introduced earlier in [4,5,8]. In particular, this fixes the coefficient in front of (2.17). The stress tensor (2.17) also agrees with the one suggested in [13].[3] We see that the conformal anomaly is not affected by the presence of the Chern-Simons term. Taking that conformal symmetry on the boundary of AdS appears as part of bulk diffeomorphisms [7] which are broken by the gravitational Chern-Simons term, it is rather nontrivial that the conformal anomaly remains unchanged. On the other hand, the gravitational anomaly (2.18) is entirely due to the Chern-Simons.

The holographic stress tensor is defined as variation of the on-shell gravitational action (which may include also the boundary counterterms) with respect to the boundary metric. In a theory with covariant and gauge invariant action, the tensor defined this way is covariantly conserved. The theory described by the gravitational action (2.1) is however not gauge invariant. The Chern-Simons (CS) term is invariant under the local Lorentz transformations only up to the boundary terms, the noninvariance of the action thus is concentrated on the boundary, and this leads to the anomalous nonconservation of the holographic stress tensor. This is most obvious in the case when the CS term is expressed in terms of the Christofell symbols rather than in terms of spin connection. The CS action then is Lorentz invariant but not coordinate invariant. Actually, looking at how the action changes under the coordinate transformations in this case, one can get the explicit structure of the nonconservation of the stress tensor. This is an alternative to the approach in this paper. This analysis was done by Kraus and Larsen [13] and is not repeated here. Expressing the CS term in terms of the spin connection as in (2.3), one has the action which is coordinate invariant but not invariant under the local Lorentz transformations. One then gets the stress tensor with either the gravitational anomaly or the Lorentz anomaly depending on whether the stress tensor in question is the metric stress tensor or a *zweibein* stress tensor.

The stress tensor T_{ij} which appears in (2.17) and (2.18) is what is usually called the metric stress tensor defined as variation of the action with respect to the boundary metric g^{ij}. Since we have at our disposal the objects h^a_i (and their inverse h^i_a) which are the "square root" of metric, $g^{ij} = h^i_a h^{aj}$, we can define what might be called the *zweibein* stress tensor $T^a{}_i$ considering variation of the action with respect to the *zweibein* h^i_a. Obviously, we have $\delta/(\delta h^i_a) = 2h^{aj}\frac{\delta}{\delta g^{ij}}$ and hence $T^a{}_i = 2h^{aj}T_{ij}$. In the Lorentz invariant case antisymmetric part $T^{[a,b]}$, where $T^{ab} = h^{bi}T^a{}_i$, vanishes. For the price of losing the local Lorentz symmetry the *zweibein* stress tensor $T^a{}_i$ can be redefined so that the new stress tensor would be covariantly conserved. Indeed, a new stress tensor

$$\hat{T}^a{}_i = T^a{}_i + \frac{\beta l}{16\pi G_N}\epsilon^a{}_i R, \qquad \nabla^j \hat{T}^a{}_j = 0 \tag{2.19}$$

is covariantly conserved. However, anomaly does not disappear. It reappears as the local Lorentz anomaly. Indeed, we have for the new tensor

$$\epsilon_a{}^i \hat{T}^a{}_i = \frac{\beta}{8\pi G_N} R \tag{2.20}$$

that is a clear violation of the local Lorentz symmetry under which $\delta h^i_a = \delta\phi \epsilon_a{}^b h^i_b$. This is of course well known: the coordinate invariance can be restored for the price of losing the local Lorentz invariance.

It is of obvious interest to analyze the gravitational anomaly which may appear in higher dimension $d = 4n + 2$ when gravitational Chern-Simons term[4] is added to the $(d+1)$-dimensional Einstein-Hilbert action. This is studied in [20].

III. REMARKS ON ANOMALIES IN TWO DIMENSIONS

A. Local counterterms, conformal and Lorentz anomalies

Once the Lorentz symmetry is broken anyway, it is allowed to add local counterterms to the boundary action that are not Lorentz invariant. Appropriate counterterms depend on the *zweibein* h^a_i, $a = 1, 2$ rather than on the metric. It is interesting that by adding such local counterterms we can shift the value of the conformal anomaly—the possibility which we did not have when dealing only with metric. The counterterm of this sort was suggested in [21]

$$I_{\rm ct} = \frac{1}{4}\int d^2x h C^a_{ij} C^{ij}_a, \tag{3.1}$$

where $C^a_{ij} = \partial_i h^a_j - \partial_j h^a_i$ is the anholonomity object for the *zweibein* h^a_i, $a = 1, 2$ on the boundary and $h = \det h^a_i$. Notice that this term added on the regulated boundary (at fixed value of radial coordinate r) is finite in the limit when r is infinite.

The Lorentz group in two dimensions is Abelian so that the Lorentz connection has only one component,

$$\omega^a{}_{b,i} = \epsilon^a{}_b \omega_i, \qquad \omega_i = \tfrac{1}{2}\omega_{ab,i}\epsilon^{ab}.$$

Under local Lorentz and conformal transformation $\delta h^a_i = \delta\sigma h^a_i + \delta\phi \epsilon^a{}_b h^b_i$, the Lorentz connection transforms as

$$\delta\omega_i = \partial_i \delta\phi + \epsilon_i{}^j \partial_j \delta\sigma. \tag{3.2}$$

[3]Note that our coupling β differs from the one used in [13], the exact relation being $\beta = 32\pi G_N \beta_{\rm KL}$.

[4]We mean here the Chern-Simons term for the local Lorentz group SO(1, d). Other possible Chern-Simons terms, for instance for group SO(2, d), do not seem to produce gravitational anomaly on the boundary of AdS [12].

The counterterm (3.1) changes as follows:

$$\delta I_{\rm ct} = \frac{1}{2} \int d^2 x h [\delta\sigma R + \delta\phi K]. \tag{3.3}$$

R is the two-dimensional Ricci scalar which can be expressed in terms of the Hodge dual to the Lorentz connection one-form

$$R = 2\nabla_i(\tilde{\omega}^i), \qquad \tilde{\omega}^i = \epsilon^i{}_j \omega^j.$$

The quantity K that appears in (3.3) has a similar expression in terms of the Lorentz connection itself:

$$K = 2\nabla_i(\omega^i). \tag{3.4}$$

It is invariant under conformal transformations and changes under the local Lorentz transformations. There is a certain similarity between R and K well discussed in [21]. It is important that both R and K may appear in conformal and/or Lorentz anomaly. Obviously, adding (3.1) with an appropriate coefficient to the boundary effective action, we can always shift the value of the conformal anomaly and even remove it completely. As a price for that, the quantity K would appear in the Lorentz anomaly.

B. Stress-energy tensor from anomalies

The two-dimensional black hole can be put on the boundary of the three-dimensional anti-de Sitter, the two-dimensional Hawking effects then would be encoded in the bulk three-dimensional geometry [5]. It is well known that conformal anomaly plays an important role in two dimensions and eventually is responsible for the Hawking effect. An important element in this demonstration [22] is the observation that the conformal anomaly can be integrated to determine the covariantly conserved stress-energy tensor. In this subsection, we analyze whether this is still true when the gravitational anomaly is present. Thus, we would like to see whether the equations

$$T_{ij}g^{ij} = aR, \qquad \nabla_j T^j{}_i = -b\epsilon_i{}^j\partial_j R, \tag{3.5}$$

where a and b are some constants, can be integrated and determine the stress-energy tensor T_{ij}. Constants a and b can be further related to the central charge in left- and right-moving sectors of two-dimensional theory as we discuss in Sec. V. The exact relation is $a = \frac{c_+}{24\pi}$ and $b = \frac{c_-}{48\pi}$, $c_\pm = (c_L \pm c_R)/2$.

We start with the two-dimensional static metric in the Schwarzschild-like form

$$ds^2 = -g(x)dt^2 + \frac{1}{g(x)}dx^2, \tag{3.6}$$

where $g(x)$ is some function of the spatial coordinate x. The only nonvanishing Christoffel symbols for this metric are

$$\Gamma^t_{tx} = \frac{g'}{2g}, \qquad \Gamma^x_{tt} = \frac{gg'}{2},$$

where $g' = \partial_x g$, and the scalar curvature takes the simple form $R = -g''(x)$. Assuming that components of the stress tensor T_{ij} do not depend on time t, we get that Eqs. (3.5) are equivalent to a set of differential equations:

$$T^x_x + T^t_t = -ag'' \qquad \partial_x T^x_x + \frac{g'}{2g}(T^x_x - T^t_t) = 0 \qquad \partial_x T^x_t = bgg'''. \tag{3.7}$$

We chose orientation in which $\epsilon^{tx} = +1$ when deriving (3.7). These equations can be solved and the solution reads

$$T^t_t = a\left(-g'' + \frac{g'^2}{4g} + \frac{C_1}{g}\right) \qquad T_{xt} = b\left(g'' - \frac{g'^2}{2g} + \frac{C_2}{g}\right), \tag{3.8}$$

where C_1 and C_2 are integration constants. The Hawking temperature of the two-dimensional black hole is $T_H = g'(x_+)/4\pi$, where x_+ is the location of horizon defined as a simple root of function $g(x_+) = 0$. The condition of regularity (see for instance [23]) of T^t_t and T_{xt} at horizon fixes the constants $C_1 = -\frac{1}{4}g'^2(x_+)$ and $C_2 = \frac{1}{2}g'^2(x_+)$. The stress tensor thus can be uniquely reproduced from the anomaly equations (3.5). We see that the gravitational anomaly shows up only in the component T_{xt} which is now nonvanishing and proportional to b. If the two-dimensional space-time is asymptotically flat, i.e. $g(x) \to 1$ when $x \to \infty$, then (3.8) describes at infinity a nonvanishing flow

$$T_{tt} = c_+ \frac{\pi}{6} T_H^2, \qquad T_{xt} = c_- \frac{\pi}{6} T_H^2 \tag{3.9}$$

due to the Hawking particles radiated by the black hole.

IV. BLACK HOLE ENTROPY FROM GRAVITATIONAL CHERN-SIMONS TERM

The gravitational Chern-Simons term is a legitimate action for gravitational field. It produces covariant field equations which might have sensible solutions.[5] In particular, the constant curvature space-time is always the solution of these equations and remains to be a solution when the gravitational dynamics is governed, as in (2.1), by the sum of Einstein-Hilbert action and the Chern-Simons term. The BTZ black hole is thus a solution to the equations (2.7), as was first noted in [24]. On the other hand, it is well known that the expression for the Bekenstein-Hawking entropy is modified if gravitational action is nonlinear or even a nonlocal function of curvature. In general, the entropy is not just a quarter of the horizon area but depends also on the way the horizon is embedded in the space-time. It is

[5]The corresponding equations of motion $C_{\mu\nu} = 0$ are satisfied for any conformally flat 3D metric.

thus an interesting question whether the gravitational Chern-Simons (2.3) leads to any modifications of the entropy. The tricky point here is that the Chern-Simons is defined with respect to the Lorentz connection so that one might worry whether the corresponding entropy is gauge invariant. In this section we analyze this issue.

There are various ways to compute the entropy for a given gravitational action. The most popular is the Wald's Noether charge method [25]. It is however an on-shell method which is valid on the equations of motion. Below we use another method which is universal and does not rely on the equations of motion. This is the method of conical singularity [23,26,27]. The idea is to allow the black hole to have a temperature different from the Hawking one. In the Euclidean description this leads to the appearance of deficit angle $\delta = 2\pi(1-\alpha)$, $\alpha = T_H/T$ at horizon Σ. The geometry of manifolds with conical singularities was analyzed in detail in [27]. In particular, it was found that components of the Riemann tensor contain a singular, delta-function–like, part

$$R^{\alpha\beta}{}_{\mu\nu} = (R^{\alpha\beta}{}_{\mu\nu})_{\rm reg} + 2\pi(1-\alpha)[(n^\alpha n_\mu)(n^\beta n_\nu) - (n^\alpha n_\nu)(n^\beta n_\mu)]\delta_\Sigma, \tag{4.1}$$

where $(R^{\alpha\beta}{}_{\mu\nu})_{\rm reg}$ is the nonsingular part of the curvature; $(n^\alpha n_\mu) = n_1^\alpha n_\mu^1 + n_2^\alpha n_\mu^2$, n_1 and n_2 is a pair of vectors normal to Σ and orthogonal to each other. Obtained originally in [27] for the static nonrotating metric, this formula was later shown in [28] to be correct in the case of the stationary metric.

Taking into account (4.1), the gravitational action in question is now a function of α. The entropy then is defined as

$$S = \left(\alpha\frac{\partial}{\partial\alpha} - 1\right)\Big|_{\alpha=1} I_{\rm gr}(\alpha). \tag{4.2}$$

Applying this formula to the Chern-Simons term (2.3), we get[6]

$$S_{\rm CS} = -\frac{\beta}{8G_N}\int_\Sigma \omega_{ab,\sigma}h^a_\alpha h^b_\beta \epsilon^{\mu\nu\sigma}(n^\alpha n_\mu)(n^\beta n_\nu) \tag{4.3}$$

for the entropy. Note that indices a, b run values from 1 to 3. In the case of the $(2+1)$-dimensional black hole, the horizon Σ is a circle. Suppose φ is the angular coordinate on Σ, then vector ∂_φ is orthogonal to n_1 and n_2. We assume that vector ∂_φ together with vector ∂_τ form a pair of Killing vectors at horizon. (Outside the horizon, the Killing vectors are linear combinations of these two vectors.) It follows that the integrand in (4.3) is nonvanishing only if index $\sigma = \varphi$. Introducing $\hat\epsilon^{\alpha\beta} = \epsilon^{\mu\nu\varphi}(n^\alpha n_\mu)(n^\beta n_\nu)$, expression (4.3) can be rewritten as

$$S_{\rm CS} = -\frac{\beta}{8G_N}\int_\Sigma \omega_{ab,\varphi}h^a_\alpha h^b_\beta \hat\epsilon^{\alpha\beta}. \tag{4.4}$$

As far as we are aware, the result (4.3) and (4.4) for the Chern-Simons entropy is new. Under local Lorentz transformations parametrized by Ω_{ab}, the expression (4.4) changes as

$$\delta S_{\rm CS} = -\frac{\beta}{8G_N}\int_\Sigma [\partial_\varphi(\Omega_{ab})\hat\epsilon^{ab}]\gamma d\varphi, \tag{4.5}$$

where γ is an induced measure on Σ. Since ∂_φ is the Killing vector the quantity $\hat\epsilon^{ab} = h^a_\alpha h^b_\beta\hat\epsilon^{\alpha\beta}$, being considered on Σ, does not depend on φ. Therefore, integrating by parts in (4.5) we find that $\delta S_{\rm CS} = 0$, i.e. entropy (4.4) is Lorentz invariant in spite of the fact that the Lorentz connection enters explicitly in (4.4).

A. The BTZ black hole

The BTZ black hole is an important, and in fact the only one known, example of black hole in three dimensions.[7] Therefore it is interesting to see how our formulas work in this case. The orthonormal basis $e^a = h^a_\mu dx^\mu$ for the BTZ metric is

$$e^1 = \sqrt{f(r)}dt, \qquad e^2 = \frac{1}{\sqrt{f(r)}}dr, \qquad e^3 = r(d\varphi + N(r)dt), \tag{4.6}$$

where

$$f(r) = \frac{r^2}{l^2} - \frac{j^2}{r^2} - m = \frac{(r^2-r_+^2)(r^2-r_-^2)}{l^2r^2}, \qquad N(r) = -\frac{j}{r^2}. \tag{4.7}$$

We have that

$$m = \frac{r_+^2 + r_-^2}{l^2}, \qquad j = \frac{r_+r_-}{l}. \tag{4.8}$$

Here we work in the Lorentzian signature. The analytic continuation to the Euclidean signature was analyzed in [30,31]. The vectors orthogonal to the horizon are

$$n_1 = \frac{1}{\sqrt{f}}(\partial_t - N\partial_\varphi), \qquad n_2 = \sqrt{f}\partial_r \tag{4.9}$$

so we have that

[6]For spin connection, one has that $\omega = \omega_{\rm reg} + \omega_{\rm sing}$ so that $R_{\rm reg} = d\omega_{\rm reg} + \cdots$ is regular and $R_{\rm sing} = d\omega_{\rm sing}$ is the singular part in (4.1). Therefore, schematically, one has $\int\omega R = \int\omega_{\rm reg}R_{\rm reg} + 2\int\omega_{\rm reg}R_{\rm sing}$. This gives an extra factor of 2 when (4.1) is applied to action (2.3).

[7]For a recent review on the BTZ black hole, conformal field theory, and three-dimensional gravity, see [29].

$$(n^t n_t) = (n^r n_r) = 1, \qquad (n^\phi n_t) = -N(r_+),$$
$$\hat{\epsilon}^{tr} = \frac{1}{r_+}, \qquad \hat{\epsilon}^{\varphi r} = \frac{N(r_+)}{r_+}, \qquad \hat{\epsilon}^{12} = \frac{1}{r_+}, \qquad (4.10)$$
$$\hat{\epsilon}^{13} = \hat{\epsilon}^{23} = 0.$$

Taking into account that measure $\gamma = r_+$ on Σ, we find that the expression for entropy takes a simple form:

$$S_{\rm CS} = -\frac{\beta}{4G_N}\int_0^{2\pi} \omega_{12,\varphi} d\varphi. \qquad (4.11)$$

Explicit calculation, making use of Eq. (2.5), shows that

$$\omega_{12,\varphi} = \frac{j}{r_+}. \qquad (4.12)$$

The contribution to the entropy due to the Chern-Simons term

$$S_{\rm CS} = -\frac{\beta}{4G_N}\frac{2\pi r_-}{l} \qquad (4.13)$$

is thus proportional to the area $2\pi r_-$ of the inner horizon. That is a curious property of the gravitational Chern-Simons term. Its entropy is apparently due to degrees of freedom at inner horizon rather that at the horizon which may be seen by an external observer. We will comment on this interesting feature later in the paper. Summing the contributions to the entropy that come from each term in the gravitational action (2.1), the total entropy of the BTZ black hole is

$$S_{\rm BH} = \frac{2\pi r_+}{4G_N} - \frac{\beta}{l}\frac{2\pi r_-}{4G_N}. \qquad (4.14)$$

Depending on the sign of β, the contribution of the Chern-Simons term to the entropy may be negative. This is not a problem as soon as the total entropy (4.14) is positive. This imposes a certain bound on possible values of β. We discuss this in the next section.

V. THE BOUNDARY THEORY CALCULATION

In this section we use the representation when the Lorentz symmetry is broken but the theory is diffeomorphism invariant, so that the stress-energy tensor of the dual theory is covariantly conserved. The boundary theory in question is characterized by different values of the central charge for holomorphic and antiholomorphic fields. The *zweibein* stress tensor of the theory has both conformal and Lorentz anomalies. Summarizing our analysis in Sec. 4, we have

$$h^i_a \hat{T}^a{}_i = \frac{c_+}{12\pi}R, \qquad \epsilon_a{}^i \hat{T}^a{}_i = \frac{c_-}{12\pi}R, \qquad (5.1)$$

where $c_\pm = \frac{1}{2}(c_L \pm c_R)$ and c_L (c_R) is central charge for the left(right)-moving sector. Expressions (2.18) and (2.20) give precise values for the central charge in each sector[8]

[8] A similar shift in central charge was observed in [32] within the Brown-Henneaux approach.

$$c_L = \frac{3}{2}\frac{(l+\beta)}{G_N}, \qquad c_R = \frac{3}{2}\frac{(l-\beta)}{G_N}. \qquad (5.2)$$

The BTZ black hole corresponds to the sector in the boundary theory characterized by conformal weights [1]

$$h_L = \frac{Ml - J}{2}, \qquad h_R = \frac{Ml + J}{2} \qquad (5.3)$$

that are determined by mass M and angular momentum J of the black hole. These two parameters are the integrals

$$M = l\int_0^{2\pi} d\varphi T_{tt}, \qquad J = -l\int_0^{2\pi} d\varphi T_{t\varphi}$$

of the components of the stress tensor defined in (2.16) and (2.17). The coefficients in the Fefferman-Graham expansion of BTZ metric are collected in Appendix C. These are needed for computing the stress tensor using (2.16) and (2.17). We then get

$$M = M_0 - \frac{\beta}{l^2}J_0, \qquad J = J_0 - \beta M_0, \qquad (5.4)$$

where quantities

$$M_0 = \frac{r_+^2 + r_-^2}{8G_N l^2}, \qquad J_0 = \frac{r_+ r_-}{4G_N l} \qquad (5.5)$$

are values of mass and angular momentum in the absence of the Chern-Simons term. The shift (5.4) has been recently found in [13]. In fact it was known for some time (see [32,33]) that mass and angular momentum in topologically massive gravity are linear combinations of mass and angular momentum obtained in pure GR.

The entropy in the boundary theory is computed by the Cardy formula

$$S_{\rm CFT} = 2\pi\left(\sqrt{\frac{c_L h_L}{6}} + \sqrt{\frac{c_R h_R}{6}}\right). \qquad (5.6)$$

Plugging here the known values for the central charge and conformal weight in each sector, we find

$$S_{\rm CFT} = \frac{2\pi r_+}{4G_N} - \frac{\beta}{l}\frac{2\pi r_-}{4G_N} \qquad (5.7)$$

that is in perfect agreement with the black hole entropy (4.14) computed in the previous section. When the bulk theory is pure GR the agreement is well known [34]. The gravitational and CFT entropies still agree when the gravitational Chern-Simons term is added in the bulk, as we have just shown. Notice that this bulk theory is much richer than GR since it now contains propagating degrees of freedom.

Apparently, large values (of any sign) of the coupling β are not allowed in the theory. There are two obvious signals of instability for large β. Central charge in either of the two sectors may become negative. Also, entropy becomes negative when β is "too large." These bad things do not happen if parameter β is within the range

$$|\beta| \leq l. \tag{5.8}$$

This "stability bound" guarantees that both the bulk theory with the gravitational Chern-Simons term and the boundary CFT with $c_L \neq c_R$ are well defined.

VI. DOES THE CHERN-SIMONS TERM LOOK DEEP INTO THE BLACK HOLE?

In theories of gravity involving higher powers of Riemann tensor, the black hole entropy is no longer the usual $A/4G_N$ and is always modified. This is well known and quite well understood. We refer the reader to [35] for the Noether charge calculation and to [27] for the calculation that uses the conical singularity method. For black holes arising in string theory this issue was much studied, see review in [36]. For the BTZ black hole and higher-dimensional black holes that reduce to BTZ, this issue was studied in [37] and recently in [38]. We would like to discuss here some interesting peculiarities of higher curvature modifications of the entropy of the BTZ black hole.

The general action of the local theory of gravity with higher derivatives can be represented as a power series in Riemann curvature. This in fact is true also for a nonlocal theory, however each term in such an expansion then would contain nonlocal factors. Keeping theory local, the quadratic term in our action would be something like this:

$$W = \int \left(\frac{a_1}{24\pi} R^2 + \frac{a_2}{16\pi} R^2_{\mu\nu} + \frac{a_3}{16\pi} R^2_{\alpha\beta\mu\nu} \right). \tag{6.1}$$

The corresponding contribution (see [27,35] for more detail) to the entropy is

$$S = -\int_\Sigma \left(\frac{a_1}{3} R + \frac{a_2}{4} R_{\mu\nu}(n^\mu n^\nu) + \frac{a_3}{2} R_{\mu\nu\alpha\beta}(n^\mu n^\alpha)(n^\nu n_\beta) \right) \tag{6.2}$$

as can be easily obtained using the method outlined in Sec. IV. Applying this to the BTZ black hole, we notice that the BTZ metric is locally AdS and hence the Riemann tensor factorizes $R_{\alpha\beta\mu\nu} = \frac{1}{l^2}(G_{\beta\mu}G_{\alpha\nu} - G_{\alpha\mu}G_{\beta\nu})$. This factorization and that vectors n_1 and n_2 are orthonormal lead to an interesting conclusion that nothing in the integrand in (6.2) depends on the parameters of the black hole. Those parameters enter (6.2) only via the area of Σ, i.e. via r_+,

$$S = (a_1 + a_2 + a_3)\frac{2\pi r_+}{l}. \tag{6.3}$$

Obviously this property remains in place when higher powers of curvature are included in the action. In fact we can state that any local theory of gravity that is nonlinear in curvature results in the entropy which takes the form

$$S_{\rm non} = \mu(a_i, l)\frac{2\pi r_+}{l}, \tag{6.4}$$

where $\mu(a_i, l)$ is some function of higher curvature couplings a_i and the AdS scale l but not of the parameters of black hole. A similar result was recently derived in [38]. The higher derivative theory of gravity, provided it is formulated in terms of gauge invariant objects, i.e. the Riemann tensor, thus sees only the radius r_+ of outer horizon of the BTZ black hole and leaves r_- unnoticed.[9]

The gravitational Chern-Simons term, as we have seen in Sec. V, shows radically different behavior. Its entropy is proportional to the area $2\pi r_-$ of the inner horizon so that it is r_+ that is now unnoticed. This is despite the fact that the entropy is actually given by integral (4.3) over the outer horizon. This is an interesting feature of the gravitational Chern-Simons term that it seems to see the interior of the black hole. The Lorentz connection apparently does the trick. Most dramatically this feature manifests itself when the gravitational action contains the Chern-Simons term only. The BTZ metric is still a solution to the field equation. Its temperature, mass, and angular momentum are nonvanishing and, hence thermodynamically, there must be some entropy and that entropy is precisely $S_{\rm CS}$ defined in (4.13) and determined by the area of inner horizon. Notice that it constitutes the entire entropy of the black hole in this case. This observation poses interesting questions that may be challenging to our present understanding of the black hole entropy. The obvious one is whether there should be some degrees of freedom associated with the inner (rather than with the outer) horizon which would be responsible for this entropy? We leave this and other questions for the future.

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APPENDIX A: CURVATURE COMPONENTS AND THEIR EXPANSION

Components of the Riemann tensor are

$$\begin{aligned} R^r{}_{irj} &= \tfrac{1}{2}[-g'' + \tfrac{1}{2}g'g^{-1}g']_{ij} \\ R^r{}_{ikj} &= -\tfrac{1}{2}[\nabla_k g'_{ij} - \nabla_j g'_{ik}] \\ R^l{}_{ikj} &= R^l{}_{ikj}(g) - \tfrac{1}{4}g'_{ij}g^{ln}g'_{nk} + \tfrac{1}{4}g'_{ik}g^{ln}g'_{nj}, \end{aligned} \tag{A1}$$

where $g' \equiv \partial_r g$. Components of Ricci tensor are

[9] In general, this may be different in the case of the nonlocal theory of gravity. Such a theory may produce logarithmic terms in the entropy and both $\ln r_+$ and $\ln r_-$ are *a priori* possible. The concrete calculation in [31] however shows that to the leading order such an entropy is determined by r_+ only, r_- appearing in the subleading terms.

$$
\begin{aligned}
R_{ij} &= R_{ij}(g) - \tfrac{1}{2}g''_{ij} - \tfrac{1}{4}g'_{ij}\,\mathrm{Tr}(g^{-1}g') + \tfrac{1}{2}(g'g^{-1}g')_{ij}\\
R_{ri} &= \tfrac{1}{2}[\nabla^k(g^{-1}g')_{ki} - \nabla_i\,\mathrm{Tr}(g^{-1}g')]\\
R_{rr} &= -\tfrac{1}{2}\mathrm{Tr}(g^{-1}g'') + \tfrac{1}{4}\mathrm{Tr}(g^{-1}g'g^{-1}g')
\end{aligned} \tag{A2}
$$

and the Ricci scalar is

$$
\begin{aligned}
R = R(g) - \mathrm{Tr}(g^{-1}g'') - \tfrac{1}{4}[\mathrm{Tr}(g^{-1}g')]^2\\
+ \tfrac{3}{4}\mathrm{Tr}(g^{-1}g'g^{-1}g').
\end{aligned} \tag{A3}
$$

The leading terms in the Fefferman-Graham expansion of the curvature tensors are

$$
\begin{aligned}
R_{ri} &= [-\nabla_n g^n_{(2)i} + \partial_i\,\mathrm{Tr}g_{(2)}]e^{-2r} + \cdots\\
R^k{}_i &= -2\delta^k_i + [R^k_i(g_{(0)}) + \delta^k_i\,\mathrm{Tr}g_{(2)}]e^{-2r} + \cdots\\
R_{rr} &= -2 + [-4\,\mathrm{Tr}g_{(4)} + \mathrm{Tr}g^2_{(2)}]e^{-4r} + \cdots\\
R &= -6 + [R(g_{(0)}) + 2\,\mathrm{Tr}g_{(2)}]e^{-2r} + \cdots.
\end{aligned} \tag{A4}
$$

For the constant curvature $R = -6$ metric, we have a constraint

$$
\mathrm{Tr}\, g_{(2)} = -\tfrac{1}{2}R_{(0)}. \tag{A5}
$$

APPENDIX B: COMPONENTS OF THE COTTON TENSOR AND THEIR EXPANSION

In space-time with constant Ricci scalar $R = -6$, the Cotton tensor is defined as

$$
C_{\alpha\beta} = \epsilon_\alpha{}^{\mu\nu}\nabla_\mu R_{\nu\beta}. \tag{B1}
$$

For the Levi-Cività symbol we have that $\epsilon^{rij} = \epsilon^{ij}$ where ϵ^{ij} is defined for the 2D metric $g_{ij}(r, x)$.

In terms of $g_{ij}(r, x)$, we get for the components of (B1)

$$
\begin{aligned}
C_{ri} &= -\epsilon_n{}^k\nabla_k R^n_i + \tfrac{1}{2}\epsilon^{kn}g'_{ki}R_{rn}\\
C_{rr} &= \epsilon^{ij}[\nabla_i R_{rj} - \tfrac{1}{2}(g^{-1}g')^k_i R_{kj}]\\
C_{ij} &= -\epsilon_i{}^k[\partial_r R_{kj} - \tfrac{1}{2}(g^{-1}g')^n_j R_{kn} - \nabla_k R_{rj} - \tfrac{1}{2}g'_{kj}R_{rr}].
\end{aligned} \tag{B2}
$$

Taking into account the constraint (A5), we find the following expansion for the components of the Cotton tensor:

$$
\begin{aligned}
C_{ri} &= \epsilon_i{}^j[-\nabla_k g^k_{(2)j} + \partial_j\,\mathrm{Tr}g_{(2)}]e^{-2r} + \cdots\\
C_{rr} &= -\epsilon^{ij}\nabla_i\nabla_k g^k_{(2)j}e^{-4r} + \cdots\\
C_{ij} &= 0 + O(e^{-2r}),
\end{aligned} \tag{B3}
$$

where the leading term (of order e^{0r}) in the expansion of C_{ij} vanishes due to constraint (A5).

APPENDIX C: THE BTZ METRIC IN NORMAL COORDINATES

The BTZ metric can be brought to the normal coordinates in the form (2.10) as follows:

$$
\begin{aligned}
ds^2 = dr^2 - \left(\frac{r_+^2}{l^2}\sinh^2\frac{r}{l} - \frac{r_-^2}{l^2}\cosh^2\frac{r}{l}\right)dt^2\\
+ \left(\frac{r_+^2}{l^2}\cosh^2\frac{r}{l} - \frac{r_-^2}{l^2}\sinh^2\frac{r}{l}\right)d\varphi^2 - \frac{2r_+r_-}{l}\,dtd\varphi,
\end{aligned} \tag{C1}
$$

where r_+ (r_-) is radius of outer (inner) horizon. The coefficients in the expansion (2.11) of this metric are

$$
\begin{aligned}
g^{(0)}_{tt} = -\frac{1}{l^2}g^{(0)}_{\varphi\varphi} = -\frac{(r_+^2 - r_-^2)}{4l^2},\\
g^{(2)}_{tt} = \frac{1}{l^2}g^{(2)}_{\varphi\varphi} = \frac{(r_+^2 + r_-^2)}{2l^2}, \qquad g^{(2)}_{t\varphi} = -\frac{r_+r_-}{l}.
\end{aligned} \tag{C2}
$$

We choose orientation in which $\epsilon_t{}^\varphi = -1/l$.

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