Conformally flat noncircular spacetimes

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The general metric for conformally flat stationary cyclic symmetric noncircular spacetimes is explicitly derived. In spite of the complexity (due to the noncircular contributions) of the partial differential equations arising from the conformal flatness constraints, they allow for their complete integration in terms of elementary functions. A canonical coordinate representation of the search metric structure is given. Conditions for the existence of a rotation axis (axisymmetry) are established; these conditions coincide with those which restrict the studied noncircular class of spacetimes to be static. As a consequence, a previously known theorem by Collinson is just a part of a more general result: any conformally flat stationary cyclic symmetric spacetime (even a noncircular one) is additionally axisymmetric if and only if it is also static. Recent astrophysical motivations point in the direction of considering stationary noncircular configurations to describe magnetized neutron stars.

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I. INTRODUCTION

One of the questions still open in general relativity is the determination of enough general interior configurations describing isolated rotating bodies supporting their corresponding exterior gravitational fields. Usually, the description of these rotating configurations is done by means of circular spacetimes, i.e., stationary axisymmetric spacetimes where the metric, in addition to being time and rotation-angle independent, also possesses as isometries the inversions of time and of the rotation angle. It is worthwhile to mention that almost all the interior stationary axisymmetric configurations reported in the literature belong to this class, and most of them suffer from some deficiency.

Nevertheless, there is no room for the circular idealization when considering rotating neutron stars [1] surrounded by strong toroidal magnetic fields ranging from $\sim 10^{16}$ to 10^{17} G; see also [2,3] and references therein. The circularity condition is a very severe restriction, which fails to hold in spacetimes allowing the existence of toroidal magnetic fields and meridional flows [3]. Thus, to deal with such astrophysical configurations one has to abandon the fulfillment of the circularity condition and consider in consequence the wider noncircular class of spacetimes.

Besides the above astrophysically motivated reasons to study noncircular configurations, there are also purely theoretical ones. As soon as Schwarzschild published his exterior spherically symmetric static solution, he was able to determine its interior solution modeled through a perfect fluid with homogeneous density. Later, in light of the Petrov classification, it was established that the Schwarzschild solution belongs to Petrov type D, while the interior Schwarzschild solution falls in the conformally flat family. In 1973 Kerr reported his famous stationary axisymmetric gravitational field corresponding to a field created by a rotating body; this solution also belongs to Petrov type D. The search for the interior solution to the exterior Kerr metric began after that time. Collinson established that a conformally flat stationary axisymmetric spacetime is necessarily static [4]. Some stationary axisymmetric Petrov-D metrics (coupled to perfect fluid distributions) have been reported in the literature, but none of them allows for the matching with the Kerr metric. Recently, the results of Ref. [5] indicate that the matching of an interior noncircular spacetime with an exterior circular one is at least technically possible. This fact opens up the possibility of searching for interior solutions within the noncircular class.

Moreover, the general metric for conformally flat stationary cyclic symmetric circular metric has been reported recently [6]; see, in this respect, also [7,8]. For this metric, being cyclic but not axisymmetric, because of the lack of a rotation axis, the circularity theorem [9] does not hold. The next step in complexity, which is the main goal of the present work, is to determine the metric for conformally flat stationary cyclic symmetric noncircular spacetimes. In particular, from this more general result one is able to derive the particular circular branch, and if one requires the existence of an axis of symmetry one recovers the staticity property of the considered class of metrics, thus the Collinson theorem is just included within a more general result.

In the next section the mathematical preliminaries needed in order to study the spacetimes under consideration are introduced. Specifically, the physical and geomet-

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rical details behind the concepts of stationarity, cyclic symmetry, axisymmetry, circularity, and staticity are clearly stated in order to make the work self-contained. In Sec. III the conformal flatness conditions, consisting in the vanishing of the complex Weyl components, are fully integrated for any stationary cyclic symmetric noncircular spacetime. Section IV is devoted to revising the conditions guaranteeing staticity of the obtained spacetimes in their canonical representation. In Sec. V the conditions for the existence of the axis of symmetry for the derived spacetimes are analyzed. It is concluded that both properties—the staticity and existence of a rotation axis—restrict the metric to become circular and static; these conclusions are stated in Sec. VII.

II. STATIONARY CYCLIC SYMMETRIC SPACETIMES

In this section we characterize stationary cyclic symmetric spacetimes; see, for example, Ref. [10] for the original definitions. A spacetime is *stationary* if it admits an asymptotically timelike Killing field. A spacetime is called *cyclic symmetric* if it is invariant under the action of the one-parameter group $SO(2)$; it is assumed that the corresponding Killing field m with closed integral curves is spacelike. A cyclic symmetric spacetime is named *axisymmetric* if the fixed point set of the $SO(2)$ action, i.e., the *rotation axis*, is nonempty. A spacetime is called *stationary cyclic symmetric (axisymmetric)* if it is both stationary and cyclic symmetric (axisymmetric) and if the Killing fields k and m commute.

A stationary cyclic symmetric (axisymmetric) spacetime is said to be *circular* if the 2-dimensional surfaces orthogonal to the Killing fields k and m are integrable. This is equivalent to satisfying the Frobenius integrability conditions

$$m \wedge k \wedge dk = 0, \qquad k \wedge m \wedge dm = 0. \tag{1}$$

The circularity property means that locally the gravitational field is not only independent of time and the rotation angle, but, it is also invariant under the simultaneous inversion of time and the angle. Almost all the literature related to stationary cyclic symmetric (axisymmetric) spacetimes concerns only the circular case. This is, in part, for simplicity, since in this case it is possible to use the Lewis-Papapetrou ansatz for the metric.

As a last definition, a stationary spacetime is said to be *static* if the Killing field k is hypersurface orthogonal. This occurs if and only if it satisfies

$$k \wedge dk = 0, \tag{2}$$

and it is equivalent to demanding that locally the gravitational field is not only time independent but it is also invariant under time reversal.

In this work we are interested in noncircular spacetimes, i.e., general stationary cyclic symmetric spacetimes not necessarily restricted to satisfying the Frobenius integrability conditions (1). The metric of such spacetimes can be written as

$$g = e^{-2Q}\Big[-\tfrac{1}{a+b}(d\tau + a d\sigma + \mathrm{Im}(M dz))(d\tau - b d\sigma - \mathrm{Im}(N dz)) + e^{-2P} dz d\bar{z}\Big], \tag{3}$$

where a, b, P, and Q are real functions and M and N are complex ones. Here the bar means complex conjugation, and Im (respectively, Re) denotes the imaginary (respectively, real) part of a complex quantity. Because the Killing fields realizing the stationary and cyclic isometries are $k = \partial_\tau$ and $m = \partial_\sigma$, all functions appearing above depend on the coordinates z and $\bar{z}$ only. The above metric has eight independent real functions; hence, the diffeomorphism invariance still allows for two remaining gauge choices. Notice that this metric (3) is invariant under the coordinate transformations

$$(\tau, \sigma, z, \bar{z}) \rightarrow \Big(\tau, \sigma, \int \gamma(z) dz, \int \bar{\gamma}(\bar{z}) d\bar{z}\Big), \tag{4}$$

together with the redefinitions

$$P \rightarrow P + \ln\sqrt{\gamma(z)\bar{\gamma}(\bar{z})}, \qquad M \rightarrow \frac{M}{\gamma(z)}, \qquad N \rightarrow \frac{N}{\gamma(z)}. \tag{5}$$

This gauge of freedom will be exploited in the forthcoming section to bring the final result into an optimal form.

The noncircularity property of the metric (3) becomes apparent from the nonvanishing of the following quantities:

$$*(m \wedge k \wedge dk) = \frac{e^{2(P-Q)}}{2(a+b)} \mathrm{Re}\Big(\frac{\partial(M-N)}{\partial\bar{z}} - \frac{M+N}{a+b}\frac{\partial(a-b)}{\partial\bar{z}}\Big), \tag{6}$$

$$*(k \wedge m \wedge dm) = \frac{e^{2(P-Q)}}{2(a+b)} \mathrm{Re}\Big(a\frac{\partial M}{\partial\bar{z}} + b\frac{\partial N}{\partial\bar{z}} - \frac{M+N}{a+b}\frac{\partial(ab)}{\partial\bar{z}}\Big), \tag{7}$$

where the star stands for the Hodge dual operation.

To evaluate the Weyl tensor components, it is more convenient to use the Newman-Penrose formalism. Writing the metric (3) as

$$g = 2e^1e^2 - 2e^3e^4, \tag{8}$$

with respect to the complex null tetrad basis

$$e^1 = \frac{1}{\sqrt{2}} e^{-Q-P} dz, \tag{9a}$$

$$e^2 = \frac{1}{\sqrt{2}} e^{-Q-P} d\bar{z}, \tag{9b}$$

$$e^3 = \frac{1}{\sqrt{2}} \frac{e^{-Q}}{\sqrt{a+b}} \left(d\tau - b d\sigma - \frac{N dz - \bar{N} d\bar{z}}{2i} \right), \tag{9c}$$

$$e^4 = \frac{1}{\sqrt{2}} \frac{e^{-Q}}{\sqrt{a+b}} \left(d\tau + a d\sigma + \frac{M dz - \bar{M} d\bar{z}}{2i} \right), \tag{9d}$$

one evaluates, via well-known standard procedures, the corresponding Weyl complex components:

$$\Psi_0 = \frac{2e^{2(Q+P)}}{a+b} \left[\frac{\partial^2 a}{\partial z^2} + 2 \frac{\partial P}{\partial z} \frac{\partial a}{\partial z} - \frac{2}{a+b} \left(\frac{\partial a}{\partial z} \right)^2 \right],$$

$$\bar{\Psi}_4 = \frac{2e^{2(Q+P)}}{a+b} \left[\frac{\partial^2 b}{\partial z^2} + 2 \frac{\partial P}{\partial z} \frac{\partial b}{\partial z} - \frac{2}{a+b} \left(\frac{\partial b}{\partial z} \right)^2 \right],$$

$$2\Psi_1 = -\frac{i e^{2Q+P}}{\sqrt{a+b}} \left[\frac{\partial}{\partial z} (e^{2P} \mathcal{M}) - \frac{e^{2P}}{a+b} (\mathcal{M} - 3\mathcal{N}) \frac{\partial a}{\partial z} \right],$$

$$2\bar{\Psi}_3 = \frac{i e^{2Q+P}}{\sqrt{a+b}} \left[\frac{\partial}{\partial z} (e^{2P} \mathcal{N}) - \frac{e^{2P}}{a+b} (\mathcal{N} - 3\mathcal{M}) \frac{\partial b}{\partial z} \right],$$

$$6\Psi_2 = \frac{2e^{2(Q+P)}}{(a+b)^2} \left[2(a+b)^2 \frac{\partial^2 P}{\partial z \partial \bar{z}} + 5 \frac{\partial a}{\partial z} \frac{\partial b}{\partial \bar{z}} - \frac{\partial a}{\partial \bar{z}} \frac{\partial b}{\partial z} \right] - \frac{4e^{2Q+4P}}{a+b} \mathcal{M}\mathcal{N}, \tag{10}$$

where for further convenience the following auxiliary functions have been introduced:

$$\mathcal{M} := \mathrm{Re}\left(\frac{\partial M}{\partial \bar{z}} - \frac{M+N}{a+b} \frac{\partial a}{\partial \bar{z}} \right),$$
$$\mathcal{N} := \mathrm{Re}\left(\frac{\partial N}{\partial \bar{z}} - \frac{M+N}{a+b} \frac{\partial b}{\partial \bar{z}} \right). \tag{11}$$

Moreover, these functions can be used to describe the noncircularity properties of the metric; from Eq. (6) one obtains

$$*(m \wedge k \wedge dk) = \frac{e^{2(P-Q)}}{2(a+b)} (\mathcal{M} - \mathcal{N}),$$
$$*(k \wedge m \wedge dm) = \frac{e^{2(P-Q)}}{2(a+b)} \left(a\mathcal{M} + b\mathcal{N} + \frac{1}{2} \mathrm{Re}\left(\frac{M+N}{a+b} \frac{\partial}{\partial \bar{z}} (a-b)^2 \right) \right). \tag{12}$$

For vanishing functions M and N, which in turn yield $\mathcal{M}$ and $\mathcal{N}$ equal to zero, one gets back the circular case.

III. SOLVING THE CONFORMAL FLATNESS CONSTRAINTS

In order to determine the general class of conformally flat stationary cyclic symmetric metrics, we demand the vanishing of all the Weyl complex components $\Psi_0 = \Psi_4 = \Psi_2 = \Psi_1 = \Psi_3 = 0$.

The complex components Ψ_0 and Ψ_4 occur to be equal to the corresponding ones of the circular case ($M = 0 = N$) studied in Refs. [4,6]. Hence, one can follow the same strategy applied in [6] to integrate the equations for Ψ_0 and Ψ_4; the vanishing of the combinations

$$\Psi_0 - \bar{\Psi}_4 = 2(a+b) e^{2Q} \frac{\partial}{\partial z} \left(\frac{e^{2P}}{(a+b)^2} \frac{\partial (a-b)}{\partial z} \right) = 0,$$
$$\Psi_0 \frac{\partial b}{\partial z} + \bar{\Psi}_4 \frac{\partial a}{\partial z} = 2(a+b) e^{2(Q-P)} \frac{\partial}{\partial z} \left(\frac{e^{4P}}{(a+b)^2} \frac{\partial a}{\partial z} \frac{\partial b}{\partial z} \right) = 0 \tag{13}$$

gives rise to the following first order partial differential equations for the structural functions $a(z)$ and $b(z)$:

$$\frac{\partial a}{\partial z} - \frac{\partial b}{\partial z} = \bar{g}(\bar{z})(a+b)^2 e^{-2P}, \tag{14}$$

$$\frac{\partial a}{\partial z} \frac{\partial b}{\partial z} = \bar{h}(\bar{z}) \bar{g}^2(\bar{z})(a+b)^2 e^{-4P}, \tag{15}$$

where $g(z)$ and $h(z)$ are integration functions.

Because of the real character of the functions a, b, and P, Eq. (14) implies

$$g(z) \frac{\partial}{\partial z} (a-b) = \bar{g}(\bar{z}) \frac{\partial}{\partial \bar{z}} (a-b). \tag{16}$$

Using the freedom of the coordinate z, Eq. (4), together with the redefinition of P from Eq. (5), by choosing $\gamma = g$, without any loss of generality one can set $g(z) = 1$ in Eq. (14), and consequently the same substitution can be done in Eq. (16), which yields

$$a(z, \bar{z}) - b(z, \bar{z}) = F(z + \bar{z}). \tag{17}$$

Substituting the function $b(z, \bar{z})$ from Eq. (17) into the imaginary part of the component Ψ_2,

$$\mathrm{Im}\,(\Psi_2) = \frac{e^{2(Q+P)}}{i(a+b)^2} \left(\frac{\partial a}{\partial z} \frac{\partial b}{\partial \bar{z}} - \frac{\partial a}{\partial \bar{z}} \frac{\partial b}{\partial z} \right) = 0, \tag{18}$$

one arrives at

$$\frac{dF}{d(z+\bar{z})} \left(\frac{\partial a}{\partial z} - \frac{\partial a}{\partial \bar{z}} \right) = 0; \tag{19}$$

therefore, if $F \neq$ const then necessarily $a = a(z + \bar{z})$, and $b = b(z + \bar{z})$, and as a consequence of Eq. (14) one concludes that $P = P(z + \bar{z})$. As far as the function $\bar{h}(\bar{z})$ appearing in Eq. (15) is concerned, because of the dependence of a, b, and P on the variable $z + \bar{z}$, one easily establishes, using Eq. (15), that $\bar{h}(\bar{z}) = h(z) = \text{const} \equiv \epsilon k^2$, where $\epsilon \equiv \pm 1$ just encodes the signs of the constant. The relevance of both signs of ϵ was pointed out in Ref. [6], and they were used to establish new results.

Introducing the real and imaginary parts of $z = x + iy$ as coordinates, and the notations for the real and imaginary parts of $M(x, y) = M_R(x, y) + iM_I(x, y)$ and $N(x, y) = N_R(x, y) + iN_I(x, y)$, the metric (3) can be written as

$$g = e^{-2Q}\Big[-\frac{1}{a+b}(d\tau + ad\sigma + (M_R dy + M_I dx))(d\tau - bd\sigma - (N_R dy + N_I dx)) + e^{-2P}(dx^2 + dy^2)\Big]. \tag{20}$$

Now, we apply the same strategy of Ref. [6] to solve Eqs. (14) and (15), which are now expressed as

$$\frac{da}{dx} - \frac{db}{dx} = (a+b)^2 e^{-2P}, \tag{21}$$

$$\frac{da}{dx}\frac{db}{dx} = \epsilon k^2 (a+b)^2 e^{-4P}. \tag{22}$$

First we redefine the functions a and b through X and Y by

$$\begin{aligned} a+b &= 2kY, \qquad a = k(Y+X),\\ a-b &= 2kX, \qquad b = k(Y-X). \end{aligned} \tag{23}$$

Using these new functions X and Y, Eqs. (21) and (22) are rewritten correspondingly as

$$\frac{dX}{dx} = 2kY^2 e^{-2P}, \tag{24}$$

$$\Big(\frac{dY}{dx}\Big)^2 - \Big(\frac{dX}{dx}\Big)^2 = 4\epsilon k^2 Y^2 e^{-4P}. \tag{25}$$

Equation (24) suggests choosing a new coordinate x' defined by

$$x' = 2k\int Y^2 e^{-2P} dx. \tag{26}$$

Consequently,

$$dx = \frac{1}{2k}Y^{-2}e^{2P}dx', \qquad \frac{d}{dx} = 2kY^2 e^{-2P}\frac{d}{dx'}. \tag{27}$$

Thus Eqs. (24) and (25) become

$$\frac{dX}{dx'} = 1, \tag{28}$$

$$\Big(\frac{dY}{dx'}\Big)^2 = \frac{\epsilon + Y^2}{Y^2}. \tag{29}$$

The general solutions of Eqs. (28) and (29) are $X(x') = x'$ and $Y(x') = \sqrt{(x' - x'_0)^2 - \epsilon}$. Without any loss of generality one can equate x'_0 to zero by replacing $x' \to x' + x'_0$ and $\tau \to \tau - \sigma x'_0$. Hence,

$$X(x') = x', \qquad Y(x') = \sqrt{x'^2 - \epsilon}. \tag{30}$$

Returning to the complex conformal coefficients, from the vanishing of $\mathrm{Re}(\Psi_1)$ and $\mathrm{Re}(\Psi_3)$,

$$\begin{aligned} \mathrm{Re}(2\bar{\Psi}_3) &= \frac{ie^{2Q+3P}}{2\sqrt{a+b}}\Big[2\mathcal{N}\Big(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}}\Big)P - \Big(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}}\Big)\mathcal{N} - \frac{\mathcal{N} - 3\mathcal{M}}{a+b}\Big(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}}\Big)b\Big],\\ \mathrm{Re}(2\Psi_1) &= -\frac{ie^{2Q+3P}}{2\sqrt{a+b}}\Big[2\mathcal{M}\Big(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}}\Big)P - \Big(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}}\Big)\mathcal{M} - \frac{\mathcal{M} - 3\mathcal{N}}{a+b}\Big(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}}\Big)a\Big], \end{aligned} \tag{31}$$

one concludes that

$$\mathcal{M} = \mathcal{M}(z + \bar{z}), \qquad \mathcal{N} = \mathcal{N}(z + \bar{z}). \tag{32}$$

Introducing two new auxiliary functions

$$\alpha = \frac{e^{2P}}{a+b}\mathcal{M}, \qquad \beta = \frac{e^{2P}}{a+b}\mathcal{N}, \tag{33}$$

from the vanishing of the full expression of Ψ_1 and Ψ_3, one obtains first order differential equations for the functions $\alpha(z + \bar{z})$ and $\beta(z + \bar{z})$, correspondingly,

$$\begin{aligned} (a+b)\frac{\partial\alpha}{\partial z} + \alpha\frac{\partial b}{\partial z} + 3\beta\frac{\partial a}{\partial z} &= 0,\\ (a+b)\frac{\partial\beta}{\partial z} + \beta\frac{\partial a}{\partial z} + 3\alpha\frac{\partial b}{\partial z} &= 0. \end{aligned} \tag{34}$$

In terms of the variable x' [Eq. (26)] and functions $X(x')$ and $Y(x')$ [Eq. (30)] one has

$$\begin{aligned} 2Y\frac{d\alpha}{dx'} + \alpha\frac{d}{dx'}(Y - X) + 3\beta\frac{d}{dx'}(Y + X) &= 0,\\ 2Y\frac{d\beta}{dx'} + \beta\frac{d}{dx'}(Y + X) + 3\alpha\frac{d}{dx'}(Y - X) &= 0. \end{aligned} \tag{35}$$

Explicitly,

$$\begin{aligned} 2Y^2\frac{d\alpha}{dx'} + \alpha(x' - Y) + 3\beta(x' + Y) &= 0,\\ 2Y^2\frac{d\beta}{dx'} + \beta(x' + Y) + 3\alpha(x' - Y) &= 0. \end{aligned} \tag{36}$$

Introducing a pair of new functions $A(x')$ and $B(x')$ through

$$\alpha(x') = A(x') + B(x'), \qquad \beta(x') = A(x') - B(x'), \tag{37}$$

by adding and subtracting the above differential equations, one arrives at the simple system

$$Y^2\frac{d}{dx'}A(x') + 2x'A(x') - 2YB(x') = 0, \tag{38}$$

$$Y^2\frac{d}{dx'}B(x') - x'B(x') + YA(x') = 0, \tag{39}$$

which can be solved by using the method of increasing the order of the differential equations; differentiating Eq. (39) one arrives at

$$Y^2 \frac{d^2}{dx'^2} B + x' \frac{d}{dx'} B - B + Y \frac{d}{dx'} A + \frac{x'}{Y} A = 0. \quad (40)$$

Substituting into this equation the derivative $\frac{d}{dx'} A$ from Eq. (38) followed by the substitution of $A(x')$ from Eq. (39), one gets

$$(x'^2 - \epsilon)^2 \frac{d^2}{dx'^2} B + x'(x'^2 - \epsilon) \frac{d}{dx'} B - \epsilon B = 0, \quad (41)$$

which allows for a further simplification by replacing $B(x')$ as $B(x') = V(x')/\sqrt{x'^2 - \epsilon}$, namely

$$\frac{d^2}{dx'^2} V(x') = 0, \qquad \rightarrow V(x') = C_1 x' + C_0. \quad (42)$$

Consequently, using Eq. (39) to evaluate $A(x')$, one has

$$B(x') = \frac{C_1 x' + C_0}{\sqrt{x'^2 - \epsilon}}, \qquad A(x') = \frac{(x'^2 + \epsilon) C_1 + 2x' C_0}{x'^2 - \epsilon}. \quad (43)$$

From Eq. (37) one determines the search functions α and β, namely

$$\begin{aligned} \alpha(x') &= \left(\frac{x'^2 + \epsilon}{x'^2 - \epsilon} + \frac{x'}{\sqrt{x'^2 - \epsilon}} \right) C_1 \\ &\quad + \left(\frac{2x'}{x'^2 - \epsilon} + \frac{1}{\sqrt{x'^2 - \epsilon}} \right) C_0, \\ \beta(x') &= \left(\frac{x'^2 + \epsilon}{x'^2 - \epsilon} - \frac{x'}{\sqrt{x'^2 - \epsilon}} \right) C_1 \\ &\quad + \left(\frac{2x'}{x'^2 - \epsilon} - \frac{1}{\sqrt{x'^2 - \epsilon}} \right) C_0. \end{aligned} \quad (44)$$

On the other hand, from the equation

$$\begin{aligned} \mathrm{Re}(3\Psi_2) = 2 \frac{e^{2Q+2P}}{a+b} \Bigg[& (a+b) \frac{\partial^2 P}{\partial z \partial \bar{z}} \\ & + \frac{1}{a+b} \left(\frac{\partial a}{\partial z} \frac{\partial b}{\partial \bar{z}} + \frac{\partial a}{\partial \bar{z}} \frac{\partial b}{\partial z} \right) - e^{2P} \mathcal{M} \mathcal{N} \Bigg] = 0, \end{aligned} \quad (45)$$

one establishes that

$$Y^2 \frac{d^2}{dx'^2} (Y e^{-2P}) + \frac{4}{k} \alpha(x') \beta(x') = 0. \quad (46)$$

The integral of the above equation yields

$$e^{-2P(x')} = \frac{K_1 x' + K_0}{Y} - \frac{4}{kY} \iint \frac{\alpha \beta}{Y^2} dx' dx'. \quad (47)$$

Replacing $\alpha(x')$ and $\beta(x')$ from Eq. (44) one arrives at

$$e^{-2P(x')} = \frac{K_1 x' + K_0}{Y} - \frac{2}{kY} \frac{2x' C_0 C_1 + \epsilon C_1^2 + C_0^2}{x'^2 - \epsilon}, \quad (48)$$

which, by replacing $K_0 \rightarrow K_0 - 2C_1^2/k$, can be brought to the form

$$e^{-2P(x')} = \frac{K_1 x' + K_0}{Y(x')} - \frac{2}{kY(x')} B(x')^2. \quad (49)$$

Still it remains to integrate the first order equations contained in $\mathcal{M}$ and $\mathcal{N}$ for the real functions M_R, N_R, M_I, and N_I, which explicitly amount to

$$\begin{aligned} 2\mathcal{M} &= 4kY e^{-2P} \alpha(x') \\ &= 2kY^2 e^{-2P(x} \left[\frac{\partial}{\partial x'} M_R - \frac{1}{2Y} \left(\frac{x'}{Y} + 1 \right) (M_R + N_R) \right] \\ &\quad - \frac{\partial}{\partial y} M_I, \\ 2\mathcal{N} &= 4kY e^{-2P} \beta(x') \\ &= 2kY^2 e^{-2P} \left[\frac{\partial}{\partial x'} N_R - \frac{1}{2Y} \left(\frac{x'}{Y} - 1 \right) (M_R + N_R) \right] \\ &\quad - \frac{\partial}{\partial y} N_I. \end{aligned} \quad (50)$$

At this stage it will be useful to introduce the definitions

$$\tilde{N}_I := e^{2P(x')} \frac{N_I}{2kY}, \qquad \tilde{M}_I := e^{2P(x')} \frac{M_I}{2kY}. \quad (51)$$

Substituting in the metric (20) the primed coordinate x', $dx' = 2kY^2 e^{-2P} dx$, and using Eq. (51), dropping the primes in what follows , the general conformally flat metric is rewritten as

$$\begin{aligned} g = e^{-2Q} \Bigg[& -\frac{1}{a+b} (d\tau + a d\sigma + (M_R dy + \tilde{M}_I dx)) \\ & \times (d\tau - b d\sigma - (N_R dy + \tilde{N}_I dx)) \\ & + \frac{1}{4k^2} Y^{-4} e^{2P} dx^2 + e^{-2P} dy^2 \Bigg]. \end{aligned} \quad (52)$$

The remaining equations for the unknown functions $M_R(x, y)$, $N_R(x, y)$ and $\tilde{N}_I(x, y)$, $\tilde{N}_I(x, y)$ are given by

$$\begin{aligned} & \frac{\partial}{\partial x} M_R - \frac{1}{2Y} \left(\frac{x}{Y} + 1 \right) (M_R + N_R) - \frac{\partial}{\partial y} \tilde{M}_I \\ & \quad = \frac{2}{Y} (A(x) + B(x)), \\ & \frac{\partial}{\partial x} N_R - \frac{1}{2Y} \left(\frac{x}{Y} - 1 \right) (M_R + N_R) - \frac{\partial}{\partial y} \tilde{M}_I \\ & \quad = \frac{2}{Y} (A(x) - B(x)). \end{aligned} \quad (53)$$

Thus, there are various possibilities for searching for the solutions of this system.

A. Nondiagonal (xy)-metric sector

Let us start by considering the functions $M_R(x, y)$ and $N_R(x, y)$ as the ones to be determined. Introducing the functions

$$2F_+(x,y) := M_R + N_R, \qquad 2F_-(x,y) := M_R - N_R, \tag{54}$$

and adding Eqs. (53), one obtains

$$(x^2-\epsilon)\frac{\partial}{\partial x}\frac{F_+}{\sqrt{x^2-\epsilon}} = \left(2A(x) + \frac{\partial}{\partial y}\tilde{M}_I + \frac{\partial}{\partial y}\tilde{N}_I\right), \tag{55}$$

with the general solution

$$F_+(x,y) = -2B(x) + \sqrt{x^2-\epsilon}f_+(y) + \sqrt{x^2-\epsilon}\int\frac{1}{x^2-\epsilon}\left(\frac{\partial}{\partial y}\tilde{M}_I + \frac{\partial}{\partial y}\tilde{N}_I\right)dx. \tag{56}$$

Next, subtracting Eqs. (53) one arrives at

$$\sqrt{x^2-\epsilon}\frac{\partial}{\partial x}F_- = 2B(x) + F_+ + \frac{\partial}{\partial y}\tilde{M}_I - \frac{\partial}{\partial y}\tilde{N}_I. \tag{57}$$

Using the expression of F_+ from Eq. (56) and integrating, one gets

$$F_-(x,y) = f_+(y)x + \int\frac{1}{\sqrt{x^2-\epsilon}}\left(\frac{\partial}{\partial y}\tilde{M}_I - \frac{\partial}{\partial y}\tilde{N}_I\right)dx + \iint\frac{1}{x^2-\epsilon}\left(\frac{\partial}{\partial y}\tilde{M}_I + \frac{\partial}{\partial y}\tilde{N}_I\right)dxdx. \tag{58}$$

Notice that the function $f_+(y)$, entering in M_R as $f_+(y)a(x)$ and in N_R as $-f_+(y)b(x)$, can be eliminated from the metric by a shifting of the coordinate σ, $d\sigma \to d\sigma - f_+(y)dy$. Therefore the structural functions M_R and N_R are

$$M_R(x,y) = -2B(x) + Y\int\frac{1}{Y}\left(\frac{\partial}{\partial y}\tilde{M}_I + \frac{\partial}{\partial y}\tilde{N}_I\right)dx + \int\frac{1}{Y}\left(\frac{\partial}{\partial y}\tilde{M}_I - \frac{\partial}{\partial y}\tilde{N}_I)dx + \iint\frac{1}{Y^2}\left(\frac{\partial}{\partial y}\tilde{M}_I + \frac{\partial}{\partial y}\tilde{N}_I\right)dxdx, \tag{59}$$

$$N_R(x,y) = -2B(x) + Y\int\frac{1}{Y}\left(\frac{\partial}{\partial y}\tilde{M}_I + \frac{\partial}{\partial y}\tilde{N}_I\right)dx - \int\frac{1}{Y}\left(\frac{\partial}{\partial y}\tilde{M}_I - \frac{\partial}{\partial y}\tilde{N}_I\right)dx - \iint\frac{1}{Y^2}\left(\frac{\partial}{\partial y}\tilde{M}_I + \frac{\partial}{\partial y}\tilde{N}_I\right)dxdx \tag{60}$$

which, together with the functions

$$Y(x) = \sqrt{x^2-\epsilon}, \qquad a(x) = k(Y+x), \qquad b(x) = k(Y-x), \qquad e^{-2P(x)} = \frac{K_1x+K_0}{Y(x)} - \frac{2}{kY(x)}\left(\frac{C_1x+C_0}{\sqrt{x^2-\epsilon}}\right)^2, \tag{61}$$

determine completely the conformally flat noncircular metric

$$g = e^{-2Q}\left[-\frac{1}{a+b}(d\tau + ad\sigma + (M_Rdy + \tilde{M}_Idx)) \times (d\tau - bd\sigma - (N_Rdy + \tilde{N}_Idx)) + \frac{1}{4k^2}Y^{-4}e^{2P}dx^2 + e^{-2P}dy^2\right], \tag{62}$$

where the functions $\tilde{M}_I(x,y)$ and $\tilde{N}_I(x,y)$ still remain free.

A second possible representation can be achieved by integrating Eq. (53) for the imaginary parts $\tilde{M}_I(x,y)$ and $\tilde{N}_I(x,y)$ of $M(x,y)$ and $N(x,y)$, respectively, arriving at

$$\tilde{M}_I(x,y) = -2\frac{\alpha(x)}{Y}y + m(x) + \int\left[\frac{\partial}{\partial x}M_R - \frac{1}{2Y}\left(\frac{x}{Y}+1\right)(M_R+N_R)\right]dy,$$
$$\tilde{N}_I(x,y) = -2\frac{\beta(x)}{Y}y + n(x) + \int\left[\frac{\partial}{\partial x}M_R - \frac{1}{2Y}\left(\frac{x}{Y}-1\right)(M_R+N_R)\right]dy, \tag{63}$$

where $m(x)$ and $n(x)$ are integration functions, while the functions $M_R(x',y)$ and $N_R(x',y)$ remain as free functions. Via an appropriate choice of new σ and τ coordinates, one can set $m(x)$ and $n(x)$ to zero. Therefore, in this coordinate gauge, the search metric structure can be represented by the metric (62) with structural functions $\alpha(x)$ and $\beta(x)$ given by unprimed Eqs. (44) and structural functions $a(x)$, $b(x)$, $P(x)$, and $Y(x)$ given by Eq. (61), together with $\tilde{M}_I(x,y)$ and $\tilde{M}_I(x,y)$ from Eqs. (63), while the structural functions $Q(x,y)$, $M_R(x,y)$, and $N_R(x,y)$ remain as free functions.

The noncircularity properties, guaranteed by the nonvanishing of Eqs. (12), can be written in terms of $\alpha(x)$ and $\beta(x)$.

According to a general theorem of geometry, any 2-dimensional metric $g(x,y)$ can be locally brought to the diagonal form, i.e., to its canonical representation, by means of coordinate transformations involving both x and y coordinates.

B. Canonical diagonal (xy) sector; $g_{xy}=0$

One might choose a canonical gauge of coordinates x and y for which the metric (xy) sector assumes a diagonal form, that is, with vanishing metric component g_{xy}.

This representation can also be achieved from the above general metric expression Eq. (62) by equating $\tilde{M}_I(x,y)$ and $\tilde{N}_I(x,y)$ to zero. In this way one arrives at the canonical expression

$$g = e^{-2Q}\left[-\frac{1}{a+b}(d\tau + ad\sigma + M_Rdy)(d\tau - bd\sigma - N_Rdy) + \frac{1}{4k^2}Y^{-4}e^{2P}dx^2 + e^{-2P}dy^2\right], \tag{64}$$

with structural functions

$$
\begin{aligned}
&Y(x)=\sqrt{x^2-\epsilon}, \qquad B(x)=\frac{C_1x+C_0}{\sqrt{x^2-\epsilon}},\\
&a=k(Y+x), \qquad b=k(Y-x),\\
&M_R(x)=-2B(x), \qquad N_R(x)=-2B(x),\\
&e^{-2P(x)}=\frac{K_1x+K_0}{Y(x)}-\frac{2}{kY(x)}B(x)^2.
\end{aligned} \tag{65}
$$

As a last step, in order to write the obtained metric in a simple form, we carry out the following coordinate transformation, $(\tau,\sigma,x,y)\rightarrow(\tau/\sqrt{2},\sigma/(\sqrt{2}k),x,y/(2k))$, together with the following redefinition of the conformal factor, $Q\rightarrow Q-\frac{1}{2}\ln(4k^2Y)$, and also simple redefinitions of the involved constants, $(C_1,C_0)\rightarrow(kC_1/\sqrt{2},kC_0/\sqrt{2})$. The general canonical form of the conformally flat stationary cyclic symmetric metric can be given as

$$
\begin{aligned}
g=e^{-2Q(x,y)}\bigg(&-k(d\tau^2+2xd\tau d\sigma+\epsilon d\sigma^2)\\
&+\frac{dx^2}{(K_0+K_1x)(x^2-\epsilon)-k(C_0+C_1x)^2}\\
&-2k(C_0+C_1x)d\sigma dy+(K_0+K_1x)dy^2\bigg).
\end{aligned} \tag{66}
$$

It is easy to note that for $C_0=0=C_1$ we recover the circular metrics of Ref. [6]. Instead, for $C_0\neq0$ and $C_1\neq0$ the above metric is noncircular as follows from the nonvanishing quantities,

$$
\begin{aligned}
*(m\wedge k\wedge dk)&=k^2e^{-2Q}(C_0+C_1x),\\
*(k\wedge m\wedge dm)&=k^2e^{-2Q}(\epsilon C_1+C_0x).
\end{aligned} \tag{67}
$$

Since one can always achieve the canonical representation of the studied metric structures, it will be enough to establish the properties of staticity and axisymmetry, if any, with respect to the canonical metric (66).

IV. STATICITY

As it was defined in Sec. II, the spacetimes derived in the previous section would be static if there exists a timelike linear combination of the Killing fields,

$$
k_s=A_0\frac{\partial}{\partial\tau}+B_0\frac{\partial}{\partial\sigma}, \tag{68}
$$

satisfying the staticity condition (2). For metric (66) such a condition becomes

$$
\begin{aligned}
0&=*(k_s\wedge dk_s)\\
&=ke^{-2Q(x,y)}[kB_0(xC_b+C_a)d\tau-kA_0(xC_b+C_a)d\sigma\\
&\quad+((K_1x+K_0)C_c+kB_0(C_1x+C_0)C_b)dy],\\
C_a&:=C_0A_0-\epsilon C_1B_0, \qquad C_b:=C_1A_0-C_0B_0,\\
C_c&:=A_0^2-\epsilon B_0^2.
\end{aligned} \tag{69}
$$

It is straightforward to realize that we are in the presence of static configurations only if $C_a=0=C_b=C_c$, which yields

$$
\epsilon=1, \qquad A_0=\pm B_0, \quad \text{and} \quad C_1=\pm C_0. \tag{70}
$$

In such case, the hypersurface orthogonal Killing fields are proportional to $k_s=\partial/\partial\tau\pm\partial/\partial\sigma$.

V. AXISYMMETRY

Now we turn our attention to the existence of a rotation axis, i.e., if there are conformally flat stationary axisymmetric configurations within the class (66). It follows from the definition of Sec. II that the rotation axis is the spacetime region where the cyclic Killing field m vanishes. For metric (66), its general stationary and cyclic Killing fields are a linear combination of the vectors ∂_τ and ∂_σ. Hence, performing the transformation ($\tau=\alpha t+\beta\phi,\sigma=\gamma t+\delta\phi$), where $\alpha\delta-\beta\gamma\neq0$, the Killing fields are written as $k=\partial_t$ and $m=\partial_\phi$, respectively. In terms of the new coordinates, all the metric components $g_{\phi\mu}=g(m,\partial_\mu)$ must vanish on the axis, which implies the following set of algebraic equations:

$$
g_{\phi t}=-ke^{-2Q}[(\alpha\delta+\beta\gamma)x+\alpha\beta+\epsilon\gamma\delta]=0, \tag{71a}
$$

$$
g_{\phi\phi}=-ke^{-2Q}(\beta^2+2\beta\delta x+\epsilon\delta^2)=0, \tag{71b}
$$

$$
g_{\phi y}=ke^{-2Q}\delta(C_0+C_1x)=0. \tag{71c}
$$

Isolating x from Eq. (71b) and inserting the result in Eq. (71a), we obtain

$$
\frac{(\alpha\delta-\beta\gamma)(\beta^2-\epsilon\delta^2)}{\beta\delta}=0. \tag{72}
$$

Since $\alpha\delta-\beta\gamma\neq0$ the above equation has nontrivial solutions only if $\epsilon=1$. Using the above conditions in the remaining equation (71c), we obtain

$$
\delta(C_1\mp C_0)=0, \tag{73}
$$

which implies that the related rotation axis must be located at

$$
x=\mp1. \tag{74}
$$

Summarizing, metric (66) describes a stationary axisymmetric spacetime only for $\epsilon=1$ and $C_1=\pm C_0$, i.e., the conditions for the existence of the rotation axis are the same ones that guarantee the static character of the spacetime; see conditions (70).

As a consequence, the previously known theorem by Collinson [4,6] is not only generalized to include configurations which are not necessarily circular, but, it is part of a more general statement: any conformally flat stationary cyclic symmetric spacetime (even a noncircular one) is additionally axisymmetric if and only if it is also static.

VI. CASE $a = b$; THE STATIC METRIC

This section deals with the remaining case, $a - b = F = \text{const}$. By shifting the timelike coordinate and at the same time redefining the function a or b, the constant F can be equated to zero and hence this case is equivalent to having $a = b$.

From the list of the Ψ complex components, Eqs. (10), we see that now $\Psi_0 = \bar{\Psi}_4$. Thus, from the vanishing of the Ψ_0 component one arrives at

$$\Psi_0 = e^{2Q}\frac{\partial}{\partial z}\left(e^{2P}\frac{\partial}{\partial z}\ln a\right) = 0 \Rightarrow \frac{\partial a}{\partial z} = \bar{g}(\bar{z})a e^{-2P}, \quad (75)$$

which, taking into account the real character of a and P, yields

$$\frac{1}{\bar{g}(\bar{z})}\frac{\partial a}{\partial z} = \frac{1}{g(z)}\frac{\partial a}{\partial \bar{z}}. \quad (76)$$

Again, just applying the coordinate transformation [Eqs. (4)] together with the relevant redefinitions of the functions P, M, and N [Eqs. (5)], by choosing $\gamma = g$ one can set $g(z) = 1$. Therefore, by virtue of Eq. (76) the function a depends on $z + \bar{z}$ only, $a(z + \bar{z})$, and as a consequence of Eq. (75) one concludes that P is a function of $z + \bar{z}$ too, or in terms of the real coordinates $x = (z + \bar{z})/2$ and $y = -i(z - \bar{z})/2$, one has $a = a(x)$ and $P = P(x)$. With all generality, we can rewrite Eq. (75) as

$$\frac{da}{dx} = a e^{-2P} \Rightarrow dx = \frac{da}{a}e^{2P}, \quad (77)$$

thus a can be considered as a new spatial coordinate instead of x.

From the vanishing of the imaginary parts of Ψ_1 and Ψ_3 one arrives at

$$\mathcal{M} = \mathcal{M}(z + \bar{z}) = \mathcal{M}(x), \quad \mathcal{N} = \mathcal{N}(z + \bar{z}) = \mathcal{N}(x). \quad (78)$$

In terms of the variable a, from their definitions through the structural functions $M(x, y) = M_R(x, y) + iM_I(x, y)$ and $N(x, y) = N_R(x, y) + iN_I(x, y)$, Eq. (11), one has

$$\begin{aligned} 2\mathcal{M}(a) &= a e^{-2P}\frac{\partial M_R}{\partial a} - \frac{1}{2}(M_R + N_R)e^{-2P} - \frac{\partial M_I}{\partial y} \\ &=: 4a\alpha(a)e^{-2P}, \\ 2\mathcal{N}(a) &= a e^{-2P}\frac{\partial N_R}{\partial a} - \frac{1}{2}(M_R + N_R)e^{-2P} - \frac{\partial N_I}{\partial y} \\ &=: 4a\beta(a)e^{-2P}, \end{aligned} \quad (79)$$

which are constrained to the equations arising from the vanishing of the full expression of Ψ_1 and Ψ_3, namely

$$2a\frac{d\alpha}{dx} + \alpha + 3\beta = 0, \qquad 2a\frac{d\beta}{dx} + \beta + 3\alpha = 0, \quad (80)$$

which allow a straightforward integration,

$$\alpha(a) = C_0 a^{-2} + C_1 a, \qquad \beta(a) = C_0 a^{-2} - C_1 a, \quad (81)$$

where C_0 and C_1 are constants of integration.

The equation for the function P arising from the vanishing of Ψ_2 reads

$$a^4\frac{d^2}{da^2}e^{-2P} + a^3\frac{d}{da}e^{-2P} - a^2 e^{-2P} = -16\left(\frac{C_0^2}{a} - C_1^2 a^5\right), \quad (82)$$

with the general solution

$$e^{-2P} = \frac{K_0}{a} + K_1 a - 2\frac{C_0^2}{a^3} + 2C_1^2 a^3. \quad (83)$$

Finally, introducing the definitions

$$\tilde{M}_I(a, y) = \frac{e^{2P}}{a}M_I(a, y), \qquad \tilde{N}_I(a, y) = \frac{e^{2P}}{a}N_I(a, y), \quad (84)$$

the studied metric is rewritten as

$$\begin{aligned} g = e^{-2Q}\Big[&-\frac{1}{2a}(d\tau + a d\sigma + (M_R dy + \tilde{M}_I da)) \\ &\times (d\tau - a d\sigma - (N_R dy + \tilde{N}_I da)) \\ &+ \frac{1}{a^2}e^{2P}da^2 + e^{-2P}dy^2\Big], \end{aligned} \quad (85)$$

and the integration of Eqs. (79) for M_R and N_R yields

$$\begin{aligned} M_R(a, y) = &-2a\beta + aF_1(y) + F_0(y) \\ &- \frac{1}{2}\iint \frac{1}{a^2}\left(\frac{\partial}{\partial y}\tilde{M}_I(a, y) - \frac{\partial}{\partial y}\tilde{N}_I(a, y)\right)dada \\ &+ \iint \frac{1}{a}\frac{\partial^2}{\partial a \partial y}\tilde{M}_I(a, y)dada, \\ N_R(a, y) = &-2a\alpha + aF_1(y) - F_0(y) \\ &+ \frac{1}{2}\iint \frac{1}{a^2}\left(\frac{\partial}{\partial y}\tilde{M}_I(a, y) - \frac{\partial}{\partial y}\tilde{N}_I(a, y)\right)dada \\ &+ \iint \frac{1}{a}\frac{\partial^2}{\partial a \partial y}\tilde{M}_I(a, y)dada. \end{aligned} \quad (86)$$

It becomes apparent that by shifting the τ and σ coordinates, $d\tau + F_0(y)dy \to d\tau$ and $d\sigma + F_0(y)dy \to d\sigma$, one can eliminate the functions $F_1(y)$ and $F_0(y)$; thus, with all generality, one can set $F_1(y) = 0 = F_0(y)$.

Finally, scaling the Killing coordinates and constants according to $(\tau, \sigma, C_0, C_1) \to (\tau/\sqrt{2}, \sigma/\sqrt{2}, C_0/\sqrt{2}, C_1/\sqrt{2})$, the metric in canonical (ay) coordinates $(g_{ay} = 0)$ for a stationary cyclic symmetric spacetime with $a = b$ amounts to

$$g = e^{-2Q(x,y)}\bigg[-\frac{d\tau^2}{a} + a d\sigma^2 + \frac{1}{a^2}\frac{da^2}{K_0/a + K_1 a - C_0^2/a^3 + C_1^2 a^3} + 2\bigg(C_1 a d\tau - \frac{C_0}{a}d\sigma\bigg)dy + \bigg(\frac{K_0}{a} + K_1 a\bigg)dy^2\bigg]. \quad (87)$$

Notice that the noncircular character of this metric is guaranteed by the nonvanishing of

$$*(m \wedge k \wedge dk) = -2aC_1 e^{-2Q}, \qquad *(k \wedge m \wedge dm) = 2\frac{C_0}{a}e^{-2Q}. \quad (88)$$

For $C_0 = 0 = C_1$ we recover the static metric of Ref. [6]. The above metric describes a static spacetime for $C_1 = 0$. Additionally, it allows for the existence of an axis of rotation, hence this class is in agreement with Collinson's theorem.

VII. CONCLUSIONS

In this paper we study all the stationary cyclic symmetric spacetimes which are at the same time conformally flat. In contrast to previous works on the subject, we also consider noncircular configurations. The conformal flatness condition is imposed on the metric structural functions by demanding the vanishing of the Weyl tensor. The resulting constraints are quite involved due to the inclusion of noncircular contributions. However, one can still accomplish the complete integration of the system of ten nonlinear partial differential equations, as has been shown in detail in Sec. III. The class of obtained spacetimes is fully determined up to an arbitrary conformal factor and it is endowed with several constants. In particular, two constants characterize the noncircular behavior of these spacetimes; when they vanish, we recover the circular configurations obtained in Ref. [6].

The conditions allowing for the existence of a rotation axis in the resulting configurations are also established. Moreover, the static branch of spacetimes is considered too. The set of values of the constant is the same for both physical situations; the resulting spacetimes are axisymmetric if and only if they are also static. Hence, one of the main results of the paper can be summarized by the following statement.

Theorem—Any conformally flat stationary cyclic symmetric spacetime (even a noncircular one) is additionally axisymmetric if and only if it is also static.

Within the properly cyclic symmetric class of metrics (with no rotation axis and, as a consequence of the above theorem, necessarily nonstatic), it should be of interest to investigate what kind of sources can solve Einstein equations. In the case that one could retain the noncircular contributions when matter is present, the derived configurations should be of particular interest for the description of astrophysical relevant objects. Moreover, the present coordinate gauge and null tetrad choice could permit a successful treatment of other Petrov types because of the highly symmetric representation of the Weyl complex components; a research program in this direction is in progress.

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