Why gravitational contraction must be accompanied by emission of radiation in both Newtonian and Einstein gravity

Abhas Mitra*

MPI fur Kernphysik, Saupfercheckweg 1, D-67117 Heidelberg, Germany

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By using virial theorem, Helmholtz and Kelvin showed that the contraction of a bound self-gravitating system must be accompanied by release of radiation energy *irrespective of the details* of the contraction process. This happens because the total Newtonian energy of the system E_N (and not just the Newtonian gravitational potential energy E_g^N) decreases for such contraction. In the era of general relativity (GR) too, it is justifiably believed that gravitational contraction must release radiation energy. However no GR version of (Newtonian) Helmholtz- Kelvin (HK) process has ever been derived. Here, for the first time, we derive the GR version of the appropriate virial theorem and Helmholtz Kelvin mechanism by simply equating the well known expressions for the *gravitational mass* and the *inertial mass* of a spherically symmetric static fluid. Simultaneously, we show that the GR counterparts of global ''internal energy'', ''gravitational potential energy'' and ''binding energy'' are actually different from what have been used so far. Existence of this GR HK process asserts that, in Einstein gravity too, gravitational collapse must be accompanied by emission of radiation irrespective of the *details of the collapse process*.

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I. INTRODUCTION

It is generally believed that General Relativistic (GR) gravitational collapse must be accompanied by the emission of radiation such as photons and neutrinos. There have been many studies on radiative spherical GR collapse and we may recall here only a few of them [1]. Though all such studies vastly differ in their details, nevertheless, all of them indicate that the fluid becomes hotter during the collapse while net amount of radiated energy steadily increases. Surprisingly, though, such effects seem obvious at first sight, one would wonder whether there is a fundamental reason for occurrence of such physical effects during gravitational collapse. Put another way, whether, irrespective of solution of actual collapse equations with their associated assumptions and simplifications, one can demand from a general perspective, that actual gravitational collapse must be accompanied by both emission of radiation and heating up of the fluid. From thermodynamics perspective, one can ask whether the phenomenon of occurrence ''negative specific heat'', known for Newtonian gravity for a long time, must be valid in Einstein gravity too. To appreciate this, let us recall that the specific heat is defined through $C = dQ/dT$ where dQ is the amount of heat injected into the system and dT is the corresponding increment of temperature. Gravitational compression raises the temperature so that $dT > 0$. A negative C would then demand $dQ < 0$ and vice-versa. Hence a negative dQ means loss of heat (radiation) from the system and vice-versa.

We emphasize here that, in a strict sense, this phenomenon of ''negative specific heat'' is known only for weak Newtonian gravity. Intuitively such an effect is expected to be more pronounced for a fluid subject to much stronger Einstein gravity. But the actual fact is that, there is no proper GR theorem which can assert that global Einstein gravity too is marked by the same ''negative specific heat''.

To highlight this, in Sec. I, we will first review the case in the Newtonian gravitation. This would show why Newtonian collapse must be accompanied by radiation howsoever small it may be and why global gravitation is characterized by ''negative specific heat'' in Newtonian case. Then it would be emphasized that a corresponding GR derivation is non existent and accordingly we shall present an exact GR counterpart of this Newtonian process. We would then automatically arrive at global definitions GR Self-Gravitational energy and Binding Energy from the perspective of global energy conservation of a static spherically symmetric fluid. The entire exercise will show why, irrespective of the details, GR collapse must be accompanied by emission of radiation and an increasing fluid temperature.

II. NEWTONIAN GRAVITATIONAL COLLAPSE

As per Chandrasekhar [2], von Helmholtz first proposed in 1854 that contraction of self-gravitating bodies should emit radiation. Few years later, in 1861, Kelvin [3] elaborated on this process of energy generation which may be called as Helmholtz-Kelvin (HK) process. Without going into further historical account, the basic physics behind the H-K process is reviewed below from a relatively modern perspective [4,5]:

If we consider a spherically symmetrical static isotropic fluid in hydrodynamical equilibrium, it follows that

$$E_g^N + 3 \int p dV = 4\pi R^3 p_b \qquad (1)$$

*Electronic address: Abhas.Mitra@mpi-hd.mpg.de;
Also at Theoretical Astrophysics Section, Bhabha Atomic Research Center, Mumbai-400085, India

where, E_g^N is the Newtonian gravitational potential energy, $p = p(r)$ is the isotropic pressure, p_b is the pressure at the boundary $r = R$, and the volume element $dV = 4\pi r^2 dr$. For a laboratory gas sphere, it is possible to have $p(r) \approx$ uniform $\approx p_b$. In such a case, Eq. (1) would reduce to

$$E_g^N + 4\pi R^3 p_b \approx 4\pi R^3 p_b \tag{2}$$

and which would express the obvious fact that for a laboratory gas $E_g^N \approx 0$. Note that it is necessary to have $p_b > 0$ in such a case.

In contrast, if the fluid is assumed to be self-contained, i.e., *bound* by its own gravity then one expects

$$p_b = 0 \tag{3}$$

and obtains the better known form of Eq. (1):

$$E_g^N + 3\int p dV = 0 \tag{4}$$

This is known as static and scalar virial theorem. If the adiabatic index of the fluid is γ, then

$$p = (\gamma - 1)e \tag{5}$$

where e is the internal energy density. For some ideal fluids, γ may also be considered as the "ratio of specific heats". For example, for a monoatomic ideal gas having an equation of state (EOS) $p = nkT$, where n is number density of the monoatomic molecules and k is the Boltzmann constant γ is the ratio of specific heats with a unique value $\gamma \equiv 5/3$. For an ultrarelativistic gas with particle momenta $\to \infty$ or for a pure photon gas, γ is again the ratio of specific heats having the unique value $\gamma \equiv 4/3$ [2]. The internal energy for the entire fluid is

$$U = \int e dV \tag{6}$$

Using Eqs. (5) and (6) in (4), and assuming γ to be uniform, we have

$$E_g^N + 3(\gamma - 1)U = 0 \tag{7}$$

so that

$$U = \frac{-1}{3(\gamma - 1)} E_g^N \tag{8}$$

The total Newtonian energy of the fluid is

$$E_N = E_g^N + U = \frac{3\gamma - 4}{3(\gamma - 1)} E_g^N \tag{9}$$

Note that for the above mentioned ideal gas having an EOS $p = nkT$, Eq. (7) reduces to the more familiar form $E_g^N + 2U = 0$ because here $\gamma = 5/3$. However Eqs. (1)–(9) are valid for any fluid obeying relation (5) and not necessarily by an "ideal gas" alone, as long as the fluid may be assumed to obey an EOS of the form (5). Thus, in principle, γ is arbitrary here subject to general thermodynamical constraints.

In Eq. (9), E_N is the Newtonian binding energy of the fluid and must be negative for a fluid which is already assumed to be *bound*. For attractive gravity, for *any case, bound or unbound*, one must have $E_g^N < 0$. In the present case, as soon as we set $p_b = 0$, we imply the system to be self-bound, and one must have $E_N < 0$. Thus from Eq. (9), it transpires that, one must have $\gamma > 4/3$. A limiting case of $\gamma = 4/3$ would signify a transition to unbound systems. For "unbound systems" one expects to have, $E_N > 0$. If the system would be unbound, one would have $p_b > 0$ and further the system would not be in hydrostatic equilibrium. In such a case, one needs to use dynamical form of virial theorem to study it. Also, by noting Eq. (7), one might think that for unbound systems, one might have $\gamma < 4/3$. But this would be an incorrect conclusion, because as explained above, for unbound systems, Eq. (7) would cease to be valid. It may be borne in mind that γ is an inherent thermodynamical parameter and cannot be dictated by gross global hydrodynamical behavior of the fluid.

A limiting value of $\gamma = 4/3$ corresponds to a situation when the momenta of the constituent particles of the fluid, $P = \infty$, and thus, cannot be strictly realized except for singular situations [6]. For instance note that the critical ultrarelativistic White Dwarf of Chandrasekhar has $R = 0$ because it strictly corresponds to a fluid having $\gamma = 4/3$ [2].

If, *additionally*, the fluid obeys a polytropic equation of state

$$p = K\rho^{\gamma_p} \tag{10}$$

where, K and γ_p are uniform over the fluid, it follows that

$$E_g^N = \frac{-3}{5 - n} \frac{GM^2}{R} \tag{11}$$

where $\gamma_p \equiv 1 + 1/n$ and G is the gravitational constant. Note that, in general, $\gamma \neq \gamma_p$ and only if the fluid is considered to undergo adiabatic change, one would have $\gamma = \gamma_p$. Further, as we would see, all contraction processes are expected to be accompanied by emission of radiation, and they must be nonadiabatic in a strict sense. It may be also mentioned that, in the Newtonian case, the fluid density appearing in Eq. (10) essentially means rest mass density: $\rho = \rho_0$.

If one differentiates Eq. (8), for slow contraction, one will have

$$\frac{dU}{dt} = \frac{-1}{3(\gamma - 1)} \frac{dE_g^N}{dt} \tag{12}$$

Also, from Eq. (11), we see that

$$\frac{dE_g^N}{dt} = \frac{3}{5 - n} \frac{GM^2}{R^2} \dot{R} \tag{13}$$

Since $\dot{R} < 0$ for contraction, while the value of E_g^N decreases during contraction its absolute value $|E_g^N|$ increases. From Eq. (12), we find that, as $|E_g^N|$ increases during such contraction, so does U. However, the amount of total gravitational energy released by the contraction, $|dE_g^N|$, is not fully accounted for by the gain in the value of U:

$$dU = \frac{1}{3(\gamma - 1)}|dE_g^N| < |dE_g^N| \tag{14}$$

For overall energy conservation, it is therefore necessary that the rest of the energy gain

$$\left[1 - \frac{1}{3(\gamma - 1)}\right]|dE_g^N| = \frac{3\gamma - 1}{3(\gamma - 1)}|dE_g^N| = dQ \tag{15}$$

is radiated away by the system. This could have been found directly by differentiating Eq. (9):

$$\frac{dE_N}{dt} = \frac{3\gamma - 4}{3(\gamma - 1)}\frac{dE_g^N}{dt} \tag{16}$$

Using Eq. (13) into above Eq., we find that

$$\frac{dE_N}{dt} = \frac{3\gamma - 4}{3(\gamma - 1)}\frac{3}{5 - n}\frac{GM^2}{R^2}\dot{R} < 0 \tag{17}$$

In case of a gas confined in a laboratory by physical inclosure, $p_b > 0$. One can also imagine the physical inclosure to be a perfect insulator and one may conceive of a radiationless adiabatic contraction for arbitrary $\gamma \geq 4/3$. But in an astrophysical context, there is neither any physical inclosure nor any perfect insulating surrounding. Hence, it appears from Eq. (15) that a strictly adiabatic contraction ($dQ = 0$) would be possible only for the idealized case of $\gamma = 4/3$. And Eq. (9) would show that, in such a case, one would already have $E_N = 0$, i.e., they system would be unbound. In reality, one has $\gamma = 4/3$ only for pure radiation or when the energy of the particles per unit rest mass $E^* = \infty$, which is possible only for a singular situation in case the "gas" is not already a pure radiation [6].

Thus, Eq. (17) shows that the total (Newtonian) energy of the system decreases for contraction and it could be so only if the system radiates appropriate amount of energy. Since U increases, the fluid become hotter while it radiates ($dQ < 0$). Therefore a self-gravitating fluid has a *negative specific heat* and this fact is well known. Note that this result follows from the Newtonian HK process and does not depend on the details of either the physical properties of the fluid or the collapse process.

Since the above result is of generic nature, it is expected to be qualitatively valid even in case of strong gravity. In fact, even after the introduction of General Relativity (GR) into astrophysics, the idea that gravitational contraction must result into radiation output is naturally and justifiably used [1]. While considering the configuration of static fluid spheres in GR, Buchdahl [7] posed the question whether the amount total radiation emitted by gravitational contraction can *even exceed* the initial value of $E = Mc^2$ itself. The answer was in the negative. Yet, for Einstein gravity *no exact counterpart of* Eqs. (12)–(17) *exists*. Thus, we cannot assert, as a principle, that self-gravitating matter has "negative specific heat" in Einstein gravity too. And we want to address this precise aspect in the present paper.

III. STATIC FLUIDS IN GENERAL RELATIVITY

Let us consider a static self-gravitating fluid sphere described by the metric

$$ds^2 = A^2(r)dt^2 - B^2(r)dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{18}$$

Here we have taken $G = c = 1$. Recall that one often uses the symbols $A^2 = e^\nu = g_{00}$ and $B^2 = e^\lambda = -g_{rr}$. The radial coordinate r is the luminosity distance and as before, the coordinate volume element is $dV = 4\pi r^2 dr$. The proper volume element however is

$$d\mathcal{V} = B(r)dV = 4\pi r^2 B(r)dr \tag{19}$$

Note that since $B(r) > 1$, $d\mathcal{V} > dV$, and this might be seen as the effect of "stretching" of space by gravity. The energy momentum tensor of the body in mixed tensor form is

$$T_k^i = (p + \rho)u^i u_k + p g_k^i \tag{20}$$

where u^i is the fluid 4-velocity, ρ is the total mass-energy density (excluding contribution due to self-gravitation), and p is the isotropic pressure.

The total mass-energy density of the fluid (excluding negative self-gravitational energy) is

$$\rho = \rho_0 + e \tag{21}$$

where $\rho_0 = m_N n$ is the proper rest mass-energy density, m_N is nucleon rest mass, n is nucleon proper number density (not to be confused with polytropic index), e is the internal energy density, and g_k^i is the metric tensor. The explicit form of $B(r)$ is [8]

$$B(r) = [1–2M(r)/r]^{-1/2} = (1–2m/r)^{-1/2} \tag{22}$$

where

$$M(r) = m = \int_0^r 4\pi\rho r^2 dr \tag{23}$$

The total energy of the system as perceived by a distant inertial observer, S_∞, i.e., the gravitational mass of the fluid is

$$M = \int_0^R \rho dV = \int_0^R (\rho/B) d\mathcal{V} \tag{24}$$

This total energy is also known as gravitational mass or *Schwarzschild mass* of the fluid although actually this should have been named by the name of Hilbert. Note that

$$M = \int_0^R (\rho/B) d\mathcal{V} \neq \int_0^R \rho d\mathcal{V} \tag{25}$$

as one might have expected. The reason for this is that total mass energy includes not only local contributions from ρ but also the *negative* global contribution of self-gravitational energy. It is this negative latter contribution which reduces the effective net proper energy density from ρ to $\rho/B(r)$.

On the other hand, the total *proper* energy content of the sphere, by excluding any negative self-energy contribution, i.e., the energy obtained by merely adding individual local energy packets, is

$$M_{\rm proper} = \int_0^R \rho d\mathcal{V} \tag{26}$$

The *conventional* definition GR self- gravitational energy of the body is [7,8]

$$E_G = M - M_{\rm proper} = \int_0^R (1 - \sqrt{-g_{rr}}) \rho dV \tag{27}$$

Since $B(r) > 1$ in the presence of mass energy, E_G is a $-$ve quantity, as is expected. Note however that the E_G defined by Eq. (27) is the sum of appropriate local (proper) quantities somewhat like the definition of $M_{\rm proper}$ in Eq. (26); and is not defined with respect to the inertial observer S_∞. Note also that, in contrast, gravitational mass M is indeed the mass-energy measured by S_∞.

The *proper* rest mass energy of the fluid is

$$M_0 = \int \rho_0 d\mathcal{V} \tag{28}$$

If there are no initial antibaryons or antileptons, $M_0 = m_N N$, where N is the total number of baryons, and is therefore a conserved quantity.

The *proper* internal energy of the fluid is

$$U = \int e d\mathcal{V} \tag{29}$$

As before, note that, neither M_0 nor U is defined with respect to S_∞. By using Eqs. (20)–(28), it can be verified that

$$E_G + U = M - M_0 \tag{30}$$

From the above equation, it may *appear* that the GR equivalent of E_N, the binding energy, is

$$E_{GR} = M - M_0 = E_G + U \tag{31}$$

Suppose the gas was initially infinitely dispersed to infinity. Further, let the gas molecules be rest with respect to S_∞. Under such a case, the initial mass energy of the cloud is just the rest mass energy:

$$M(t = 0) = M_0 \tag{32}$$

And if the contraction of the cloud into a finite size would indeed release energy, one must have $M - M_0 < 0$. But unlike the Newtonian case, as noted by Tooper [9], it is not apparent from the definitions of E_G and U that $E_{GR} = E_G + U < 0$! This is so because while in the Newtonian case, E_g^N and U are related through Eq. (7), *there is no known relationship* between E_G and U in the GR case. Further there might be examples when the occurrence of a negative E_{GR} "*is not a sufficient condition for instability of the system against an expansion to infinity*" [9]. This means that E_{GR} may not be the correct GR equivalent of a "binding energy". Had E_{GR} been the true GR equivalent of E_N, probably, one would have had equations similar to (8) and (9) involving E_G, U and E_{GR}. But no such equations exist. Thus, although, intuitively, one expects GR contraction to release radiation energy, one really cannot show it unlike the Newtonian case developed in previous section.

In Eq. (30), we may note that, while, E_G, U and M_0 *are not* defined with respect to (w.r.t.) S_∞, M, on the other hand *is defined* w.r.t. S_∞. And this may be the fundamental reason that E_{GR} defined in Eq. (31) may not be the true "binding energy" of the fluid.

IV. ANOTHER DEFINITION OF FLUID MASS

For any stationary gravitational field, total four momentum of matter plus gravitational field is conserved and independent of the coordinate system used [10,11]:

$$P^i = \int (T^{i0} + t^{i0}) dV \tag{33}$$

where t^{ik} is the energy momentum pseudotensor associated with the gravitational field. Further, the inertial mass (same as gravitational mass), i.e, the time component of the 4-momentum of any given body in GR can be expressed as [10–12]

$$M = \int_0^\infty (T_0^0 - T_1^1 - T_2^2 - T_3^3) \sqrt{-g} d^3x \tag{34}$$

where

$$g = -r^4 A(r) B(r) \sin^2\theta \tag{35}$$

is the determinant of the metric tensor g_{ik} and $d^3x = dr d\theta d\phi$. Since

$$T_1^1 = T_2^2 = T_3^3 = -p; \qquad T_0^0 = \rho \tag{36}$$

it follows that [11,12]

$$M = \int_0^\infty (\rho + 3p) A(r) B(r) dV = \int_0^\infty (\rho + 3p) A(r) d\mathcal{V} \tag{37}$$

When the body is *bound* and $p_b = \rho_b = 0$ for $r \geq R$, then, the foregoing integrals shrink to [11,12]

$$M = \int_0^R (\rho + 3p) A(r) B(r) dV = \int_0^R (\rho + 3p) A(r) d\mathcal{V} \tag{38}$$

Although our result would not depend on splitting of the foregoing equation, (since we would simply equate the "total" expression of "inertial" and "gravitational" mass), we might nevertheless do so

$$M = \int_0^R \rho A(r) d\mathcal{V} + \int_0^R 3pA(r) d\mathcal{V} \tag{39}$$

for the sake of obtaining physical insight.

Global Definitions

The Newtonian virial theorem (4) is essentially a statement of energy conservation involving negative self-gravitational energy and positive thermodynamic energy of the fluid. In the Newtonian case, there exists global inertial frames and a statement of energy conservation can be made in a trivial way. But in GR, even for this simplest case of a static fluid sphere, there is no global inertial frame. Thus the exercise of having global definitions of related energies and to enact their conservation is a highly nontrivial task. As is well known, for stationary systems, however global energy can be defined in a meaningful way for asymptotically flat spacetimes. Further, when the energy is defined with reference to an observer at a spatial infinity (S_∞), we obtain the so-called ADM Mass [13]. Also global energy conservation can be meaningfully defined only with reference to S_∞.

Note that E_G and U are essentially summation over *local* appropriate values and *not* over the corresponding quantities measured by S_∞. It may be mentioned that, if *any* locally measured energy is ϵ, then the energy measured by the far away inertial observer is the redshifted quantity

$$\epsilon_\infty = \sqrt{g_{00}}\epsilon = A(r)\epsilon \tag{40}$$

Accordingly, the total mass-energy content of the fluid, excluding any self-energy, as measured by an inertial frame such as the distant observer, S_∞ is *different* from $M_{\rm proper}$ and is *given by* [8] (Eq. 11.1.19):

$$M_{\rm matter} = \int_0^R A(r)\rho d\mathcal{V} = \int A(r)B(r)\rho dV \tag{41}$$

Physically this means that local energy content in a given cell $\rho d\mathcal{V}$ is measured (*redshifted*) as $A(r)\rho d\mathcal{V}$ by the inertial observer S_∞.

In fact, the *inertial* observer S_∞ would see the rest mass energy, i.e., the proper energy, of a nucleon too to be reduced by the same factor $\sqrt{g_{00}}$. Hence, when the body is finite and not dispersed to infinity, the total rest mass energy of the body, as reckoned by the inertial observer S_∞, is

$$\tilde{M}_0 = \int_0^R \rho_0 A(r) d\mathcal{V} \tag{42}$$

Similarly, the *redshifted* global internal energy of the fluid as measured by S_∞ is

$$\tilde{U} = \int_0^R e(r)A(r) d\mathcal{V} \tag{43}$$

We can, now, quickly identify the 1st term on the RHS of Eq. (39) as $M_{\rm matter}$. Again bear in mind the fact that our eventual result would not depend on such identification or splitting because it would be obtained by equating the total expressions for inertial and gravitational masses.

Further, we see that, the 2nd term on the RHS of Eq. (39) is the global energy associated with pressure as perceived by S_∞. Accordingly, we rewrite Eq. (39) as:

$$M = M_{\rm matter} + M_{\rm pressure} \tag{44}$$

Again recall that M too is defined only with respect to S_∞. Having done this splitting, we are in a position to obtain the GR Virial theorem, which is essentially an accounting of global energies involved in the problem. And since global energies, in GR, can be defined only w.r.t. the inertial observer S_∞, all relevant integrals must be defined w.r.t. the same observer. And this is what we have just done.

V. GR HELMHOLTZ KELVIN PROCESS

Let us simply transpose Eq. (39) (irrespective of its splitting) as

$$\int_0^R \rho A(r) d\mathcal{V} - M + \int_0^R 3pA(r) d\mathcal{V} = 0 \tag{45}$$

to reexpress as

$$\tilde{E}_g + \int_0^R 3pA(r) d\mathcal{V} = 0 \tag{46}$$

Or,

$$\tilde{E}_g + \int 3p\sqrt{-g_{00}g_{rr}} dV = 0 \tag{47}$$

where

$$\tilde{E}_g = M - M_{\rm matter} = \int (AB - 1)\rho dV \tag{48}$$

Or else,

$$\tilde{E}_g = \int (\sqrt{-g_{00}g_{rr}} - 1)\rho dV \tag{49}$$

Since $AB < 1$ in the presence of mass-energy, we have $\tilde{E}_g < 0$ as is expected. Clearly, we have, obtained, now an equation similar to (4). It appears then that the above defined $\tilde{E}_g$ rather than the previously defined E_G is the true measure of self-gravitational energy as perceived by an inertial observer S_∞. This is so because E_G is not defined with respect to S_∞, the accountant for global energy. In contrast, both the components of $\tilde{E}_g$, namely, M and $M_{\rm matter}$ are defined w.r.t. S_∞.

Further, recall that, in GR, the effect of "gravitational potential" is conveyed by g_{00}. But E_G (Eq. (27)) *does not* *contain* g_{00} at all. In contrast, $\tilde{E}_g$ indeed involves gravitational potential term $g_{00} = A^2$ (Eq. (49)).

What would be the value of true global GR self-gravitational energy $\tilde{E}_g$ in the Newtonian limit?

To see this we consider a sphere with ρ = constant for which [8]

$$A(r) = \frac{1}{2}[3(1 - 2M/R)^{1/2} - B(r)^{-1}] \tag{50}$$

Using Eq. (22) in (50), we further see that

$$A(r)B(r) = \frac{1}{2}[3(1 - 2M/R)^{1/2}B(r) - 1] \tag{51}$$

Again using Eq. (22) in (51), we obtain

$$A(r)B(r) = \frac{1}{2}[3(1 - 2M/R)^{1/2}(1 - 2m/r)^{-1/2} - 1] \tag{52}$$

Now if we proceed to linearized gravity limit with $M/R \ll 1$ and $m/r \ll 1$, we will have

$$A(r)B(r) = 1 - \frac{3}{2}(m/r - M/R)] \tag{53}$$

Using Eq. (52) in (48), we see that

$$\tilde{E}_g = -\frac{3}{2}\int (M/R - m/r)\rho dV \tag{54}$$

When we carry out this above integration with ρ = constant, we obtain

$$\tilde{E}_g = \frac{-3}{5}\frac{M^2}{R} \tag{55}$$

Therefore, in the Newtonian limit, $\tilde{E}_g = E_G = E_g^N$, though in general $\tilde{E}_g$ and E_g are different.

Now, using the thermodynamical relation (5) and (43) in Eq. (46), as before, we will have

$$\tilde{E}_g + 3(\gamma - 1)\tilde{U} = 0 \tag{56}$$

By direct comparison with Eqs. (4) and (7), we can easily identify Eqs. (48) and (56) as the appropriate GR version of static Virial Theorem. Note that Eqs. (46) and (56) naturally reduce to their Newtonian forms, Eqs. (1) and (7) for sufficiently weak gravity with $g_{00} \approx -g_{rr} \approx 1$. Thus we may interpret the existence of the Newtonian virial theorem too as due to equivalence of "gravitational mass" and "inertial mass".

From Eq. (56), we obtain

$$\tilde{U} = \frac{-1}{3(\gamma - 1)}\tilde{E}_g \tag{57}$$

If the fluid undergoes quasistatic contraction and Δ denotes the associated changes in relevant quantities, then we will have

$$\Delta\tilde{U} = \frac{-1}{3(\gamma - 1)}\Delta\tilde{E}_g = \frac{+1}{3(\gamma - 1)}|\Delta\tilde{E}_g| \tag{58}$$

Here we have used the fact that since $\tilde{E}_g < 0$ it must decrease for contraction. Thus as is expected, the internal energy of the fluid must increase for gravitational contraction. If the appropriately averaged value of $A(r) = \bar{g_{00}}$ during this contraction, the increase in proper internal energy would be

$$\Delta U = \frac{\Delta\tilde{U}}{\bar{g}_{00}} = \frac{1}{3(\gamma - 1)}\frac{|\Delta E_g|}{\bar{g}_{00}} > 0 \tag{59}$$

Since $g_{00} < 1$ in the presence of gravity, this means that, the *rate of increase in proper internal energy would be higher than in the corresponding Newtonian case*.

As we look back at Eq. (58), the increase in the value of $|\tilde{E}_g|$, namely $|\Delta\tilde{E}_g|$ is not fully accounted for by the increase in the value of $\tilde{U}$. Note that in the absence of initial antiparticles, the contribution of rest mass-energy is unaffected during the process. Therefore, for the sake of global energy conservation, as reckoned by the inertial observer S_∞, the fluid *must* radiate out an amount of energy $+\Delta Q$ given by

$$\Delta Q = \left[1 - \frac{1}{3(\gamma - 1)}\right]|\Delta\tilde{E}_g| = \frac{3\gamma - 4}{3(\gamma - 1)}|\Delta\tilde{E}_g| \tag{60}$$

in order to be able to contract. Note, as before, that $\gamma > 4/3$ in a strict sense, as long as particle momenta are finite. To see that in the GR context too, that $\gamma = 4/3$ implies singular situation, look at the *Ist row of* Table I of [14] which shows that in this case again $R_{\max} = 0$, where $R_{\max}$ is the maximum possible radius of the configuration. Further, for $\gamma = 4/3$, the next entry in the same row shows that $R_0 = 0$ where $R_0 = 2GM/c^2$ is the Schwarzschild radius. This implies that for $\gamma = 4/3$, one has $R_0 = 0$. This latter fact implies that total mass energy $Mc^2 = 0$ just as $E_N = 0$ for $\gamma = 4/3$ (Eq. (9)). Occurrence of $R_{\max} = 0$ implies a fluid sphere that has collapsed to a singular point. And as per Ref. [14], the configuration then would have zero mass energy. Incidentally, in the 2nd classic paper of Ref. [13], Arnowitt, Deser and Misner too found that a neutral "point particle" has zero "clothed mass". Then Chandrasekhar's exercise [14], in addition, suggests that if the fluid would attain such a singular state, the value of $\gamma \to 4/3$. This is also in perfect agreement with the notion that a singular state should be infinitely hot with complete domination of radiation energy over rest mass energy [6].

In fact, in one would misconceive of a situation with $\gamma < 4/3$, irrespective of whether it is a Newtonian or a GR case, one would have to ensure injection of energy into the system to let it collapse. This would mean that in the absence of external injection of energy, the system would not contract/evolve at all in defiance of basic tenet of global gravitation.

Thus, one must indeed have $\gamma > 4/3$ and, in a strict sense, *there cannot be any adiabatic gravitational contraction*. Thermodynamically, the global specific heat of the contracting fluid is negative because $dQ < 0$ while the temperature and internal energy of the fluid increase.

VI. GENERAL RELATIVISTIC BINDING ENERGY

Equation (60) suggests that, we might isolate a quantity

$$\tilde{E} = \frac{3\gamma - 4}{3(\gamma - 1)} \tilde{E}_g \tag{61}$$

as the total energy of the system excluding any rest mass contribution. This is actually the "Binding Energy" of the gravitating system, defined by a given distribution of mass-energy.

By using Eq. (57), it is seen that

$$\tilde{E} = \tilde{E}_g + \tilde{U} \tag{62}$$

Further using Eqs. (39), (42), (43), and (48), it also transpires that

$$\tilde{E} = \tilde{E}_g + \tilde{U} = M - \tilde{M}_0 \tag{63}$$

We may see that, *unlike in* Eq. (30), all the quantities involved in Eq. (63) are defined w.r.t. S_∞ and which suggests that $\tilde{E}$ is indeed the true binding energy of the fluid. The reader is again requested here to appreciate the subtle point why the true binding energy of the fluid is given by Eq. (63) rather than by Eq. (31).

Though M_0 is a conserved quantity, once the fluid is contracted into a finite size, it gets dissociated from the inertial frame S_∞. And S_∞, who is doing the global energy accounting, sees the locally defined rest mass energy to be redshifted to $\tilde{M}_0$ rather than as M_0. On the other hand, the total mass-energy of the fluid, again defined w.r.t. S_∞ is M. Therefore, the global binding energy of the fluid, i.e., the difference between the total mass energy and rest mass energy as seen by the same inertial observer S_∞ is $M - \tilde{M}_0$. Note that the existence of Eqs. (62) and (63) does not depend on such interpretations because they, in any case, crept up spontaneously.

Although, in a Newtonian case, the notion of a binding energy always existed, to the best of our knowledge, such a notion was never before properly derived in the GR context. A relativistic "bound system" may thus be defined as one having $\tilde{E} < 0$, and an "unbound system" will have $\tilde{E} > 0$.

While, in the Newtonian case, "Total Energy" is the binding energy E_N, in GR, total global energy, as measured by S_∞, always is $E = Mc^2$.

VII. DISCUSSIONS

The important idea of Helmholtz and Kelvin, developed in the 19th century, that gravitational contraction should both raise the internal energy and cause the fluid to radiate was always expected to be valid irrespective of the strength of the gravity. However, the original derivation to this effect was made in the framework of extremely weak, i.e., Newtonian gravity. We showed here that even for arbitrary, strong gravity, this process indeed remains valid. In fact, as shown by Eq. (59), the process becomes even *more* effective compared to the Newtonian case as gravity increases. Pictorially, we may think that stronger gravity churns out more radiation from matter even in the absence of chemical or thermonuclear energy sources. A similar conclusion is supported by a recent work which shows that the ratio of radiation energy density to rest mass-energy density of a self-luminous contracting object is proportional to its surface redshift z [6]. When $z \ll 1$, the self-gravitating contracting object is "matter dominated", i.e., $\rho_0 \gg \rho_r$, but when, $z \gg 1$, the object becomes radiation dominated : $\rho_r \gg \rho_0$ like the very early Universe [6]. The present study provided an additional explanation for this result. These studies show that the actual fate of radiative physical gravitational collapse could be radically different from traditional pictures of continued gravitational collapse inspired by the pressureless dust collapse where a Black Hole (BH) or a Naked Singularity is catastrophically formed in a finite comoving proper time. Traditional GR collapse studies are usually done by (i) assuming dust models with $p \equiv 0$ even when the fluid is supposed to have collapsed to singularity, or (ii) considering pressure but neglecting all heat transport, $dQ \equiv 0$. But as shown by Eqs. (58) and (60), $dQ = 0$ implies (a) $d\tilde{U} = 0$ and (b) $d\tilde{E}_g = 0$. The condition (a) is satisfied only for dust and thus despite a formal consideration of existence of pressure, in the context of collapse, a fluid satisfying condition (a) becomes similar to a pressureless, internal energyless dust. The condition (b) is not satisfied even for a dust unless it has $M = U = E_g = \text{fixed} = 0$ at the beginning of the collapse. But no isolated fluid with finite size can have $M = 0$ (the Universe may, however, have $M = 0$ even being of finite or infinite extent). In several numerical studies of supposed radiative collapse, one implicitly or explicitly assumes $Q \ll M_0 c^2$. At the advanced stage of collapse, this assumption fails [6] and such cases effectively become similar to case (ii) of adiabatic continued collapse valid for $M = 0$.

In contrast, in a breakthrough research on physical gravitational collapse, Herrera and Santos [1] have shown that the force exerted on the collapsing fluid by the outward propagating heat/radiation may stall the continued collapse process and formation of either a finite mass BH or Naked Singularity may be averted. Herrera, Di Prisco and Barreto [1] have successfully made a numerical model of continued collapse to substantiate this pathbreaking idea. Such ideas are consistent with the model independent generic studies [6] which show that continued catastrophic collapse indeed degenerates into a radiation pressure supported hot quasistatic state called eternally collapsing objects because of outward force due to collapse generated radiation at extremely deep gravitational potential wells, $z \gg 1$ where z is the surface gravitational redshift of the collapsing object. It is because of the resultant reduction in the value of M due to continuous radiation outpour that no apparent horizon or event horizon is formed until $M = R = 0$ [6].

Finally, the GR definition of "binding energy" of a static fluid is $M - \tilde{M}_0$ rather than $M - M_0$.

Newtonian HK process is a direct sequel of static Newtonian virial theorem. Similarly, we needed to derive the exact GR version of the static virial theorem. This GR virial theorem involved globally defined quantities measured w.r.t. the *same* inertial observer S_∞. In general, the notion of global energy is far from transparent and unique in GR. For example, for nonstatic and nonspherically symmetric systems or charged systems, there could be various notions of "energy" and "mass"; to appreciate this one may have a look at a recent long review paper [15]. However the present paper must not be confused with such studies. The aim of this paper was not at all to define any new definition of either mass or "global energy" from any preferred theoretical perspective, correct or incorrect. This is so because, unlike a generic case, the definition of "global mass energy" of a chargeless static spherically symmetric fluid (measured by S_∞) is very well known since long [10,12]. And we just appealed to the Principle of Equivalence to equate the *already well known* expressions for "gravitational mass" (Schwarzshild mass) and "inertial mass"—the time component of linear 4-momentum. It is this simple operation which yielded the GR virial theorem and GR HK mechanism (Eqs. (12)–(17)). It is the same principle of equivalence which demanded that, from energy conservation considerations, all the relevant globally defined energies are defined w.r.t. the *unique inertial* frame S_∞ rather than w.r.t. a series of (infinite) proper frames. To the best of our knowledge, this is the maiden derivation of HK process using GR. And this is also the maiden proper GR explanation for the intuitive notion that a self-gravitating object has effective global "negative specific heat" in Einstein gravity too.

For further appreciation of the "global quantities" involving $\sqrt{g_{00}}$ in our study which arose spontaneously and not imposed from new theoretical perspective, we recall that the "Poisson's Equation" in GR has the form [16]

$$\nabla^2 \sqrt{g_{00}} = 4\pi G \sqrt{g_{00}}(\rho + 3p) \qquad (64)$$

And only when one moves to weak gravity with $GM/r \ll 1$ and $\sqrt{g_{00}} \approx 1 - GM/r \approx 1$, one obtains the more familiar form

$$\nabla^2 \phi = 4\pi G(\rho + 3p) \qquad (65)$$

where the weak "gravitational potential $\phi \sim GM/r$. Of course, Eq. (64) would also degenerate into Eq. (65) in a *local* free falling frame where $g_{00} = 1$. But for fluids having finite pressure there cannot be any such *global* free falling frame and therefore Eq. (65), in the present context, can be recovered only for weak gravity (in any case we are dealing with a *static* fluid). Formally only a pressureless "dust" with $p = U \equiv 0$ can undergo a strict adiabatic collapse, though, physically, in this case, the gravitational mass of the fluid $M = 0$.

Since virial theorem is important for the study of compact objects and gravitational contraction, the *exact* relativistic virial theorem obtained here could be useful for relativistic astrophysics, either now or in future.

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