Challenges for scaling cosmologies

Luca Amendola,[1,*] Miguel Quartin,[2,†] Shinji Tsujikawa,[3,‡] and Ioav Waga[2,§]

[1]*INAF/Osservatorio Astronomico di Roma, Via Frascati 33, I-00040 Monte Porzio Catone, Rome, Italy*
[2]*Universidade Federal do Rio de Janeiro, Instituto de Física, CEP 21941-972, Rio de Janeiro, Brazil*
[3]*Department of Physics, Gunma National College of Technology, Gunma 371-8530, Japan*

A cosmological model that aims at solving the coincidence problem should show that dark energy and dark matter follow the same scaling solution from some time onward. At the same time, the model should contain a sufficiently long matter-dominated epoch that takes place before acceleration in order to guarantee a decelerated epoch and structure formation. So a successful cosmological model requires the occurrence of a sequence of epochs, namely, a radiation era, a matter-dominated era, and a final accelerated scaling attractor with $\Omega_\varphi \simeq 0.7$. In this paper we derive the generic form of a scalar-field Lagrangian that possesses scaling solutions in the case where the coupling Q between dark energy and dark matter is a free function of the field φ. We then show, rather surprisingly, that the aforementioned sequence of epochs cannot occur for a vast class of generalized coupled scalar-field Lagrangians that includes, to our knowledge, all scaling models in the current literature.

I. INTRODUCTION

The unexpected discovery of the accelerated expansion of the universe opened a Pandora's box of new issues and questions. Many of these are related to the nature of dark energy and to its role within the particle physics model (see Ref. [1] for reviews). Other questions arise because of the so-called coincidence problem: why two components that are completely unrelated and scale with time in a different way, namely, dark energy and matter, appear to have roughly the same energy density just now and only now (''now'' here means within the last one or two e-folding times).

It is possible that once we know the fundamental nature of dark energy the problem of coincidence will be automatically and naturally explained. On the other hand, the reverse could be true as well: understanding the origin of the coincidence could shed light on the nature of dark energy and its relation to the rest of the world. This is the footpath that we intend to pursue in this paper.

This work rests on a fundamental assumption: a complete solution of the coincidence problem requires that dark energy and matter follow the same evolution with time, at least from some time onward. Otherwise it is clear that the occurrence of coincidence will always depend on the initial conditions of the system: changing the ratio dark energy/matter at some initial time will always imply a displacement in time of the coincidence epoch. In other words, we can explain the coincidence only if we show that it is not a coincidence at all, but rather that energy and matter always (or from some time onward) shared a similar

fraction of the total budget. From a phase-space perspective, explaining the coincidence requires showing that our *present* universe has already attained its final attractor solution. A solution in which matter and dark energy densities are both finite and have a constant ratio is denoted in literature a *scaling* solution [2–4] or, if stable and accelerated, a *stationary* solution [5].

Once one accepts this assumption, then follows the first immediate consequence: if we require that the energy density of dark energy (DE) is proportional to that of matter (i.e., $\rho_{\rm DE}/\rho_m = {\rm const}$) and at the same time we require that the dark energy equation of state parameter is less than $-1/3$ to get an accelerated expansion, then one needs either to assume that matter has the same negative equation of state as dark energy or that there is an interaction between the two components, so that ρ_m does not scale as a^{-3}. The first possibility is clearly to be ruled out because such a modified matter equation of state would profoundly affect the growth of perturbations. In fact, for any good model it is not enough to require present acceleration: we also need the universe to pass through a decelerated matter-dominated epoch in the past in order to have a well-behaved epoch of structure formation. A successful cosmological model should therefore admit for a sequence of epochs: a radiation era, a sufficiently long matter-dominated era, and a final stable accelerated scaling solution. This paper aims at searching for such a ''good scaling cosmology.''

The assumption of a stable accelerated scaling solution requires therefore the existence of an interaction between dark energy and matter [6]. If dark energy is modeled as a scalar field then the interaction with matter has to be a scalar force additional to gravity. That is, our model has to be a scalar-tensor gravitational theory [7] or, equivalently, an Einsteinian theory with an explicit coupling between matter and field. These models have been studied many

*Electronic address: amendola@mporzio.astro.it
†Electronic address: mquartin@if.ufrj.br
‡Electronic address: shinji@nat.gunma-ct.ac.jp
§Electronic address: ioav@if.ufrj.br

times in the past and several important properties have been discussed [8–10]. It is known, for instance, that a standard scalar-tensor model with an exponential potential has a stable scaling solution and that the scaling solution can be accelerated [11]. However, in this case, it is possible to show that no matter phase precedes the acceleration. In other words, after the radiation dominated era the system enters directly the accelerated regime. This is in contrast with observations, as has been shown in Ref. [6].

The fact that the simplest case does not work is the main motivation for us to look further. A simple generalization of the scalar-field Lagrangian is to consider the so-called k-essence Lagrangian [12], such that the Lagrangian is a function $p(X, \varphi)$ of the field φ and of the kinetic term $X = -(1/2)g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi$. This form is relatively simple, being still second order in the field, and has been already investigated several times. We note that this type of Lagrangian also covers a wide variety of dark energy models such as quintessence [13], tachyons [14], phantoms [15], and (dilatonic) ghost condensates [16,17]. If $\partial p/\partial X < 0$, it has been shown that a phantom behavior ($w_{DE} < -1$) occurs [9,18] and that matter feels a repulsive scalar force [19]. Moreover, it has been shown also that if the system contains a scaling solution then p can be cast in the form [17,20]

$$p(X, \varphi) = Xg(Y), \tag{1}$$

where $g(Y)$ is any function of the argument $Y = Xe^{\lambda\varphi}$ with λ being a constant. For instance, $g(Y) = 1 - c/Y$ is in fact the standard Lagrangian with an exponential potential ($p = X - ce^{-\lambda\varphi}$). The above form for p was shown to be valid for uncoupled dark energy and for the case in which the coupling

$$Q \equiv \frac{-1}{\rho_m\sqrt{-g_M}}\frac{\delta S_m}{\delta\varphi} \tag{2}$$

is a constant (here S_m is the action for matter and g_M is the metric determinant).

In this paper we perform a search of a good scaling cosmology in three steps. First, we show that the Lagrangian (1) extends also to the case of variable coupling $Q(\varphi)$ up to a field redefinition. This is an interesting result in itself since it unifies some sporadic results obtained in different ways in literature (e.g. [21]). Then, assuming as a model for g a polynomial

$$g = \sum_n c_n Y^{-n}, \tag{3}$$

with both positive and negative integer powers of n, we derive the critical points of the system. Finally, we show that within this class of models there is no way to obtain a sequence of a matter phase followed by a stable accelerated scaling solution. When a kinetic scaling matter-dominated era exists, this stage is generally followed by a scalar-field dominated attractor ($\Omega_\varphi = 1$) instead of an accelerated

scaling attractor. Although our proof of absence of two scaling regimes does not extend to any possible $g(Y)$, we believe that it seriously undermines the real possibility of realizing such an ideal cosmology. This negative result opens the challenge: is there *any* case in which a successful sequence can be realized? In the final section we will comment on the possibility of obtaining a good scaling cosmology with a fractional power Lagrangian $g = c_0 - cY^{-u}$, in which $0 < u < 1$. Nevertheless, we leave a more complete study of this, perhaps very exotic, case to future work.

Beside being a way to approach the problem of coincidence, a scaling cosmology also provides us with a useful alternative to standard dark energy scenarios. The behavior of the background cosmology and of its linear perturbations is in fact radically different in scaling cosmology with respect to most other models. Let us just mention three basic differences (see e.g., Refs. [5,22,23]). First, in a scaling cosmology the acceleration could start at any epoch in the past. Second, the perturbations can keep growing even during the accelerated regime. Third, the amount of dark energy does not necessarily become negligible at high redshifts. All three features radically distinguish scaling cosmologies from usual dark energy models which only focus on dark energy itself and not on its relation with matter. As such, scaling cosmologies may serve as a useful testing ground for observations.

Before passing to the actual calculations, we should make note of the local gravity constraints on scalar forces. In principle, the coupling we introduce is severely constrained by local gravity experiments on scalar-tensor theories. However, these can be escaped at least in three ways: First, by designing a potential with a large mass and, consequently, a short interaction range [24]; second, by building a model that happens to satisfy the constraints now, but not in the past; third, by assuming that the baryons are actually uncoupled to the scalar field [6]. The first two solutions change the potential and affect the global evolution and therefore will not in general satisfy the requisite for a cosmology that solves the coincidence problem. The third case on the contrary can be implemented without affecting the potential of the scalar field.

In general, a component of uncoupled baryons can dominate in the past, even if their abundance now is very small [5]. In this case a matter phase does exist, but it is a *baryonic* matter epoch instead of a dark matter epoch. This raises many problems on its own. For instance, the baryonic perturbations are almost erased on small scales due to the coupling to radiation and therefore, without the support from dark matter, would hardly grow to the observed amplitude; moreover, the baryonic era would finish early in the past, at redshifts quite larger than 1, and the subsequent accelerated regime would be too long to be in fair agreement with both the supernovae experiments (although here the discrepancy is marginal, see Ref. [22])

and with the integrated Sachs-Wolfe effect [5]. In any case, if a standard (dark) matter phase exists, the baryons would never dominate, as we will argue later on. Hence, the search for a "good scaling cosmology" can simply neglect the small baryon component and this is what will be done in the present work. Finally, since the two matter components have a different coupling, one has to choose a physical frame in which baryons are conserved (otherwise particle masses will be time-varying) but dark matter is not. We will work therefore in this frame, which is the so-called Einstein frame.

II. THE GENERAL LAGRANGIAN FOR SCALING SOLUTIONS WITH ARBITRARY COUPLING

We shall derive here the general Lagrangian admitting scaling solutions in the case where the coupling between dark energy and dark matter depends upon the field φ. This is the generalization of the works [17,20]. Let us start by considering the following action, written in the Einstein frame:

$$S = \int d^4x \sqrt{-g_M}\left[\frac{M_P^2}{2}R + p(X, \varphi)\right] + S_m[\varphi, \psi_i, g_{\mu\nu}],$$

(4)

where M_P is the reduced Planck mass, R is the Ricci scalar, φ is a scalar field, $X = -(1/2)g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi$ and ψ_i are the various matter fields. Notice that we allow for an arbitrary coupling between the matter fields and the scalar field φ. As mentioned in the Introduction, in order to cope with current observations, we assume that φ couple only to dark matter. We will also suppose that the dark matter component dominates over any other baryonic form of matter.

We are interested in cosmological scaling solutions in a spatially flat Friedmann-Robertson-Walker (FRW) background metric with a scale factor $a(t)$:

$$ds^2 = -dt^2 + a^2(t)d\mathbf{x}^2.$$

(5)

The Friedmann equation in Einstein gravity is given by

$$3H^2 = M_P^{-2}\rho_T,$$

(6)

where $M_P^{-2} = 8\pi G$ with G being the gravitational constant, and ρ_T is the total energy density of the universe. In what follows we shall set $M_P = 1$.

Our focus will be on solutions with constant equation of state parameter $w_\varphi \equiv p(X, \varphi)/\rho_\varphi$ in the scaling regime and in which the universe is filled only by two components: a barotropic fluid (such that $w_m \equiv p_m/\rho_m$) and the scalar field φ. Rewriting the Klein-Gordon equation for the field φ (in the above metric) in terms of its energy density, $\rho_\varphi = 2X\partial p/\partial X - p$, one gets [17]

$$\frac{d\rho_\varphi}{dN} + 3(1 + w_\varphi)\rho_\varphi = -Q\rho_m\frac{d\varphi}{dN},$$

(7)

where $N \equiv \ln a$ and Q is defined by Eq. (2).

If one starts from scalar-tensor theories [7] or a mass-varying neutrino scenario [25], we have

$$\frac{d\rho_\varphi}{dN} + 3(1 + w_\varphi)\rho_\varphi = -\tilde{Q}(1 - 3w_m)\rho_m\frac{d\varphi}{dN}.$$

(8)

This case reduces to Eq. (7) if one relates the coupling $\tilde{Q}$ to the coupling Q as $\tilde{Q}(1 - 3w_m) = Q$. In what follows we shall derive the condition for the existence of scaling solutions by using Eq. (7). Note that the energy density of a barotropic fluid satisfies

$$\frac{d\rho_m}{dN} + 3(1 + w_m)\rho_m = Q\rho_m\frac{d\varphi}{dN}.$$

(9)

We shall define the fractional densities of ρ_φ and ρ_m as

$$\Omega_\varphi \equiv \frac{\rho_\varphi}{3H^2}, \qquad \Omega_m \equiv \frac{\rho_m}{3H^2}.$$

(10)

These satisfy $\Omega_\varphi + \Omega_m = 1$ from Eq. (6). Scaling solutions are characterized by the condition $\rho_\varphi/\rho_m = $ const, in which case Ω_φ is a constant. Using these relations, together with Eqs. (7) and (9), and following the procedure of Refs. [17,20] but dropping the assumption of a constant coupling $Q(\varphi)$, one finds the relations

$$\frac{d\ln p_\varphi}{dN} = \frac{d\ln\rho_\varphi}{dN} = \frac{d\ln\rho_m}{dN} = -3(1 + w_{\text{eff}}),$$

(11)

and

$$\frac{d\varphi}{dN} = \frac{3\Omega_\varphi}{Q(\varphi)}(w_m - w_\varphi) \propto \frac{1}{Q(\varphi)},$$

(12)

where we introduced an effective quantity:

$$w_{\text{eff}} \equiv w_m\Omega_m + w_\varphi\Omega_\varphi.$$

(13)

From these equations and the definition of X, one arrives at

$$2X = H^2\left(\frac{d\varphi}{dN}\right)^2 \propto \frac{\rho_\varphi}{Q^2} \propto \frac{p(X, \varphi)}{Q^2},$$

(14)

and thus

$$\frac{d\ln X}{dN} = -3(1 + w_{\text{eff}}) - 2\frac{d\ln Q}{dN}.$$

(15)

Making use of Eqs. (11), (12), and (15), we arrive at the following generalized "master equation" for the Lagrangian $p(X, \varphi)$:

$$\left[1 + \frac{2}{\lambda Q^2}\frac{dQ(\varphi)}{d\varphi}\right]\frac{\partial\ln p}{\partial\ln X} - \frac{1}{\lambda Q}\frac{\partial\ln p}{\partial\varphi} = 1,$$

(16)

where

$$\lambda \equiv \frac{1 + w_m - \Omega_\varphi(w_m - w_\varphi)}{\Omega_\varphi(w_m - w_\varphi)}.$$

(17)

Equation (16) reduces to the one found in Ref. [17] when $Q(\varphi)$ is constant.

Solving Eq. (16), one gets

$$p(X, \varphi) = XQ^2(\varphi)g(XQ^2(\varphi)e^{\lambda\psi(\varphi)}), \qquad (18)$$

where g is an arbitrary function and

$$\psi(\varphi) \equiv \int^{\varphi} Q(\xi)\mathrm{d}\xi. \qquad (19)$$

See Appendix A for the derivation of Eq. (18). In the case of constant coupling both Q^2 terms in Eq. (18) can be absorbed in the definition of g, so our solution reduces to that in Ref. [17]. In a nutshell, what Eq. (18) means is that any Lagrangian that allows scaling solutions with constant w_φ can always be cast in the above form by a convenient field redefinition. The standard kinetic case corresponds therefore to $p = XQ^2 - e^{-\lambda\psi}$. Another example is provided by a coupling $Q = 1/\varphi$ as in Ref. [21]. By using Eq. (19) we find that the $e^{\lambda\psi}$ term in Eq. (18) is given by $e^{\lambda\psi} = \varphi^\lambda$. In this case the Lagrangian (18) becomes $p = Xg(X\varphi^{\lambda-2})/\varphi^2$. When $\lambda = 2$ this simplifies to $p = \bar{g}(X)/\varphi^2$, where $\bar{g}(X) \equiv Xg(X)$ is an arbitrary function of X. This form of p corresponds to the choice given in Ref. [21].

Now let us make the following field redefinition: $\varphi \rightarrow \psi(\varphi)$, with $\psi(\varphi)$ as defined in (19). This in turn implies $X \rightarrow X_\psi = XQ^2(\varphi)$, and Eq. (18) becomes

$$p(X_\psi, \psi) = X_\psi g(X_\psi e^{\lambda\psi}), \qquad (20)$$

which is the same functional form found in the constant coupling scenario [17]. At the same time the relation between S_m and the coupling becomes

$$1 = \frac{-1}{\rho_m\sqrt{-g_M}} \frac{\delta S_m}{\delta\psi}. \qquad (21)$$

We have thus shown that the case of a constant coupling ($Q = 1$) is the most general one. In other words, if one is interested in scaling solutions, one can always work with a Lagrangian in the above form, no matter what kind of coupling one has in mind.

In order to follow the current notation in the literature (e.g. [17,20]) we will use, instead of ψ, the field $\bar{\varphi}(\psi)$ defined by $\bar{\varphi}(\psi) \equiv \psi/\bar{Q}$, where $\bar{Q}$ is a constant. In this case one can absorb the $\bar{Q}$ term that appears in the exponential in the argument of g into the definition of λ. Hence, in what follows, we shall always consider the Lagrangian density (dropping the bars on Q and φ)

$$p = Xg(Xe^{\lambda\varphi}), \qquad (22)$$

where now λ is given by

$$\lambda \equiv Q\frac{1 + w_m - \Omega_\varphi(w_m - w_\varphi)}{\Omega_\varphi(w_m - w_\varphi)}. \qquad (23)$$

It is important at this stage to realize that the system is invariant under a simultaneous change of sign of Q and λ.

We can therefore without loss of generality consider only the case $\lambda > 0$.

In Appendix B we generalize the above results for a more general cosmological background, in which Eq. (6) is replaced by $H^2 \propto \rho_\mathrm{T}^q$. In subsequent sections, though, we shall restrict the analysis to Einstein gravity ($q = 1$).

III. PHASE-SPACE EQUATIONS

So far we have derived the most general Lagrangian that possesses scaling solutions. In addition to scaling solutions there exist other fixed points for the system (22) characterized by $\Omega_\varphi = 1$. In what follows we shall derive the autonomous equations taking into account radiation to find the general behavior of the solutions. As we mentioned in the Introduction, we are interested in searching for a good scaling cosmology, namely, a sequence composed of a radiation epoch, a matter-dominated era, and an accelerated scaling attractor.

Many results will be shown to hold for any $g(Y)$. However, we carry out our search assuming as a reference model a polynomial expansion in positive and negative *integer* powers:

$$g = c_0 + \sum_{n>0} c_n Y^{-n} + \sum_{n'<0} c_{n'} Y^{-n'}, \qquad (24)$$

where c_0, c_n, and $c_{n'}$ are constants. Note that if we have $c_0 = 1$ and all other c_n, except c_1, being zero, our case reduces to that of an ordinary scalar field with an exponential potential [3].

For the Lagrangian density (22) in the presence of pressureless dust and radiation, we obtain the following equations:

$$3H^2 = X(g + 2g_1) + \rho_m + \rho_\mathrm{rad}, \qquad (25)$$

$$2\dot{H} = -\left[2X(g + g_1) + \rho_m + \tfrac{4}{3}\rho_\mathrm{rad} \right], \qquad (26)$$

$$\ddot{\varphi} + 3AH(g + g_1)\dot{\varphi} + \lambda X[1 - A(g + 2g_1)] + AQ\rho_m = 0, \qquad (27)$$

where $A \equiv (g + 5g_1 + 2g_2)^{-1}$ and

$$g_n \equiv Y^n \partial^n g/\partial Y^n. \qquad (28)$$

The speed of sound, c_s, is related to the quantity A by [10,17]

$$c_s^2 = A(g + g_1). \qquad (29)$$

When $A^{-1} = 0$ the speed of sound diverges. Hence, no physically acceptable evolution can cross the border $A^{-1} = 0$.

In order to study the dynamics of the above system it is convenient to introduce the following dimensionless quantities:

$$x = \frac{\dot{\varphi}}{\sqrt{6}H}, \qquad y = \frac{e^{-\lambda\varphi/2}}{\sqrt{3}H}, \qquad z = \frac{\sqrt{\rho_{\text{rad}}}}{\sqrt{3}H}. \quad (30)$$

Then Y is written as $Y = x^2/y^2$. Equation (25) gives the following constraint equation:

$$\Omega_m \equiv \frac{\rho_m}{3H^2} = 1 - \Omega_\varphi - z^2, \quad (31)$$

where

$$\Omega_\varphi = x^2(g + 2g_1). \quad (32)$$

It is important to note that, in principle, Ω_φ could as well be negative. From Eq. (26) we find

$$\frac{1}{H}\frac{dH}{dN} = -\frac{1}{2}(3 + 3gx^2 + z^2). \quad (33)$$

By using Eqs. (27), (31), and (33), we obtain the autonomous equations:

$$\frac{dx}{dN} = \frac{3}{2}x\left[1 + gx^2 - 2A(g + g_1) + \frac{1}{3}z^2\right]$$
$$+ \frac{\sqrt{6}}{2}[A(Q + \lambda)(g + 2g_1)x^2 - \lambda x^2 + QA(z^2 - 1)], \quad (34)$$

$$\frac{dy}{dN} = \frac{y}{2}(3 - \sqrt{6}\lambda x + 3gx^2 + z^2), \quad (35)$$

$$\frac{dz}{dN} = \frac{z}{2}(-1 + 3gx^2 + z^2). \quad (36)$$

It is useful to notice the relations

$$\rho_\varphi = X(g + 2g_1), \qquad w_\varphi = \frac{g}{g + 2g_1}, \quad (37)$$

and also

$$w_\varphi = -1 + \frac{2x^2}{\Omega_\varphi}(g + g_1), \qquad \Omega_\varphi w_\varphi = gx^2. \quad (38)$$

This means that $w_\varphi > -1$ for $g + g_1 > 0$ and $w_\varphi < -1$ for $g + g_1 < 0$. From Eq. (33) the effective equation of state parameter of the system is given by

$$w_{\text{eff}} = -1 - \frac{2}{3}\frac{\dot{H}}{H^2} = gx^2 + \frac{1}{3}z^2. \quad (39)$$

Then, in the absence of radiation ($z = 0$), one has $w_{\text{eff}} = \Omega_\varphi w_\varphi$ [see Eq. (38)].

IV. CRITICAL POINTS

In this section we shall derive the fixed points for the above autonomous system in the absence of radiation ($z = 0$). The critical points for $z \neq 0$ are irrelevant to our study. They are listed in Appendix C, for the sake of completeness. The fixed points for $z = 0$ are derived by setting $dx/dN = 0$ and $dy/dN = 0$ in Eqs. (34) and (35). From

Eq. (35) we find two distinct classes of solutions, either for $3 - \sqrt{6}\lambda x + 3gx^2 = 0$ or for $y = 0$. The former case gives both scalar-field dominated and scaling solutions [10]. In fact in this case we have

$$x = \frac{\sqrt{6}(1 + w_\varphi\Omega_\varphi)}{2\lambda}, \quad (40)$$

and, inserting this into Eq. (34), we find two cases:

(i) Point A: a scalar-field dominated solution with

$$\Omega_\varphi = 1. \quad (41)$$

(ii) Point B: a scaling solution with

$$\Omega_\varphi = -\frac{Q}{w_\varphi(Q + \lambda)}. \quad (42)$$

Let us remind that by definition a scaling solution corresponds to a situation in which Ω_φ equals neither 1 nor 0.

The properties of points A and B will be discussed in Secs. IV A and IV B, respectively. In Sec. IV C we shall discuss the second class of solutions, in which $y = 0$. Since these solutions exist in the limit of a vanishing potential, we denote them as *kinetic solutions*.

A. Point A: Scalar-field dominated solutions

When $\Omega_\varphi = 1$, we have the following relations [10]:

$$g(Y_A) = \frac{\sqrt{6}\lambda x_A - 3}{3x_A^2}, \qquad g_1(Y_A) = \frac{6 - \sqrt{6}\lambda x_A}{6x_A^2}. \quad (43)$$

Specifying the model $g(Y)$, one obtains Y_A and the fixed point (x_A, y_A) by using Eq. (43) and the relation $Y = x^2/y^2$. From Eq. (38) we find

$$w_{\text{eff}} = w_\varphi = -1 + \frac{\sqrt{6}\lambda}{3}x_A. \quad (44)$$

A general Lagrangian could have in principle many different classes of point A. The number of such critical points is known by solving Eq. (43).

The stability of fixed points can be analyzed by considering linear perturbations around them. This was carried out in Ref. [10] for a general $g(Y)$ for positive values of Q and λ. The eigenvalues of the matrix for perturbations are given by

$$\mu_+ = -3 + \sqrt{6}(Q + \lambda)x_A, \qquad \mu_- = -3 + \frac{\sqrt{6}}{2}\lambda x_A. \quad (45)$$

The fixed point is a stable node if $\mu_+ < 0$ and $\mu_- < 0$. Allowing negative values of Q as well, the fixed point A is stable when

$$\begin{cases} x_A < \frac{\sqrt{6}}{2(Q+\lambda)}, & \text{if } Q > -\lambda/2, \\ x_A < \frac{\sqrt{6}}{\lambda}, & \text{if } -\lambda < Q \le -\lambda/2, \\ \frac{\sqrt{6}}{2(Q+\lambda)} < x_A < \frac{\sqrt{6}}{\lambda}, & \text{if } Q < -\lambda. \end{cases} \quad (46)$$

For negative x_A, which corresponds to a phantom equation of state ($w_\varphi < -1$) from Eq. (44), the first two conditions in Eq. (46) are automatically satisfied. Hence, when $Q > -\lambda$ the phantom fixed point is always classically stable. On the other hand, the stability of nonphantom fixed points ($x_A > 0$) depends upon the values of Q and λ. Since x_A is given by $x_A = \lambda/[\sqrt{6}(g + g_1)]$, the stability condition (46) for nonphantom fixed points is expressed as

$$\begin{cases} g + g_1 > \lambda(Q+\lambda)/3, & \text{if } Q > -\lambda/2, \\ g + g_1 > \lambda^2/6, & \text{if } Q \le -\lambda/2. \end{cases} \quad (47)$$

B. Point B: Scaling solutions

The scaling solution satisfies the relation (42). Then Eq. (40) gives

$$x_B = \frac{\sqrt{6}}{2(Q+\lambda)}. \quad (48)$$

We also obtain the following relations valid for all g [10]:

$$g(Y_B) = -\frac{2Q(Q+\lambda)}{3}, \quad (49)$$

$$w_{\text{eff}} = -\frac{Q}{Q+\lambda}, \quad (50)$$

$$w_\varphi = -\frac{Q(Q+\lambda)}{Q(Q+\lambda) + 3(g+g_1)}, \quad (51)$$

$$\Omega_\varphi = \frac{Q(Q+\lambda) + 3(g+g_1)}{(Q+\lambda)^2}. \quad (52)$$

Again, once the function $g(Y)$ is specified, one obtains Y_B and $y_B = |x_B|/\sqrt{Y_B}$ as a function of Q, λ. The condition for an accelerated expansion corresponds to $w_{\text{eff}} < -1/3$ and this gives us

$$Q > \lambda/2 \quad \text{or} \quad Q < -\lambda, \quad (53)$$

which are again independent of the form of $g(Y)$. Note that the latter case corresponds to the effective phantom ($w_{\text{eff}} < -1$) as we see from Eq. (50).

The eigenvalues of the matrix for perturbations around the fixed point B are given by [10]

$$\mu_\pm = \xi_1[1 \pm \sqrt{1 - \xi_2}], \quad (54)$$

where

$$\xi_1 = -\frac{3(2Q+\lambda)}{4(Q+\lambda)}, \quad (55)$$

$$\xi_2 = \frac{8(1-\Omega_\varphi)(Q+\lambda)^3[\Omega_\varphi(Q+\lambda) + Q]}{3(2Q+\lambda)^2}A. \quad (56)$$

This point is stable if $\xi_1 < 0$ and $\xi_2 > 0$. We find that negative ξ_1 corresponds to $Q > -\lambda/2$ or $Q < -\lambda$. Hence, when the condition for an acceleration (53) is satisfied, ξ_1 is automatically negative. In what follows we shall consider a realistic situation in which the acceleration condition (53) is imposed. Then point B is stable when

$$-\frac{Q}{Q+\lambda} \le \Omega_\varphi < 1 \quad \text{and} \quad A > 0. \quad (57)$$

The second condition is satisfied if we avoid the ultraviolet instability of quantum fluctuations [17], which is the case for a nonphantom scalar field. From Eq. (52) the condition $-Q/(Q+\lambda) \le \Omega_\varphi$ corresponds to $-2Q(Q+\lambda) \le 3(g+g_1)$. This is automatically fulfilled for a nonphantom fixed point ($g + g_1 > 0$) under the condition (53). The most crucial condition for the stability of point B is $\Omega_\varphi < 1$, i.e.,

$$g + g_1 < \lambda(Q+\lambda)/3. \quad (58)$$

For a nonphantom fixed point this is not satisfied if $Q < -\lambda$ but can be satisfied if $Q > \lambda/2$. Hence, when $Q > \lambda/2$ there exist stable, accelerated, and nonphantom fixed points B provided that $\Omega_\varphi < 1$ (whose condition is actually required to get a viable scaling solution). Note that when $Q > \lambda/2$ the stability condition given in Eq. (47) has an opposite equality to that in Eq. (58). Hence, the stability of points A and B is divided by the border $g + g_1 = \lambda(Q+\lambda)/3$, which means that the final attractor is either point A or point B depending on the values of Q and λ. When a nonphantom scaling solution with positive Q exists in the region (57), it is the only stable attractor point for any $g(Y)$,[1] so that scaling solutions have the crucial property of being *global attractors*.

As a last remark, we note that a general form of g could in principle exhibit several scaling solutions, all of them with the same x_B, $g(Y_B)$, and w_{eff}, but with different y_B.

C. Points C and D: Kinetic solutions

Now we study the second class of solutions of Eq. (35), i.e., the case of $y = 0$. These points exist only if $g = g(x^2/y^2)$ is nonsingular, i.e., only if one can expand g in positive powers of y^2/x^2,

$$g = c_0 + \sum_{n>0} c_n \left(\frac{y^2}{x^2}\right)^n, \quad (59)$$

in which case one has

$$g_n(y \to 0) = 0 \quad (n > 0). \quad (60)$$

---

[1]An exception to this rule exists in the case of the fractional power-law Lagrangian (69) with $0 < u < 1$, in which a phantom attractor may also coexist.

In this case Eq. (34) is simply given by

$$\frac{dx}{dN} = \frac{1}{2}(3c_0 x + \sqrt{6}Q)\left(x^2 - \frac{1}{c_0}\right) = 0. \qquad (61)$$

For $c_0 = 0$ this equation gives no real solutions. For $c_0 \neq 0$ we get the following fixed points:

(iii) Point C: a φ-matter-dominated era or φMDE (see Ref. [6])

$$(x_C, y_C) = \left(-\frac{\sqrt{6}Q}{3c_0}, 0\right). \qquad (62)$$

In this case Eqs. (32) and (37) give

$$\Omega_\varphi = \frac{2Q^2}{3c_0}, \qquad w_{\text{eff}} = \frac{2Q^2}{3c_0}, \qquad w_\varphi = 1. \quad (63)$$

Hence Ω_φ and c_0 always have the same sign. When $c_0 > 0$ the solutions are decelerated, and the requirement of the condition $\Omega_\varphi < 1$ gives

$$|Q| < \sqrt{\frac{3}{2}c_0}. \qquad (64)$$

We note that the φMDE also corresponds to the same class of scaling solutions of point B. In fact, setting $g(Y_C) = c_0 = -2Q(Q + \lambda)/3$ in Eq. (49) and eliminating λ in Eqs. (50)–(52), we obtain the results in Eq. (63). Also, we remark that for $c_0 > 0$, $w_{\text{eff}} > 0$ and therefore dark matter density dilutes as $a^{-3(1+w_{\text{eff}})}$, i.e. faster than baryons. This ensures that baryons did not dominate in the past. Since during acceleration baryons dilute faster than dark energy, we can safely assume that baryons never contributed a large portion of cosmic energy. For $c_0 < 0$ this is not necessarily true but these cases will be ruled out for other reasons presented later.

(iv) Point D: pure kinetic solutions

$$(x_D, y_D) = (\pm 1/\sqrt{c_0}, 0), \qquad (65)$$

which exists only for positive c_0. In this case we have that matter is absent and

$$\Omega_\varphi = 1, \qquad w_{\text{eff}} = 1, \qquad w_\varphi = 1. \qquad (66)$$

Let us now consider linear perturbations δx and δy about the generic kinetic fixed point $(x, y) = (x_k, 0)$. In this case the 2×2 matrix $\mathcal{M}$ for perturbations [3] is diagonal and its eigenvalues are given by

$$\mu_1 = -\tfrac{3}{2} + \tfrac{9}{2}c_0 x_k^2 + \sqrt{6}Q x_k, \qquad (67)$$

$$\mu_2 = \frac{3}{2}\left(1 + c_0 x_k^2 - \frac{\sqrt{6}}{3}\lambda x_k\right). \qquad (68)$$

Hence, in the case of the φMDE solution the eigenvalues are $\mu_1 = (Q^2/c_0) - 3/2$ and $\mu_2 = 3/2 + Q(Q + \lambda)/c_0$. When $c_0 > 0$, μ_1 is negative under the condition (64) whereas $\mu_2 > 0$ for the values of Q satisfying Eq. (53). This shows that the φMDE corresponds to a saddle point for all the relevant cases when $c_0 > 0$. When c_0 is negative, it can be a stable point if $Q(Q + \lambda) > 3|c_0|/2$.

In the case of pure kinetic solutions (which exist only for $c_0 > 0$), one has $\mu_1 = 3 \pm \sqrt{6/c_0}Q$ and $\mu_2 = 3 \mp \sqrt{6/c_0}\lambda/2$ for $x_k = \pm 1/\sqrt{c_0}$. Thus, for $Q > 0$, in both cases at least one of the eigenvalues is positive, which means that the solutions are either unstable nodes or saddle points depending on the values of Q or λ. When $Q < 0$, the point $x_k = 1/\sqrt{c_0}$ is stable when $Q < -\sqrt{3c_0/2}$ and $\lambda > \sqrt{6c_0}$, whereas the point $x_k = -1/\sqrt{c_0}$ is an unstable node.

D. Summary of fixed points

In Table I we summarize the property of fixed points for the Lagrangian density (20). The scalar-field dominated fixed point A and the scaling solution B exist for any form of $g(Y)$ as long as they satisfy the condition of existence given in Table I. Both fixed points can be used for late-time acceleration, since the effective equation of state w_{eff} can be smaller than $-1/3$ depending upon the values of Q and λ. For a nonphantom case, the final attractor is either A or B depending on the values of Q and λ. The scaling solution B is a global attractor provided that the condition (57) is satisfied.

TABLE I. The properties of critical points for the Lagrangian density (22) in the presence of a pressureless dust ($w_m = 0$). Specifying the form of $g(Y)$, x_A and Y_A are determined by solving Eq. (43) whereas Y_B is known by Eq. (49). The kinetic fixed points C and D exist when g is given by Eq. (59).

Point	x	y	Existence	Stability	Ω_φ	w_{eff}
A	x_A	$(x_A^2/Y_A)^{1/2}$	$y_A \neq 0$ and $\Omega_\varphi = 1$	Stable node under conditions (46)	1	$-1 + \sqrt{\frac{2}{3}}\lambda x_A$
B	$\frac{\sqrt{6}}{2(Q+\lambda)}$	$(x_B^2/Y_B)^{1/2}$	$3(g + g_1) < (Q + \lambda)\lambda$	Stable node for $-\frac{Q}{Q+\lambda} \leq \Omega_\varphi < 1$ and $A > 0$	$\frac{Q(Q+\lambda)+3(g+g_1)}{(Q+\lambda)^2}$	$-\frac{Q}{Q+\lambda}$
C	$-\frac{\sqrt{6}Q}{3c_0}$	0	$|Q| < \sqrt{(3c_0/2)}$ or $c_0 < 0$	Saddle point for $c_0 > 0$ / Stable node for $c_0 < 0$ and $Q(Q + \lambda) > 3|c_0|/2$	$\frac{2Q^2}{3c_0}$	$\frac{2Q^2}{3c_0}$
D	$\pm\frac{1}{\sqrt{c_0}}$	0	$c_0 > 0$	Unstable node or saddle for $Q > 0$ / Stable node for $Q < -\sqrt{3c_0/2}$	1	1

The existence of the kinetic fixed points C and D depends on the form of the scalar-field Lagrangian. They appear when g is expanded into positive powers of y^2/x^2, i.e., Eq. (59). An ordinary scalar field with an exponential potential ($g = 1 - c/Y$) belongs to this class, while for instance a dilatonic ghost condensate model [17] ($g = -1 + cY$) does not. The fixed point C corresponds to a saddle point for $c_0 > 0$ with $\Omega_\varphi = w_{\mathrm{eff}} = 2Q^2/3c_0$. Hence, one can have a temporary scalar-field matter-dominated era (φMDE) in the presence of the coupling Q. We note that $\Omega_\varphi < 0$ when c_0 is negative. The fixed point D appears only for positive c_0 and corresponds to $\Omega_\varphi = 1$ with no acceleration ($w_{\mathrm{eff}} = 1$). Hence, this is neither viable for the matter-dominated era nor for the dark energy dominated era and it will not be considered further.

V. CAN WE HAVE TWO SCALING REGIMES?

As we anticipated in the Introduction, we search now for the occurrence of a two-stage cosmology: a decelerated matter epoch and an accelerated scaling regime. This amounts to searching for two distinct fixed points for the same set of parameters $\{Q, \lambda\}$. Clearly, the matter point has to be a saddle point in order to give way to the final accelerated stable attractor. Since in general during the matter epoch there will be a non-negligible contribution of the scalar field, this point is, in general, a scaling point. Therefore we search for two subsequent scaling regimes. It is in principle possible to obtain an approximate matter epoch without an associated fixed point but this would require a fine-tuning of the initial condition, so we exclude this possibility here.

As we have shown in the previous section, there are two possibilities which lead to an accelerated expansion at late times—using either the scalar-field dominated fixed point A or the scaling solution B. The φMDE fixed point C appears prior to the accelerated epoch for the models given by Eq. (59). For an ordinary scalar field with an exponential potential it was found that the φMDE is followed by the attractor point A [6] (or by point B but in this case without acceleration). In this case the present universe ($\Omega_\varphi \simeq 0.7$) would be finally dominated by the energy density of the scalar field ($\Omega_\varphi = 1$). Conversely, if the present accelerated universe corresponds to a scaling attractor B, it was shown that the matter-dominated epoch, if any, is not sufficiently long to form large-scale structure. This is associated with the fact that we require a large coupling Q to obtain an accelerated scaling attractor, but in this case the solution quickly approaches the attractor after the end of a radiation era since there is no saddle matter point. Hence, one cannot have two scaling regimes (the decelerated point C and the accelerated point B) at the same time for the standard scalar field with an exponential potential.

In this section we investigate whether two scaling solutions can be realized for the general Lagrangian (20) with g given by Eq. (24). We scan all the parameter space $\{Q, \lambda\}$ in search of a successful scaling cosmology. Since the cosmological dynamics is different depending on the sign of c_0, we shall consider three cases (i) $c_0 > 0$, (ii) $c_0 < 0$, and (iii) $c_0 = 0$ separately. We shall also look into the alternative, fractional power-law Lagrangian given by

$$g(Y) = c_0 - cY^{-u}, \qquad (69)$$

where u, as opposed to n, is not limited to integer values.

A. Case of $c_0 > 0$

The function g given in Eq. (24) is composed by positive and negative powers of Y. We shall first show that the case of positive powers of Y in Eq. (24) is not cosmologically viable and then proceed to the case of negative values of n.

1. Positive powers of Y

Let us first consider the function g given by

$$g = c_0 + \sum_{n<0} c_n Y^{-n}. \qquad (70)$$

In this case all the critical points with $y = 0$ disappear because of the singularity. Then, the only possibilities giving rise to a matter-dominated phase corresponds to either $x = 0$ or $g = g_1 = 0$ [see Eqs. (32) and (39)], for which indeed $w_{\mathrm{eff}} = 0$ and $\Omega_\varphi = 0$. For $x \to 0$, however, one has $g \to c_0$ and $g_1, g_2 \to 0$ from Eq. (70). Then it is immediate to see that $dx/dN(x \to 0) = -\sqrt{6}Q/(2c_0)$ from Eq. (34), so we do not have a fixed point unless of course $Q = 0$ [in which case the scaling solution B is not accelerated, as can be seen from Eq. (53)]. If $g = g_1 = 0$, the situation is the same and again we only have fixed points when $Q = 0$. Thus we do not have a successful cosmological scenario for the function given by Eq. (70). Note also that for the model $g = c_0 - cY^{-u}$ with negative u the scalar-field energy fraction Ω_φ for point B has to be negative from Eq. (75) when the condition (53) for an acceleration is imposed.

2. Negative powers of Y

Since we have seen that all positive powers of Y in the general polynomial form of g are discarded, let us then focus on polynomials with negative power of Y, i.e.,

$$g = c_0 + \sum_{n>0} c_n Y^{-n}. \qquad (71)$$

We will prove that when $c_n \neq 0$ for $n \geq 1$, it is impossible to have two viable scaling regimes which satisfy observational constraints.

From Eqs. (48) and (62) we see that the φMDE decelerated solution C and the accelerated point B have always opposite signs of x. In fact, requiring the acceleration at B,

we have either $Q > \lambda/2 > 0$ or $Q < -\lambda < 0$. In the former case, one has $x_B > 0$ and $x_C < 0$, whereas in the latter case $x_B < 0$ and $x_C > 0$. However, the function g given in Eq. (71) is singular at $x = 0$ (except for power laws $u \leq 1$, see below), which implies that the sequence of the solution from C to B is prevented. In what follows we will provide a more detailed analysis for the possibility of getting two scaling regimes.

Let us first consider a single power-law function of $g(Y)$ given in Eq. (69). In the limit $x \to 0$ with a nonzero value of y (which can be $y \ll 1$), the gx^2y term on the right-hand side (rhs) of Eq. (35) exhibits a divergence for $u > 1$ together with a divergence of the gx^3 term on the rhs of Eq. (34) for $u > 3/2$. In fact, for $u \neq 1$ we have that

$$\left| \frac{dy/dN}{dx/dN} \right|_{x \to 0} \to \infty. \tag{72}$$

Hence, when $u \neq 1$ the solutions cannot pass the line given by $x = 0$. Since the signs of x_B and x_C are always different, it is inevitable to hit this singularity for $u > 1$ if the solutions move from the φMDE point C to the scaling solution B. This shows that a sequence of solutions from C to B is forbidden because of the singularity at $x = 0$.

In Fig. 1 we plot a phase space for the model (69) with $u = 2$, $c = c_0 = 1$, $\lambda = 4$, and $Q = 0.7$ together with the fixed points of the system. The phase space is characterized by another singularity in Eq. (34), associated with the divergence of the speed of sound. This appears when the quantity, $A^{-1} = c_0 - c(u-1)(2u-1)Y^{-u}$, becomes

equal to zero, i.e.,

$$y = \pm \left(\frac{c_0}{c(u-1)(2u-1)} \right)^{1/2u} x. \tag{73}$$

For positive c, it exists for $u > 1$ or $0 < u \leq 1/2$ but disappears for $1/2 < u \leq 1$. When $c < 0$ the converse is true.

We note that for the model (69) the fixed point B corresponds to

$$x_B = \frac{\sqrt{6}}{2(Q+\lambda)}, \quad y_B = \left(\frac{2Q(Q+\lambda)+3c_0}{3c} \right)^{1/2u} x_B. \tag{74}$$

When the condition (53) for acceleration is satisfied, we require $c > 0$ for the existence of point B. Then in what follows, we shall only consider the case of positive c. When $u > 3/2$, point B does not satisfy the condition $A > 0$, i.e., $y < (\frac{c_0}{c(u-1)(2u-1)})^{1/2u}|x|$. This can be checked in Fig. 1 in which the scaling solution B exists in the region $A < 0$. When $1 < u < 3/2$, it is possible to obtain positive values of A. However, we still need another condition: $\Omega_\varphi^{(B)} < 1$, which gives a more severe constraint. Unless u is close to 1, it is not easy for the critical point B to fulfill the conditions $A > 0$ and $\Omega_\varphi^{(B)} < 1$. One example satisfying these conditions is $u = 1.1$, $Q = 2$, $\lambda = 2$ with $c_0 = c_1 = 1$, as plotted in Fig. 2. In this case, however, the φMDE point exists in the region $\Omega_\varphi^{(C)} > 1$. More importantly, the trajectories cannot move from point C to point B because of the singularities at $x = 0$ and also at $A^{-1} = 0$.
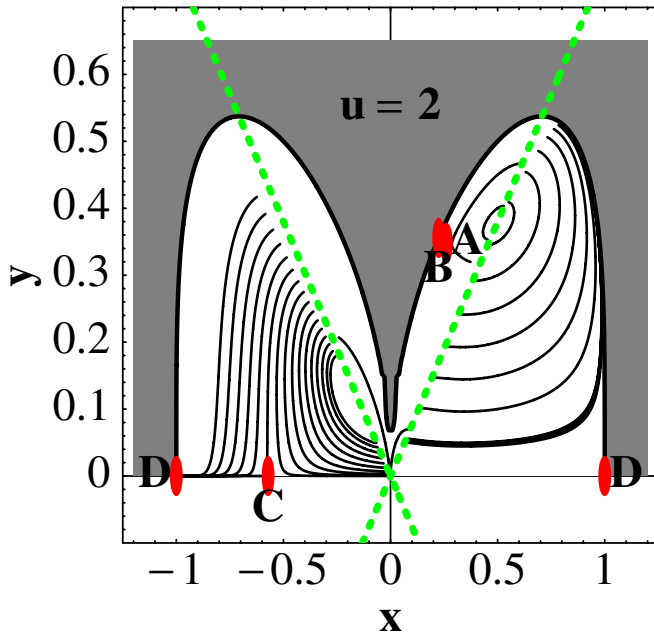

FIG. 1 (color online). Phase space for the model (69) with $u = 2$, $c = c_0 = 1$, $\lambda = 4$, and $Q = 0.7$ together with the fixed points A, B, C, and D. Here and in the following figures, the gray area represents the region where $\Omega_\varphi > 1$. The dotted line corresponds to the singularity given by (73) at which the speed of sound diverges.
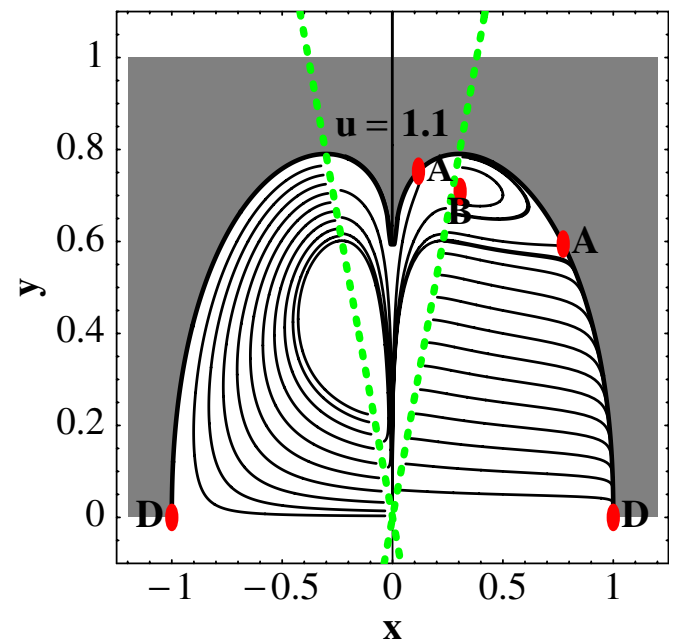

FIG. 2 (color online). Phase space for the model (69) with $u = 1.1$, $c = c_0 = 1$, $\lambda = 2$, and $Q = 2$ together with the fixed points A, B, and D (point C lies in the $\Omega_\varphi > 1$ region). The dotted line corresponds to the singularity given by (73) at which the speed of sound diverges.

The above discussion shows that when $u > 1$ one cannot realize two scaling regimes. On the other hand, the $u = 1$ case (an ordinary scalar field with an exponential potential) is free from both singularities at $x = 0$ and $A^{-1} = 0$. This case however has been already ruled out as a successful cosmological model [6]. The argument is as follows; we leave u as a free parameter to clarify some interesting aspects of the more general case. The relevant quantities for the fixed points B and C are given in Eqs. (50) and (63) and by the following relation:

$$\Omega_\varphi^{(B)} = \frac{(2u - 1)Q(Q + \lambda) + 3uc_0}{(Q + \lambda)^2}. \tag{75}$$

Let us impose the observational constraints that during the phase C $\Omega_\varphi^{(C)} < 1$, and during the phase B $w_{\text{eff}}^{(B)} < -1/3$ and $\Omega_\varphi^{(B)} < 1$ (notice that the supernovae observed value w_{DE} is in reality defined through the standard Friedmann equation so it cannot be directly used here; we have shown elsewhere [22] that the best value for $w_{\text{eff}}^{(B)}$ is in fact around -0.6 ± 0.1). In reality observations require quite more stringent constraints than this. For instance, supernovae observations constrain $\Omega_\varphi^{(B)} = 0.7 \pm 0.2$ and $w_{\text{eff}}^{(B)} < -0.6 \pm 0.1$. Moreover, too much dark energy during phase C leads to a weak growth of perturbations and serious conflicts with the cosmic microwave background, so a conservative limit would be $\Omega_\varphi^{(C)} < 0.2$ (see e.g., Ref. [6]).

The condition for the acceleration of the scaling solution B ($w_{\text{eff}}^{(B)} < -1/3$) requires either the condition $Q > \lambda/2 > 0$ or $Q < -\lambda < 0$. In the latter case it is easy to find that $\Omega_\varphi^{(B)}$ becomes larger than 1 for $u > 1$. In the former case the condition $3Q > Q + \lambda$ gives

$$\Omega_\varphi^{(B)} > \frac{uc_0}{3Q^2} + \frac{2u - 1}{3}, \tag{76}$$

while the condition $|Q| < \sqrt{3c_0/2}$ for the existence of the φMDE implies

$$\Omega_\varphi^{(B)} > \frac{8u - 3}{9}. \tag{77}$$

This shows explicitly that for any $u \geq 3/2$ the existence/acceleration of point B is in contradiction with the existence of point C. When $u = 1$ it is possible to have two scaling solutions C and B, but we do not get values smaller than $w_{\text{eff}}^{(B)} = -0.4$, $\Omega_\varphi^{(B)} = 0.9$, and $\Omega_\varphi^{(C)} = 0.7$. This certainly excludes the $u = 1$ case from the range of viable cosmological models. The values of the parameters that match this limit are $\lambda = 1.54$ and $Q = 1.02$. The phase-space plot in this case is presented in Fig. 3, from which it is clear that the φMDE fixed point C is indeed followed by the scaling solution B without singularities, although such solutions are not cosmologically viable. When $1 < u < 3/2$, one cannot satisfy the observational constraints either, in addition to the impossibility of reaching point B from C.
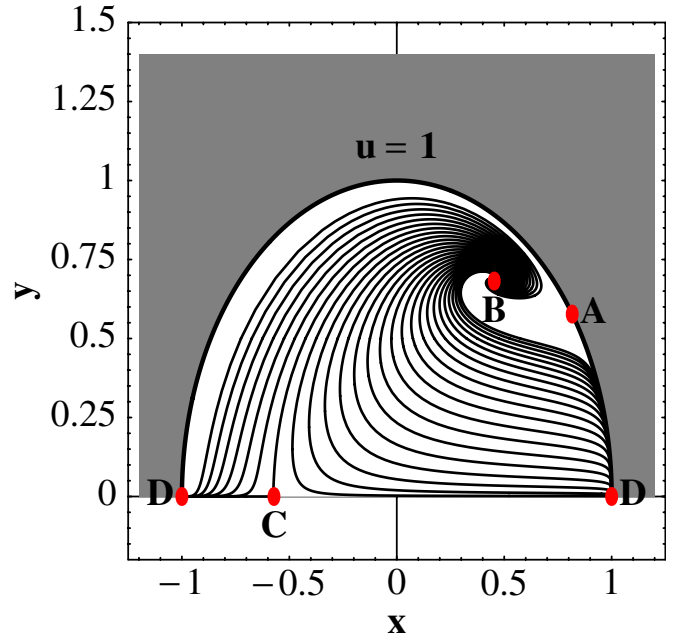

FIG. 3 (color online). Phase space for the model (69) with $u = 1$, $c = c_0 = 1$, $\lambda = 1.54$, and $Q = 1.02$ together with the fixed points A, B, C, and D.

In Fig. 2 we plot a phase space for the model (69) with $u = 1.1$, $\lambda = 2$, and $Q = 2$ together with the fixed points of the system.

We have thus shown that it is not possible to obtain two ideal scaling regimes for the function (69) with $u \geq 1$. Now we extend this proof to a polynomial form of $g(Y)$ with negative powers. The problem as we have seen is mainly associated with the fact that two scaling solutions B and C are separated by singularities at $x = 0$ and $A^{-1} = 0$. The latter can disappear by considering the sum of the powers given in Eq. (71) with the adjustment of the co-efficients c_n. However, if the polynomial includes any power n larger than 1, this leads to a singularity at $x = 0$ even when the singularity at $A^{-1} = 0$ is not present. Hence, if the function g possesses at least one term whose power n is larger than 1, the polynomials (71) are excluded as an ideal scaling cosmology. This completes our proof of the impossibility of obtaining two scaling solutions in the case of negative powers of Y and positive c_0.

Let us conclude this subsection with a brief discussion of the case $u < 1$ in Eq. (69). When $u < 1$ the line $x = 0$ is no longer singular; however, Eq. (72) still holds and the phase space is again separated into positive and negative abscissa subspaces. Moreover, the singularity at $A^{-1} = 0$ disappears for $1/2 < u < 1$ (remember we are only interested in $c > 0$). In this case it is also possible to have another nearly matter-dominated phase. This corresponds to a situation in which one takes a limit $x \to 0$ with a nonzero but small y in Eqs. (34) and (35). Then we can have a matter-dominated era in the region $x > 0$ followed by the scaling solution B with $x_B > 0$ (when Q is positive). This situation

is similar also in the case $0 < u \leq 1/2$; as long as the system is in the region $x > 0$ and $A > 0$ initially, the solutions reach the scaling attractor B without any singularity. It is interesting to remark that when $u < 1$ and $c > 0$, every class A point with $x_{A1} > 0$ is accompanied by a second point A: a phantom attractor ($x_{A2} < 0$). Hence, it can happen that the φMDE be followed by a point A without a singularity even for $Q > 0$ ($x_C < 0$). The $u < 1$ "fractional Lagrangians" are then promising but clearly for these models to work there are several other observational and theoretical issues that should be considered and we leave them to a future work.

B. Case of $c_0 < 0$

We shall next consider the case of negative c_0. The positive powers of Y given in Eq. (70) are excluded as viable cosmological scenarios by arguments similar to those presented in the previous subsection. Then let us focus on the negative powers of Y given in Eq. (71). In this case, one has the φMDE solution (62) with a negative Ω_φ [see Eq. (63)].

In addition to the fact that this may be unphysical, we are also faced with another problem to obtain two scaling regimes C and B. Since the φMDE satisfies the condition (60), one has $A^{-1} = g + 5g_1 + 2g_2 = c_0$, which means that A is negative. On the other hand, in order to get a stable scaling solution B with $0 \leq \Omega_\varphi < 1$, we require A positive. Then, to reach point B from point C, one needs to cross either the singularity at $A^{-1} = 0$ (which is not allowed) or go through $A = 0$ ($x = 0$). The latter can only be accomplished in the alternative model (69) with $u = 1$. Therefore one cannot realize a good scaling cosmology when c_0 is negative.

C. Case of $c_0 = 0$

The case $c_0 = 0$ is also easy to dispose of. In fact in this case there are no kinetic solutions and therefore no matter eras with $y = 0$ [see Eqs. (62) and (65)]. One can have a matter era also for $x \rightarrow \infty$ with $n \geq 1$ or for $x \rightarrow 0$ with fractional powers less than one. However, in both cases A is singular and therefore there are no fixed points. Finally, if both y and x go to zero so that $Y = $ const, then one can verify that dx/dN does not vanish and therefore the point $(0, 0)$ is not a solution.

This completes our proof. Although the discussion has been rather long and technical, the conclusion is straightforward: we have shown that no cosmologically viable scaling solutions exist for the general class of integer polynomial field Lagrangians with variable coupling.

VI. CONCLUSIONS

In this paper we have addressed a number of interesting aspects of cosmological scaling solutions and derived the following results.

(i) We have identified the most general form of second-order scalar-field Lagrangian given in Eq. (18) with a coupling to matter that is a completely arbitrary function of φ (but does not depend on X) under the condition that the system exhibits scaling solutions. This is the generalization of the works [17,20] in which a similar form of Lagrangian was obtained in the case of a constant coupling.

(ii) We have classified the phase-space topology for the scaling Lagrangian and obtained four classes of fixed points: (A) scalar-field dominated points with $\Omega_\varphi = 1$, (B) scaling solution with $\Omega_\varphi = -Q/[w_\varphi(Q + \lambda)]$, (C) a φMDE solution, and (D) pure kinetic solutions. Points of the first two classes may exist for any scaling Lagrangian and can lead to an accelerated expansion. The accelerated scaling attractor B, when it exists in the region $-Q/(Q + \lambda) \leq \Omega_\varphi < 1$, is the only global attractor apart from the case in which another phantom attractor A is present. Points C and D appear when the function g can be expanded in the form (59). The φMDE solution C is another scaling solution (always decelerated in the cases of interest).

(iii) We have addressed the possibility of finding a sequence of matter and scaling acceleration and found that this is *impossible* for any scaling Lagrangian which can be approximated as a polynomial with both positive and negative integer powers of its argument Y. This is essentially due to the fact that a scaling Lagrangian is always singular either along the x-axis or the y-axis of the phase space, thereby either preventing the matter-dominated era or isolating the region with a viable matter era from the region where the scaling acceleration occurs.

It is rather remarkable that the sequence of two scaling regimes cannot be realized for such a vast class of scalar-field Lagrangians [although, to be fair, we did not investigate thoroughly the consequences of Eq. (24) having infinite terms]. This emphasizes how difficult it is to solve the problem of coincidence: although cosmological scaling solutions have been studied for over a decade now, no successful case has been identified and this paper shows that even a large generalization of the models does not help. The search for a good scaling cosmology is not over yet, though. In fact we have also shown that a possible exception exists in the $0 < u < 1$ sector of the function $g(Y)$ given in Eq. (69). A detailed investigation of this type of fractional Lagrangian is underway.

ACKNOWLEDGMENTS

APPENDIX A: DERIVATION DETAILS

In order to solve Eq. (16), we first rewrite it using as a field λ times Eq. (19). That is, $\partial/\partial\varphi = \lambda Q(\varphi)\partial/\partial\psi$:

$$\frac{\partial \ln p}{\partial \ln X}\left[1 + \frac{2}{Q(\varphi)}\frac{dQ(\varphi(\psi))}{d\psi}\right] - \frac{\partial \ln p}{\partial \psi} = 1. \quad (A1)$$

We then decompose $p(X, \varphi)$ into

$$\ln p(X, \varphi) = -\psi + \ln f(X, \psi), \quad (A2)$$

thus arriving at

$$\frac{\partial \ln f}{\partial \ln X}\left[1 + \frac{2}{Q(\varphi(\psi))}\frac{dQ(\varphi(\psi))}{d\psi}\right] - \frac{\partial \ln f}{\partial \psi} = 0. \quad (A3)$$

This last equation can be solved by Fourier analysis:

$$\ln f(X, \psi) \equiv \frac{1}{\sqrt{2\pi}}\int e^{i\omega \ln X}F(\omega, \psi)d\omega. \quad (A4)$$

Equation (A3) then becomes (with $Q_\psi \equiv Q \circ \varphi$, where we use $\circ$ to denote a composite function)

$$i\omega F\left[1 + \frac{2}{Q_\psi}\frac{dQ_\psi}{d\psi}\right] = \frac{\partial F}{\partial \psi}, \quad (A5)$$

which has as a solution

$$\ln F = \int^\psi i\omega\left[1 + \frac{2}{Q_\psi(z)}\frac{dQ_\psi(z)}{dz}\right]dz + B(\omega), \quad (A6)$$

where $B(\omega)$ is an arbitrary function. Undoing the Fourier transformation, we get

$$\ln f = \frac{1}{\sqrt{2\pi}}\int d\omega B(\omega)\Big\{i\omega\Big[\ln X$$
$$+ \underbrace{\int^\psi\Big(1 + \frac{2}{Q_\psi(z)}\frac{dQ_\psi(z)}{dz}\Big)dz}_{=\psi+2\ln Q_\psi(\psi)+\text{const}}\Big]\Big\}.$$

Finally, recalling (A2) and noting that the composite function $Q_\psi \circ \psi = Q$, we arrive at

$$p(X, \varphi) = e^{-\psi(\varphi)}\tilde{g}(Xe^{\psi(\varphi)}Q^2(\varphi)), \quad (A8)$$

from which Eq. (18) follows immediately by a redefinition of the arbitrary function $\tilde{g}$ and by setting $\psi_{\text{new}} \to \psi_{\text{old}}/\lambda$. Notice that the arbitrariness of the function B is absorbed into that of $\tilde{g}$.

APPENDIX B: MORE GENERAL COSMOLOGICAL BACKGROUND

For completeness we also derive the scaling Lagrangian in an effective FRW equation which is given by

$$H^2 = \beta_q^2\rho_{\text{T}}^q, \quad (B1)$$

where β_q and q are constants. General relativity, Randall-Sundrum braneworlds [26], Gauss-Bonnet braneworlds [27], and Cardassian cosmology [28] correspond to $q = 1$, $q = 2$, $q = 2/3$, and $q = 1/3$, respectively. Equations (7) and (9) are unchanged even for the background (B1). The definitions of Ω_φ and Ω_m are modified as

$$\Omega_\varphi \equiv \frac{\rho_\varphi}{(H/\beta_q)^{2/q}}, \qquad \Omega_m \equiv \frac{\rho_m}{(H/\beta_q)^{2/q}}, \quad (B2)$$

which satisfies $\Omega_\varphi + \Omega_m = 1$ from Eq. (B1).

While Eq. (11) holds for $q \neq 1$ as well, Eq. (14) is subject to the change:

$$2X = H^2\left(\frac{d\varphi}{dN}\right)^2 \propto \frac{\rho_\varphi^q}{Q^2} \propto \frac{p^q(X, \varphi)}{Q^2}. \quad (B3)$$

Then we obtain the following master equation:

$$q\left[1 + \frac{2}{q\lambda Q^2}\frac{dQ(\varphi)}{d\varphi}\right]\frac{\partial \ln p}{\partial \ln X} - \frac{1}{\lambda Q}\frac{\partial \ln p}{\partial \varphi} = 1, \quad (B4)$$

where λ is defined in Eq. (17). The integration of this equation gives

$$p(X_\psi, \psi) = (XQ^2(\varphi))^{1/q}g(XQ^2(\varphi)e^{q\lambda\psi}), \quad (B5)$$

where ψ is defined in Eq. (19). For constant Q this reproduces the result obtained in Ref. [20].

APPENDIX C: FIXED POINTS FOR $z \neq 0$

We shall derive here the fixed points for $z \neq 0$. In this case one has $z^2 = 1 - 3gx^2$ from Eq. (36). Then from Eq. (39) the effective equation of state always corresponds to $w_{\text{eff}} = 1/3$, which means that the scale factor evolves as $a \propto t^{1/2}$. Hence, we cannot use the fixed points with $z \neq 0$ to get a matter-dominated era or an accelerated expansion. By substituting the relation $z^2 = 1 - 3gx^2$ for Eq. (35), we find the following two cases: (i) $y = 0$ and (ii) $x = 4/(\sqrt{6}\lambda)$.

Case (i) is similar to the kinetic solutions discussed in Sec. IV. Then by considering the function $g(Y)$ given in Eq. (59) one gets $dx/dN = -x(1 + \sqrt{6}Qx) = 0$, which gives $x = 0$ or $x = -1/\sqrt{6}Q$. Thus, for $y = 0$ we have two fixed points: (a) $(x, y, z) = (0, 0, 1)$ and (b) $(x, y, z) = (-1/\sqrt{6}Q, 0, \sqrt{1 - c_0/2Q^2})$. Point (a) corresponds to a standard radiation dominated era with $\Omega_\varphi = 0$, whereas for point (b) there is an energy fraction of the scalar field given by $\Omega_\varphi = c_0/6Q^2$.

In case (ii) we have $x = 4/\sqrt{6}\lambda$ and $z^2 = 1 - 8g/\lambda^2$, while y is only determined by the specific form of g and could as well be zero. From Eq. (34) we obtain the relation $A(\lambda + 4Q)(g - g_1) = 0$, which leads to three different fixed points: (c) $g = g_1$, (d) $\lambda = -4Q$, and (e) $A = 0$. Solving $g = g_1$ by specifying the form of g, we obtain

the value $y = y_c (\neq 0)$, i.e., the fixed point (c) $(x, y, z) = (4/\sqrt{6}\lambda, y_c, \sqrt{1 - 8g/\lambda^2})$. The special case where $\lambda = -4Q$ (i.e., $Q < 0$) is an interesting one because y_d and z_d are not specified even after the form of g is given. In fact, given the form of g, the *critical curve* (d) is found by

solving the relation $z_d = \sqrt{1 - 8g(8/(3\lambda^2 y_d^2))}$. Finally, point (e) demands that y is zero, otherwise since x is finite, g would be finite and $A = (g + 5g_1 + 2g_2)^{-1}$ would not vanish. But then z is either imaginary or infinite, so this fixed point is never realized.

---

[1] V. Sahni and A. A. Starobinsky, Int. J. Mod. Phys. D **9**, 373 (2000); V. Sahni, Lect. Notes Phys. **653**, 141 (2004); S. M. Carroll, Living Rev. Relativity **4**, 1 (2001); T. Padmanabhan, Phys. Rep. **380**, 235 (2003); P. J. E. Peebles and B. Ratra, Rev. Mod. Phys. **75**, 559 (2003); E. J. Copeland, M. Sami, and S. Tsujikawa, hep-th/0603057.

[2] P. G. Ferreira and M. Joyce, Phys. Rev. Lett. **79**, 4740 (1997); Phys. Rev. D **58**, 023503 (1998).

[3] E. J. Copeland, A. R. Liddle, and D. Wands, Phys. Rev. D **57**, 4686 (1998).

[4] A. P. Billyard, A. A. Coley, and R. J. van den Hoogen, Phys. Rev. D **58**, 123501 (1998); A. R. Liddle and R. J. Scherrer, Phys. Rev. D **59**, 023509 (1999); J. P. Uzan, Phys. Rev. D **59**, 123510 (1999); R. J. van den Hoogen, A. A. Coley, and D. Wands, Classical Quantum Gravity **16**, 1843 (1999); T. Barreiro, E. J. Copeland, and N. J. Nunes, Phys. Rev. D **61**, 127301 (2000); V. Sahni and L. M. Wang, Phys. Rev. D **62**, 103517 (2000); A. Nunes and J. P. Mimoso, Phys. Lett. B **488**, 423 (2000); C. Rubano and J. D. Barrow, Phys. Rev. D **64**, 127301 (2001); Z. K. Guo, Y. S. Piao, R. G. Cai, and Y. Z. Zhang, Phys. Lett. B **576**, 12 (2003); Z. K. Guo, Y. S. Piao, and Y. Z. Zhang, Phys. Lett. B **568**, 1 (2003); S. Mizuno, S. J. Lee, and E. J. Copeland, Phys. Rev. D **70**, 043525 (2004); E. J. Copeland, S. J. Lee, J. E. Lidsey, and S. Mizuno, Phys. Rev. D **71**, 023526 (2005); M. Sami, N. Savchenko, and A. Toporensky, Phys. Rev. D **70**, 123526 (2004); V. Pettorino, C. Baccigalupi, and F. Perrotta, J. Cosmol. Astropart. Phys. 12 (2005) 003; Y. Gong, A. Wang, and Y. Z. Zhang, Phys. Lett. B **636**, 286 (2006).

[5] D. Tocchini-Valentini and L. Amendola, Phys. Rev. D **65**, 063508 (2002).

[6] L. Amendola, Phys. Rev. D **62**, 043511 (2000).

[7] L. Amendola, Phys. Rev. D **60**, 043501 (1999); J. P. Uzan, Phys. Rev. D **59**, 123510 (1999); T. Chiba, Phys. Rev. D **60**, 083508 (1999); N. Bartolo and M. Pietroni, Phys. Rev. D **61**, 023518 (2000); F. Perrotta, C. Baccigalupi, and S. Matarrese, Phys. Rev. D **61**, 023507 (2000); B. Boisseau, G. Esposito-Farese, D. Polarski, and A. A. Starobinsky, Phys. Rev. Lett. **85**, 2236 (2000); E. Elizalde, S. Nojiri, and S. D. Odintsov, Phys. Rev. D **70**, 043539 (2004); L. Perivolaropoulos, J. Cosmol. Astropart. Phys. 10 (2005) 001; S. Capozziello, S. Nojiri, and S. D. Odintsov, Phys. Lett. B **634**, 93 (2006).

[8] M. Doran and J. Jaeckel, Phys. Rev. D **66**, 043519 (2002); D. Comelli, M. Pietroni, and A. Riotto, Phys. Lett. B **571**, 115 (2003); L. P. Chimento, A. S. Jakubi, D. Pavon, and

W. Zimdahl, Phys. Rev. D **67**, 083513 (2003); M. Sami and T. Padmanabhan, Phys. Rev. D **67**, 083509 (2003); L. Amendola and C. Quercellini, Phys. Rev. D **68**, 023514 (2003); M. B. Hoffman, astro-ph/0307350; A. V. Maccio *et al.*, Phys. Rev. D **69**, 123516 (2004); G. Huey and B. D. Wandelt, astro-ph/0407196 [Phys. Rev. D (to be published)]; S. Nojiri, S. D. Odintsov, and S. Tsujikawa, Phys. Rev. D **71**, 063004 (2005); S. Nojiri and S. D. Odintsov, Phys. Rev. D **72**, 023003 (2005); G. Olivares, F. Atrio-Barandela, and D. Pavon, Phys. Rev. D **71**, 063523 (2005); G. Calcagni, S. Tsujikawa, and M. Sami, Classical Quantum Gravity **22**, 3977 (2005); X. Zhang, Mod. Phys. Lett. A **20**, 2575 (2005); Phys. Lett. B **611**, 1 (2005); T. Koivisto, Phys. Rev. D **72**, 043516 (2005); I. Y. Aref'eva, A. S. Koshelev, and S. Y. Vernov, Phys. Rev. D **72**, 064017 (2005); B. Wang, C. Y. Lin, and E. Abdalla, Phys. Lett. B **637**, 357 (2006); S. Lee, G. C. Liu, and K. W. Ng, Phys. Rev. D **73**, 083516 (2006); B. Hu and Y. Ling, Phys. Rev. D **73**, 123510 (2006); Z. G. Huang, H. Q. Lu, and W. Fang, hep-th/0604160.

[9] B. Gumjudpai, T. Naskar, M. Sami, and S. Tsujikawa, J. Cosmol. Astropart. Phys. 06 (2005) 007.

[10] S. Tsujikawa, Phys. Rev. D **73**, 103504 (2006).

[11] C. Wetterich, Astron. Astrophys. **301**, 321 (1995).

[12] T. Chiba, T. Okabe, and M. Yamaguchi, Phys. Rev. D **62**, 023511 (2000); C. Armendáriz-Pícon, V. Mukhanov, and P. J. Steinhardt, Phys. Rev. Lett. **85**, 4438 (2000); C. Armendáriz-Pícon, V. Mukhanov, and P. J. Steinhardt, Phys. Rev. D **63**, 103510 (2001).

[13] Y. Fujii, Phys. Rev. D **26**, 2580 (1982); L. H. Ford, Phys. Rev. D **35**, 2339 (1987); C. Wetterich, Nucl. Phys. **B302**, 668 (1988); B. Ratra and J. Peebles, Phys. Rev. D **37**, 3406 (1988); Y. Fujii and T. Nishioka, Phys. Rev. D **42**, 361 (1990); E. J. Copeland, A. R. Liddle, and D. Wands, Ann. N. Y. Acad. Sci. **688**, 647 (1993); J. A. Frieman *et al.*, Phys. Rev. Lett. **75**, 2077 (1995); R. R. Caldwell, R. Dave, and P. J. Steinhardt, Phys. Rev. Lett. **80**, 1582 (1998); I. Zlatev, L. M. Wang, and P. J. Steinhardt, Phys. Rev. Lett. **82**, 896 (1999); P. J. Steinhardt, L. M. Wang, and I. Zlatev, Phys. Rev. D **59**, 123504 (1999).

[14] G. W. Gibbons, Phys. Lett. B **537**, 1 (2002); T. Padmanabhan, Phys. Rev. D **66**, 021301 (2002); J. S. Bagla, H. K. Jassal, and T. Padmanabhan, Phys. Rev. D **67**, 063504 (2003).

[15] R. R. Caldwell, Phys. Lett. B **545**, 23 (2002); R. R. Caldwell, M. Kamionkowski, and N. N. Weinberg, Phys. Rev. Lett. **91**, 071301 (2003); S. M. Carroll, M. Hoffman, and M. Trodden, Phys. Rev. D **68**, 023509 (2003); P.

Singh, M. Sami, and N. Dadhich, Phys. Rev. D **68**, 023522 (2003).

[16] N. Arkani-Hamed, P. Creminelli, S. Mukohyama, and M. Zaldarriaga, J. Cosmol. Astropart. Phys. 04 (2004) 001.

[17] F. Piazza and S. Tsujikawa, J. Cosmol. Astropart. Phys. 07 (2004) 004.

[18] L. Amendola, S. Tsujikawa, and M. Sami, Phys. Lett. B **632**, 155 (2006); S. Tsujikawa, Phys. Rev. D **72**, 083512 (2005).

[19] L. Amendola, Phys. Rev. Lett. **93**, 181102 (2004).

[20] S. Tsujikawa and M. Sami, Phys. Lett. B **603**, 113 (2004).

[21] H. Wei and R.-G. Cai, Phys. Rev. D **71**, 043504 (2005).

[22] L. Amendola, M. Gasperini, and F. Piazza, J. Cosmol. Astropart. Phys. 09 (2004) 007.

[23] L. Amendola, D. Polarski, and S. Tsujikawa, astro-ph/0603703.

[24] J. Khoury and A. Weltman, Phys. Rev. D **69**, 044026 (2004); Phys. Rev. Lett. **93**, 171104 (2004).

[25] P. Gu, X. Wang, and X. Zhang, Phys. Rev. D **68**, 087301 (2003); R. Fardon, A. E. Nelson, and N. Weiner, J. Cosmol. Astropart. Phys. 10 (2004) 005; M. Li and X. Zhang, Phys. Lett. B **573**, 20 (2003); P. Gu, X. Wang, and X. Zhang, Phys. Rev. D **68**, 087301 (2003); P. Hung and H. Pas, Mod. Phys. Lett. A **20**, 1209 (2005); X. J. Bi, B. Feng, H. Li, and X. m. Zhang, Phys. Rev. D **72**, 123523 (2005); H. Li, B. Feng, J. Q. Xia, and X. Zhang, Phys. Rev. D **73**, 103503 (2006); A. W. Brookfield, C. van de Bruck, D. F. Mota, and D. Tocchini-Valentini, Phys. Rev. Lett. **96**, 061301 (2006).

[26] L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 4690 (1999).

[27] C. Charmousis and J. F. Dufaux, Classical Quantum Gravity **19**, 4671 (2002); S. C. Davis, Phys. Rev. D **67**, 024030 (2003).

[28] K. Freese and M. Lewis, Phys. Lett. B **540**, 1 (2002).