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Fermat principle for spinning light

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Mimicking the description of spinning particles in General Relativity, the Fermat Principle is extended to spinning photons. Linearization of the resulting Papapetrou-Souriau type equations yields the semiclassical model used recently to derive the ''Optical Hall Effect'' for polarized light (alias the ''Optical Magnus Effect'').

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Light is an electromagnetic wave, whose propagation is described by Maxwell's theory. It can also be viewed, however, as a particle (a ''photon''). Here we adhere to the second approach: we describe light by a *bona fide* mechanical model in that we use a Lagrangian.

In traditional geometrical optics the spin degree of freedom is neglected, and the light rays obey the Fermat Principle [1]. In the intermediate model advocated by Landau and Lifchitz [2], the photon is polarized, but the polarization is simply carried along by the light rays, and has no influence on the trajectory of light. Recent approaches [3,4] go one step further: the feedback from the polarization deviates the trajectory from that given by the Fermat Principle. A dramatic consequence is that, for polarized light, the Snel(-Descartes) law of refraction requires correction: the plane of the refracted (or reflected) ray is shifted perpendicularly to that of the incident ray [3]. This ''Hall Effect for light'' is a manifestation of the Magnus-type interaction between the refractive medium and the photon's polarization [4]. It can be derived in a semiclassical framework, which also includes a Berry-type term [5–7].

In this Rapid Communication, we argue that the deviation of polarized light from the trajectory predicted by ordinary geometrical optics is indeed analogous to the deviation of a spinning particle from geodesic motion in General Relativity. The resulting equations are reminiscent of those of Papapetrou and Souriau [8].

In detail, the Fermat Principle of geometrical optics says that light in an isotropic medium of refractive index $n = n(\mathbf{r})$ propagates along curves that minimize the optical length. Light rays are hence geodesics of the ''optical'' metric $g_{ij} = n^2(\mathbf{r})\delta_{ij}$ of 3-space. To extend this theory to spin we consider the bundle of positively oriented orthonormal frames over a 3-manifold endowed with a Riemannian metric g_{ij}. At each point, such a ''Dreibein'' is given by three orthogonal vectors U^i, V^i, W^i of unit length that span unit volume. We stress that the [6-dimensional] orthonormal frame bundle we are using here is a mere artifact that allows us to define a variational formalism. Eliminating unphysical degrees of freedom will leave us with 4 independent physical variables.

Introducing the covariant exterior derivative associated with the Levi-Civita connection, $DU^k = dU^k + \Gamma^k_{ij} dx^i U^j$, we posit the reparametrization-invariant action

$$S = S_{\mathrm{Fermat}} + S_{\mathrm{spin}} = \kappa \int U_i \frac{dx^i}{d\tau} d\tau - s \int V_i \frac{DW^i}{d\tau} d\tau, \tag{1}$$

where τ is some parameter along the light ray. The parameters s and $\kappa > 0$ are interpreted as the spin and the color, respectively. Upon first quantization, κ becomes indeed, for a monochromatic wave, $2\pi\hbar/\lambda$, where λ is the wavelength [9]. For the photon $s = \pm\hbar$, but we keep it arbitrary for future convenience. Equation (1) is supplemented with the constraints $U_iU^i = V_iV^i = W_iW^i = 1$, and $U_iV^i = U_iW^i = V_iW^i = 0$.

The first term in (1) is [κ times] the usual optical length; the second, ''Wess-Zumino-type'' [10] term, that arises naturally in the geometric framework of [11], corresponds to the Berry connection, and is indeed analogous to the torsion term considered by Polyakov [12].

The Euler-Lagrange equations are obtained as follows. Variation of the first term in (1) yields

$$\delta S_{\mathrm{Fermat}} = \kappa \int \bigg[-\delta x_k \frac{DU^k}{d\tau} + \frac{dx^k}{d\tau} \delta_\Gamma U_k \bigg] d\tau, \tag{2}$$

where $\delta_\Gamma U^k = \delta U^k + \Gamma^k_{lm} \delta x^l U^m$ is the covariant variation

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of the vector field U^i. For the spin term, straightforward calculation yields

$$\delta S_{\rm spin} = \int\left[\delta_\Gamma U_j \frac{DU_k}{d\tau} + \frac{1}{2}\delta x^i \frac{dx^\ell}{d\tau} R_{jki\ell}\right] S^{jk} d\tau, \quad (3)$$

where $R_{jki\ell} = g_{im}(\partial_j \Gamma^m_{k\ell} - \partial_k \Gamma^m_{j\ell} + \cdots)$ is the Riemann tensor and $S^{ij} = -s(V^i W^j - W^i V^j)$ is the spin tensor. Then the variational principle $\delta S = 0$ allows us to infer the pair of equations

$$\kappa g_{ik}\frac{dx^k}{d\tau} + S_{ik}\frac{DU^k}{d\tau} = \mu U_i, \quad (4)$$

$$\kappa g_{ik}\frac{DU^k}{d\tau} - \frac{1}{2} R_{jki\ell} S^{jk} \frac{dx^\ell}{d\tau} = 0, \quad (5)$$

where μ is a Lagrange multiplier which enforces the orthogonality of U^i and $\delta_\Gamma U^i$. Inserting $DU^k/d\tau$ into Eq. (4) and redefining the parameter along the ray by $dt = (\mu/\kappa)d\tau$ yields the Papapetrou-Souriau type [8] equations,

$$\frac{dx^i}{dt} = U^i + \frac{1}{2}\frac{S^i{}_j R(S)^j{}_k U^k}{(\kappa^2 + s^2 E_{\ell m} U^\ell U^m)}, \quad (6)$$

$$s\frac{DU^i}{dt} = -\frac{1}{2\kappa} R(S)^i{}_j \frac{dx^j}{dt}, \quad (7)$$

where $E_{ij} = R_{ij} - \frac{1}{2} R g_{ij}$ is the Einstein tensor and the matrix $R(S)^\ell_k = R^\ell_{ijk} S^{ij}$ represents the interaction of spin with the curvature, responsible for tidal forces. Because of the spin-curvature coupling, the direction of the velocity differs, in general, from that of the spin vector $sU^i = -\frac{1}{2}\sqrt{g}\epsilon^{ijk} S_{jk}$.

For the optical metric, the Christoffel symbols are $\Gamma^k_{ij} = \frac{1}{n}(\partial_i n \delta^k_j + \partial_j n \delta^k_i - \partial^k n \delta_{ij})$. (The optical metric is hence not flat unless the refraction index is constant.) Putting $\mathbf{r} = (x^i)$, $u^i = nU^i$, $v^i = nV^i$, $w^i = nW^i$ introducing the momentum,

$$\mathbf{p} = n\left[\kappa \mathbf{u} + s\boldsymbol{\nabla}\left(\frac{1}{n}\right) \times \mathbf{u}\right], \quad (8)$$

and denoting the derivative w.r.t. t by a "dot", our Lagrangian in (1) can also be presented as $L = \mathbf{p}\cdot\dot{\mathbf{r}} - s\mathbf{v}\cdot\dot{\mathbf{w}}$. The equations of motion for $\mathbf{r}$ and $\mathbf{p}$ read, in this case,

$$\dot{\mathbf{r}} = aA\mathbf{p} + \frac{s^2}{n\kappa^2}\boldsymbol{\nabla}\left(\boldsymbol{\nabla}\left(\frac{1}{n}\right)\right)A\mathbf{p}, \quad (9)$$

$$\dot{\mathbf{p}} = -n(\mathbf{p}\cdot\dot{\mathbf{r}})\boldsymbol{\nabla}\left(\frac{1}{n}\right) + \frac{s}{\kappa}\boldsymbol{\nabla}\left(\boldsymbol{\nabla}\left(\frac{1}{n}\right)\right)A\mathbf{p}\times\dot{\mathbf{r}}, \quad (10)$$

where $a = 1 + (s^2/\kappa^2)((\boldsymbol{\nabla}(1/n))^2 - (1/n)\Delta(1/n))$ and

$$A\mathbf{p} = \frac{1}{1 + (s/\kappa)^2(\boldsymbol{\nabla}(\frac{1}{n}))^2} \times \left[\mathbf{p} - \frac{s}{\kappa}\boldsymbol{\nabla}\left(\frac{1}{n}\right)\times\mathbf{p} + \frac{s^2}{\kappa^2}\left(\boldsymbol{\nabla}\left(\frac{1}{n}\right)\cdot\mathbf{p}\right)\boldsymbol{\nabla}\left(\frac{1}{n}\right)\right]. \quad (11)$$

These equations describe spinning light in an inhomogeneous medium. Let us mention, for completeness, that the evolution of the spin vector, which follows from (7), is given by $s\dot{\mathbf{u}} = -n\kappa[\dot{\mathbf{r}} - \frac{s}{\kappa}\boldsymbol{\nabla}(\frac{1}{n}) \times \dot{\mathbf{r}}] \times \mathbf{u}$.

If the medium is spherically symmetric, $n = n(r)$, conserved angular momentum is readily derived using Noether's theorem. It reads

$$\mathbf{J} = \mathbf{r}\times\mathbf{p} + s\mathbf{u}. \quad (12)$$

Let us now discuss some particular cases of our general theory.

(i) For $s = 0$ we have $\mathbf{p}^2 = n^2\kappa^2$, $a = 1$ and $A\mathbf{p} = \mathbf{p}$. Introducing the elementary arc length $d\sigma = n\kappa dt$, we recognize the usual Fermat equations, $nd\mathbf{r}/d\sigma = \mathbf{p}/\kappa$, $d(\mathbf{p}/\kappa)/d\sigma = \boldsymbol{\nabla} n$ [1].

(ii) In a homogeneous medium, $n =$ const. we get, for any color, κ, and spin, s, the same equations: light propagates along straight lines parallel to $\mathbf{p} = n\kappa\mathbf{u}$. The model is invariant w.r.t. the Euclidean group SE(3) consisting of space translations and rotations. The associated conserved quantities are the linear momentum, $\mathbf{p} = n\kappa\mathbf{u}$, and the angular momentum, (12), which is now $\mathbf{J} = \mathbf{r}\times\mathbf{p} + s\mathbf{p}/p$.

(iii) In a medium with slowly varying refractive index, terms involving second-order derivatives and quadratic expressions in $\boldsymbol{\nabla}(1/n)$ can be neglected, e.g., $R_{ij} = \frac{2\partial_i n \partial_j n}{n^2} - \frac{\partial_i\partial_j n}{n} - \frac{\Delta n}{n}\delta_{ij} \approx 0$. Hence the trajectory of light is approximately tangent to the spin and the latter is approximately parallel transported,

$$\frac{dx^i}{d\tau} \approx U^i, \qquad \frac{DU^i}{d\tau} \approx 0. \quad (13)$$

In $\mathbf{p}$-terms, $\mathbf{p}^2 \approx n^2\kappa^2$, and the general Eqs. (9) and (10) are approximated by

$$\dot{\mathbf{r}} \approx \mathbf{p} - \frac{s}{\kappa}\boldsymbol{\nabla}\left(\frac{1}{n}\right)\times\mathbf{p}, \qquad \dot{\mathbf{p}} \approx -n^3\kappa^2\boldsymbol{\nabla}\left(\frac{1}{n}\right). \quad (14)$$

In the case of spherical symmetry, the general angular momentum (12) reduces, up to the approximately conserved extra term $(s^2/\kappa)\boldsymbol{\nabla}(1/n)\times\mathbf{u}$, to the expression used by Onoda *et al.* in [3], namely, to

$$\mathbf{J}^{\rm OMN} = \mathbf{r}\times\mathbf{p} + s\frac{\mathbf{p}}{p}. \quad (15)$$

Let first consider the free case, $n = 1$. The variable $\mathbf{r}$ used so far has been an arbitrary point of the light ray. Now, the ray itself can be labeled by its direction $\mathbf{u}$ and $\mathbf{q} = \mathbf{r} - (\mathbf{u}\cdot\mathbf{r})\mathbf{u}$, which is in fact the shortest vector drawn from the origin to the ray (orthogonal to the unit vector $\mathbf{u}$), and can be thought of as the "position" of the ray. The 4-dimensional manifold, $\mathcal{M}$, of light rays described by $\mathbf{u}$ and $\mathbf{q}$, has the topology of the tangent bundle of the two-sphere and can be identified with a coadjoint orbit of SE(3). The Casimir invariants of the orbit (which determine unitary irreducible representations) are $\kappa = p(=\sqrt{\mathbf{p}^2})$ and $s = \mathbf{J}\cdot\mathbf{p}/p$. The corresponding orbit, $\mathcal{M}$, is endowed with the canonical symplectic structure

$$\omega_0 = \kappa du_i \wedge dq^i - \frac{s}{2}\epsilon_{ijk}u^i du^j \wedge du^k, \qquad (16)$$

see [9]. The monopolelike term in (16) is the Berry curvature, $\frac{1}{2}s\epsilon_{ijk}p^i dp^j \wedge dp^k/p^3$. It makes the components of $\mathbf{q}$ noncommuting [11], i.e., Cartesian coordinates q_1 and q_2 have nonvanishing Poisson bracket, $\{q_1, q_2\} = s/\kappa^2$. Upon (first) quantization, in the case $s = \hbar$, the quantum position operators $\hat{q}_1$ and $\hat{q}_2$ satisfy

$$[\hat{q}_1, \hat{q}_2] = i(\lambda/2\pi)^2, \qquad (17)$$

where $\lambda = 2\pi\hbar/\kappa$. The Heisenberg uncertainty relation read, therefore, $\Delta\hat{q}_1 \cdot \Delta\hat{q}_2 \geq \frac{1}{2}(\lambda/2\pi)^2$, which provide a new interpretation of the localization defect of spinning light rays, limiting the resolving power of optical instruments to the order of the wavelength, $\Delta\hat{q} \approx O(\lambda)$.

In a nontrivial refractive medium the (exact) twoform (16) is replaced by

$$\omega = \kappa DU_i \wedge dx^i - \frac{1}{4}R(S)_{ij}dx^i \wedge dx^j - \frac{s}{2}\sqrt{g}\epsilon_{ijk}U^i DU^j \wedge DU^k \qquad (18)$$

on the orthonormal frame bundle. The Euler-Lagrange Eqs. (6) and (7) correspond in fact to the kernel of the twoform (18), see [11]. Conversely, the spin term in our Lagrangian comes from a potential for the spin terms in the twoform (18).

Now putting $\mathbf{p} \to \mathbf{p}/\kappa$ and $\tau \to \kappa\tau$, our linearized Eqs. (14) become those proposed in [4].

The relation to the model of Onoda *et al.* [3] is more subtle. In their approach, polarization is an additional variable, represented by a two-component complex vector, $z = (z_a)$ with $a = \pm$, such that $|z_+|^2 + |z_-|^2 = 1$, acted upon by su(2). Their semiclassical equations of motion can be written as

$$\mathbf{r}' = \frac{1}{nk}\mathbf{k} + \mathbf{k}' \times \mathbf{\Omega}^{ab}\bar{z}_a z_b, \qquad \mathbf{k}' = -\nabla\left(\frac{1}{n}\right)k, \qquad (19)$$

supplemented with

$$z'_a = k\nabla\left(\frac{1}{n}\right)\cdot \mathbf{\Lambda}^{ab} z_b, \qquad (20)$$

where $\mathbf{k}$ is the wave vector and $\mathbf{\Lambda}^{ab}(\mathbf{k})$ is an su(2)-valued nonabelian "Berry" vector potential. The Berry curvature can be represented by an su(2)-valued vector $\mathbf{\Omega} = \sigma_3\mathbf{k}/k^3$, a Dirac monopole in $\mathbf{k}$-space, diagonally embedded into su(2). The vector potential for $\mathbf{\Omega}$ can, therefore, be chosen as $\mathbf{\Lambda}^{ab} = i\mathbf{\Lambda}(\sigma_3)^{ab}$ where $\mathbf{\Lambda}(\mathbf{k})$ is a monopole potential, $\mathbf{rot}\mathbf{\Lambda} = \mathbf{k}/k^3$. The number of equations in (19) and (20) can be reduced to two. The polarization Eq. (20) can in fact be solved formally by parallel transport, $z_a = e^{ia\theta}z_a^0$, where the phase is given by the nonintegrable phase factor $\theta = \int k(\nabla(\frac{1}{n})\cdot\mathbf{\Lambda})d\sigma$. Then the Berry term becomes simply $\mathbf{\Omega}^{ab}\bar{z}_a z_b = s\mathbf{k}/k^3$ where $s = |z_+|^2 - |z_-|^2 = |z_+^0|^2 - |z_-^0|^2$, since the $|z_a|^2$ are separately conserved. Notice that this s is a constant of the motion, which can take any value between -1 and $+1$. Identifying the wave vector, $\mathbf{k}$, with our momentum, $\mathbf{p}$, and putting $(\cdot)' = (n^2\kappa)d/dt$ transforms finally (19) into our Eqs. (14).

We did not consider the polarization in our framework. As long as we are only interested in describing light rays, polarization is a secondary quantity, whose only role is to generate spin, which in turn deviates the trajectory from that of conventional geometrical optics. It is hence more appropriate to speak of spinning light than of polarized light. Let us nevertheless mention that first quantization along the lines of [9] of the classical model (16), with Casimirs κ and $s = \pm\hbar$, yields, in the gauge div$\mathbf{A} = 0$, the vectorial Helmholtz equation

$$(\Delta + k^2)\mathbf{A} = 0, \qquad (21)$$

where $k = \kappa/\hbar$. It follows that $\mathbf{E} = ik\mathbf{A}$ and $\mathbf{B} = \mathbf{rot}\,\mathbf{A}$ satisfy the field equations

$$\mathbf{rot}\,\mathbf{E} - ik\mathbf{B} = 0, \qquad (22)$$

$$\mathbf{rot}\,\mathbf{B} + ik\mathbf{E} = 0, \qquad (23)$$

associated with our Euclidean model. Promoting the parameter t as "time", these are indeed the vacuum Maxwell equations for the stationary fields $\mathbf{E}e^{-ikt}$ and $\mathbf{B}e^{-ikt}$, respectively, $(c = 1)$. In an isotropic medium, Eqn (23) is generalized [1] to

$$\mathbf{rot}\,\mathbf{H} + ik\mathbf{D} = 0, \qquad (24)$$

where $\mathbf{D} = \epsilon\mathbf{E}$ and $\mathbf{B} = \mu\mathbf{H}$, with ϵ and μ the permittivity and permeability, respectively. The refractive index is $n = \sqrt{\epsilon\mu}$. Remarkably, the field equations can, again, be rewritten in terms of optical metric, namely, in the form (22) and (24) above, replacing the operator **rot** by its curved-space form, $\mathbf{rot}\,\mathbf{X} \to n^{-3}\mathbf{rot}(n^2\mathbf{X})$, and rescaling the fields, $\mathbf{E} \to n^{-2}\mathbf{E}$, $\mathbf{H} \to n^{-2}\mathbf{H}$, $\mathbf{B} \to n^{-3}\mathbf{B}$, $\mathbf{D} \to n^{-3}\mathbf{D}$ and $\epsilon \to n^{-1}\epsilon$, $\mu \to n^{-1}\mu$.

Conventional geometric optics can be derived from the eikonal approximation of Maxwell's electrodynamics [1]. Here we followed the opposite way: we started with a classical model and derived the stationary Maxwell Eqs. (22)–(24) by (first) quantization. Although we have not yet been able to deduce our action (1) from taking a suitable semiclassical limit of (22)–(24), we emphasize that our model actually comes from *first principles*—but of those of Mechanics [11]. Firstly, the free model is constructed along the lines suggested by Souriau's [9], applied to the Euclidean group. The second step is *minimal gravitational coupling*, which amounts to replacing the ordinary scalar product by the one associated with the optical metric and the ordinary derivatives by covariant derivatives; this yields the twoform (18). The latter is in turn associated with first-order variational calculus on "phase space", whose Lagrangian is precisely our (1).

Our model reproduces, at first order, the phenomenological descriptions proposed in [3,4] which can, in turn, be derived by taking an improved semiclassical limit of the Maxwell equations [4]. Does a similar procedure work for our model ? The question is open.

Our theory is neither relativistic nor nonrelativistic, since it does not involve time at all; it is based on the Euclidean group—which is indeed a subgroup of both the Galilei and of the Poincaré groups. Our Euclidean model arises in fact as a *reduction* by time translations of the zero-mass spinning orbits of *both the Galilei and the Poincaré groups* [11]—which constitute the conventional descriptions of "classical light" [9,10]).

An application of our semiclassical model is the derivation of the modified laws of refraction and reflection at the interface of two homogeneous media with different refractive indices. As found by Onoda *et al.* [3], polarized light suffers, in fact, a transverse shift. This "Optical Hall effect" [3] is indeed an optical version of the spin-Hall effect [7]. Their shift formula can be rederived [11], following Souriau [9], who argues that the two "mechanical" states on both sides of the interface are related by a symplectic transformation, S, which is indeed the classical counterpart of the quantum scattering matrix. This transformation commutes with the symmetries of the optical device; in our case, this is plainly the Euclidean group generated by translations of the separating plane and rotations around its normal direction, $\mathbf{N}$. Tedious calculation provides us with the "classical scattering matrix", S [11].

Firstly, the conservation of planar linear momentum extends Snel's laws to spinning light, namely

$$n_{\rm in}\sin\theta_{\rm in} = n_{\rm out}\sin\theta_{\rm out}, \tag{25}$$

for refraction, and $\theta_{\rm in} = \pi - \theta_{\rm out}$, for reflection, respectively, where $n_{\rm in}$ resp. $n_{\rm out}$ denote the refractive indices on both sides of the interface.

Next, the conservation of the (planar) angular momentum implies that light is shifted transversally [3,11], viz.

$$\mathbf{q}_{\rm out} - \mathbf{q}_{\rm in} = \frac{[s_{\rm out}\cos\theta_{\rm out} - s_{\rm in}\cos\theta_{\rm in}]}{\kappa n_{\rm in}|\sin\theta_{\rm in}|}\frac{\mathbf{N}\times\mathbf{u}_{\rm in}}{|\mathbf{N}\times\mathbf{u}_{\rm in}|}, \tag{26}$$

where $s_{\rm in} = s_{\rm out}$ for refraction.

Notice that the shift depends in general on the wavelength, $\lambda = 2\pi\hbar/\kappa$, white light is split, in general, into colors, shifted by different amounts. For $n_{\rm out} = -n_{\rm in}$, however, Snel's laws entail that the shift (26) *vanishes*: white light is *not* decomposed and is indeed refracted following the classic Snel law, as if it had no spin!

This case is *not* of pure academic interest, owing to the existence of left-handed media (with a negative refractive index) [13]. In the ideal case, one can have $n = -1$, and a simple slab with parallel sides [14] provides us with a "perfect lens" with no chromatic aberration.

We note that the shift (26) vanishes also for a reflection, since then $s_{\rm in} = -s_{\rm out}$. This does not contradict the results in Imbert [15], which are indeed of higher-order.

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