$SU(N_c \to \infty)$ lattice data and degrees of freedom of the QCD string

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(Received 23 December 2005; published 28 July 2006)

Lattice simulation data on the critical temperature and long-distance potential, that probe the degrees of freedom of the QCD string, are critically reviewed. It is emphasized that comparison of experimental or $SU(N_c)$ lattice data, at finite number of colors N_c, with free string theory can be misleading due to string interactions. Large-N_c extrapolation of pure lattice gauge theory data, in both 3 and 4 dimensions, indicates that there are more world sheet degrees of freedom than the purely massless transverse ones of the free Nambu-Goto string. The extra variables are consistent with massive modes of oscillation that effectively contribute like $c \approx 1/2$ conformal degrees of freedom to highly excited states. As a concrete example, the highly excited spectrum of the Chodos-Thorn relativistic string in $1 + 1$ dimensions is analyzed, where there are no transverse oscillations. We find that the asymptotic density of states for this model is characteristic of a $c = 1/2$ conformal world sheet theory. The observations made here should also constrain the backgrounds of holographic string models for QCD.

DOI: 10.1103/PhysRevD.74.014025 PACS numbers: 12.38.Lg

I. INTRODUCTION

It is well known that the long-wavelength properties of confining gauge theories can be modeled quite accurately by thin, structureless relativistic strings oscillating transverse to themselves. The prototypical example is the Nambu-Goto model [1], with action proportional to world sheet area. At shorter wavelengths, additional properties or degrees of freedom are likely to play a role. A related and popular suggestion has been that the string is always thin and structureless, but that the additional effects can effectively be seen as motion in a curved higher dimension [2,3]. One would like to characterize any deviations from structureless strings in spacetime in order to determine what, if any, string theory exactly describes confining gauge theories at all length scales. In this paper, data from lattice simulations of $SU(N_c)$ pure gauge theories will be used as a guide. It is only quite recently that such data have become accurate and extensive enough to be useful in this regard. Based on this analysis, a concrete suggestion is made for additional degrees of freedom that must be included at short wavelengths—longitudinal modes of oscillation similar to those of the Chodos-Thorn massive string [4].

The gauge theory data used are the asymptotic density of states in the hadron spectrum, the (related) Hagedorn temperature, and the long-distance ground state potential energy of a winding string or pair of heavy sources. These probe the short and long wavelength regimes, respectively. The existence in gauge theory of a Hagedorn temperature in particular—which is to be distinguished from the deconfinement temperature in general—is an important indication that a string description exists on all length scales. It is not the intention to provide here a complete historical review of lattice data; only the most accurate and relevant results have been selected. Data from 3 as well as 4 spacetime dimensions are used, since pure gauge theory seems to be described by a string theory in both cases. Data for fixed, finite N_c, which are often used to address such questions, can be misleading if compared with a free string model. It is shown that string interactions of strength $1/N_c$ can mask the effects of additional world sheet degrees of freedom. To identify the latter unambiguously, one must extrapolate data to $N_c = \infty$. Large-N_c results are shown to indicate a Hagedorn temperature below that of the Nambu-Goto free- string, equivalent to adding an extra effective conformal degree of freedom with central charge $c \approx 1/2$. However, the "Luscher-term" [5,6], in the asymptotic $1/l$ expansion of the ground state energy of a gauge string with minimal allowed length l, seems to be consistent with the Nambu-Goto result and not to depend on N_c. These two statements are consistent if the additional world sheet degrees of freedom, beyond those of Nambu-Goto, are massive. Only at energies large compared to their mass is a conformal approximation valid.

In the second part of the paper, it is suggested that massive longitudinal oscillations are responsible for the extra degrees of freedom. This is hardly an original observation. However, new evidence is presented that such oscillations have roughly the right number of degrees of freedom to account for the facts displayed in the first part of the paper. The Chodos-Thorn massive relativistic string is essentially the Nambu-Goto model with additional massive pointlike insertions of energy-momentum on the string. Although this model is difficult to solve in greater than two dimensions, the asymptotic spectrum is found exactly in two dimensions, where only longitudinal oscillations remain. It is found that these are equivalent to those of a $c = 1/2$ world sheet field theory. Whether this is coincidence and we are barking up the wrong tree remains to be seen. In any event, the lattice data strongly constrain any purported string theory of QCD.

II. $SU(N_c \to \infty)$ DATA

A. Stringy observables

Let us first briefly review the physical observables used later. The bound state spectra of string theories generically have a density of states, $\rho(M)$ per unit interval of mass M, that grows exponentially for large M. If, on the world sheet of the *free* string, there is a conformal field theory of central charge c representing physical oscillations (not counting those eliminated by reparametrization invariance etc.), then as $M \to \infty$ [7]

$$\rho(M) \propto M^{-(3+D_\perp)/2} \exp\left(\frac{M}{T_H}\right), \tag{2.1}$$

where

$$T_H = \sqrt{\frac{3\sigma}{c\pi}}, \tag{2.2}$$

σ is the string tension, and $D_\perp$ is the effective number of dimensions for transverse oscillations. Note that the same asymptotic density of states can be expected even if the world sheet theory is massive, once the excitation energies are far above the relevant mass scale. The theory is effectively conformal in this regime. As is well known, the canonical partition function of the free string gas diverges above the Hagedorn temperature $T > T_H$ [8]. The physics of this point is a second order phase transition [9] if the power-law corrections in Eq. (2.1) are such that the internal energy does not diverge at T_H, driven by the entropy of strings. The transition is also signaled by the vanishing of the mass $E_c(1/T)$ of the string that winds once around the compact Euclidean time direction of circumference $1/T$ in the finite-temperature partition function [10]. All these statements have an analogue in confining gauge theories. The (real) hadronic spectrum does rise exponentially [11] and there are phase transitions in pure gauge theory at which winding Polyakov loops get a VEV [12]. When those transitions are second order, the Polyakov loop mass vanishes and one may assume this is a Hagedorn transition at $T = T_H$. If the gluon entropy is more important than the string entropy, which tends to be the case for larger N_c [13–15], a first order transition at $T_c < T_H$ will occur first. However, it has been shown by Teper and Bringoltz [16] that one may study the approach to T_H in the metastable superheated phase above T_c; in fact, this phase should become stable when $N_c \to \infty$.

Another measure of the degrees of freedom of string theory results from the asymptotic $1/l$ expansion [5] of the ground state energy $E_o(l)$ of an open string with endpoints at fixed separation l:

$$E_o(l) = \sigma L + \text{const.} - \frac{\pi\tilde{c}}{24l} + \cdots. \tag{2.3}$$

The coefficient of the "Luscher-term," with $\tilde{c} = c - 24h$, is related to the Casimir energy of free massless fields [6] of central charge c and the lowest dimension h of primary field that propagates on the world sheet. There is a similar expression for a closed string [17] that winds once around a compact spatial direction of circumference l:

$$E_c(l) = \sigma L + \text{const.} - \frac{\pi\tilde{c}}{6l} + \cdots. \tag{2.4}$$

The analogues of these configurations in pure gauge theory are the potential between heavy sources in the fundamental representation of $SU(N_c)$ and the Polyakov loop mass. As well as the $l \to \infty$ limit, we will also be interested in the minimum allowed l, since $E_c(1/T_H) = 0$ can be taken to define the Hagedorn temperature.

B. Why $3 \neq \infty$

The original connection between string world sheets and confining gauge theories was made by 't Hooft [18], in the case of weak gauge coupling g expansion, and by Wilson [19], in the case of strong gauge coupling expansion. In both cases, the string coupling g_s, by which we mean the (logarithm of the) coupling to the world sheet topological invariant

$$\int dx^0 dx^1 R\sqrt{-G}, \tag{2.5}$$

is $1/N_c$. Here, $G = \det G_{\alpha\beta}$, where the induced world sheet metric is

$$G_{\alpha\beta} = \frac{\partial X^\mu}{\partial x^\alpha}\frac{\partial X_\mu}{\partial x^\beta}, \tag{2.6}$$

R its Gaussian curvature, and X^μ the world sheet embedding coordinate in space-time of dimension D, $\mu \in \{0, \cdots D-1\}$, and $\{x^0, x^1\}$ the intrinsic coordinates on the world sheet. Equation (2.5) governs the splitting and joining interactions of strings. In the modern era also, the ADS/CFT correspondence [2] and its nonconformal nonsupersymmetric generalizations are between $SU(N_c)$ gauge theories and fundamental string theory with coupling $g_s \propto 1/N_c$ in the $N_c \to \infty$ limit. If we are trying to phenomenologically determine the rest of the world sheet action that must be added to (2.5) in gauge theories, the presence of string interactions greatly complicates matters. Most results for string theories have been derived to leading orders in the string coupling expansion. In particular, the spectrum of string states is known exactly for several string actions, but only for free strings.

The Nambu-Goto model adds to the world sheet action the area term:

$$-\sigma \int dx^0 dx^1 \sqrt{-G}. \tag{2.7}$$

By light cone gauge fixing of reparametrization invariance and Fourier decomposition of transverse coordinates ($i \in \{1, \cdots D-2\}$) for open strings

$$X^i = X_0^i + P^i x^0 + \sum_{n\neq 0} \alpha_n^i e^{-inx^0} \cos nx^1, \qquad (2.8)$$

free strings ($g_s = 0$) in the quantized model have mass spectrum operator given by a sum of harmonic oscillators [20]

$$M^2 = 2\pi\sigma \sum_{n=1}^{\infty} \alpha_{-n}^i \alpha_n^i, \qquad (2.9)$$

$$[\alpha_n^i, \alpha_m^j] = n\delta_{nm}\delta^{ij}. \qquad (2.10)$$

The Nambu-Goto model is a classical conformal field theory in two dimensions with $c = D - 2 = D_\perp$, $h = 0$. Therefore, it predicts

$$\tilde{c} = D - 2, \qquad T_H = \sqrt{\frac{3\sigma}{(D-2)\pi}}. \qquad (2.11)$$

In particular

$$T_H \approx 0.98\sqrt{\sigma} \quad (D = 3), \qquad (2.12)$$

$$T_H \approx 0.69\sqrt{\sigma} \quad (D = 4). \qquad (2.13)$$

Although this model is not Lorentz invariant for $D \neq 26$, the action should probably be understood as the long distance approximation of a more general one describing gauge theory strings. In principle one can include further dimensionful interactions between the transverse degrees of freedom X^i or add further degrees of freedom. This procedure can be made formal through the $1/l$ expansion [5], and there is some indication that consistency problems can be fixed order-by-order [21]. If the additional degrees of freedom are massive, they could be integrated out systematically to leave higher order corrections to the Nambu-Goto action.

At first sight, it appears that the simple Nambu-Goto model, in the case of free strings, is in excellent agreement with data on the observables mentioned above in $SU(3)$ gauge theory (the case relevant for QCD). First, recent precision lattice data on the value of the Luscher coefficient, indicate that $\tilde{c} = D - 2$ [22,23]. Indeed, this also appears to be the case for other gauge groups, including Z_2 [24,25], $SU(2)$ [25,26], and $SU(N_c \to \infty)$ [27], suggesting the $1/N_c$ string interactions do not affect the simple Casimir effect argument for the coefficient of the Luscher term. In principle, it is possible that there are additional massless degrees of freedom, $c > D - 2$, but still $\tilde{c} = D - 2$. Although this cannot technically be ruled out, it seems an unlikely coincidence and, moreover, the ground state is usually not a scalar if $h > 0$. More generally, the result would be consistent with additional dimensionful world sheet interactions or additional massive degrees of freedom, since these should not contribute to the long-distance l properties of the string.

Pure lattice gauge theory simulations of the Polyakov loop correlators enable accurate determinations of the transition temperature, with the loop expectation value as order parameter. For $D = 3$, $SU(3)$ pure lattice gauge theory the transition is second order and agrees with the $D = 3$ Hagedorn temperature of the simple Nambu-Goto model (2.12) [28,29]. For $D = 4$ and $SU(3)$, the deconfinement transition temperature T_c is slightly below the Nambu-Goto Hagedorn temperature (2.13) [13]. But it is weakly first order, so the actual gauge theory Hagedorn temperature is probably slightly higher than T_c. Although it is difficult to study the density of hadron states directly in lattice gauge theory, except in some effective theories [30], this data can be extracted from experiment. The meson resonance spectrum shows a clear exponential rise. A study by Dienes and Cudell [31] established that the best fit to the data was for $c = D_\perp = 2$, in agreement with the Nambu-Goto model.

Given this evidence, one might be led to conclude that the string theory relevant for QCD is just the Nambu-Goto one. Of course, this cannot be true because the model is not even consistent in 3 and 4 dimensions. The spectrum of the $SU(3)$ heavy-source potential shows clear deviations from the Nambu-Goto prediction for small separations [32]. Therefore it would be surprising if the Hagedorn temperature were correctly predicted. Results for other gauge groups, however, imply a more complicated picture. For $D = 3$ pure gauge theory, the transition is second order for $N_c = 2, 3, 4$ (or at worst very weakly first order for $SU(4)$, so the Hagedorn point is very close by), and first order for higher N_c [29,33–35]. The corresponding temperatures are plotted versus $1/N_c$ in Fig. 1. If we are correct in identifying the $N_c = 2, 3, 4$ points as Hagedorn transitions, there is clearly a strong dependence of T_H on N_c. For $D = 4$ pure gauge theory, the transition is second order only for $SU(2)$ [15]. However, Teper and Bringoltz [16] have recently been able to follow the Polyakov loop mass into the metastable phase above the first order transitions for $N_c > 2$, identifying the Hagedorn temperature from $E_c(1/T_H) = 0$. Figure 2 shows again that T_H depends upon N_c. Recalling that this temperature in a free resonance gas is dictated by the asymptotic density of states, it is natural to conclude that the $1/N_c$ string interactions alter the spectrum so as to reduce the density of states; they will also give states a width, but this is not usually noticed in a lattice simulation. In fact, this thinning of the spectrum can be seen already in the low-lying glueball spectrum. Figure 3 compares the lowest glueball masses for $SU(2)$ and $SU(N_c \to \infty)$ in 3 dimensions [36]. For $\rho(M)$ to decrease, the mass shift of the tth glueball should increase with M_t. A similar qualitative effect can be discerned in the spectrum for $D = 4$ [37], although fewer accurate glueball masses are available in this case.

The results for the $SU(3)$ Hagedorn temperature thus appear to be accidentally close to those for the free Nambu-

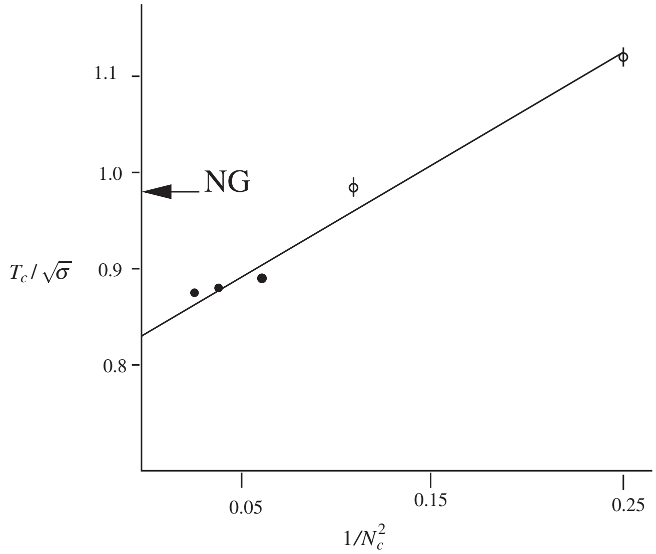

FIG. 1. The variation of the transition temperature T_c, in units of the zero-temperature string tension σ, with the number of colors N_c for pure $SU(N_c)$ gauge theory in $D = 3$ dimensions. Open circles [29] and filled circles [34] are from different simulations. The linear fit is to the second order transitions at $N_c = 2, 3, 4$, where we identify $T_c = T_H$. NG indicates the free-string Nambu-Goto prediction.

Goto string. The strings of a $SU(3)$ gauge theory are not free, but have $1/N_c$ interactions which mask world sheet effects not accounted for by the Nambu-Goto model. To

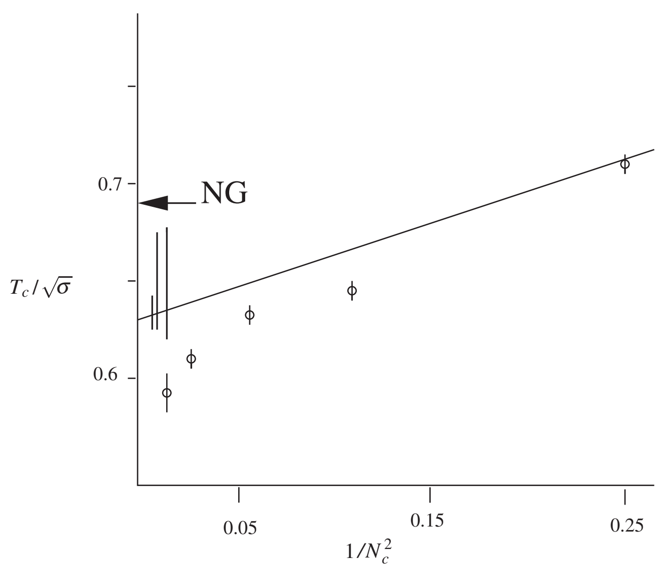

FIG. 2. Open circles show the variation of the deconfinement temperature T_c, in units of the zero-temperature string tension σ, with the number of colors N_c for pure $SU(N_c)$ gauge theory in $D = 4$ dimensions [13,15]. The solid bars represent the range of Hagedorn temperatures obtained in Ref. [16] by different fits to the Polyakov loop mass. The straight line fit is to these and the second order $SU(2)$ transition, where we identify $T_c = T_H$. NG indicates the free-string Nambu-Goto prediction.

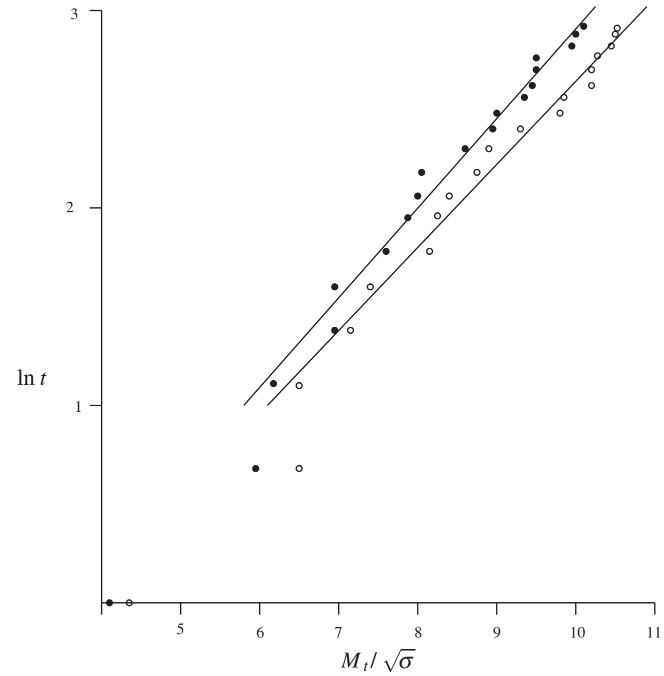

FIG. 3. The mass M_t of the tth glueball in the spectrum of pure $SU(2)$ (open circles) and $SU(\infty)$ (filled circles) gauge theory in $D = 3$ dimensions [36]. The lowest 19 states are shown. The straight lines are to guide the eye only.

compare like-with-like, one should only use free-string formulas when comparing with the $N_c \to \infty$ limit of gauge theory. What do such extrapolations predict? In pure gauge theory, one expects physical observables to be expandable as an asymptotic series in $1/N_c^2$ [18]. If one naively extrapolates the Hagedorn temperatures of pure gauge theory to $N_c = \infty$, using fits $A + B/N_c^2$, one finds

$$T_H = 0.82(7)\sqrt{\sigma} \quad (D = 3) \tag{2.14}$$

$$T_H = 0.63(1)\sqrt{\sigma} \quad (D = 4). \tag{2.15}$$

Note that, unlike the papers from which the data are sourced, the fit is only to the second-order transitions. Since they arise from a separate physical mechanism, there is no reason for the first-order points to be analytically related to them. In general, there is also no theoretical reason why one should truncate the fit at a simple $1/N_c^2$ correction; this is dictated only by the paucity of data. Comparing these results with Eq. (2.1), they predict an effective central charge

$$c = (D - 2) + \delta \tag{2.16}$$

of the world sheet theory contributing to highly excited states, where

$$\delta = 0.42(24) \quad (D = 3) \tag{2.17}$$

$$\delta = 0.41(8) \quad (D = 4). \tag{2.18}$$

Given that the Luscher coefficient seems unaffected by string interactions and agrees with the Nambu-Goto result, whatever degrees of freedom give rise to δ must be massive, although effectively conformal at high energies.

Given the volatility of numerical lattice results, this neat picture could be upset, of course. There is already some inconsistency in the $D = 3$ results for transition temperatures, since the extrapolation of second-order transitions at $N_c = 2, 3, 4$ seems to lie below the first-order transitions at $N_c = 5, 6$ (also shown on Fig. 1), which cannot be right. A study of the superheated phase at higher N_c would be useful in this respect. The first-order transitions for $D = 4$ and $N_c > 2$, on the other hand, lie correctly below the "Hagedorn line" in Fig. 2. While δ may vary in size and accuracy in the light of further simulations and fits, present data are accurate enough to exclude $\delta = 0$ (Nambu-Goto) and $\delta = 1$ (one extra bosonic degree of freedom). In both in 3 and 4 dimensions, $\delta \approx 1/2$, which leads to the interesting suggestion that $\delta = 1/2$. With this in mind, we analyze a specific string model in the next section that could give rise to this scenario.

III. LONGITUDINAL STRING OSCILLATIONS

A. Spectrum

In this section, we present evidence that massive longitudinal string oscillations may account for the shift in the effective central charge in the highly excited spectrum. Longitudinal degrees of freedom are obviously not a new idea. In a sense, the Nambu-Goto model already possesses them if $D \neq 26$. Both the covariant quantization [38] and the original Polyakov formulation (Liouville theory) [39] lead to an extra $c = 1$ (massless) degree of freedom corresponding to longitudinal oscillation. This is too much; it will add to the Luscher coefficient; it will produce too many degrees of freedom for the asymptotic spectrum (in the Polyakov case, the Liouville zero mode produces a continuous spectrum). But massive longitudinal modes may be expected. For example, they occur naturally in the Nielsen-Olesen vortex solution of the Abelian Higgs model [40]. Also the Polchinski-Strominger effective string action indicates that the Liouville field gets a mass [21]. Here, a specific model with longitudinal oscillations will be analyzed—the excited spectrum of the Chodos-Thorn [4] massive relativistic string. Unfortunately, this model is difficult to solve above 2 dimensions. But the longitudinal modes of interest are present in 2 dimensions and the asymptotic spectrum can be found exactly in this case. Assuming that coupling of longitudinal and transverse degrees of freedom in higher dimensions does not drastically alter things, this will provide a measure of the number of longitudinal degrees of freedom.

The action generalizes the Nambu-Goto one (2.7) by the insertion of massive particle degrees of freedom on the string

$$- \int dx^0 dx^1 (\mu \sqrt{\partial_0 X^\mu \partial_0 X_\mu} + \sigma \sqrt{-G}). \qquad (3.1)$$

In principle, the mass distribution could be continuous along the string, but we will consider the case when it is discrete. Such a string is likely to have the power-law falloff of high energy scattering amplitudes characteristic of particle field theory, which does not occur with conventional strings in general [41]. In addition to transverse oscillations, there are now oscillations associated with motion of the point masses along the length of the string. A semiclassical analysis of the free-string spectrum of this theory in $D = 2$ dimensions was performed by Bardeen *et al.* [42]. In the sector with pointlike insertions at positions X_j, $j \in \{1, \cdots, N\}$, in the interior of an open string, and insertions at each end X_0, X_{N+1} (see Fig. 4), they found the light cone gauge Hamiltonian

$$P^- = \sum_{j=0}^{N+1} \frac{\mu^2}{2P_j^+} + \sum_{j=0}^{N} \sigma |X_j^- - X_{j+1}^-|. \qquad (3.2)$$

In two dimensions, the entire light cone momentum of the string is carried by the insertions

$$P^+ = \sum_{j=0}^{N+1} P_j^+. \qquad (3.3)$$

Finding the normal modes of classical solutions in the $\mu \to 0$ limit, Bardeen *et al.* then impose Bohr-Sommerfeld quantization conditions to obtain a spectrum of masses

$$M^2 = 2\pi\sigma \sum_{n=1}^{N+1} n(l_n + \text{const.}), \qquad (3.4)$$

where $l_n \in \{0, 1, 2, \cdots\}$. Note that the massless limit $\mu \to 0$ is *not* the Nambu-Goto model, provided we allow points on the string, where there were masses, to move at the speed of light. Although the actions are the same, the Hamiltonians are not [42].

An important question is whether one should include sectors of different N independently in the spectrum (3.4). In the $\mu \to 0$ limit, the classical solutions of the equations of motion for an open string with N insertions contain the solutions for $N' < N$ insertions also. This happens because

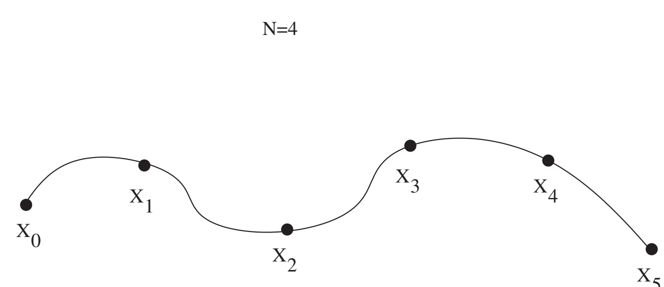

FIG. 4. Open string with particle insertions ($N = 4$ in this example).

a subset of the normal-mode motions occur with some of the insertions at coincident points. For example, one of the $N = 1$ classical solutions has the "interior" point always attached to one or other end of the open string, $X_1 = X_0$ or $X_1 = X_2$. Such a motion is geometrically indistinguishable from the classical solution for $N = 0$. Therefore, one might expect that in this case the entire spectrum can be obtained by taking $N \to \infty$ in Eq. (3.4). If one compares this spectrum with Eq. (2.9) and recalls that there are no transverse oscillations for $D = 2$, the longitudinal oscillations in this model as $N \to \infty$ appear to asymptotically give the same spectrum as a physical $c = 1$ world sheet degree of freedom (like one extra transverse dimension). However, once a mass μ is introduced, however small, the indistinguishability is lost; the total mass of the insertion is significant. For this reason, the sectors for each fixed N give rise to distinguishable states in the spectrum, resulting in many more states than just the $N \to \infty$ sector. In fact, if all N sectors are allowed, the presence of modes with $l_n = 0$ would lead to each energy level in the spectrum being infinitely degenerate in the $\mu \to 0$ limit [neglecting the constant in (3.4)].

The resolution of this difficulty lies in the fact that one expects the semiclassical spectrum (3.4) to be a valid description of the full quantum theory only for large quantum numbers. To gain a better understanding of this, one can perform a different kind of semiclassical analysis motivated by two-dimensional large-N_c gauge theory. The expression (3.2) is isomorphic to the light cone Hamiltonian of two-dimensional large-N_c gauge theory minimally coupled to fundamental and adjoint matter particles at the endpoints and interior points, respectively. The origin of the linear string potential is then the Coulomb force in two dimensions. Introducing the light cone wave function $\phi_N(x_0, x_1, \cdots, x_{N+1})$ for the sector with N interior points, with $x_j P^+ = P_j^+$, this linear potential between bosonic particles j and $j+1$ becomes [43]

$$\frac{\sigma}{4\pi P^+}\int_0^{x_j+x_{j+1}} \frac{(x_j+y)(x_j+2x_{j+1}-y)}{(x_j-y)^2\sqrt{yx_jx_{j+1}(x_j+x_{j+1}-y)}} \times\{\phi_N(x_0,\cdots,x_{N+1})-\phi_N(\cdots,x_{j-1},y,x_j+x_{j+1}-y,\cdots)\}dy+\left[\frac{\sigma}{4P^+\sqrt{x_jx_{j+1}}}\right]\phi_N(x_0,\cdots,x_{N+1}) \tag{3.5}$$

(In the gauge theory there would also be particle number changing interactions that are not present in the first quantized string theory.) The highly excited spectrum consists of wave functions ϕ_N that oscillate rapidly and, following 't Hooft [44] and Kutasov [45], one may simplify the integrand above in this limit. The integrals average to zero except near the singularities of the integrand $x_j \approx y$, in which case they may be effectively replaced by

$$\frac{\sigma}{\pi P^+}\int_{-\infty}^{\infty}\frac{dz}{z^2}\{\phi_N(x_0,\cdots,x_{N+1}) - \phi_N(\cdots,x_{j-1},x_j+z,x_{j+1}-z,\cdots)\}. \tag{3.6}$$

In the same regime, one can also assume that the mass terms are negligible compared to the excitation energy. However, they do impose the boundary condition that $\phi_N = 0$ whenever any $x_j = 0$. Thus, in the $N = 0$ sector the spectral eigenvalue equation for highly excited states becomes

$$M^2\phi_0 = \frac{2\sigma}{\pi}\int_{-\infty}^{\infty} dz \frac{\phi_0(x_0,x_1)-\phi_N(x_0+z,x_1-z)}{z^2}. \tag{3.7}$$

The solutions are of the form

$$\phi_0(x_0,x_1) = \sin s\pi x_0 \tag{3.8}$$

$$x_1 = 1 - x_0 \tag{3.9}$$

$$M^2 = 2\pi\sigma s \tag{3.10}$$

for large *positive* integers s. Comparing with Eq. (3.4), we can identify $l_1 \equiv s$ in the $N = 0$ sector. Note, however, that the new analysis excludes the troublesome $l_1 = 0$ solution.

Proceeding in the same way for $N = 1$, the corresponding eigenvalue equation for highly excited states becomes

$$M^2\phi_1 = \frac{2\sigma}{\pi}\int_{-\infty}^{\infty}\frac{dz}{z^2}\{\phi_1(x_0,x_1,x_2)-\phi_1(x_0+z,x_1-z,x_2) + \phi_1(x_0,x_1,x_2)-\phi_1(x_0,x_1+z,x_2-z)\}. \tag{3.11}$$

The solutions in this case which respect the boundary conditions are

$$\phi_1(x_0,x_1,x_2) = \sin s_1\pi x_0 \sin s_2\pi x_2 \pm \sin s_1\pi x_2 \sin s_2\pi x_0 \tag{3.12}$$

$$x_0 + x_1 + x_2 = 1 \tag{3.13}$$

$$M^2 = 2\pi\sigma(s_1 + s_2) \tag{3.14}$$

for large positive integers s_1 and s_2 such that $s_2 > s_1$. The plus solution occurs when $s_2 - s_1$ is odd, the minus solution when it is even, corresponding to states of opposite parity under orientation reversal of the open string. The identification with Eq. (3.4) is made by $l_2 = s_1$, $l_1 + l_2 = s_2$. Again, the solutions $l_1 = 0$ and $l_2 = 0$ are excluded, but this time one can use large s_1 and s_2, where the analysis is valid, to exclude the former.

B. Asymptotic density of states

The natural generalization of these results is that in the N-sector the excited spectrum is

$$M^2 = 2\pi\sigma(s_1 + s_2 + \cdots + s_{N+1}) \tag{3.15}$$

for large positive integers $s_{N+1} > s_N > \cdots > s_1$, with

$$s_j = \sum_{n=N-j+2}^{N+1} l_n. \tag{3.16}$$

This matches (3.4) provided $l_n = 0$ is excluded. Note that a similar asymptotic spectrum has been derived for closed strings of bosonic and fermionic adjoint matter in Refs. [45,46], although only the solutions even under orientation reversal were found. Numerical solutions for the low-lying spectrum of two-dimensional large-N_c gauge theory coupled to adjoint matter have also been obtained [43]. They provide additional support for the argument that each N-sector should contribute independently to the spectrum, even in the $\mu \to 0$ limit, since no degeneracies across different N-sectors are observed.

The asymptotic density of states corresponding to the spectrum (3.15), including all N-sectors, can be obtained in a standard way from the generating function

$$G(w) \equiv \sum_{n=1}^{\infty} d_n w^n = \prod_{m=1}^{\infty} (1 + w^m) \tag{3.17}$$

where d_n is the number of states at level $M^2 = 2\pi\sigma n$. The large n behavior is obtained from the limit $w \to 1$:

$$\begin{aligned} \ln G &= -\sum_{m,q=1}^{\infty} \frac{(-w^m)^q}{q} \\ &= -\sum_{q=1}^{\infty} \frac{(-w)^q}{q(1-w^q)} \\ &\to \frac{1}{1-w} \sum_{q=1}^{\infty} \frac{(-w)^q}{q^2} \quad (\text{as } w \to 1) \\ &= \frac{\pi^2}{12(1-w)}. \end{aligned} \tag{3.18}$$

Then d_n can be obtained from the saddle point approximation to the integral

$$d_n = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{G(w)}{w^{n+1}} \tag{3.19}$$

where the contour $\mathcal{C}$ encircles the origin. The result

$$d_n \sim \exp\left[M\sqrt{\frac{\pi}{6\sigma}}\right] \tag{3.20}$$

is characteristic of a $c = 1/2$ conformal field theory [Eq. (2.1)].

In fact, one could have guessed this result without further calculation by rewriting the spectrum (3.15), including all N-sectors, as

$$M^2 = 2\pi\sigma \sum_{n=1}^{\infty} n t_n \tag{3.21}$$

where $t_n \in \{0, 1\}$. This is the spectrum of a tower of fermionic harmonic oscillators. The nonlinear massive bosonic longitudinal oscillations of this string model contribute, in the asymptotic spectrum, just as would a single free massless Majorana world sheet fermion field. This is approximately the right number of degrees of freedom to account for the lowering of the Hagedorn temperature observed in large-N_c lattice simulations.

IV. DISCUSSION

Lattice data that may be used to constrain any purported string theory of QCD have been critically reviewed. In general, one must extrapolate data to $N_c = \infty$ before comparing with free-string formulas. From data on the Luscher coefficient and Hagedorn temperature, in addition to the usual massless degrees of freedom associated with oscillations in $D - 2$ transverse dimensions, it appears that further massive modes contribute to the asymptotic spectrum. Subject to the numerical accuracy of the data, they contribute effectively like a $c \approx 1/2$ conformal world sheet field. Present data are accurate enough to imply that such degrees of freedom beyond the Nambu-Goto model must exist, but rule out $c = 1$ for the extra variables.

One obvious place to look for the extra modes is in the longitudinal direction. The asymptotic spectrum of an "old" massive relativistic string model in two dimensions was rederived and clarified. It possesses only longitudinal oscillations. Despite being a bosonic model, the counting of highly excited states matches that of a $c = 1/2$ world sheet conformal field, which is approximately the right number of extra degrees of freedom seen in lattice simulations. The observations made in this paper are also relevant for "new" approaches to string theory. Longitudinal modes of oscillation will exist in the bulk for holograhic models based on the ADS/CFT correspondence [47]. The constraints from lattice data may help to identify the correct background that corresponds to confining gauge theory. In this and other string models, there still remains much to understand concerning the usual unitarity, spacetime symmetry, and ground state stability expected of gauge theory.

To test the hypothesis suggested in this paper, it would be useful to have further lattice data at large-N_c for more detailed observables. For example, the thermodynamic pressure may vary rapidly enough close to the Hagedorn temperature for a quantitative comparison to be made with the result based on the density of states (2.1) with $c = (D-2) + 1/2$ and $D_\perp = 2$. In order to make general statements, in this paper we have essentially been considering data from the large-l and minimum-l behavior of the ground state functions $E_c(l)$, $E_o(l)$ in Eqs. (2.3) and (2.4). Of course, it would be interesting to compare the predictions of specific string models and large-N_c gauge theory at general l and for low excited states of these systems [32].

ACKNOWLEDGMENTS

The work is supported by PPARC Grant No. PP/D507407/1. I thank B. Lucini for helpful discussions.

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