

Pion-pion scattering amplitude. II. Improved analysis above $\bar{K}K$ threshold

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We improve, in the energy region between the $\bar{K}K$ threshold and ~ 1.4 GeV, the energy-dependent phase shift analysis of $\pi\pi$ scattering presented in a previous paper. For the S0 wave, we have included more data above the $\bar{K}K$ threshold and we have taken into account systematically the elasticity data on the reaction $\pi\pi \rightarrow \bar{K}K$. We here made a coupled channel fit. For the D0 wave, we have considered information on low energy parameters, and imposed a better fit to the f_2 resonance. For both waves the expressions we now find are substantially more precise than the previous ones. We also provide slightly improved D2 and P waves, including the estimated inelasticity for the first, and a more flexible parametrization between 1 and 1.42 GeV for the second. The accuracy of our amplitudes is now such that it requires a refinement of the Regge analysis, for $s^{1/2} \geq 1.42$ GeV, which we also carry out. We show that this more realistic input produces $\pi\pi$ scattering amplitudes that satisfy better forward dispersion relations, particularly for $\pi^0\pi^0$ scattering.

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I. INTRODUCTION

In a recent paper by two of us [1] (J. R. P. and F. J. Y.) that we will consistently denote by PY05, we have presented a set of fits to the data on $\pi\pi$ scattering phase shifts and inelasticities, and we also checked how well forward dispersion relations are satisfied by the different $\pi\pi$ scattering phase shift analyses (including our own). These various sets differ on the values of the S0 phase shifts below the $\bar{K}K$ threshold. We found that some of the most frequently used sets of phase shifts fail to satisfy forward dispersion relations and we then presented a consistent energy-dependent phase shift analysis of $\pi\pi$ scattering amplitudes that satisfies well forward dispersion relations for energies below ~ 1 GeV. Above this energy, we found a certain mismatch between the real parts of the scattering amplitudes, calculated from phase shifts and inelasticities, and the result of the dispersive evaluations, particularly for $\pi^0\pi^0$ scattering. This we attributed to imperfect experimental information in the region $1 \text{ GeV} \leq s^{1/2} \leq 1.4$ GeV. In the present paper we improve our analysis of the S0 wave, the D0 wave and, to a lesser extent, the D2 and P waves¹ in the energy range around and above the $\bar{K}K$ threshold (for the D0 wave, we also slightly improve the low energy region). For the S0 wave we take into account systematically the elasticity data from the reaction $\pi\pi \rightarrow$

$\bar{K}K$; for the D0 wave we include information on low energy parameters, and we improve the fit to the $f_2(1270)$ resonance, to describe better its width and inelasticity. These two parametrizations are more accurate than what we had in PY05; not only in that they include more data, but also because they have *smaller* errors. A slight improvement for the P wave (using a more flexible parametrization) between the $\bar{K}K$ threshold and 1.42 GeV is also presented and, for the D2 wave, we improve on PY05 by including its estimated inelasticity above ~ 1 GeV. We have also found convenient to reconsider the Regge analysis, for the energy region above 1.42 GeV, particularly in view of the accuracy of the present parametrizations. This we do by taking into account more precise values for the intercepts $\alpha_\rho(0)$ and $\alpha_{\rho'}(0)$ than those used in PY05. Although the changes this induces are very small, and indeed quite unnoticeable below 1 GeV, the verification of the dispersion relation for exchange of isospin 1 above the $\bar{K}K$ threshold is sensitive to this Regge improvement. We then also show that, with this more accurate input in the phase shift analysis, the forward $\pi^0\pi^0$ dispersion relation is much better satisfied than with the amplitudes in PY05, particularly for energies above 1 GeV. The $\pi^0\pi^+$ dispersion relation is also improved, but only a little. Finally, the dispersion relation for exchange of isospin unity is practically unchanged below 1 GeV, and deteriorates slightly above. The new, improved parametrizations therefore provide a very precise and reliable representation of pion-pion amplitudes at all energies:

¹We will use consistently the self-explanatory notation S0, S2, P, D0, D2, F, ... for the $\pi\pi$ partial waves.

the average fulfillment of the dispersion relations is at the level of 1.05σ , for energies below 0.93 GeV, and of 1.27σ for energies up to 1.42 GeV.

II. THE S0 WAVE AT HIGH ENERGY

In PY05 we provided fits to data for the S0 wave that satisfied forward dispersion relations reasonably well below 0.925 GeV, as well as an improved parametrization constrained to satisfy forward dispersion relations below this energy and to fit data. Here we will concentrate on a parametrization at higher energies, taking care to match it to the low energy one, which we do at 0.92 GeV. The information on the S0 wave at high energy ($s^{1/2} > 0.92$ GeV) comes from two sources: $\pi\pi$ scattering experiments [2–6] and, above $\bar{K}K$ threshold, also from $\pi\pi \rightarrow \bar{K}K$ scattering [7]. The second provides reliable measurements of the elasticity parameter,² $\eta_0^{(0)}(s)$: since there are no isospin 2 waves in $\pi\pi \rightarrow \bar{K}K$ scattering, and the $\pi\pi - \bar{K}K$ coupling is very weak for P and D0 waves, it follows that measurements of the differential cross section for $\pi\pi \rightarrow \bar{K}K$ give directly the quantity $1 - [\eta_0^{(0)}]^2$ with good accuracy, so long as the multipion cross section is small; see the discussion of this below. Below the $\bar{K}K$ threshold we fit data between 0.929 and 0.970 GeV from Hyams *et al.* [2], Protopopescu *et al.* [4], and from Grayer *et al.* [3], as composed in PY05 [1]:

$$\begin{aligned}\delta_0^{(0)}(0.929^2 \text{ GeV}^2) &= 112.5 \pm 13^\circ; \\ \delta_0^{(0)}(0.935^2 \text{ GeV}^2) &= 109 \pm 8^\circ; \\ \delta_0^{(0)}(0.952^2 \text{ GeV}^2) &= 126 \pm 16^\circ; \\ \delta_0^{(0)}(0.965^2 \text{ GeV}^2) &= 134 \pm 14^\circ; \\ \delta_0^{(0)}(0.970^2 \text{ GeV}^2) &= 141 \pm 18^\circ.\end{aligned}\tag{2.1a}$$

As explained in PY05, these errors cover the *systematic* uncertainties, which are large. We also add the recent data of Kamiński *et al.* [5] and we include in the fit the value

$$\delta_0^{(0)}(4m_K^2) = 205 \pm 8^\circ\tag{2.1b}$$

obtained in the constant K-matrix fit of Hyams *et al.* [2], which is compatible with the other data used here. Finally, we include two values that follow from the low energy analysis in PY05 from the global data fit (i.e., before imposing forward dispersion relations, to avoid correlations with other waves),

$$\begin{aligned}\delta_0^{(0)}(0.900^2 \text{ GeV}^2) &= 101.0 \pm 3.7^\circ, \\ \delta_0^{(0)}(0.920^2 \text{ GeV}^2) &= 102.6 \pm 4^\circ.\end{aligned}\tag{2.1c}$$

To fit the data above the $\bar{K}K$ threshold, we notice that

²In the present paper we refer to η as the elasticity, or elasticity parameter. The inelasticity is $\sqrt{1 - \eta^2}$.

analyses based on $\pi\pi$ scattering experiments only determine a combination of phase shift and inelasticity and, indeed, different results are obtained for the S0 wave in the various analyses. For this wave we only fit data sets whose inelasticity is *compatible* with what is found in $\pi\pi \rightarrow \bar{K}K$ scattering [7], in the region $4m_K^2 \leq s \leq (1.25 \text{ GeV})^2$. This includes the solution³ (— — —) of Hyams *et al.* [6], the data of Hyams *et al.* [2] (or⁴ of Grayer *et al.* [3]), and the data of Kamiński *et al.* [5].

For the elasticity parameter, $\eta_0^{(0)}$, we will improve on the analysis of PY05 by including more data⁵ (especially, $\pi\pi \rightarrow \bar{K}K$ data) and being more realistic in the parametrization. First of all, we remark that the modulus squared of the S0 amplitude for $\pi\pi \rightarrow \bar{K}K$ scattering is proportional to $\frac{1}{4}(1 - [\eta_0^{(0)}]^2)$, provided the two-channel approximation is valid. This is known to be the case experimentally for $s^{1/2} \leq 1.25$ GeV for such waves as have been measured, and will very likely be also true for our case (as we verify in Appendix B). In this range, the $\pi\pi \rightarrow \bar{K}K$ scattering experiments give the more reliable measurements of the parameter $\eta_0^{(0)}$. Therefore, in the region $s^{1/2} \leq 1.25$ GeV we fit $\pi\pi \rightarrow \bar{K}K$ data [7] and, among the $\pi\pi \rightarrow \pi\pi$ data sets, only those whose inelasticity is compatible with that from $\pi\pi \rightarrow \bar{K}K$ below ~ 1.25 GeV. This includes the data sets of Hyams *et al.* [2] (or Grayer *et al.* [3]); the data from Ref. [6], solution (— — —); and the data of Ref. [5]. We however do *not* include in the fits the data of Protopopescu *et al.* [4], since they are quite incompatible with the $\pi\pi \rightarrow \bar{K}K$ information.

A convenient way to fit phase shift and inelasticity is to use the K-matrix formalism. This has the advantage over the method of polynomial fits, used in PY05 (see also Appendix B here), that the relations that occur at threshold between $\delta_0^{(0)}$ and $\eta_0^{(0)}$, given in Appendix A [Eq. (A6)], are automatically fulfilled. The method, however, presents the drawback that it is not possible to take into account the existence of other channels unless one introduces an excessive number of parameters. This is why we present an alternate polynomial fit in Appendix B. Fortunately, the fits given in Appendix B show that the contribution of such multiparticle channels is rather small; in fact, within the errors of the two-channel fit (this smallness is probably due to the fact that, because of its quantum numbers, the first quasi-two-body channel that contributes is $\rho\rho$). So we would expect that neglecting those other channels will

³(— — —) is the preferred solution in the original reference. Unfortunately, this reference only provided numbers for the statistical uncertainties. We add to these 5° as estimated systematic error, in agreement with an analysis similar to that of PY05.

⁴The data of Grayer *et al.* in Ref. [3], and those of Hyams *et al.* in Ref. [2] come from the same experiment.

⁵In all cases we add an estimated error of 0.04 to data that only give statistical errors.

not produce an excessive bias, being anyway covered by our uncertainties.

To perform the fit, we consider $\delta_0^{(0)}$ and $\eta_0^{(0)}$ to be given in terms of the K-matrix elements by the expressions (cf. Appendix A)

$$\tan \delta_0^{(0)}(s) = \begin{cases} \frac{k_1 |k_2| \det \mathbf{K} + k_1 K_{11}}{1 + |k_2| K_{22}}, & s \leq 4m_K^2, \\ \frac{1}{2k_1 [K_{11} + k_2^2 K_{22} \det \mathbf{K}]} \{ k_1^2 K_{11}^2 - k_2^2 K_{22}^2 + k_1^2 k_2^2 (\det \mathbf{K})^2 - 1 \\ + \sqrt{(k_1^2 K_{11}^2 + k_2^2 K_{22}^2 + k_1^2 k_2^2 (\det \mathbf{K})^2 + 1)^2 - 4k_1^2 k_2^2 K_{12}^4} \}, & s \geq 4m_K^2; \end{cases} \quad (2.2)$$

and

$$\eta_0^{(0)}(s) = \sqrt{\frac{(1 + k_1 k_2 \det \mathbf{K})^2 + (k_1 K_{11} - k_2 K_{22})^2}{(1 - k_1 k_2 \det \mathbf{K})^2 + (k_1 K_{11} + k_2 K_{22})^2}}, \quad s \geq 4m_K^2. \quad (2.3)$$

Then we write a standard diadic expansion for \mathbf{K} , with a constant background (like, e.g., in Hyams *et al.* [2]):

$$K_{ij}(s) = \frac{\mu \alpha_i \alpha_j}{M_1^2 - s} + \frac{\mu \beta_i \beta_j}{M_2^2 - s} + \frac{1}{\mu} \gamma_{ij} \quad (2.4a)$$

and μ is a mass scale, that we take $\mu = 1$ GeV. The powers of μ have been arranged so that the α_i , β_i , γ_{ij} are dimensionless; they are also assumed to be constant. The pole at M_1^2 simulates the left-hand cut of \mathbf{K} , and the pole at M_2^2 is connected with the phase shift crossing 270° around 1.3 GeV; both poles are necessary to get a good fit.

We fit simultaneously all data, above as well as below the $\bar{K}K$ threshold for the phase shift. For $\eta_0^{(0)}$, we also fit all data, $\pi\pi \rightarrow \pi\pi$ and $\pi\pi \rightarrow \bar{K}K$, over the whole range, which is justified since we are neglecting other inelastic channels. We require perfect matching with the lower energy determination of the phase shift at 0.920 GeV, as obtained in PY05. We find a $\chi^2/\text{d.o.f.} = 0.6$ and the values of the parameters are

$$\begin{aligned} \alpha_1 &= 0.727 \pm 0.014, & \alpha_2 &= 0.19 \pm 0.04, \\ \beta_1 &= 1.01 \pm 0.08; & \beta_2 &= 1.29 \pm 0.03, \\ M_1 &= 0.909 \pm 0.007 \text{ GeV}, & M_2 &= 1.324 \pm 0.006 \text{ GeV}; \\ \gamma_{11} &= 2.87 \pm 0.17, & \gamma_{12} &= 1.93 \pm 0.18, \\ \gamma_{22} &= -6.44 \pm 0.17; & \delta_0^{(0)}((0.92 \text{ GeV})^2) &= 102.6 \pm 4.6^\circ. \end{aligned} \quad (2.4b)$$

Note that M_1 indeed lies near the beginning of the left-hand cut for $\bar{K}K \rightarrow \pi\pi$ scattering, located at 0.952 GeV. The parameters in (2.4b) are strongly correlated. In fact, we have verified that there exists a wide set of minima, with very different values of the parameters. This is not surprising, since we do not have sufficiently many observables to determine the three K_{ij} on an energy independent basis. Nevertheless, the corresponding values of $\delta_0^{(0)}$ and $\eta_0^{(0)}$ vary very little in all these minima, so that (2.4b) can be considered a faithful representation of the S0 wave for $\pi\pi$

scattering, albeit very likely with somewhat underestimated errors due to our neglecting other channels, like $\pi\pi \rightarrow 4\pi$. The corresponding phase shift and elasticity

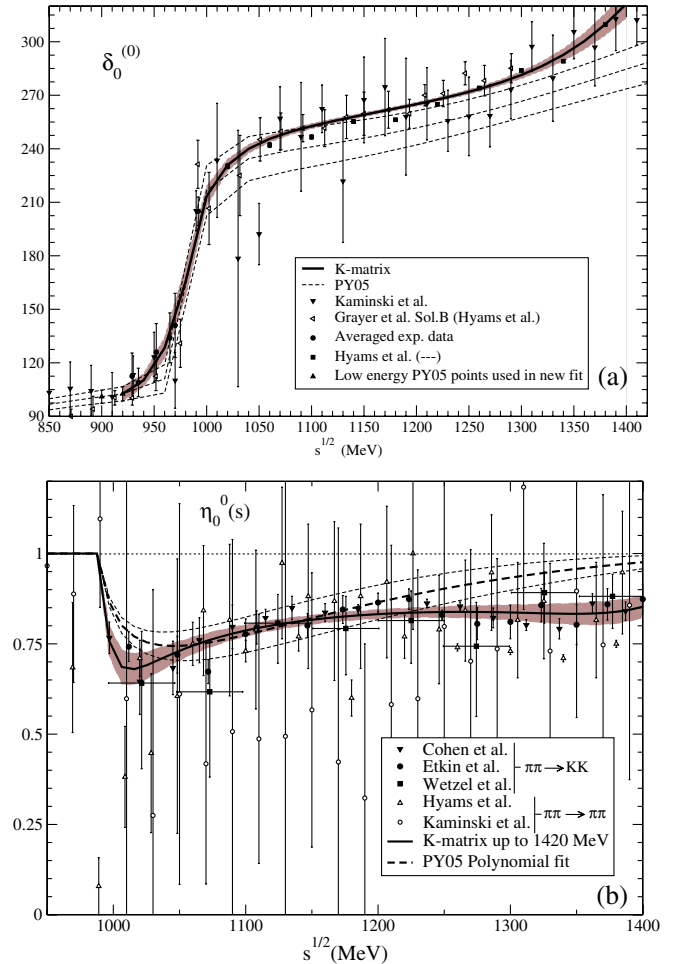


FIG. 1 (color online). (a) K-matrix fit to $\delta_0^{(0)}$ (solid line and dark area). Dotted lines: the fit in PY05. (b) K-matrix fit to $\eta_0^{(0)}$ using (2.4), with error given by the shaded area. The dotted lines represent the central values and error limits of the old fit in PY05.

parameter are shown in Figs. 1(a) and 1(b). Both phase shift and elasticity clearly improve what we had in PY05. It is mainly because of the use of the phase shift in (2.4a) and, above all, the smaller inelasticity driven by $\pi\pi \rightarrow \bar{K}K$ data, that we find a substantial improvement in forward $\pi^0\pi^0$ dispersion relations above 1 GeV (see below), as already remarked in PY05.

III. THE IMPROVED D0, D2 WAVES

A. The D0 wave

The experimental data on the D0 wave are of poor quality; different experiments give very incompatible results, particularly below 0.93 GeV. Above 1.1 GeV, and although the data of Refs. [2,6] (but not Ref. [3]) are compatible, it is better to use directly the Particle Data Table's information [8] on the f_2 resonance, which summarizes the existing experimental data.

The reliable information on this wave is then of three kinds. First, in the range around 1.27 GeV, we have the referred very precise measurements of the $f_2(1270)$ resonance parameters, which give [8] a mass $M_{f_2} = 1275.4 \pm 1.2$ MeV, a width $\Gamma_{f_2} = 185.1 \pm 3.4$ MeV, and a $\pi\pi$ branching ratio of $84.7 \pm 2.4\%$. Second, the Froissart-Gribov representation allows an accurate determination of the scattering length, $a_2^{(0)}$, and effective range parameter, $b_2^{(0)}$, as shown in PY05;⁶ one finds [1]

$$\begin{aligned} a_2^{(0)} &= (18.7 \pm 0.4) \times 10^{-4} M_\pi^{-5}, \\ b_2^{(0)} &= (-4.2 \pm 0.3) \times 10^{-4} M_\pi^{-7}. \end{aligned} \quad (3.1)$$

This helps us to fix the phase shift at low energy. Third, we have the 1973 data of Hyams *et al.* [2], Protopopescu *et al.* [4], and solution (---) of Hyams *et al.* [6] in the range $0.935 \text{ GeV} \leq s^{1/2} \leq 1.1 \text{ GeV}$, which are reasonably compatible among themselves; see Fig. 2. We denote them by, respectively, H73, P, and H(---). The data we include thus are

E , in GeV	$\cot\delta_2^{(0)}$	source	
0.935	10.4 ± 2	P	
0.950	10.2 ± 5.5	H73	
0.965	7.3 ± 2	P	
0.970	4.0 ± 2.2	H73	
0.990	4.8 ± 1.4	H73	(3.2a)

and, above $\bar{K}K$ threshold,

⁶The values given below in (3.1) are those obtained in PY05, with the old parametrizations. We have verified that they do not change, within the accuracy of (3.1), if recalculating the Froissart-Gribov representation with the parametrizations in the present paper. This of course occurs because the new parametrizations only change the amplitudes significantly above 1 GeV, a region to which $a_2^{(0)}$ and $b_2^{(0)}$ are almost not sensitive.

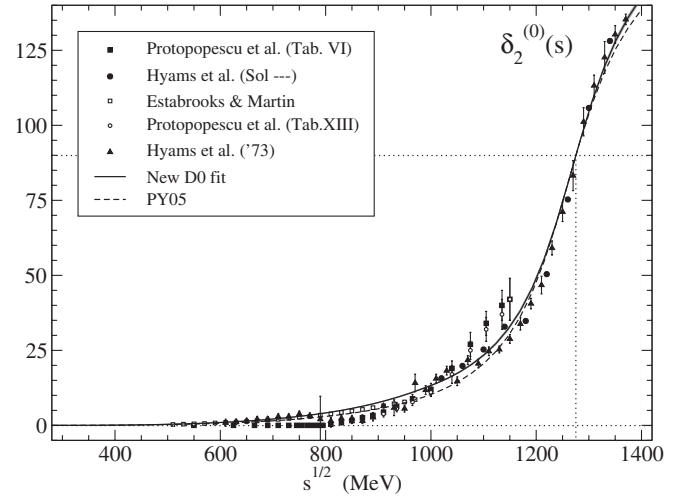


FIG. 2. The D0 phase shift as determined here (continuous line; the error is like the thickness of the line) and that from PY05 (broken line). The experimental data points are also shown. Note that the high energy fit is tightly constrained by the $f_2(1270)$ mass and width.

E , in GeV	$\cot\delta_2^{(0)}$	source	
1.00	5.1 ± 2	P	
1.01	3.6 ± 0.8	H73	
1.02	3.6 ± 0.8	H(---)	
1.03	3.0 ± 0.6	H73	
1.04	3.3 ± 0.8	P	(3.2b)

E , in GeV	$\cot\delta_2^{(0)}$	source	
1.05	3.8 ± 0.9	H73	
1.06	2.8 ± 0.8	H(---)	
1.07	2.5 ± 0.4	H73	
1.09	2.7 ± 0.45	H73	
1.10	2.1 ± 0.8	H(---)	

A few words must be said about the errors in (3.2). Since H(---) do not give errors, we take them as equal to those of P. We also multiplied *all* errors by a factor 2, to take into account the estimated systematic errors (for e.g. P, estimated as the difference between the fits XIII and VI in Ref. [4]).

We present the details of the fits. To take into account the analyticity structure, we fit with different expressions for energies below and above the $\bar{K}K$ threshold, requiring however exact matching at $s = 4m_K^2$. Below the $\bar{K}K$ threshold we take into account the existence of non-negligible inelasticity above 1.05 GeV, which is near the $\omega\pi$ or $\rho\pi\pi$ thresholds, by choosing a conformal variable w appropriate to a plane cut for $s > (1.05 \text{ GeV})^2$. So we write

$$\begin{aligned} \cot\delta_2^{(0)}(s) &= \frac{s^{1/2}}{2k^5} (M_{f_2}^2 - s) M_\pi^2 \{B_0 + B_1 w\}, \quad s < 4m_K^2; \\ w &= \frac{\sqrt{s} - \sqrt{\hat{s}} - s}{\sqrt{s} + \sqrt{\hat{s}} - s}, \quad \hat{s}^{1/2} = 1.05 \text{ GeV}. \end{aligned} \quad (3.3a)$$

The mass of the f_2 we fix at $M_{f_2} = 1275.4$ MeV; no error is taken for this quantity, since it is negligibly small (1.2 MeV) when compared with the other errors. We fit the values of $a_2^{(0)}$ and $b_2^{(0)}$ given in (3.1) and the data in (3.2a). We find the values of the parameters

$$B_0 = 12.47 \pm 0.12; \quad B_1 = 10.12 \pm 0.16. \quad (3.3b)$$

We note that the series shows good convergence. Above the $\bar{K}K$ threshold we use the following formula for the phase shift:

$$\cot \delta_2^{(0)}(s) = \frac{s^{1/2}}{2k^5} (M_{f_2}^2 - s) M_\pi^2 \{B_{h0} + B_{h1} w\}, \quad s > 4m_K^2; \\ w = \frac{\sqrt{s} - \sqrt{s_h - s}}{\sqrt{s} + \sqrt{s_h - s}}; \quad s_h^{1/2} = 1.45 \text{ GeV}. \quad (3.4a)$$

This neglects inelasticity below 1.45 GeV, which is approximately the $\rho\rho$ threshold; inelasticity will be added by hand, see below. We then fit the values for the width of $f_2(1270)$, as given above, and the set of data in (3.2b). We get

$$B_{h0} = 18.77 \pm 0.16; \quad B_{h1} = 43.7 \pm 1.8. \quad (3.4b)$$

As stated, we have required exact matching of high and low energy at the $\bar{K}K$ threshold, where our fits give $\cot \delta_2^{(0)}(4m_K^2) = 4.42 \pm 0.04$. This matching implies that there is a relation among the four B_i s, so there are in effect only three free parameters. The overall chi-squared of the fit is very good, $\chi^2/\text{d.o.f.} = 9.8/(18 - 3)$.

$$\eta_2^{(0)}(s) = \begin{cases} 1, & s < 4m_K^2, \\ 1 - \epsilon \frac{k_2(s)}{k_2(M_{f_2}^2)}, & s > 4m_K^2; \end{cases}$$

This probably only provides a fit to the elasticity parameter on the average, but we have not been able to find a clear improvement on this. The corresponding elasticity parameter is shown⁷ in Fig. 3. Two important properties of the new fit are that it reproduces better than the one in PY05 the width and inelasticity of the f_2 resonance, which is the more salient feature of the D0 wave, and that it is more precise than what we had in PY05. This improvement of the D0 wave, although it does not give a phase shift very different from that in PY05, also contributes a non-negligible amount to the improved fulfillment of the $\pi^0\pi^0$ dispersion relations.

B. The D2 wave

In PY05 we fitted the D2 wave with a single parametrization over the whole energy range up to 1.42 GeV, and neglected inelasticity. We wrote

⁷Although there is nothing new in this fit, we show the picture because we had not shown it in PY05.

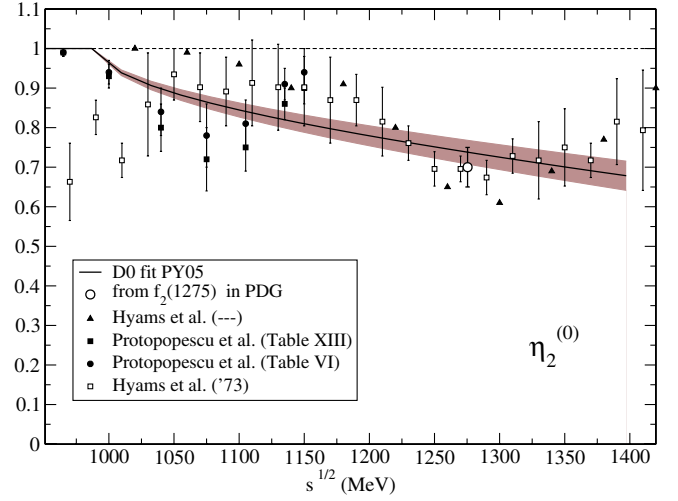


FIG. 3 (color online). Fit to $\eta_2^{(0)}$ (continuous line and dark area that covers the uncertainty) from PY05. Data from Refs. [2–4,6]. The elasticity on the $f_2(1270)$, from the PDT [8], is also shown (large white dot).

We note that, although not included in the fit, our new D0 phase shift fits better than the old PY05 one the data points of Hyams *et al.* [6], solution (---), above the f_2 resonance: see Fig. 2. The data for the inelasticity are not sufficiently good to improve significantly the fit in PY05; so we simply write, as in Ref. [1],

$$\epsilon = 0.262 \pm 0.030, \quad k_2 = \sqrt{s/4 - m_K^2}. \quad (3.4c)$$

$$\cot \delta_2^{(2)}(s) = \frac{s^{1/2}}{2k^5} \{B_0 + B_1 w(s) + B_2 w(s)^2\} \frac{M_\pi^4 s}{4(M_\pi^2 + \Delta^2) - s} \quad (3.5a)$$

with Δ a free parameter fixing the zero of the phase shift near threshold, and

$$w(s) = \frac{\sqrt{s} - \sqrt{s_0 - s}}{\sqrt{s} + \sqrt{s_0 - s}}, \quad s_0^{1/2} = 1450 \text{ MeV}.$$

Since the data on this wave are not accurate we included extra information. To be precise, we incorporated in the fit the value of the scattering length that follows from the Froissart-Gribov representation (PY05),

$$a_2^{(2)} = (2.78 \pm 0.37) \times 10^{-4} M_\pi^{-5},$$

but *not* that of the effective range parameter,

$$b_2^{(2)} = (-3.89 \pm 0.28) \times 10^{-4} M_\pi^{-7}.$$

We got a mediocre fit, $\chi^2/\text{d.o.f.} = 71/(25 - 3)$, and the

values of the parameters were

$$\begin{aligned} B_0 &= (2.4 \pm 0.3) \times 10^3, & B_1 &= (7.8 \pm 0.8) \times 10^3, \\ B_2 &= (23.7 \pm 3.8) \times 10^3, & \Delta &= 196 \pm 20 \text{ MeV}. \end{aligned} \quad (3.5b)$$

The corresponding numbers for the scattering length and for the effective range parameter $b_2^{(2)}$ that follow from this are

$$\begin{aligned} a_2^{(2)} &= (2.5 \pm 0.9) \times 10^{-4} M_\pi^{-5}; \\ b_2^{(2)} &= (-2.7 \pm 0.8) \times 10^{-4} M_\pi^{-7}. \end{aligned}$$

The last is a bit away from what one has from the Froissart-Gribov representation, but still is compatible at the 2σ level. The low quality of the fit may be traced to the fact that the various data sets are not very compatible among themselves. Therefore, there is no chance to improve the fit as we did for the D0 wave (where we had the very precise data on the f_2 resonance). We here merely improve the treatment of this wave by including the inelasticity by hand. To get an estimate of the inelasticity, we have two possible methods: we can take the inelasticity to be similar to that of the D0 wave or we can make a model calculation; for example, that of Ref. [9], in which the authors assume inelasticity to go via rho intermediate states, fixing the coupling parameters to reproduce the properties of the better known waves. Both methods yield negligible inelasticity below $\rho\pi\pi$ threshold, and something around 5% inelasticity at the highest energy considered, 1.42 GeV. For the elasticity parameter we thus simply write, above 1.05 GeV,

$$\begin{aligned} \eta_2^{(2)}(s) &= 1 - \epsilon(1 - \hat{s}/s)^3, & \hat{s}^{1/2} &= 1.05 \text{ GeV}, \\ \epsilon &= 0.2 \pm 0.2; \end{aligned} \quad (3.5c)$$

this is negligible up to 1.25 GeV and, above that, covers both what was estimated in Ref. [9], and the fact that experiments fail to detect inelasticity.

IV. THE IMPROVED P WAVE BETWEEN THE $\bar{K}K$ THRESHOLD AND 1.42 GeV

We next fit the P wave above $2m_K \simeq 0.992$ GeV, incorporating in the fit the data from solution (---) of Hyams *et al.* [6], besides the data from Protopopescu *et al.* [4] (the last is the one more compatible with what one finds from the pion form factor). We have added estimated errors of 2° to the phase shift and 0.04 to the elasticity parameter for the data of solution (---) in Hyams *et al.* [6], since no errors are provided in this reference. We now use one more parameter both for the phase shift and for the elasticity parameter than what we had in PY05, writing

$$\begin{aligned} \delta_1(s) &= \lambda_0 + \lambda_1(\sqrt{s/4m_K^2} - 1) + \lambda_2(\sqrt{s/4m_K^2} - 1)^2, \\ \eta_1(s) &= 1 - \epsilon_1\sqrt{1 - 4m_K^2/s} - \epsilon_2(1 - 4m_K^2/s); \quad s > 4m_K^2. \end{aligned} \quad (4.1)$$

The phase at the low energy edge, $\delta_1(0.992^2 \text{ GeV}^2) = 153.5 \pm 0.6^\circ$, is obtained from the fit to the form factor of the pion (Ref. [10]; see also Ref. [1]). This fixes the value of λ_0 . The fits are reasonable; we get $\chi^2/\text{d.o.f.} = 0.6$ for the phase and $\chi^2/\text{d.o.f.} = 1.1$ for the elasticity. We find the parameters

$$\begin{aligned} \lambda_0 &= 2.687 \pm 0.008, & \lambda_1 &= 1.57 \pm 0.18, \\ \lambda_2 &= -1.96 \pm 0.49; & \epsilon_1 &= 0.10 \pm 0.06, \\ \epsilon_2 &= 0.11 \pm 0.11. \end{aligned} \quad (4.2)$$

The only noticeable differences with the fit in PY05 is that the phase shift and elasticity parameter are now less rigid, that we match the low and high energy expressions at $\bar{K}K$ threshold, and that the *inelasticity* is now somewhat larger than what we had in PY05. This improved solution, together with that in PY05, are shown in Fig. 4. Another matter is the contribution of the $\phi(1020)$ resonance. This can be included in the standard way, by adding to the P wave a resonant piece

$$\hat{f}_1(s) \rightarrow \hat{f}_1(s) + \hat{f}_1^\phi(s), \quad (4.3a)$$

where $\hat{f}_1(s)$ is normalized so that, in the elastic case,

$$\hat{f}_1 = \sin\delta_1 e^{i\delta_1}$$

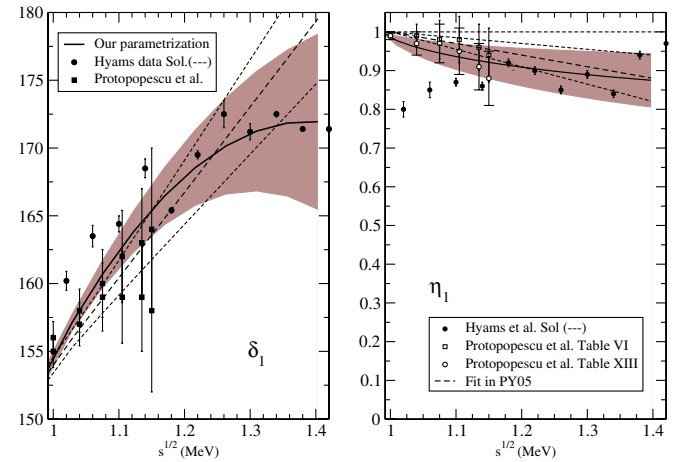


FIG. 4 (color online). The fit to the P wave above the $\bar{K}K$ threshold (continuous line and dark area that covers the uncertainty), with data from solution (---) of Hyams *et al.* [6] and of Protopopescu *et al.* [4]. Note that the errors shown for the data are only the *statistical* errors; systematic errors, estimated as in the text, about double them. The broken lines are the phase shift and elasticity parameter of PY05. The effect of the $\phi(1020)$ resonance is not shown in this figure.

and

$$\hat{f}_1^\phi(s) = \frac{\frac{M_\phi}{s^{1/2}} \left[\frac{k_2}{k_2(M_\phi^2)} \right]^3 M_{\phi\phi}}{M_\phi^2 - s - i \frac{M_\phi}{s^{1/2}} \left[\frac{k_2}{k_2(M_\phi^2)} \right]^3 M_{\phi\phi}} B_{\pi\pi}, \quad s \geq 4m_K^2; \quad (4.3b)$$

here $k_2 = \sqrt{s/4 - m_K^2}$, and the width and $\pi\pi$ branching ratio of the $\phi(1020)$ resonance are $\phi = 4.26 \pm 0.05$ MeV and $B_{\pi\pi} = (7.3 \pm 1.3) \times 10^{-5}$. Something similar could be done for the contribution of the ω . The influence of these resonances is totally negligible and, in fact, we will *not* include them in our calculations of dispersion relations below.

V. IMPROVEMENT OF THE REGGE INPUT

To evaluate the dispersion relations, we need an estimate for the high energy ($s^{1/2} \geq 1.42$ GeV) scattering amplitudes. This is furnished by the Regge model. We have here three amplitudes, one for each of the exchange of isospin 0, 1, and 2. We first take, for these Regge amplitudes, the results of the fits in Ref. [11]; see also PY05, Appendix B. Then we will consider improvement of the Regge parameters.

The expressions for the amplitudes for exchange of isospin 1 and 0 are, respectively,

$$\text{Im } F^{(I=1)}(s, 0) \underset{s \rightarrow \infty}{\simeq} \beta_\rho(0)(s/\hat{s})^{\alpha_\rho(0)}, \quad s \geq (1.42 \text{ GeV})^2, \quad (5.1a)$$

and

$$\text{Im } F_{\pi\pi}^{(I=0)}(s, 0) \underset{s \rightarrow \infty}{\simeq} P(s, 0) + P'(s, 0), \quad s \geq (1.42 \text{ GeV})^2; \\ P(s, 0) = \beta_P(s/\hat{s})^{\alpha_P(0)}, \quad P'(s, 0) = \beta_{P'}(s/\hat{s})^{\alpha_{P'}(0)}. \quad (5.1b)$$

In both expressions $\hat{s} = 1 \text{ GeV}^2$. The values of the parameters are (Refs. [1,11], Appendix B)

$$\beta_\rho(0) = 1.02 \pm 0.11; \quad \alpha_\rho(0) = 0.52 \pm 0.02 \quad (5.2)$$

and

$$\beta_P = 2.54 \pm 0.03, \quad \beta_{P'} = 1.05 \pm 0.02; \\ \alpha_{P'}(0) = \alpha_P(0), \quad \alpha_P(0) = 1. \quad (5.3)$$

For exchange of isospin 2, which is very small, we also take the amplitude of Ref. [11]:

$$\text{Im } F^{(I=2)}(s, 0) \underset{s \rightarrow \infty}{\simeq} \beta_2(s/\hat{s})^{2\alpha_\rho(0)-1}, \\ \beta_2 = 0.2 \pm 0.2; \quad s \geq (1.42 \text{ GeV})^2. \quad (5.4)$$

The first two, however, will now be improved: as we have seen in previous sections, the precision of our new parametrizations in the intermediate energy range (~ 1 to ~ 1.4 GeV) is such that one is sensitive to small details of

the Regge amplitudes; so, it is convenient to reassess the derivation of the values for the Regge parameters in Eqs. (5.2) and (5.3). The expressions (5.2) and (5.3) were obtained in Ref. [11] and PY05 as follows. We fixed $\alpha_\rho(0)$ as the average between what is found in deep inelastic scattering [12], $\alpha_\rho = 0.48$, and in the analysis of hadron collisions by Rarita *et al.* [13], who get $\alpha_\rho = 0.56$. We also imposed degeneracy, so that the intercept of ρ and P' were forced to be the same. We then fitted experimental $\pi\pi$ cross sections, which gives $\beta_\rho(0) = 1.0 \pm 0.3$, and improved this result demanding fulfillment of a crossing sum rule. For isospin zero exchange, the expression (5.3) was obtained requiring simultaneous fits to $\pi\pi$, πN , and NN data, using factorization, fixing the intercept of the P' to 0.52 (as already stated). However, more complete fits [14] than that of Rarita *et al.* [13] have been performed in the past years; especially, for the rho trajectory, individual data on pp , $\bar{p}p$, and np have been included in the fits, which permits improvement of the determination of the rho parameters using factorization. These fits, in particular, allowed a relaxation of the exact degeneracy condition $\alpha_\rho(0) = \alpha_{P'}(0)$, and yield central values for the rho intercept $\alpha_\rho = 0.46$, more in agreement with the result from deep inelastic scattering. For $\alpha_{P'}$, one finds a value higher than for the rho intercept: $\alpha_{P'} = 0.54$. We may then repeat the analysis of Ref. [11], but fixing now the intercepts of rho and P' trajectories to the likely more precise values

$$\alpha_\rho(0) = 0.46 \pm 0.02, \quad \alpha_{P'}(0) = 0.54 \pm 0.02, \quad (5.5a)$$

with conservative errors. We also here improve the error estimate for the rho residue β_ρ with the crossing sum rule, as we did in PY05 to get (5.2). This sum rule we calculate using the new phase shifts and inelasticities we have evaluated in the present paper. We then find

$$\beta_\rho = 1.22 \pm 0.14, \quad \beta_P = 2.54 \pm 0.04, \\ \beta_{P'} = 0.83 \pm 0.05. \quad (5.5b)$$

The errors are slightly larger now, which is due to the fact that we do not impose the exact degeneracy relation $\alpha_\rho(0) = \alpha_{P'}(0)$. For the amplitude with exchange of isospin 2, we still keep (5.4) since no new information is available. The difference between what we have now, (5.5), and what was used in PY05 is much smaller than what would appear at first sight; in fact, because the $\alpha(0)$ and β are strongly correlated, the changes in one quantity are compensated by those in the other: the amplitudes described by (5.2), (5.3), and (5.5) are very similar in the energy region of interest (cf. Fig. 5). However, these amplitudes differ in some details. So, the rho amplitude described by (5.5) is tilted with respect to that given by (5.2): the amplitude described by (5.5) is slightly larger than that described by (5.3) below ~ 5 GeV, where they cross over, and is larger above this energy. Likewise, for exchange of isospin zero (5.5) gives a smaller amplitude at

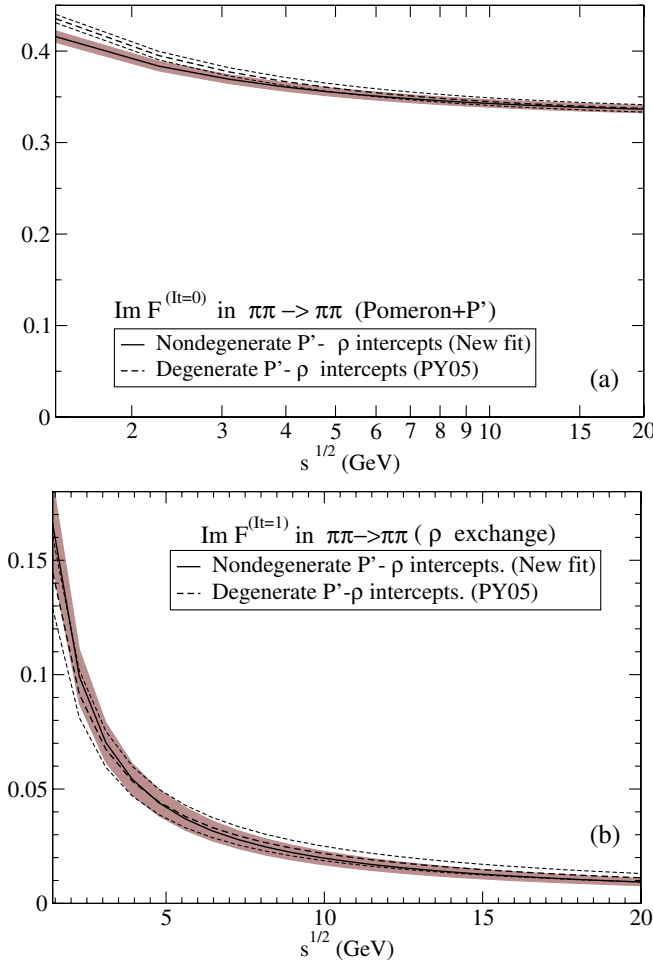


FIG. 5 (color online). (a) The scattering amplitude $\text{Im} F^{(l=1)}(s, 0)$ as described by (5.2), broken line, and (5.5), solid line with error included (gray band). (b) The scattering amplitude $\text{Im} F^{(l=0)}(s, 0)$ as described by (5.3), broken line, and (5.5), solid line with error included (gray band).

low energy, which then crosses over the amplitude given by (5.3) at higher energy.

As just stated, these changes induced by using (5.5) do almost compensate each other and, indeed, they have only a minute effect in dispersion relations below 1 GeV. At the level of precision attained by our parametrizations in the region above 1 GeV, however, the dispersion relations are sensitive to the details of the Regge behavior; because of this, we will evaluate the dispersion relations with both (5.2) and (5.4) and with (5.5). A last question related to the high energy, $s^{1/2} \geq 1.42$ GeV, input is the matching of the Regge amplitudes to the amplitude obtained below 1.42 GeV with our phase shift analyses. Although we have verified that the low and high energy amplitudes are compatible, within errors, at 1.42 GeV, we have *not* required exact matching. The reason for this is that at the lower energy Regge range, say below ~ 1.8 GeV, some amplitudes still present structure; for example, for the

amplitudes with isospin unity, the structure associated with the $\rho(1450)$, $\rho(1700)$, and $\rho_3(1690)$ resonances. It is true that these resonances couple weakly to $\pi\pi$, but, at the level of precision required in the present paper, this is not negligible: as happens in the case of $\pi^+ p$ scattering (see e.g. Fig. 2 in Ref. [11]), one expects the Regge amplitude to provide only a fit in the mean. This mismatch produces distortions near the boundary, $s^{1/2} = 1.42$ GeV, clearly seen in some of the dispersion relation calculations below; particularly, for $\pi^0 \pi^+$ scattering, where the P and, to a lesser extent, the F waves are important. We have done nothing to correct this distortion which, anyway, only affects the points very near 1.42 GeV. The alternate possibility, which would be to use phase shift analyses up to higher energies, say 1.8 GeV, would only make matters worse since it would have to contend with the nonuniqueness and unreliability of the experimental data in that region, as discussed, for example, in Ref. [15].

VI. FORWARD DISPERSION RELATIONS

In this section we will evaluate forward dispersion relations for the three independent $\pi\pi$ scattering amplitudes. For these calculations we will take the parameters for all partial waves from the fits to data⁸ in Ref. [1] (PY05), *except* for the S0 and P waves above 0.92 GeV, where we use the expressions found in the present paper (for the S0 wave, with the K-matrix fit), and for the D2 wave, where we take into account the inelasticity above 1.05 GeV. For the D0 wave we use the expressions given in the present paper all the way from threshold. To measure the fulfillment of the dispersion relations we calculate the average chi-squared, $\bar{\chi}^2$. This is defined as the sum of the squares of the real part minus the result of the dispersive integral, divided by the (correlated) errors squared; this we do at energy intervals of 25 MeV, and divide by the number of points. Note however, that this average $\bar{\chi}^2$ does *not* come from a fit to the dispersion relations, but is simply a measure of how well the forward dispersion relations are satisfied by the data fits, which are *independent* for each wave, and independent of dispersion relations. When calculating this $\bar{\chi}^2$, we first use the parameters for phase shifts and inelasticities in PY05; then, we replace the relevant waves by the ones in the present paper; and, finally, we also replace the PY05 Regge parameters with the ones in Eq. (5.5).

⁸In PY05 we gave two sets of phase shifts and inelasticities: one by fitting directly the various sets of experimental data (Sec. 2 in Ref. [1]); and a set obtained by requiring, besides fit to data, fulfillment of dispersion relations (summarized in Appendix 1 of Ref. [1]). In the present paper we of course *only* use the amplitudes obtained in PY05 by fitting data, since the ones improved with dispersion relations use a high energy ($s^{1/2} > 0.92$ GeV) input that is superseded by our calculations in the present paper.

A. The $\pi^0\pi^0$ and $\pi^0\pi^+$ dispersion relations

We first evaluate the forward dispersion relation for $\pi^0\pi^0$ scattering, the one that was worse verified in PY05 and the one for which the improvement due to the new parametrizations is more marked. We write

$$\begin{aligned} \text{Re}F_{00}(s) - F_{00}(4M_\pi^2) \\ = \frac{s(s - 4M_\pi^2)}{\pi} \text{P.P.} \int_{4M_\pi^2}^{\infty} ds' \\ \times \frac{(2s' - 4M_\pi^2) \text{Im}F_{00}(s')}{s'(s' - s)(s' - 4M_\pi^2)(s' + s - 4M_\pi^2)}. \end{aligned} \quad (6.1)$$

The result of the calculation is shown in Fig. 6, where the

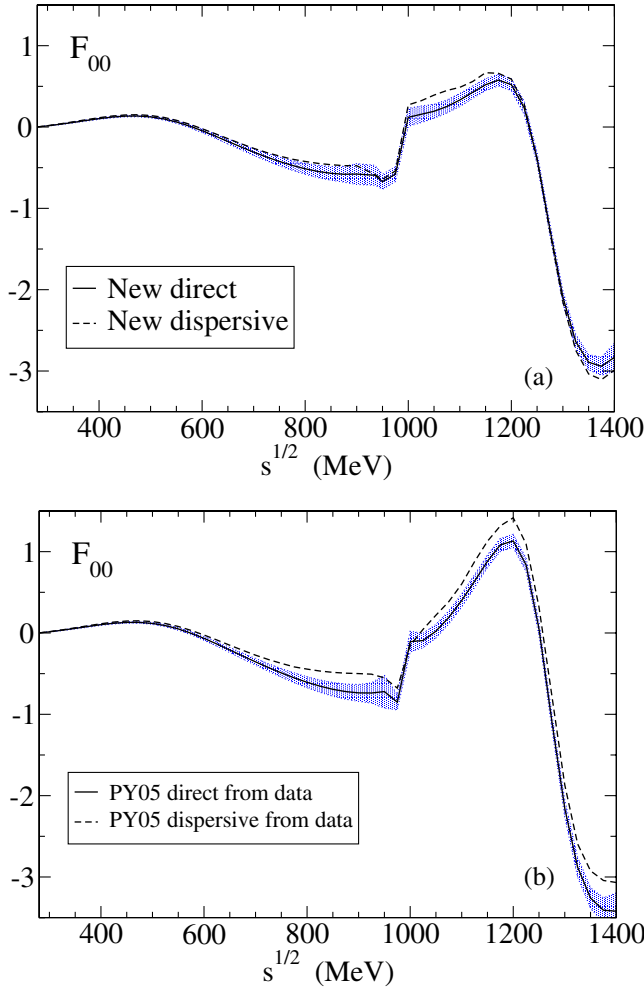


FIG. 6 (color online). (a) The $\pi^0\pi^0$ dispersion relation with the new S0, D0, and D2 waves. Continuous line: real part, evaluated directly with the parametrizations (the gray band covers the error). Dashed line: the result of the dispersive integral, with the Regge parameters of (5.5). (b) The $\pi^0\pi^0$ dispersion relation with the old, PY05 S0, D0, and D2 waves. Continuous line: real part, evaluated directly with the parametrizations. Dashed line: the result of the dispersive integral, with the Regge parameters as in PY05.

continuous curve is the real part evaluated from the parametrizations, and the broken curve is the result of the dispersive integral, i.e., the right-hand side of (6.1). The fulfillment of this dispersion relation improves substantially what we had in PY05:⁹ the changes in the average chi-squared are

$$\begin{aligned} \pi^0\pi^0: \text{PY05} \quad \text{New phase sh.} \quad \text{New Regge} \\ \bar{\chi}^2 = 3.8 \rightarrow 1.52 \quad \rightarrow 1.41, \quad \text{for } s^{1/2} \leq 930 \text{ MeV}, \\ \bar{\chi}^2 = 4.8 \rightarrow 1.76 \quad \rightarrow 1.63, \quad \text{for } s^{1/2} \leq 1420 \text{ MeV}. \end{aligned} \quad (6.2)$$

Here and in similar expressions below, “New phase sh.” means that we use the new, improved phase shifts (and inelasticities) of the present paper; “New Regge” means that we also use the new Regge parameters in (5.5). In both cases we use the K-matrix fit for the S0 wave, Eqs. (2.4).

The improvement obtained for $\pi^0\pi^0$ when using the new phase shifts is more impressive if we remember that the errors we have now for the S0 wave above 0.92 GeV, and for the D0 wave in the whole range, are substantially smaller than what we had in PY05. It is also noteworthy that the improvement in the dispersion relation is due almost exclusively to the use of the new phase shifts and inelasticities in the range ~ 1 to 1.42 GeV; the improvement due to introducing the Regge behavior (5.5) is much more modest. The dispersion relation for $\pi^0\pi^+$ scattering reads, with $F_{0+}(s)$ the forward $\pi^0\pi^+$ amplitude,

$$\begin{aligned} \text{Re}F_{0+}(s) - F_{0+}(4M_\pi^2) \\ = \frac{s(s - 4M_\pi^2)}{\pi} \text{P.P.} \int_{4M_\pi^2}^{\infty} ds' \\ \times \frac{(2s' - 4M_\pi^2) \text{Im}F_{0+}(s')}{s'(s' - s)(s' - 4M_\pi^2)(s' + s - 4M_\pi^2)}. \end{aligned} \quad (6.3)$$

In Fig. 7 we show the fulfillment of (6.3), both with what we had in PY05 and with the new phase shifts and Regge parameters. The forward dispersion relation for $\pi^0\pi^+$ scattering was already very well satisfied with the parameters in PY05; it becomes slightly better satisfied now. The changes in the average chi-squared are

$$\begin{aligned} \pi^0\pi^+: \text{PY05} \quad \text{New phase sh.} \quad \text{New Regge} \\ \bar{\chi}^2 = 1.7 \rightarrow 1.75 \quad \rightarrow 1.60, \quad \text{for } s^{1/2} \leq 930 \text{ MeV}, \\ \bar{\chi}^2 = 1.7 \rightarrow 1.60 \quad \rightarrow 1.44, \quad \text{for } s^{1/2} \leq 1420 \text{ MeV}. \end{aligned} \quad (6.4)$$

The improvement here, although existing, is rather small: not surprisingly as the corresponding amplitude does not contain the S0 or D0 waves. The amelioration is due only to use of the new Regge parameters from Eq. (5.5). The fact

⁹Of course, we here compare with the results obtained using the fits to data, *before* improving them by requiring fulfillment of the dispersion relations at low energy

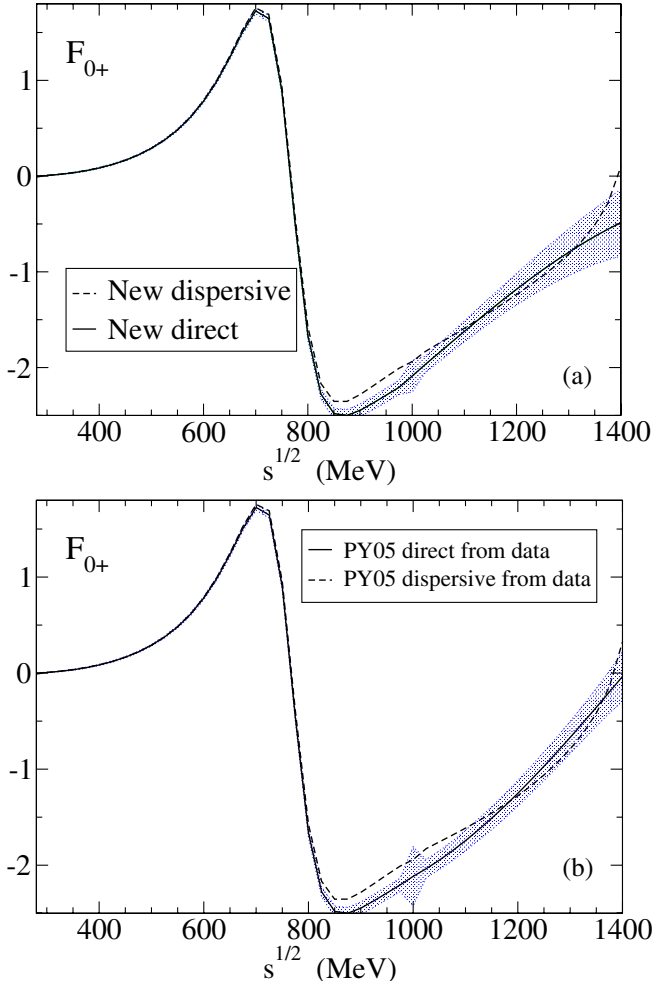


FIG. 7 (color online). (a) The $\pi^0\pi^+$ dispersion relation with the new P and D2 waves. Continuous line: real part, evaluated directly with the parametrizations. Dashed line: the result of the dispersive integral, with the Regge parameters of (5.5). (b) The $\pi^0\pi^+$ dispersion relation with the old, PY05 P and D2 waves. Continuous line: real part, evaluated with the parametrizations. Dashed line: the result of the dispersive integral, with the old, PY05 Regge parameters.

that both the dispersion relations for $\pi^0\pi^0$ and $\pi^0\pi^+$ improve with the present parameters for the P' trajectory confirms the correctness of the procedure for determining it which we developed in Sec. V.

B. The dispersion relation for the $I_t = 1$ scattering amplitude

The dispersion relation for the $I_t = 1$ scattering amplitude does not require subtractions, and reads

$$\text{Re}F^{(I_t=1)}(s, 0) = \frac{2s - 4M_\pi^2}{\pi} \text{P.P.} \int_{4M_\pi^2}^{\infty} ds' \times \frac{\text{Im}F^{(I_t=1)}(s', 0)}{(s' - s)(s' + s - 4M_\pi^2)}. \quad (6.5)$$

The result of the calculation is shown in Fig. 8.

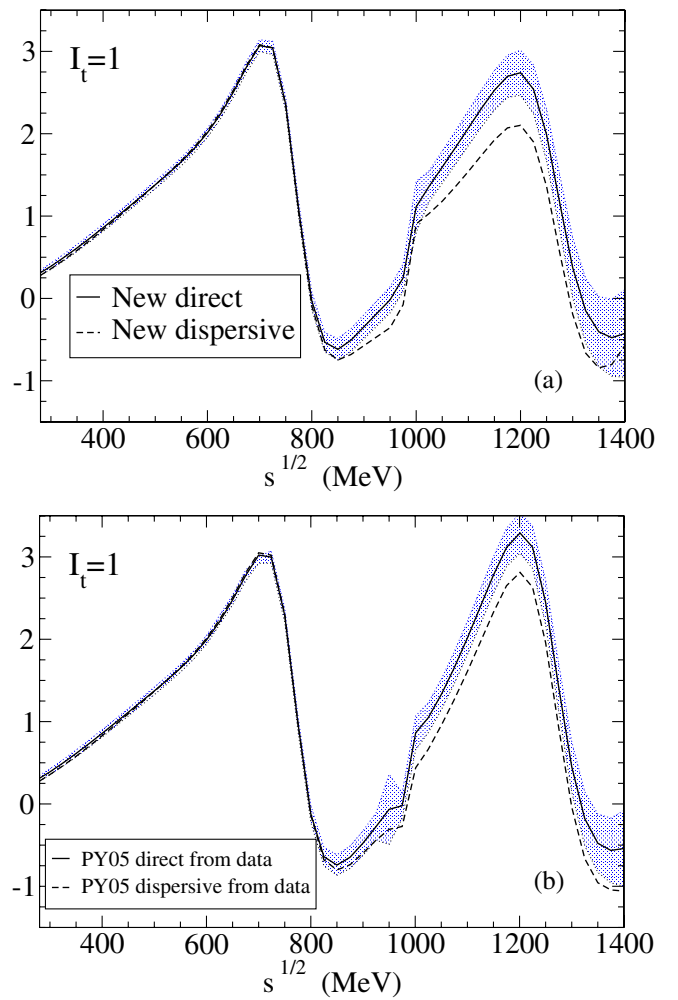


FIG. 8 (color online). (a) The dispersion relation for the $I_t = 1$ amplitude, calculated with the new amplitudes. Continuous line: real part and error (shaded area) evaluated directly with the parametrizations. Dashed line: the result of the dispersive integral, with the Regge parameters given in (5.5). (b) The dispersion relation for the $I_t = 1$ amplitude with the old, PY05 S0, D0, and P waves. Continuous line: real part and error (shaded area) evaluated directly. Dashed line: the result of the dispersive integral, with the PY05 Regge parameters for the rho.

In this case the contribution of the Regge piece is very important, although the details only matter in the region above 1 GeV. Here the fulfillment of the dispersion relation becomes entangled with which Regge behavior one uses; particularly since we now have S0 and D0 amplitudes with very small errors above 1 GeV, which is where the detailed shape of the Regge amplitude has more influence. The changes in the $\bar{\chi}^2$ from what we had in PY05 are

$$\begin{aligned} I_t = 1: \text{PY05 New phase sh. New Regge} \\ \bar{\chi}^2 = 0.2 \rightarrow 0.57 & \rightarrow 0.32 \quad \text{for } s^{1/2} \leq 930 \text{ MeV}, \\ \bar{\chi}^2 = 1.4 \rightarrow 2.32 & \rightarrow 1.76 \quad \text{for } s^{1/2} \leq 1420 \text{ MeV}. \end{aligned} \quad (6.6)$$

The conventions are like in (6.2) above. The dispersion relation deteriorates a little, which indicates that the rho Regge parameters may still be improved. In fact, it is remarkable that the simple change of (5.5) in place of (5.2) and (5.3) improves so clearly the dispersion relation above 0.9 GeV for exchange of isospin unity, while leaving it almost unchanged below this energy for all processes. This confirms that the Regge parameters are much better determined for exchange of isospin zero than for exchange of isospin 1, and indicates that a complete treatment of dispersion relations (in particular, using them to improve the scattering amplitudes) may require simultaneous consideration of the Regge parameters and of the parameters of the phase shift analyses, as in fact was done in PY05. We will leave this for a forthcoming paper, where we will also study the improvement of our parametrizations using the dispersion relations as well as Roy equations. Finally, we mention here that, although the improvements in the present paper only affected the various waves above ~ 1 GeV (with the exception of the very small change of the D0 wave below $\bar{K}K$ threshold), there is a systematic improvement of the dispersion relations also below that energy, which is a nontrivial test of the consistency of the parametrizations below and above the $\bar{K}K$ threshold.

VII. A BRIEF DISCUSSION

The results of the present article show that, if we improve the scattering amplitudes above ~ 1 GeV using more reliable data sets than those we had in PY05, the ensuing amplitudes verify much better forward dispersion relations, especially above the $\bar{K}K$ threshold; but also below it. Forward dispersion relations, particularly for $\pi^0\pi^0$ and $\pi^0\pi^+$ scattering, which (as discussed in PY05) have important positivity properties, constitute a very stringent filter when used to discriminate against spurious parametrizations or calculations, as discussed in PY05 and Ref. [15]. The fact that, with the small errors we have now, *all* values for the $\bar{\chi}^2$ are below the 1.8 level, implies that a small change in the parameters would ensure complete fulfillment (within errors). However, it is clear that, although small, some alterations are to be expected of the various parameters if we require the amplitudes to verify dispersion relations at the $\bar{\chi}^2 = 1$ level, which we will do in a forthcoming article. These changes are forced by the fact that the dispersion relations are not yet perfectly satisfied. With respect to this, we have three suspects here. First of all, we have that the experimental data for the D2 wave (which contributes to all processes) are of such a kind that our fit cannot be very reliable for the phase shift above 1 GeV, and is almost pure guesswork for the inelasticity. In fact, already in PY05 we discovered that requiring fulfillment of the dispersion relations, within errors, forces a change by more than 1σ in the phase shift parameters for this D2 wave. The second possible culprit is the inelasticity for the D0 wave. Although it fits (by

construction) that of the f_2 resonance, the expression we have used is, probably, too rigid. There is unfortunately very little one can do here, since the quality of the data does not allow an accurate treatment. The final possible culprit is the isospin 1 Regge amplitude: there is perhaps room for improvement here. The same is true, albeit to a lesser extent, for the amplitudes for P' and for exchange of isospin 2. (Alternatively, it may turn out that, once the D2 wave is improved, any change in the Regge parameters is unnecessary.) Finally, it is clear that one cannot improve our amplitudes much, since they are quite good to begin with. However, and based on the preliminary results that we have at present, we expect to show, in a forthcoming article, that it is still possible to hone our amplitude analysis by requiring fulfillment of the Roy equations and, especially, of forward dispersion relations over the whole energy range.

ACKNOWLEDGMENTS

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APPENDIX A: THE K-MATRIX FORMALISM

The phase shift δ_π and elasticity parameter η for the S0 partial wave, for $\pi\pi$ scattering, are defined as

$$\begin{aligned} \hat{f}_{11}(s) &= \sin\delta_\pi e^{i\delta_\pi}, & s < 4m_K^2; \\ \hat{f}_{11}(s) &= \frac{\eta e^{2i\delta_\pi} - 1}{2i}, & s > 4m_K^2. \end{aligned} \quad (A1)$$

We have changed a little the notation with respect to the main text; thus, δ_π is what we called $\delta_0^{(0)}$ before, \hat{f}_{11} was called $\hat{f}_0^{(0)}$ in the main text, etc. The index (11) in \hat{f}_{11} is a channel index; see below. Also, we do not write angular momentum or isospin indices explicitly. We assume here, as in the main text, that there are only two channels open (which is likely a good approximation below ~ 1.25 GeV, and not too bad up to 1.42 GeV):

$$(11): \pi\pi \rightarrow \pi\pi; \quad (12): \pi\pi \rightarrow \bar{K}K; \quad (22): \bar{K}K \rightarrow \bar{K}K.$$

Because of time reversal invariance, the channels $\pi\pi \rightarrow \bar{K}K$ and $\bar{K}K \rightarrow \pi\pi$ are represented by the same amplitude. We then form a matrix, with elements \hat{f}_{ij} , $i, j = 1, 2$, $\hat{f}_{11} = \hat{f}_{\pi\pi \rightarrow \pi\pi}$, etc.:

$$\mathbf{f} = \begin{pmatrix} \hat{f}_{\pi\pi \rightarrow \pi\pi} & \hat{f}_{\pi\pi \rightarrow \bar{K}K} \\ \hat{f}_{\pi\pi \rightarrow \bar{K}K} & \hat{f}_{\bar{K}K \rightarrow \bar{K}K} \end{pmatrix} = \begin{pmatrix} \frac{\eta e^{2i\delta_\pi} - 1}{2i} & \frac{1}{2}\sqrt{1 - \eta^2} e^{i(\delta_\pi + \delta_K)} \\ \frac{1}{2}\sqrt{1 - \eta^2} e^{i(\delta_\pi + \delta_K)} & \frac{\eta e^{2i\delta_K} - 1}{2i} \end{pmatrix}. \quad (\text{A2})$$

δ_K is the phase shift for $\bar{K}K \rightarrow \bar{K}K$ scattering. Below the $\bar{K}K$ threshold, the elasticity parameter is $\eta(s) = 1$; above the $\bar{K}K$ threshold one has the bounds $0 \leq \eta \leq 1$. We write \mathbf{f} as

$$\tan \delta_\pi = \begin{cases} \frac{k_1 |k_2| \det \mathbf{K} + k_1 K_{11}}{1 + |k_2| K_{22}}, & s \leq 4m_K^2, \\ \frac{1}{2k_1 [K_{11} + k_2^2 K_{22} \det \mathbf{K}]} \{k_1^2 K_{11}^2 - k_2^2 K_{22}^2 + k_1^2 k_2^2 (\det \mathbf{K})^2 - 1 + \sqrt{(k_1^2 K_{11}^2 + k_2^2 K_{22}^2 + k_1^2 k_2^2 (\det \mathbf{K})^2 + 1)^2 - 4k_1^2 k_2^2 K_{12}^4}\}, & s \geq 4m_K^2; \end{cases} \quad (\text{A4a})$$

one also has

$$\eta = \sqrt{\frac{(1 + k_1 k_2 \det \mathbf{K})^2 + (k_1 K_{11} - k_2 K_{22})^2}{(1 - k_1 k_2 \det \mathbf{K})^2 + (k_1 K_{11} + k_2 K_{22})^2}}, \quad s \geq 4m_K^2. \quad (\text{A4b})$$

The sign in the surd in (A4a) is to be taken positive if, as happens in our case, $K_{11}(k_2^2 = 0) > 0$. From the relation between the phase shift above and below threshold, and also with the elasticity, it may appear that one could write an expansion for $\delta_\pi(s)$ below threshold and from it, deduce corresponding expressions for $\eta_\pi(s)$ and for $\delta_\pi(s)$ above threshold. This comes about as follows. Let us define $\delta_\pi^b(s)$ to be the phase shift below threshold, and $\delta_\pi^a(s)$ that above threshold, both as given in (A4a). Write the Taylor expansion

$$\delta_\pi^b(s) = \sum_0^\infty a_n \kappa^n / m_K^n, \quad a_0 \equiv d_0 \quad (\text{A5})$$

and $\kappa = |k_2|$ below threshold. Substituting this into the expression, valid below threshold,

$$\hat{f}_{\pi\pi \rightarrow \pi\pi} = \frac{e^{2i\delta_\pi} - 1}{2i}, \quad (s \leq 4m_K^2)$$

and continuing this across the cut in the variable $\kappa = -ik_2$ above the threshold we find the expression, valid for $s \geq 4m_K^2$,

$$\begin{aligned} \hat{f}_{\pi\pi \rightarrow \pi\pi} &= \frac{e^{2(a_1 k_2 / m_K - a_3 k_2^3 / m_K^3 + \dots)} e^{2i(d_0 - a_2 k_2^2 / m_K^2 + a_4 k_2^4 / m_K^4 + \dots)} - 1}{2i}, \\ &(s \geq 4m_K^2). \end{aligned}$$

$$\mathbf{f} = \{\mathbf{k}^{-1/2} \mathbf{K}^{-1} \mathbf{k}^{-1/2} - \mathbf{i}\}^{-1}, \quad \mathbf{k} = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}. \quad (\text{A3})$$

k_i are the momenta, $k_1 = \sqrt{s/4 - M_\pi^2}$, $k_2 = \sqrt{s/4 - m_K^2}$. Then, analyticity and unitarity imply that \mathbf{K} is analytic in s through the $\bar{K}K$ threshold; hence, it only depends on k_2^2 : $K_{ij} = K_{ij}(k_2^2)$. This is the well-known K-matrix formalism, which the reader may find developed in detail in the standard textbook of Pilkuhn [16] or, perhaps more accessible, in the lecture notes by one of us [17], and, applied to the S0 wave in $\pi\pi$ scattering, in Ref. [2]. Because of (A2) and (A3), one can express δ_π and η in terms of the K_{ij} as

On comparing with the expression above threshold given in (A1) we find

$$\begin{aligned} \delta_\pi^a(s) &= d_0 - a_2 k_2^2 / m_K^2 + a_4 k_2^4 / m_K^4 + \dots; \\ \eta(s) &= e^{2(a_1 k_2 / m_K - a_3 k_2^3 / m_K^3 + \dots)}, \quad (s \geq 4m_K^2). \end{aligned} \quad (\text{A6})$$

However, the convergence of (A6) can only be guaranteed in a disk touching the left-hand cut of the K-matrix, a cut due to the left-hand cut in $\bar{K}K \rightarrow \pi\pi$ scattering,¹⁰ that runs up to $s = 4(m_K^2 - M_\pi^2)$: therefore, only for $|k_2| < M_\pi$. From a practical point of view, we have checked numerically that fitting with (A5) and (A6) represents reasonably well $\delta_\pi^b(s)$ and $\delta_\pi^a(s)$ (this one with unrealistic errors) but does certainly not represent $\eta(s)$, in the region away from $s = 4m_K^2$, unless one adds an inordinately large number of parameters. On the other hand, it is clear that all three $\delta_\pi^b(s)$, $\delta_\pi^a(s)$, and $\eta(s)$ are continuous functions of, respectively, κ , k_2^2 and k_2 . Therefore they can be approximated by polynomials in these variables over the whole range, even if they are not one the continuation of the other.

¹⁰It is not difficult to check that, although $\hat{f}_{11}(s)$ or $\hat{f}_{22}(s)$ have no left-hand cut above $s = 0$, $\eta(s)$ and $\delta^a(s)$ do. For e.g. the first, we use (A1) and find

$$\eta^2 = \frac{(2i\hat{f}_{11} + 1)(2i\hat{f}_{22} + 1)}{(2i\hat{f}_{11} + 1)(2i\hat{f}_{22} + 1) + 4\hat{f}_{12}^2}$$

from which it is obvious that η inherits the left-hand cut of \hat{f}_{12} .

APPENDIX B: POLYNOMIAL FIT

We present here a polynomial fit to phase shift and elasticity parameter in which the three quantities, phase shift below $\bar{K}K$ threshold, phase shift above this threshold, and elasticity, are fitted separately. Although this fit is less

reliable than the K-matrix one, especially near the $\bar{K}K$ threshold, it will allow us to test the importance of multi-body channels, not taken into account in the K-matrix fit. For the phase we write

$$\delta_0^{(0)}(s) = \begin{cases} d_0 + a \frac{|k_2|}{m_K} + b \frac{|k_2|^2}{m_K^2}, & (0.92 \text{ GeV})^2 < s < 4m_K^2; \\ d_0 + B \frac{k_2^2}{m_K^2}, & 4m_K^2 < s < (1.42 \text{ GeV})^2; \end{cases} \quad k_2 = \sqrt{s/4 - m_K^2}. \quad (\text{B1})$$

The parameters d_0 , a , and b are strongly correlated. One can get parameters with low correlation by eliminating the parameter b in favor of the phase shift d_1 at a low energy point, that we conveniently take $s^{1/2} = 0.92 \text{ GeV}$. We thus rewrite the parametrization as

$$\delta_0^{(0)}(s) = \begin{cases} d_0 + a \frac{|k_2|}{m_K} + \frac{|k_2|^2}{|k_2(0.92^2 \text{ GeV}^2)|^2} \left\{ d_1 - d_0 - a \frac{|k_2(0.92^2 \text{ GeV}^2)|}{m_K} \right\}, & (0.92 \text{ GeV})^2 < s < 4m_K^2; \\ d_0 + B \frac{k_2^2}{m_K^2}, & 4m_K^2 < s < (1.42 \text{ GeV})^2; \end{cases} \quad k_2 = \sqrt{s/4 - m_K^2}. \quad (\text{B2a})$$

In the previous Appendix A, we presented a discussion about these expansions. From it it follows that, while the expansion below threshold can be considered as convergent in the range of interest here, $0.92 \text{ GeV} \leq s^{1/2} \leq 2m_K$, the expansion above threshold (both for $\delta_0^{(0)}$ and $\eta_0^{(0)}$, see below) should be taken as purely phenomenological. In particular, we do *not* impose the equality $b = -B$ that would follow if we took (B2a) to be a Taylor expansion [see Eq. (A6) in the Appendix]. It is possible to fit requiring $b = -B$, at the cost of adding an extra parameter in (B1), $c|k_2|^3/m_K^3$. The resulting fit is not satisfactory: it presents excessively small errors for $s > 4m_K^2$, due to the forced relation $b = -B$, which should only be effective near threshold, the only region where the expansion converges.

We fit separately data above and below the $\bar{K}K$ threshold. The fit returns a $\chi^2/\text{d.o.f.} = 0.4$ below threshold, and $\chi^2/\text{d.o.f.} = 0.9$ above threshold; the values of the parameters are

$$d_0 = 218.3 \pm 4.5^\circ, \quad a = -537 \pm 41^\circ, \quad (\text{B2b})$$

$$d_1 = \delta_0^{(0)}(0.920^2 \text{ GeV}^2) = 102.6 \pm 4^\circ$$

and

$$B = 96 \pm 3^\circ. \quad (\text{B2c})$$

The resulting phase shift is shown in Fig. 9(a), compared with the K-matrix fit. Above 1.25 GeV, the two-channel formalism is spoiled by the appearance of new channels, notably $\pi\pi \rightarrow 4\pi$, so one does not have an exact connection between the data on $\pi\pi \rightarrow \bar{K}K$ and $\eta_0^{(0)}$. In fact, the numbers one gets for $\eta_0^{(0)}$ from $\pi\pi \rightarrow \pi\pi$, and those that follow from $\pi\pi \rightarrow \bar{K}K$, assuming only two channels, are slightly different; see below. We may take this into account by using a polynomial fit (instead of a K-matrix one, as we

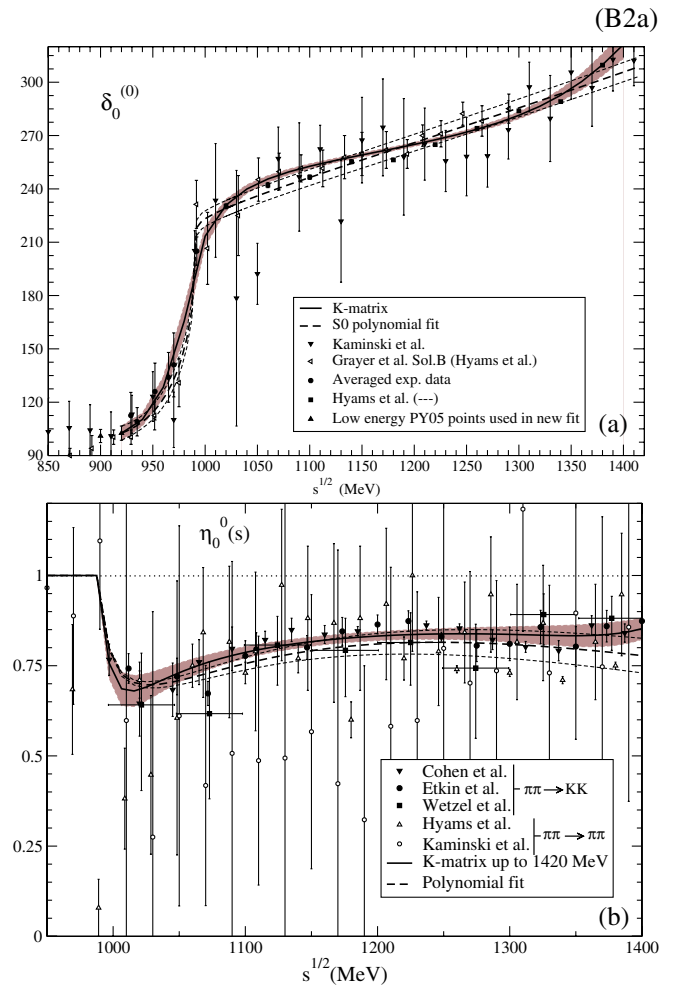


FIG. 9 (color online). (a) Comparison of the fits to $\delta_0^{(0)}$: polynomial [Eq. (B2)] given by the dashed lines, and with the K-matrix [Eq. (2.4)] (solid line and dark area). (b) Comparison of the fits to $\eta_0^{(0)}$: polynomial [Eq. (B3)] given by the dotted lines, and with the K-matrix [Eq. (2.4)] (solid line and dark area).

did in the main text). We next make a polynomial fit to the elasticity parameter writing

$$\eta_0^{(0)} = 1 - \left(\epsilon_1 \frac{k_2}{s^{1/2}} + \epsilon_2 \frac{k_2^2}{s} + \epsilon_3 \frac{k_2^3}{s^{3/2}} \right). \quad (\text{B3a})$$

In principle, the values of the ϵ_i are related to the a, c, \dots of (B1); see Appendix A, Eq. (A6). However, we will *not* impose such relations, but will consider the ϵ_i as phenomenological parameters, completely free. The reason is that the expansion $2(ak_2/m_k + ck_2^3/m_k^3 + \dots)$ is very poorly convergent above ~ 1.2 GeV: something that is a disaster for $\eta_0^{(0)}$, since the expansion appears in an *exponent* (the reason for this divergence, that can be traced to the left-hand cut in $\pi\pi \rightarrow \bar{K}K$ scattering, may be found in

Appendix A). Therefore, we would need to add extra phenomenological terms, very large, to compensate for that: it is more reasonable to make the fit phenomenological from the beginning. What we lose by so doing is that we are overestimating the value of $\eta_0^{(0)}(s)$ for s very near threshold, say for $2m_K < s^{1/2} \lesssim 0.997$ GeV, a reasonable price to pay to get a good description of the elasticity in the rest of the range.

We will consider the following possibilities: (a) To fit only $\pi\pi \rightarrow \pi\pi$ data above 1.25 GeV, in principle the more reliable option; (b) To fit *also* $\pi\pi \rightarrow \bar{K}K$ data above 1.25 GeV; (c) To fit only $\pi\pi \rightarrow \bar{K}K$ data, in the whole range. Of course, below 1.25 GeV we include both $\pi\pi \rightarrow \bar{K}K$ and $\pi\pi \rightarrow \pi\pi$ data in the fits (a) and (b). We find

$$\begin{array}{llll} \epsilon_1 = 5.45 \pm 0.04, & \epsilon_2 = -30.0 \pm 0.15, & \epsilon_3 = 46.3 \pm 0.5; & \chi^2/\text{d.o.f.} = 1.1 \quad (\text{a}); \\ \epsilon_1 = 5.27 \pm 0.04, & \epsilon_2 = -28.2 \pm 0.15, & \epsilon_3 = 42.2 \pm 0.5; & \chi^2/\text{d.o.f.} = 1.1 \quad (\text{b}); \\ \epsilon_1 = 5.77 \pm 0.05, & \epsilon_2 = -32.9 \pm 0.2, & \epsilon_3 = 51.1 \pm 0.5; & \chi^2/\text{d.o.f.} = 0.2 \quad (\text{c}). \end{array}$$

Note that the errors given here are purely nominal, as the parameters are very strongly correlated, while they were here treated as uncorrelated. Note also that the three fits are less separated than it would seem, precisely because of that correlation. Finally, we remark that the value of $\eta_0^{(0)}$ that follows from (c) is *larger* than what follows from (a) or (b); and (b) also slightly above (a). These two features constitute very nice consistency tests, since taking $\eta_0^{(0)}$ to be given from $\pi\pi \rightarrow \bar{K}K$ as if only two channels were present must surely *underestimate* the inelasticity; particularly above ~ 1.2 GeV, where the process $\pi\pi \rightarrow 4\pi$ is expected to become non-negligible.

We have verified that one may cover the two fits (a) and (b) [and even overlap (c), at the edge of the error region] by taking as central value that of the fit (a) above and slightly enlarging the errors. We then get our best result:

$$\begin{array}{ll} \epsilon_1 = 5.45 \pm 0.06, & \epsilon_2 = -30.0 \pm 0.2, \\ \epsilon_3 = 46.3 \pm 0.8; \end{array} \quad (\text{B3b})$$

the errors may now be taken as *uncorrelated*. The resulting elasticity may be seen in Fig. 9(b), compared with what we found with the K-matrix fit. The fact that both determinations overlap is a good test of the correctness of our assumption, for the K-matrix fit, that the contribution of multiparticle channels is comparable to the error of the fit itself.

The fulfillment of dispersion relations with these polynomial fits is just as good as with the K-matrix fit; however, the errors of the K-matrix fit are smaller than what we find with the polynomial fit: the fulfillment of said dispersion relations may therefore be considered to be marginally better with the K-matrix formalism, which is why we only gave results for the dispersion relations with the K-matrix fit.

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