# Bethe-Salpeter approach for relativistic positronium in a strong magnetic field

A. E. Shabad*
*P.N. Lebedev Physics Institute, Moscow, Russia*

V. V. Usov†
*Center for Astrophysics, Weizmann Institute of Science, Rehovot 76100, Israel*
(Received 7 March 2006; published 26 June 2006)

We study the electron-positron system in a strong, constant, and homogeneous magnetic field using the differential Bethe-Salpeter equation in the ladder approximation. We derive the fully relativistic two-dimensional form that the four-dimensional Bethe-Salpeter equation takes in the limit of an asymptotically strong magnetic field. A maximum value for the magnetic field is determined, which provides the full compensation of the positronium rest mass by the binding energy in the maximum symmetry state and the vanishing of the energy gap separating the electron-positron system from the vacuum. The compensation becomes possible owing to the falling-to-the-center phenomenon that occurs in a strong magnetic field because of the dimensional reduction. The solution to the Bethe-Salpeter equation corresponding to the vanishing energy momentum of the electron-positron system is obtained.

DOI: 10.1103/PhysRevD.73.125021 PACS numbers: 11.10.St, 03.65.Pm, 11.10.Jj, 12.20.−m

## I. INTRODUCTION

It is well known that the structure of atoms (positronium included) is drastically modified by a magnetic field $\mathbf{B}$ if the field strength $B = |\mathbf{B}|$ exceeds the characteristic atomic value $B_a = m^2e^3c/\hbar^3 \simeq 2.35 \times 10^9$ G [1,2], where $m$ is the electron mass and $e$ the absolute value of its charge. In a strong magnetic field ($B \gg B_a$) the usual perturbative treatment of the magnetic effects (such as Zeeman splitting of atomic energy levels) is not applicable, and, instead, the Coulomb forces act as a perturbation to the magnetic forces. For positronium in such a field the characteristic size of the electron and positron wave function across $\mathbf{B}$ is the Larmour radius

$$L_B = (eB)^{-1/2} = a_0(B_a/B)^{1/2} \tag{1}$$

that decreases with an increase of the field strength. Here $a_0$ is the Bohr radius, $a_0 = (\alpha m)^{-1}$, $\alpha = 1/137$. Henceforth, we set $\hbar = c = 1$ and refer to the Heaviside-Lorentz system of units, where the fine structure constant is $\alpha = e^2/4\pi$.

The properties of positronium in a strong magnetic field ($B \gg B_a$) are interesting for astrophysics because such fields are observed now for several kinds of astronomical compact objects (pulsars, powerful x-ray sources, soft gamma-ray repeaters, etc.). Besides, some of these objects are the sources of electron-positron pairs produced in their vicinities by various mechanisms [3]. At least part of these pairs may be bound. For instance, at the surface of radio pulsars identified with rotation-powered neutron stars the field strength is in the range from $\sim 10^9$ G to $\sim 10^{14}$ G [4]. A common point of all available models of pulsars is that electron-positron pairs dominate in the magnetosphere plasma [5]. These are formed by the single-photon production process in a strong magnetic field, $\gamma + B \to e^+ + e^- + B$. If the field strength is higher than $\sim 4 \times 10^{12}$ G the pairs created are mainly bound [6]. Much more intense magnetic fields have been conjectured to be involved in several astrophysical phenomena. For instance, superconductive cosmic strings, if they exist, may have magnetic fields up to $\sim 10^{47}$–$10^{48}$ G in their vicinities [7]. Electron-positron pairs may be produced near such strings [8].

In magnetic fields larger than $B_a$, the Coulomb force becomes more efficient in binding the positronium because the charged constituents are confined to the lowest Landau level and hence to a narrow region stretching along the magnetic field ($L_B \ll a_0$). Notwithstanding this effect, the binding energy of positronium $\Delta E$ is still very small as compared with the rest mass, $\Delta E \ll 2m$, even for the fields as high as Schwinger's critical value $B_0 = m^2/e \simeq 4.4 \times 10^{13}$ G, i.e., the positronium remains an internally nonrelativistic system. The binding energy of the ground state, as calculated nonrelativistically,

$$\Delta E \simeq \frac{m\alpha^2}{4}\left(\ln\frac{B}{B_0}\right)^2, \tag{2}$$

increases with an increase of $B$, and the relativistic effects, for extremely large fields, should be expected to become essential. The unrestricted growth of the binding energy (2) with the magnetic field is a manifestation of the fact that the Coulomb attraction force becomes supercritical in the one-dimensional Schrödinger equation, to which the nonrelativistic problem is reduced in the high-field limit [1], and the falling-to-the-center phenomenon occurs in the limit $B = \infty$.

Relativistic properties of positronium in a strong magnetic field were studied based on the Bethe-Salpeter equa-

*Electronic address: shabad@lpi.ru
†Electronic address: fnusov@wicc.weizmann.ac.il

 

tion [9,10]. The nontrivial energy dependence upon the transverse (pseudo)momentum component of the center of mass was found in [10,11]. Although the Bethe-Salpeter equation is fully relativistic, it was used within the customary "equal-time" approximation that disregards the retardation effects, so that the relative motion of the electron and positron is treated in a nonrelativistic way. In this way, the behavior (2) was reproduced for the ground state [9–11]. A completely relativistic solution for positronium in a strong magnetic field remains unknown. In this paper we study the positronium in an asymptotically strong magnetic field with not only its center-of-mass motion, but also the relative motion of the electron and positron inside it considered relativistically. We find a solution for the positronium wave function that belongs to the spectral point where all the components of the total 4-(pseudo)momentum, total energy included, are zero. Correspondingly, we point out the value of the magnetic field that causes the deepening of the positronium energy level sufficient to compensate for its whole rest mass $2m$. The vanishing of the energy gap between the electron and positron makes us refer to this field as maximum within quantum electrodynamics. Recently, a brief account of the maximum magnetic field has been given [12].

The above-mentioned deepening of the level becomes possible due to the falling-to-the-center phenomenon inherent in the two-dimensional Bethe-Salpeter equation that describes the positronium atom in the infinite magnetic field limit. Namely, the (most symmetrical) solution of this equation satisfies a Schrödinger-like equation with respect to the spacelike two-interval $s$ with the equivalent potential possessing the singular attractiveness $\sim\alpha/s^2$ near $s = 0$. The parameter equivalent to the energy of the standard Schrödinger equation has its eigenvalue spectrum unbounded from below, the probability being concentrated near the origin $s = 0$. This phenomenon is known in quantum mechanics as "falling to the center" [13], and in our problem it is responsible for the fact that the zero energy point of the positronium may belong to the spectrum, provided the magnetic field is sufficiently large. The origin of the falling to the center is in the ultraviolet singularity of the photon propagator carrying the interaction between the charged particles. The falling to the center, referred to as positronium collapse, occurs for every positive value of the fine structure constant [14].

In Sec. II we derive the fully relativistic—in two-dimensional Minkowski space-time—form that the differential Bethe-Salpeter equation in the ladder approximation takes for the positronium when the magnetic field tends to infinity. In Sec. III the maximum symmetry ultrarelativistic solution is found for the equation derived in Sec. II that corresponds to the vanishing total energy momentum of the positronium. The effects of the mass radiative corrections and of the vacuum polarization are also considered. In the concluding section, the results are summarized.

## II. TWO-DIMENSIONAL BETHE-SALPETER EQUATION FOR POSITRONIUM IN AN ASYMPTOTICALLY STRONG MAGNETIC FIELD

The view is widely accepted [1] that charged particles in a strong constant magnetic field are confined to the lowest Landau level and behave effectively as if they possessed only one spatial degree of freedom—the one along the magnetic field. Moreover, a conjecture exists [15] that the Feynman rules in the high magnetic field limit may be directly served by two-dimensional (one space + one time) form of electron propagators. As applied to the Bethe-Salpeter equation, the dimensional reduction in a strong magnetic field was considered in [9,10]. In these references the well-known simultaneous approximation to the Bethe-Salpeter equation taken in the integral form was exploited, appropriate for nonrelativistic treatment of the relative motion of the two charged particles. In the next section we shall be interested in the ultrarelativistic regime. For this reason we reject using this approximation. Besides, we find it convenient to deal only with the differential form of the Bethe-Salpeter equation.

The electron-positron bound state is described by the Bethe-Salpeter amplitude (wave function) $\chi_{\lambda,\beta}(x^{\rm e}, x^{\rm p})$ subject to the fully relativistic equation (see e.g. [16]), which in the ladder approximation in a magnetic field may be written as

$$[{\rm i}\hat{\partial}^{\rm e} - m + e\hat{A}(x^{\rm e})]_{\lambda\beta}[{\rm i}\hat{\partial}^{\rm p} - m - e\hat{A}(x^{\rm p})]_{\mu\nu}\,\chi_{\beta\nu}(x^{\rm e}, x^{\rm p}) = -{\rm i}8\pi\alpha D^{ij}(x^{\rm e} - x^{\rm p})[\gamma_i]_{\lambda\beta}[\gamma_j]_{\mu\nu}\,\chi_{\beta\nu}(x^{\rm e}, x^{\rm p}). \quad (3)$$

Here $x^{\rm e}$, $x^{\rm p}$ are the electron and positron 4-coordinates, $D^{ij}(x^{\rm e} - x^{\rm p})$ is the photon propagator, and the spinor indices $\lambda, \beta, \mu, \nu = 1, 2, 3, 4$ are explicitly written. The metric in the Minkowski space is diag $g_{ij} = (1, -1, -1, -1)$, $i, j = 0, 1, 2, 3$. The derivatives

$$\hat{\partial} = \partial^j\gamma_j = \partial^0\gamma_0 + \partial^k\gamma_k = \gamma_0\frac{\partial}{\partial x_0} + \gamma_k\frac{\partial}{\partial x_k}, \quad k = 1, 2, 3, \quad (4)$$

act on $x^{\rm e}$ or $x^{\rm p}$ as indicated by the superscripts, and $\hat{A} = A_0\gamma_0 - A_k\gamma_k$.

We consider the ladder approximation, with the photon propagator taken in the Feynman gauge. With other gauges this approximation corresponds to summation of diagrams other than the ladder ones—in agreement with the well-known fact that the ladder approximation is not gauge invariant.

We refer, if needed, to the so-called spinor representation of the Dirac $\gamma$ matrices in the block form

$$\gamma_0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \qquad \gamma_k = \begin{pmatrix} 0 & -\sigma_k \\ \sigma_k & 0 \end{pmatrix}, \quad (5)$$

where $I$ is the unit $2\times 2$ matrix and $\sigma_k$ are the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \mathrm{i}\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{6}$$

The vector potential of the constant and homogeneous magnetic field $B$, directed along axis 3 ($B_3 = B$, $B_{1,2} = 0$), is chosen in the asymmetric gauge

$$A_1(x) = -Bx_2, \qquad A_{0,2,3}(x) = 0. \tag{7}$$

With this choice, the translation invariance along the directions 0, 1, 3 holds.

Solutions to Eq. (3) may be represented in the form

$$\chi(x^{\mathrm{e}}, x^{\mathrm{p}}) = \eta(x_0^{\mathrm{e}} - x_0^{\mathrm{p}}, x_3^{\mathrm{e}} - x_3^{\mathrm{p}}, x_{1,2}^{\mathrm{e}}, x_{1,2}^{\mathrm{p}}) \times \exp\left\{\frac{\mathrm{i}}{2}[P_0(x_0^{\mathrm{e}} + x_0^{\mathrm{p}}) - P_3(x_3^{\mathrm{e}} + x_3^{\mathrm{p}})]\right\}, \tag{8}$$

where $P_{0,3}$ are the center-of-mass 4-momentum components of the longitudinal motion. Equation (8) expresses the translation invariance along the longitudinal directions (0,3). Denoting the differences $x_0^{\mathrm{e}} - x_0^{\mathrm{p}} = t$ and $x_3^{\mathrm{e}} - x_3^{\mathrm{p}} = z$, from Eqs. (3) and (8) we obtain

$$\left[\mathrm{i}\hat{\partial}_{\|} - \frac{\hat{P}_{\|}}{2} - m + \mathrm{i}\hat{\partial}_{\perp}^{\mathrm{e}} - e\gamma_1 A_1(x_2^{\mathrm{e}})\right]_{\lambda\beta}\left[-\mathrm{i}\hat{\partial}_{\|} - \frac{\hat{P}_{\|}}{2} - m + \mathrm{i}\hat{\partial}_{\perp}^{\mathrm{p}} + e\gamma_1 A_1(x_2^{\mathrm{p}})\right]_{\mu\nu}[\eta(t, z, x_{\perp}^{\mathrm{e,p}})]_{\beta\nu} = -\mathrm{i}8\pi\alpha D_{ij}(t, z, x_{1,2}^{\mathrm{e}} - x_{1,2}^{\mathrm{p}})[\gamma_i]_{\lambda\beta}[\gamma_j]_{\mu\nu}[\eta(t, z, x_{\perp}^{\mathrm{e,p}})]_{\beta\nu}, \tag{9}$$

where $x_\perp = (x_1, x_2)$, $\hat{\partial}_\perp = \gamma_1 \frac{\partial}{\partial x_1} + \gamma_2 \frac{\partial}{\partial x_2}$, $\hat{\partial}_\| = \frac{\partial}{\partial t}\gamma_0 + \frac{\partial}{\partial z}\gamma_3$, and $\hat{P}_\| = P_0\gamma_0 - P_3\gamma_3$.

### A. Fourier-Ritus expansion in eigenfunctions of the transverse motion

Let us expand the dependence of the solution of Eq. (9) on the transverse degrees of freedom into the series over the (complete set of) Ritus [17] matrix eigenfunctions $E_h(x_2)$ (the superscript "e" or "p" indicates that the corresponding function relates to the electron or positron, respectively),

$$[\eta(t, z, x_\perp^{\mathrm{e,p}})]_{\mu\nu} = \sum_{h^{\mathrm{e}}h^{\mathrm{p}}} e^{\mathrm{i}p_1^{\mathrm{e}}x_1^{\mathrm{e}}}[E_{h^{\mathrm{e}}}^{\mathrm{e}}(x_2^{\mathrm{e}})]_\mu^{\lambda^{\mathrm{e}}} \times [E_{h^{\mathrm{p}}}^{\mathrm{p}}(x_2^{\mathrm{p}})]_\nu^{\lambda^{\mathrm{p}}} e^{\mathrm{i}p_1^{\mathrm{p}}x_1^{\mathrm{p}}}[\eta_{h^{\mathrm{e}}h^{\mathrm{p}}}(t, z)]_{\lambda^{\mathrm{e}}\lambda^{\mathrm{p}}}. \tag{10}$$

Here $\eta_{h^{\mathrm{e}}h^{\mathrm{p}}}(t, z)$ denote *unknown* functions that depend on the differences of the longitudinal variables, while the Ritus matrix functions $e^{\mathrm{i}p_1x_1}E_h(x_2)$ depend on the individual coordinates $x_{1,2}^{\mathrm{e,p}}$ transverse to the field. The Ritus matrix functions and the unknown functions $\eta_{h^{\mathrm{e}}h^{\mathrm{p}}}(t, z)$ are labeled by two pairs $h^{\mathrm{e}}$, $h^{\mathrm{p}}$ of quantum numbers $h = (k, p_1)$, each pair relating to one out of the two particles in a magnetic field. The Landau quantum number $k$ takes all nonnegative integral values, $k = 0, 1, 2, 3, \ldots$, while $p_1$ is the particle momentum component along the transverse axis 1. Recall that the potential $A_\mu(x)$ (7) does not depend on $x_1$, so that $p_1$ does conserve. This quantum number is connected [13] with the orbital center coordinate $\tilde{x}_2$ along axis 2, $p_1 = -\tilde{x}_2 eB$.

Consider four eigen-bispinors $[E_h^{\mathrm{e,p}}(x_2)]_\nu^{(\sigma,\gamma)}$ of the operator $(-\mathrm{i}\hat{\partial}_\perp \pm e\hat{A})^2$, labeled by $\sigma$ and $\gamma$,

$$(-\mathrm{i}\hat{\partial}_\perp \pm e\hat{A})_{\mu\nu}^2 e^{\mathrm{i}p_1x_1}[E_h^{\mathrm{e,p}}(x_2)]_\nu^{(\sigma,\gamma)} = -2eBke^{\mathrm{i}p_1x_1}[E_h^{\mathrm{e,p}}(x_2)]_\mu^{(\sigma,\gamma)}. \tag{11}$$

Here the upper and lower signs relate to the electron and positron, respectively, while $\sigma = \pm 1$ and $\gamma = \pm 1$ are eigenvalues of the operators

$$\Sigma_3 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}, \qquad -\mathrm{i}\gamma_5 = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}, \tag{12}$$

(diagonal in the spinor representation), to which the same 4-spinors are eigen-bispinors [18],

$$-\mathrm{i}\gamma_5 E_h^{(\sigma,\gamma)} = \gamma E_h^{(\sigma,\gamma)}, \qquad \Sigma_3 E_h^{(\sigma,\gamma)} = \sigma E_h^{(\sigma,\gamma)}. \tag{13}$$

Let us place these four bispinors, as columns of four rows each, side by side to form a $4 \times 4$ matrix and unite the couple of indices $(\sigma, \gamma) = \lambda$ into one index $\lambda$, $\lambda = 1, 2, 3, 4$ according to the rule: $(+1, -1) = 1$, $(-1, -1) = 2$, $(+1, +1) = 3$, $(-1, +1) = 4$. With this convention, the equality $[E_h(x_2]_\mu^{(\sigma,\gamma)} = E_h(x_2)_\mu^\lambda$ defines the Ritus matrix function in the spinor representation. It can be dealt with as a $4 \times 4$ matrix, the united index $\lambda$ spanning a matrix space, where the usual algebra of $\gamma$ matrices may act. Correspondingly, in (10) the unknown function $[\eta_{h^{\mathrm{e}}h^{\mathrm{p}}}(t, z)]_{\lambda^{\mathrm{e}}\lambda^{\mathrm{p}}}$ is a matrix in the same space, and contracts with the Ritus matrix functions.

Following [17], the matrix functions in expansion (10) can be written in the block form as diagonal matrices

$$e^{\mathrm{i}p_1x_1}E_h^{\mathrm{e,p}}(x_2) = \begin{pmatrix} a^{\mathrm{e,p}}(h; x_{1,2}) & 0 \\ 0 & a^{\mathrm{e,p}}(h; x_{1,2}) \end{pmatrix}, \qquad a^{\mathrm{e,p}}(h; x_{1,2}) = \begin{pmatrix} a_{+1}^{\mathrm{e,p}}(h; x_{1,2}) & 0 \\ 0 & a_{-1}^{\mathrm{e,p}}(h; x_{1,2}) \end{pmatrix}. \tag{14}$$

Here $a_\sigma^{\mathrm{e,p}}(h; x_{1,2})$ are eigenfunctions of the two (for each sign $\pm$) operators $[((-\mathrm{i}\partial_\perp)_\lambda \pm eA_\lambda)^2 \mp \sigma eB]$ [we denote $(\partial_\perp)_\lambda = \partial/\partial x_\lambda$, $\lambda = 1, 2$], labeled by the two values $\sigma = 1, -1$,

$$[((-\mathrm{i}\partial_\perp)_\lambda \pm eA_\lambda)^2 \mp \sigma eB]a_\sigma^{\mathrm{e,p}}(h;x_{1,2}) = 2eBka_\sigma^{\mathrm{e,p}}(h;x_{1,2}), \tag{15}$$

namely (we omit the subscript "1" of $p_1$ in what follows),

$$a_\sigma^{\mathrm{e,p}}(h;x_{1,2}) = \mathrm{e}^{\mathrm{i}px_1}U_{k+((\pm\sigma-1)/2)}\left[\sqrt{eB}\left(x_2 \pm \frac{p}{eB}\right)\right], \quad k = 0,1,2,\ldots, \tag{16}$$

with

$$U_n(\xi) = \exp\left\{-\frac{\xi^2}{2}\right\}(2^n n!\sqrt{\pi})^{-1/2}H_n(\xi) \tag{17}$$

being the normalized Hermite functions [$H_n(\xi)$ are the Hermite polynomials]. Equation (15) is the same as (11) due to the relation

$$(\mathrm{i}\hat{\partial}_\perp \mp e\hat{A})^2 = -[(\mathrm{i}\partial_\perp)_\lambda \mp eA_\lambda]^2 \pm eB\Sigma_3 \tag{18}$$

and to (13). Simultaneously, the matrix functions (14) are eigenfunctions to the operator $-\mathrm{i}\partial_1$ that commutes with $\Sigma_3$ and $\gamma_5$ (12), and with $(\mathrm{i}\hat{\partial}_\perp \mp e\hat{A})^2_{\mu\nu}$. The corresponding eigenvalue $p_1$ does not, however, appear in the right-hand side of (15) due to the well-known degeneracy of the electron spectrum in a constant magnetic field.

The orthonormality relation for the Hermite functions

$$\int_{-\infty}^{\infty} U_n(\xi)U_{n'}(\xi)\mathrm{d}\xi = \delta_{nn'} \tag{19}$$

implies the orthogonality of the Ritus matrix eigenfunctions in the form

$$\sqrt{eB}\int E_h^*(x_2)_\mu^\lambda E_{h'}(x_2)_\mu^{\lambda'}\mathrm{d}x_2 = \delta_{kk'}\delta_{\lambda\lambda'}. \tag{20}$$

As a matter of fact, the matrix functions $E_h(x_2)$ are real, and we henceforth omit the complex conjugation sign "$*$."

The matrix functions $e^{\mathrm{i}px_1}E_h^{\mathrm{e,p}}(x_2)$, (14), commute with the longitudinal part $\pm\mathrm{i}\hat{\partial}_\parallel - \hat{P}_\parallel/2 - m$ of the Dirac operator in (9), owing to the commutativity property

$$[E_h(x_2), \gamma_{0,3}]_- = 0. \tag{21}$$

They are, in a sense, matrix eigenfunctions of the transverse part of the Dirac operator [17] [not only of its square (11)],

$$(\mathrm{i}\hat{\partial}_\perp \mp e\hat{A})e^{\mathrm{i}px_1}E_h^{\mathrm{e,p}}(x_2) = \pm\sqrt{2eBk}e^{\mathrm{i}px_1}E_h^{\mathrm{e,p}}(x_2)\gamma_1. \tag{22}$$

The Landau quantum number $k$ appears here as a "universal eigenvalue" thanks to the mechanism, easy to trace, according to which the differential operator in the left-hand side of Eq. (22) acts as a lowering or rising operator on the functions (17), whereas the matrix $\sigma_2$, involved in $\gamma_2$, interchanges the places that the functions $U_k$, $U_{k-1}$ occupy in the columns. Contrary to relations that explicitly include the variable $\sigma$, whose value is a number of a column, relations (11), (21), and (22) and the first relation in (13) retain their form if a representation of $\gamma$ matrices, other than the spinor representation, is used, and the matrix $E_h(x_2)$ may become nondiagonal.

### B. Equation for the Fourier-Ritus transform of the Bethe-Salpeter amplitude

Now we are in a position to use expansion (10) in Eq. (9). We left-multiply it by $(2\pi)^{-2}eBe^{-\mathrm{i}\bar{p}^{\mathrm{e}}x_1^{\mathrm{e}}}E^{\mathrm{e}}_{\bar{h}^{\mathrm{e}}}(x_2^{\mathrm{e}}) \times e^{-\mathrm{i}\bar{p}^{\mathrm{p}}x_1^{\mathrm{p}}}E^{\mathrm{p}}_{\bar{h}^{\mathrm{p}}}(x_2^{\mathrm{p}})$, then integrate over $\mathrm{d}^2x_{1,2}^{\mathrm{e}}\mathrm{d}^2x_{1,2}^{\mathrm{p}}$. After using (21) and (22), and exploiting the orthonormality relation (20) for the summation over the quantum numbers $h^{\mathrm{e,p}} = (k^{\mathrm{e,p}}, p_1^{\mathrm{e,p}})$, the following expression is obtained for the left-hand side of the Fourier-Ritus-transformed equation (9):

$$\left[\mathrm{i}\hat{\partial}_\parallel - \frac{\hat{P}_\parallel}{2} - m - \gamma_1\sqrt{2eBk^{\mathrm{e}}}\right]_{\lambda\lambda^{\mathrm{e}}} \times \left[-\mathrm{i}\hat{\partial}_\parallel - \frac{\hat{P}_\parallel}{2} - m + \gamma_1\sqrt{2eBk^{\mathrm{p}}}\right]_{\mu\lambda^{\mathrm{p}}}[\eta_{h^{\mathrm{e}}h^{\mathrm{p}}}(t,z)]_{\lambda^{\mathrm{e}}\lambda^{\mathrm{p}}}. \tag{23}$$

We omitted the bars over the quantum numbers.

Taking the expression

$$D_{ij}(t,z,x_{1,2}^{\mathrm{e}} - x_{1,2}^{\mathrm{p}}) = \frac{g_{ij}}{\mathrm{i}4\pi^2}[t^2 - z^2 - (x_1^{\mathrm{e}} - x_1^{\mathrm{p}})^2 - (x_2^{\mathrm{e}} - x_2^{\mathrm{p}})^2]^{-1}, \tag{24}$$

for the photon propagator in the Feynman gauge, we may write the right-hand side of the Ritus-transformed equation (9) as

$$\frac{\alpha}{2\pi^3}\int \mathrm{d}p^{\mathrm{e}}\mathrm{d}p^{\mathrm{p}}\sum_{k^{\mathrm{e}}k^{\mathrm{p}}} g_{ij}\int [E^{\mathrm{e}}_{\bar{h}^{\mathrm{e}}}(x_2^{\mathrm{e}})\gamma_i E^{\mathrm{e}}_{h^{\mathrm{e}}}(x_2^{\mathrm{e}})]_{\lambda\lambda^{\mathrm{e}}} \times [E^{p}_{\bar{h}^{p}}(x_2^{p})\gamma_j E^{\mathrm{p}}_{h^{\mathrm{p}}}(x_2^{\mathrm{p}})]_{\mu\lambda^{\mathrm{p}}}[\eta_{h^{\mathrm{e}}h^{\mathrm{p}}}(t,z)]_{\lambda^{\mathrm{e}}\lambda^{\mathrm{p}}} \times \frac{e^{\mathrm{i}(p^{\mathrm{e}}-\bar{p}^{\mathrm{e}})x_1}e^{\mathrm{i}(p^{\mathrm{p}}-\bar{p}^{\mathrm{p}})x_1}eB\mathrm{d}^2x_{1,2}^{\mathrm{e}}\mathrm{d}^2x_{1,2}^{p}}{z^2 + (x_1^e - x_1^{\mathrm{p}})^2 + (x_2^e - x_2^{p})^2 - t^2}. \tag{25}$$

Integrating explicitly the exponentials in (25) over the variable $X = (x_1^{\mathrm{e}} + x_1^{\mathrm{p}})/2$, we obtain the following expression:

$$\frac{\alpha}{\pi^2}\int \mathrm{d}p\mathrm{d}P_1\delta(\bar{P}_1 - P_1)\sum_{k^{\mathrm{e}}k^{p}} g_{ij}\int [E^{\mathrm{e}}_{\bar{h}^{\mathrm{e}}}(x_2^{\mathrm{e}})\gamma_i E^{\mathrm{e}}_{h^{\mathrm{e}}}(x_2^{\mathrm{e}})]_{\lambda\lambda^{\mathrm{e}}} \times [E^{p}_{\bar{h}^{p}}(x_2^{p})\gamma_j E^{\mathrm{p}}_{h^{\mathrm{p}}}(x_2^{\mathrm{p}})]_{\mu\lambda^{\mathrm{p}}}[\eta_{h^{\mathrm{e}}h^{\mathrm{p}}}(t,z)]_{\lambda^{\mathrm{e}}\lambda^{\mathrm{p}}} \times \frac{\exp(\mathrm{i}x(\bar{p} - p))\mathrm{d}x}{z^2 + x^2 + (x_2^{\mathrm{e}} - x_2^{\mathrm{p}})^2 - t^2}eB\mathrm{d}x_2^{\mathrm{e}}\mathrm{d}x_2^{\mathrm{p}}, \tag{26}$$

where the new integration variables $x = x_1^{\mathrm{e}} - x_1^{\mathrm{p}}$, $P_1 = p^{\mathrm{e}} + p^{\mathrm{p}}$, $p = (p^{\mathrm{e}} - p^{\mathrm{p}})/2$ and the new definitions $\bar{P}_1 = \bar{p}^{\mathrm{e}} + \bar{p}^{\mathrm{p}}$, $\bar{p} = (\bar{p}^{\mathrm{e}} - \bar{p}^{\mathrm{p}})/2$ have been introduced. The pairs of quantum numbers in (26) are

$$\bar{h}^{\mathrm{e,p}} = \left(\bar{k}^{\mathrm{e,p}}, \frac{\bar{P}_1}{2} \pm \bar{p}\right), \qquad h^{\mathrm{e,p}} = \left(k^{\mathrm{e,p}}, \frac{P_1}{2} \pm p\right). \tag{27}$$

Hence the arguments of the functions (16) in (26) are

$$\sqrt{eB}\Big(x_2^{\mathrm{e}} + \frac{\bar{P}_1 + 2\bar{p}}{2eB}\Big), \qquad \sqrt{eB}\Big(x_2^{\mathrm{e}} + \frac{P_1 + 2p}{2eB}\Big),$$
$$\sqrt{eB}\Big(x_2^{\mathrm{p}} - \frac{\bar{P}_1 - 2\bar{p}}{2eB}\Big), \qquad \sqrt{eB}\Big(x_2^{\mathrm{p}} - \frac{P_1 - 2p}{2eB}\Big), \tag{28}$$

successively as the functions $E_h(x_{1,2})$ appear in (26) from left to right. After fulfilling the integration over $\mathrm{d}P_1$ with the use of the $\delta$ function, let us introduce the new integration variable $q = p - \bar{p}$ instead of $p$, and the integration variables $\bar{x}_2^{\mathrm{e}} = x_2^{\mathrm{e}} + (\bar{P}_1 + 2\bar{p})/2eB$, $\bar{x}_2^{\mathrm{p}} = x_2^{\mathrm{p}} - (\bar{P}_1 - 2p)/2eB$ instead of $x_2^{\mathrm{e}}$ and $x_2^{\mathrm{p}}$. Then (26) may be written as

$$\frac{\alpha}{\pi^2}\int \mathrm{d}q \sum_{k^{\mathrm{e}}k^{p}} g_{ij}\int [E^{\mathrm{e}}_{\bar{h}^{\mathrm{e}}}(x_2^{\mathrm{e}})\gamma_i E^{\mathrm{e}}_{h^{\mathrm{e}}}(x_2^{\mathrm{e}})]_{\lambda\lambda^{\mathrm{e}}} \times [E^{p}_{\bar{h}^{p}}(x_2^{p})\gamma_j E^{\mathrm{p}}_{h^{\mathrm{p}}}(x_2^{\mathrm{p}})]_{\mu\lambda^{\mathrm{p}}}[\eta_{h^{\mathrm{e}}h^{\mathrm{p}}}(t,z)]_{\lambda^{\mathrm{e}}\lambda^{\mathrm{p}}} \times \int \frac{\exp(-\mathrm{i}xq)\mathrm{d}x eB \mathrm{d}\bar{x}_2^{\mathrm{e}}\mathrm{d}\bar{x}_2^{p}}{z^2 + x^2 + (\bar{x}_2^{\mathrm{e}} - \bar{x}_2^{\mathrm{p}} - \frac{\bar{P}_1 - q}{eB})^2 - t^2}. \tag{29}$$

Now the pairs of quantum numbers in (29) are

$$\bar{h}^{\mathrm{e,p}} = \Big(\bar{k}^{\mathrm{e,p}}, \frac{\bar{P}_1}{2} \pm \bar{p}\Big), \qquad h^{\mathrm{e,p}} = \Big(k^{\mathrm{e,p}}, \frac{\bar{P}_1}{2} \pm q \pm \bar{p}\Big). \tag{30}$$

Hence the arguments of the functions (16) in (29) from left to right are

$$\sqrt{eB}\bar{x}_2^{\mathrm{e}}, \quad \sqrt{eB}\Big(\bar{x}_2^{\mathrm{e}} + \frac{q}{eB}\Big), \quad \sqrt{eB}\Big(\bar{x}_2^{\mathrm{p}} - \frac{q}{eB}\Big), \quad (\sqrt{eB}\bar{x}_2^{\mathrm{p}}). \tag{31}$$

## C. Adiabatic approximation

Our next goal is to pass to the large magnetic field regime in the Bethe-Salpeter equation, with (23) as the left-hand side and (29) as the right-hand side. Defining the dimensionless integration variables $w = x\sqrt{eB}$, $q' = q/\sqrt{eB}$, $\xi^{\mathrm{e,p}} = \bar{x}_2^{\mathrm{e,p}}\sqrt{eB}$ one can write (29) in the form

$$\frac{\alpha}{\pi^2}\int \mathrm{d}q' \sum_{k^{\mathrm{e}}k^{\mathrm{p}}} g_{ij}\int [E^{\mathrm{e}}_{\bar{h}^{\mathrm{e}}}(x_2^{\mathrm{e}})\gamma_i E^{\mathrm{e}}_{h^{\mathrm{e}}}(x_2^{\mathrm{e}})]_{\lambda\lambda^{\mathrm{e}}} \times [E^{\mathrm{p}}_{\bar{h}^{\mathrm{p}}}(x_2^{\mathrm{p}})\gamma_j E^{\mathrm{p}}_{h^{\mathrm{p}}}(x_2^{\mathrm{p}})]_{\mu\lambda^{\mathrm{p}}}[\eta_{h^{\mathrm{e}}h^{\mathrm{p}}}(t,z)]_{\lambda^{\mathrm{e}}\lambda^{\mathrm{p}}} \times \int \frac{\exp(-\mathrm{i}wq')\mathrm{d}w\mathrm{d}\xi^{\mathrm{e}}\mathrm{d}\xi^{\mathrm{p}}}{z^2 + \frac{w^2}{eB} + \frac{1}{eB}(\xi^{\mathrm{e}} - \xi^{\mathrm{p}} - \frac{\bar{x}_1}{\sqrt{eB}} - q')^2 - t^2}. \tag{32}$$

The pairs of quantum numbers in (32) are

$$\bar{h}^{\mathrm{e,p}} = \Big(\bar{k}^{\mathrm{e,p}}, \frac{\bar{P}_1}{2} \pm \bar{p}\Big),$$
$$h^{\mathrm{e,p}} = \Big(k^{\mathrm{e,p}}, \frac{\bar{P}_1}{2} \pm q'\sqrt{eB} \pm \bar{p}\Big). \tag{33}$$

The arguments of the functions (16) in (32) from left to right are

$$\xi^{\mathrm{e}}, \quad \xi^{\mathrm{e}} + q', \quad \xi^{\mathrm{p}} - q', \quad \xi^{\mathrm{p}}. \tag{34}$$

When considering the large-field behavior, we admit for completeness that the difference between the centers of orbits along axis 2 $\bar{x}_2^{\mathrm{e}} - \bar{x}_2^{\mathrm{p}} = -(\bar{P}_1/eB)$ may be kept finite, in other words, that the transverse momentum $\bar{P}_1$ may grow linearly with the field. We shall see that large transverse momenta do not contradict dimensional reduction, but produce an extra regularization of the light-cone singularity.

In the region, where the 2-interval $(z^2 - t^2)^{1/2}$ essentially exceeds the Larmour radius $L_B = (eB)^{-1/2}$,

$$z^2 - t^2 \gg L_B^2, \tag{35}$$

one may neglect the dependence on the integration variables $w$ and later on $\xi^{\mathrm{e,p}}$ in the denominator. Then, the integration over $w$ produces $2\pi\delta(q')$, which annihilates the dependence on $q$ in the arguments (31) of the Hermite functions.

Let us depict the mechanism described in the previous paragraph in more detail. The explicit integration over $\mathrm{d}w$ in (32) yields

$$\int \frac{\exp(-\mathrm{i}wq')\mathrm{d}w}{z^2 - t^2 + \frac{w^2}{eB} + \frac{A^2}{eB}} = \frac{\sqrt{eB}\pi}{\sqrt{z^2 - t^2 + \frac{A^2}{eB}}}\Big[\theta(q')\exp\Big[-q'\sqrt{eB(z^2 - t^2) + A^2}\Big] + \theta(-q')\exp\Big[q'\sqrt{eB(z^2 - t^2) + A^2}\Big]\Big], \tag{36}$$

where

$$A^2 = \Big(\xi^{\mathrm{e}} - \xi^{p} - \frac{\bar{P}_1}{\sqrt{eB}} - q'\Big)^2 \tag{37}$$

and $\theta(q')$ is the step function,

$$\theta(q') = \begin{cases} 1 & \text{when } q' > 0, \\ \frac{1}{2} & \text{when } q' = 0, \\ 0 & \text{when } q' < 0. \end{cases} \tag{38}$$

Because of the decreasing exponential in (17) the effective integration range of the variables $\xi^{\mathrm{e,p}}$ does not exceed unity in the order of magnitude and hence $\xi^{\mathrm{e,p}}$ can be neglected as compared to $\bar{P}_1/\sqrt{eB}$ in (37). Unless $q'$ is large it may be neglected as compared to the same term in (37), too. Then $A^2 = \bar{P}_1^2/eB$, and after (36) is substituted in (32) and integrated over $\mathrm{d}q'$ the contribution comes only from the integration within the shrinking region $|q'| < (eB[z^2 - t^2 + \bar{P}_1^2/(eB)^2])^{-1/2}$. Then $q'$ can also be neglected in the arguments (34). If, contrary to the previous assumption, we admit that $|q'|$ is of the order of $\bar{P}_1/\sqrt{eB} \sim \sqrt{eB}$ we see that the exponentials in (36) quickly decrease with the growth of the magnetic field as $\exp[-eB(z^2 - t^2)]$, and therefore such values of $|q'|$ do not contribute to the integration. If we admit, lastly, that $|q'| \gg |\bar{P}_1/\sqrt{eB}|$, we find that the contribution $\exp[-|q'|\sqrt{eB(z^2 - t^2) + (q')^2}]$ from the integration over such values is still smaller. Thus, we have justified the disregard of the dependence on $q'$ in (37) and in (34), and also on $\xi^{\mathrm{e,p}}$ in (37).

Now we can perform the integration over $\mathrm{d}q'$ to obtain the following expression for (32):

$$\frac{2\alpha\pi^{-1}}{z^2+\frac{P_1^2}{(eB)^2}-t^2}\sum_{k^{\mathrm{e}}k^{\mathrm{p}}} g_{ii}\int[E^{\mathrm{e}}_{\bar{h}^{\mathrm{e}}}(x_2^{\mathrm{e}})\gamma_i E^{\mathrm{e}}_{h^{\mathrm{e}}}(x_2^{\mathrm{e}})]_{\lambda\lambda^{\mathrm{e}}}\mathrm{d}\xi^{\mathrm{e}} \times\int[E^{\mathrm{p}}_{\bar{h}^{\mathrm{p}}}(x_2^{\mathrm{p}})\gamma_i E^{\mathrm{p}}_{h^{\mathrm{p}}}(x_2^{\mathrm{p}})]_{\mu\lambda^{\mathrm{p}}}\mathrm{d}\xi^{\mathrm{p}}[\eta_{h^{\mathrm{e}}h^{\mathrm{p}}}(t,z)]_{\lambda^{\mathrm{e}}\lambda^{\mathrm{p}}}. \quad (39)$$

It remains yet to argue that the limit (39) is valid also when the term $\bar{P}_1/eB$ is not kept. In this case we no longer can disregard $q'$ inside $A^2$ when $q'$ is less than or of the order of unity. But we can disregard $A^2$ as compared with $eB(z^2-t^2)$ to make sure that the integration over $\mathrm{d}q'$ is restricted to the region close to zero $|q|'<[eB(z^2-t^2)]^{-1/2}$ and hence set $q'=0$ in (34). The contribution of large $q'$ is small as before.

Now that the arguments of the Hermite functions in (39) are all the same, the integration over $\xi^{\mathrm{e,p}}$ of the terms with $i=0,3$ in (39) yields the Kronecker deltas $\delta_{k^{\mathrm{e}}\bar{k}^{\mathrm{e}}}\delta_{k^{\mathrm{p}}\bar{k}^{\mathrm{p}}}$ due to the orthonormality (19) of the Hermite functions thanks to the commutativity (21) of the Ritus matrix functions (14) with $\gamma_0$ and $\gamma_3$. On the contrary, $\gamma_1$, $\gamma_2$ do not commute with (14). This implies the appearance of terms, nondiagonal in Landau quantum numbers, like $\delta_{k^{\mathrm{e}},\bar{k}^{\mathrm{e}}\pm1}$ and $\delta_{k^{\mathrm{p}},\bar{k}^{\mathrm{p}}\pm1}$, in (32), proportional to ($i=1,2$):

$$\begin{aligned}T^i_{\bar{k}_{\mathrm{e}}\pm1,\bar{k}^{\mathrm{p}}\pm1} &= \sum_{k^{\mathrm{e}}k^{\mathrm{p}}}\int[E^{\mathrm{e}}_{\bar{h}^{\mathrm{e}}}(x_2^{\mathrm{e}})\gamma_i E^{\mathrm{e}}_{h^{\mathrm{e}}}(x_2^{\mathrm{e}})]_{\lambda\lambda^{\mathrm{e}}}\mathrm{d}\xi^{\mathrm{e}} \\ &\quad\times\int[E^{\mathrm{p}}_{\bar{h}^{\mathrm{p}}}(x_2^{\mathrm{p}})\gamma_i E^{\mathrm{p}}_{h^{\mathrm{p}}}(x_2^{\mathrm{p}})]_{\mu\lambda^{\mathrm{p}}}\mathrm{d}\xi^{\mathrm{p}}[\eta_{h^{\mathrm{e}}h^{\mathrm{p}}}(t,z)]_{\lambda^{\mathrm{e}}\lambda^{\mathrm{p}}} \\ &= \sum_{k^{\mathrm{e}}k^{\mathrm{p}}}\begin{pmatrix}0 & -\Delta^i_{\bar{k}^{\mathrm{e}}k^{\mathrm{e}}}\\ \Delta^i_{\bar{k}^{\mathrm{e}}k^{\mathrm{e}}} & 0\end{pmatrix} \times\begin{pmatrix}0 & -\Delta^i_{\bar{k}^{\mathrm{p}}k^{\mathrm{p}}}\\ \Delta^i_{\bar{k}^{\mathrm{p}}k^{\mathrm{p}}} & 0\end{pmatrix}[\eta_{h^{\mathrm{e}}h^{\mathrm{p}}}(t,z)]_{\lambda^{\mathrm{e}}\lambda^{\mathrm{p}}}. \end{aligned} \quad (40)$$

Here $x_2^{\mathrm{e,p}}$ are expressed in terms of $\xi$ through the chain of the changes of variables made above starting from (25), so that all the arguments of the Hermite functions have become equal to $\xi$. Besides,

$$h^{\mathrm{e,p}}=(k^{\mathrm{e,p}},\bar{p}^{\mathrm{e,p}}),\quad \bar{h}^{\mathrm{e,p}}=(\bar{k}^{\mathrm{e,p}},\bar{p}^{\mathrm{e,p}}),\quad p^{\mathrm{e}}+p^{\mathrm{p}}=P_1, \quad (41)$$

$$\Delta^i_{\bar{k}k}=\int a'(\bar{h},x_2)\sigma_i a'(h,x_2)\mathrm{d}\xi, \qquad i=1,2, \quad (42)$$

$$\Delta^{(1)}_{\bar{k}k}=\int\begin{pmatrix}0 & a'_{+1}(\bar{h},x_2)a'_{-1}(k,x_2)\\ a'_{-1}(\bar{h},x_2)a'_{+1}(h,x_2) & 0\end{pmatrix}\mathrm{d}\xi =\begin{pmatrix}0 & \delta_{\bar{k},k-1}\\ \delta_{\bar{k},k+1} & 0\end{pmatrix}, \quad (43)$$

$$\Delta^{(2)}_{\bar{k}k}=\mathrm{i}\int\begin{pmatrix}0 & -a'_{+1}(\bar{h},x_2)a'_{-1}(k,x_2)\\ a'_{-1}(\bar{h},x_2)a'_{+1}(h,x_2) & 0\end{pmatrix}\mathrm{d}\xi =\mathrm{i}\begin{pmatrix}0 & -\delta_{\bar{\mathrm{k}},\mathrm{k}-1}\\ \delta_{\bar{\mathrm{k}},\mathrm{k}+1} & 0\end{pmatrix}. \quad (44)$$

The prime over $a$ indicates that the exponential $\exp(\mathrm{i}px_1)$ is dropped from the definitions (14) and (16). The nondiagonal Kronecker deltas appeared, because $a'_{\pm1}(\bar{h},x_2)$ are multiplied by $a'_{\mp1}(h,x_2)$ under the action of the $\sigma_{1,2}$ blocks in $\gamma_{1,2}$ (5). In the final form, the matrices in (40) are

$$\begin{pmatrix}0 & -\Delta^i_{\bar{k}k}\\ \Delta^i_{\bar{k}k} & 0\end{pmatrix}=\frac{1}{2}(\gamma_1(\pm\delta_{\bar{k},k-1}+\delta_{\bar{k},k+1}) +\mathrm{i}\gamma_2(\pm\delta_{\bar{k},k-1}-\delta_{\bar{k},k+1})), \quad (45)$$

with the upper sign relating to $i=1$ and the lower one to $i=2$. Now Eq. (9) acquires the following form:

$$\left[\mathrm{i}\hat{\partial}_\parallel-\frac{\hat{P}_\parallel}{2}-m-\gamma_1\sqrt{2eBk^{\mathrm{e}}}\right]_{\lambda\lambda^{\mathrm{e}}}\left[-\mathrm{i}\hat{\partial}_\parallel-\frac{\hat{P}_\parallel}{2}-m+\gamma_1\sqrt{2eBk^{\mathrm{p}}}\right]_{\mu\lambda^{\mathrm{p}}}[\eta_{h^{\mathrm{e}}h^{\mathrm{p}}}(t,z)]_{\lambda^{\mathrm{e}}\lambda^{\mathrm{p}}} =\frac{2\alpha\pi^{-1}}{z^2+\frac{P_1^2}{(eB)^2}-t^2}\left(\sum_{i=0,3}g_{ii}[\gamma_i]_{\lambda\lambda^{\mathrm{e}}}[\gamma_i]_{\mu\lambda^{\mathrm{p}}}[\eta_{h^{\mathrm{e}}h^{\mathrm{p}}}(t,z)]_{\lambda^{\mathrm{e}}\lambda^{\mathrm{p}}}-\sum_{i=1,2}T^{(i)}_{k^{\mathrm{e}}\pm1,k^{\mathrm{p}}\pm1}\right). \quad (46)$$

The bars over quantum numbers are omitted. This equation is degenerate with respect to the difference of the electron and positron momentum components $p=(p^{\mathrm{e}}-p^{\mathrm{p}})/2$ across the magnetic field, but does depend on its transverse center-of-mass momentum $P_1=(p^{\mathrm{e}}+p^{\mathrm{p}})$. This dependence is present, however, only for sufficiently large transverse momenta $P_1$.

At the present step of adiabatic approximation, we have come, for a high magnetic field, to the chain of Eq. (46), in which the unknown function for a given pair of Landau quantum numbers $k^{\mathrm{e}}$, $k^{\mathrm{p}}$ is tangled with the same function with the Landau quantum numbers both shifted by $\pm1$ (in contrast to the general case of a moderate magnetic field, where these numbers may be shifted by all positive and negative integers). To be more precise, the chain consists of two mutually disentangled subchains. The first one includes all functions with the Landau quantum numbers $k^{\mathrm{e}}$, $k^{\mathrm{p}}$ both even or both odd, and the second one includes their even-odd and odd-even combinations. We discuss the first subchain since it contains the lowest function with

$k^{\rm e} = k^{\rm p} = 0$. Now we argue that there exists a solution to the first subchain of Eq. (46), for which all functions $\eta_{k^{\rm e},p_1^{\rm e};k^{\rm p},p_1^{\rm p}}(t,z)$ vanish if at least one of the quantum numbers $k^{\rm e}$, $k^{\rm p}$ is different from zero. Indeed, for $k^{\rm e} = k^{\rm p} = 0$ Eq. (46) then reduces to the closed set

$$\left[\mathrm{i}\hat{\partial}_{\|} - \frac{\hat{P}_{\|}}{2} - m\right]_{\lambda\lambda^{\rm e}} \times \left[-\mathrm{i}\hat{\partial}_{\|} - \frac{\hat{P}_{\|}}{2} - m\right]_{\mu\lambda^{\rm p}} [\eta_{0,p_1^{\rm e};0,p_1^{\rm p}}(t,z)]_{\lambda^{\rm e}\lambda^{\rm p}} = \frac{2\alpha\pi^{-1}}{z^2 + \frac{P_1^2}{(eB)^2} - t^2} \sum_{i=0,3} g_{ii}[\gamma_i]_{\lambda\lambda^{\rm e}} \times [\gamma_i]_{\mu\lambda^{\rm p}} [\eta_{0,p_1^{\rm e};0,p_1^{\rm p}}(t,z)]_{\lambda^{\rm e}\lambda^{\rm p}}, \quad p_1^{\rm e} + p_1^{\rm p} = P_1. \tag{47}$$

In writing it, we have returned to the initial designation of the electron and positron transverse momenta $p_1^{\rm e,p}$. Denote for simplicity $\eta_{k^{\rm e}k^{\rm p}} = \eta_{k^{\rm e},p_1^{\rm e};k^{\rm p},p_1^{\rm p}}(t,z)$. If we consider Eq. (46) with $k^{\rm e} = k^{\rm p} = 1$, we see that $\eta_{11}$ in the left-hand side gets a nonzero contribution from the right-hand side, proportional to $\eta_{00}$, coming from $T^i_{k^{\rm e}-1,k^{\rm p}-1}$, while the other contributions—the ones from $\eta_{11}$, $\eta_{22}$, $\eta_{20}$, $\eta_{02}$—are vanishing according to the assumption. As the left-hand side of Eq. (46) now contains a term, infinitely growing with the magnetic field $B$, it can only be satisfied with the function $\eta_{11}$, infinitely diminishing with $B$ in the domain (35) as

$$[\eta_{11}]_{\lambda\mu} = -\frac{1}{2eB}\frac{\alpha\pi^{-1}}{z^2 + \frac{P_1^2}{(eB)^2} - t^2} \sum_{i=0,3} g_{ii}[\gamma_1\gamma_i]_{\lambda\lambda^{\rm e}} \times [\gamma_1\gamma_i]_{\mu\lambda^{\rm p}} [\eta_{00}]_{\lambda^{\rm e}\lambda^{\rm p}}. \tag{48}$$

Thus, the assumption that all Bethe-Salpeter amplitudes with nonzero Landau quantum numbers are zero in the large-field limit is self-consistent. We state that a solution to the closed set (47) for $\eta_{0,0}$ with all the other components $\eta_{k^{\rm e}k^{\rm p}}$, $k^{\rm e}$, $k^{\rm p} \neq 0$ equal to zero is a solution to the whole chain (46).

The derivation given in this subsection realizes in a formal way the known heuristic argumentation that, for a high magnetic field, the spacing between Landau levels is very large and hence the particles taken in the lowest Landau state remain in it. Effectively, only the longitudinal degree of freedom survives for large $B$, the space-time reduction taking place. Equation (47) is a fully relativistic two-dimensional set of equations with two space-time arguments $t$ and $z$ and two gamma matrices $\gamma_0$ and $\gamma_3$ involved. It is covariant under the Lorentz boost along the magnetic field—the two-dimensional remainder of the full Lorentz group of the initial four-dimensional Minkowski space. Since, unlike the previous works [9–11], neither the famous equal-time Ansatz for the Bethe-Salpeter amplitude [16], nor any other assumption that might imply a nonrelativistic character of the internal motion inside the positronium atom was made, the equation derived is valid for arbitrarily strong binding. It will be analyzed for the extreme relativistic case in the next section.

The two-dimensional equation (47) is valid in the space-like domain (35). This domain of validity should cover the region—call its size the Bohr length—where the wave function, i.e. the solution of Eq. (47), is mostly concentrated. Otherwise, Eq. (47) would describe only a tail of the wave function, and hence be of little use. In nonrelativistic or semirelativistic consideration it is often accepted that the wave function is concentrated within the region of the size of the standard Bohr radius $a_0 = (\alpha m)^{-1} \simeq 0.5 \times 10^{-8}$ cm, characteristic of a positronium atom placed in a small, if any, magnetic field. It is then estimated that the corresponding analog of the asymptotic equation (47) makes sense when $a_0 \gg L_B$, i.e. for the magnetic fields much larger than $B_{\rm a} = \alpha^2 m^2/e \simeq 2.35 \times 10^9$ G. This estimate, however, cannot be universal and may be applicable, at the most, to the magnetic fields close to the lower bound $B \simeq B_{\rm a}$ where the value of the Bohr radius can be borrowed from the theory without the magnetic field. For larger fields the genuine Bohr length may be smaller than $B_{\rm a}$. Generally, the question of where the wave function is concentrated should be answered *a posteriori* by inspecting a solution to Eq. (47). Therefore, one can establish how large the fields should be in order that the asymptotic equation (47) might be efficient, no sooner than its solution is investigated. We shall come back to this point when we deal with the ultrarelativistic situation.

Remember that the transverse total momentum component of the positronium system is connected with the separation between the centers of orbits of the electron and positron in the transverse plane $P_1/(eB) = \tilde{x}_2^{\rm e} - \tilde{x}_2^{p}$, so that the "potential" factor in Eq. (47) may be expressed in the following interesting form:

$$\frac{\alpha}{(x_0^{\rm e} - x_0^{\rm p})^2 - (x_3^{\rm e} - x_3^{\rm p})^2 - (\tilde{x}_2^{\rm e} - \tilde{x}_2^{\rm p})^2}, \tag{49}$$

(cf. the corresponding form of the Coulomb potential in the semirelativistic treatment of the Bethe-Salpeter equation in [10,11]—the difference between the potentials in [10,11] lies within the accuracy of the adiabatic approximation). The appearance of $P_1^2$ in the potential determines, in the end, the energy spectrum dependence upon the (pseudo)-momentum of the two-particle system across the magnetic field, like in [10,11,19].

We shall need Eq. (47) in a more convenient form. First, transcribe it as

$$\left(\mathrm{i}\overrightarrow{\hat{\partial}}_{\|} - \frac{\hat{P}_{\|}}{2} - m\right)\eta_{0,p_1^{\rm e};0,p_1^{\rm p}}(t,z)\left(-\mathrm{i}\overleftarrow{\hat{\partial}}_{\|} - \frac{\hat{P}_{\|}}{2} - m\right)^{\rm T} = \frac{2\alpha\pi^{-1}}{z^2 + \frac{P_1^2}{(eB)^2} - t^2}\sum_{i=0,3} g_{ii}\gamma_i\,\eta_{0,p_1^{\rm e};0,p_1^{\rm p}}(t,z)\,\gamma_i^{\rm T}. \tag{50}$$

Here the superscript T denotes the transposition. With the help of the relation $\gamma_i^{\mathrm{T}} = -C^{-1}\gamma_i C$, with $C$ being the charge conjugation matrix, $C^2 = 1$, and the anticommutation relations $[\gamma_i, \gamma_5]_+ = 0$, $\gamma_5^2 = -1$, we may write for a new Bethe-Salpeter amplitude $\Theta(t, z)$, defined as

$$\Theta(t, z) = \eta_{0,p_1^e;0,p_1^p}(t, z)\, C\gamma_5, \tag{51}$$

the equation

$$\left(\mathrm{i}\vec{\hat{\partial}}_\parallel - \frac{\hat{P}_\parallel}{2} - m\right)\Theta(t, z)\left(-\mathrm{i}\overleftarrow{\hat{\partial}}_\parallel - \frac{\hat{P}_\parallel}{2} - m\right) = \frac{2\alpha\pi^{-1}}{z^2 + \frac{P_1^2}{(eB)^2} - t^2}\sum_{i=0,3} g_{ii}\gamma_i\Theta(t, z)\gamma_i. \tag{52}$$

The unknown function $\Theta$ here is a $4 \times 4$ matrix, which contains, as a matter of fact, only four independent components. In order to correspondingly reduce the number of equations in the set (52), one should note that the $\gamma$-matrix algebra in two-dimensional space-time should have only four basic elements. In accordance with this fact, only the matrices $\gamma_{0,3}$ are involved in (52), indeed. Together with the matrix $\gamma_0\gamma_3$ and the unit matrix $I$ they form the basis, since $\gamma_{0,3} \cdot \gamma_0\gamma_3 = \gamma_{3,0}$, $\gamma_0^2 = -\gamma_3^2 = (\gamma_0\gamma_3)^2 = 1$, $[\gamma_0, \gamma_3]_+ = [\gamma_{0,3}, \gamma_0\gamma_3]_+ = 0$. Using this algebra and the general representation for the solution

$$\Theta = aI + b\gamma_0 + c\gamma_3 + d\gamma_0\gamma_3, \tag{53}$$

one readily obtains a closed set of four first-order differential equations for the four functions $a$, $b$, $c$, $d$ of $t$ and $z$. The same set will be obtained, if one replaces in Eqs. (52) and (53) the $4 \times 4$ matrices by the Pauli matrices (6), subject to the same algebraic relations, according to, for instance, the rule $\gamma_0 \Rightarrow \sigma_3$, $\gamma_3 \Rightarrow \mathrm{i}\sigma_2$, $\gamma_0\gamma_3 \Rightarrow \sigma_1$. Then Eq. (47) becomes a matrix equation,

$$\left(\mathrm{i}\vec{\partial}_t\sigma_3 + \vec{\partial}_z\sigma_2 - \frac{P_0}{2}\sigma_3 + \frac{P_3}{2}\mathrm{i}\sigma_2 - m\right)\vartheta(t, z) \times \left(-\mathrm{i}\overleftarrow{\partial}_t\sigma_3 - \overleftarrow{\partial}_z\sigma_2 - \frac{P_0}{2}\sigma_3 + \frac{P_3}{2}\mathrm{i}\sigma_2 - m\right) = \frac{2\alpha\pi^{-1}}{z^2 + \frac{P_1^2}{(eB)^2} - t^2}[\sigma_3\vartheta(t, z)\sigma_3 + \sigma_2\vartheta(t, z)\sigma_2] \tag{54}$$

for a $2 \times 2$ matrix $\vartheta$,

$$\vartheta = aI + b\sigma_3 + \mathrm{i}c\sigma_2 + d\sigma_1. \tag{55}$$

Here $I$ is the $2 \times 2$ unit matrix, and four functions $a$, $b$, $c$, $d$ are the same as in (53).

The two-dimensional Bethe-Salpeter equation obtained in the limit of a very strong magnetic field may be extended to include an external electric field, parallel to the magnetic field and not supposed to be strong, $E \ll B$ (see [20]). The presence of this electric field does not create an obstacle to the dimensional reduction. The corresponding generalized equation may be useful for considering an ionization of atoms and decay of positronium in a strong magnetic field under the action of an electric field.

## III. ULTRARELATIVISTIC REGIME IN A MAGNETIC FIELD

In the ultrarelativistic limit, where the positronium mass is completely compensated for by the mass defect, $P_0 = 0$, for the positronium at rest along the direction of the magnetic field $P_3 = 0$, the most general relativistic-covariant form of the solution (53) is

$$\Theta = I\Phi + \hat{\partial}_\parallel\Phi_2 + \gamma_0\gamma_3\Phi_3. \tag{56}$$

The point is that $\gamma_0\gamma_3$ is invariant under the Lorentz rotations in the plane $(t, z)$. Substituting this into (52) with $P_0 = P_3 = 0$ we get a separate equation for the singlet component of (56),

$$(-\Box_2 + m^2)\Phi(t, z) = \frac{4\alpha\pi^{-1}\Phi(t, z)}{z^2 + \frac{P_1^2}{(eB)^2} - t^2} \tag{57}$$

and the set of equations

$$\begin{aligned} (\Box_2 + m^2)\Phi_3(t, z) &= -\frac{4\alpha\pi^{-1}\Phi_3(t, z)}{z^2 + \frac{P_1^2}{(eB)^2} - t^2}, \\ (-\Box_2 + m^2)\partial_t\Phi_2 + 2m\mathrm{i}\partial_z\Phi_3 &= 0, \\ (-\Box_2 + m^2)\partial_z\Phi_2 + 2m\mathrm{i}\partial_t\Phi_3 &= 0 \end{aligned} \tag{58}$$

for the other two components. Here $\Box_2 = -\partial^2/\partial t^2 + \partial^2/\partial z^2$ is the Laplace operator in two dimensions. Note the "tachyonic" sign in front of it in the first equation (58).

Let us differentiate the second equation in (58) over $z$ and the third one over $t$ and subtract the results from each other. In this way we get that $\Box_2\Phi_3 = 0$. This, however, contradicts the first equation in (58) if $\Phi_3 \neq 0$. Therefore, only $\Phi_3 = 0$ is possible. Then, the last two equations in (58) are satisfied, provided that $(-\Box_2 + m^2)\Phi_2 = 0$. We shall concentrate on Eq. (57) in what follows.

The longitudinal momentum along $x_1$, or the distance between the orbital centers along $x_2$, plays the role of the effective photon mass and a singular potential regularizer in Eq. (57). The ground state corresponds to the zero value of the transverse total momentum $P_1 = 0$. In this case Eq. (57) for the Ritus transform of the Bethe-Salpeter amplitude finally becomes

$$(-\Box_2 + m^2)\Phi(t, z) = \frac{4\alpha\Phi(t, z)}{\pi(z^2 - t^2)}. \tag{59}$$

We consider now the consequences of the fall-down-onto-the-center phenomenon [13] inherent to Eq. (59), formally valid for an infinite magnetic field, and the alterations introduced by its finiteness.

### A. Fall-down onto the center in the Bethe-Salpeter amplitude for a strong magnetic field

In the most symmetrical case, when the wave function $\Phi(x) = \Phi(s)$ does not depend on the hyperbolic angle $\phi$ in the spacelike region of the two-dimensional Minkowski space, $t = s\sinh\phi$, $z = s\cosh\phi$, $s = \sqrt{z^2 - t^2}$, Eq. (59) becomes the Bessel differential equation

$$-\frac{d^2\Phi}{ds^2} - \frac{1}{s}\frac{d\Phi}{ds} + m^2\Phi = \frac{4\alpha}{\pi s^2}\Phi. \tag{60}$$

It follows from the derivation procedure in Sec. II that this equation is valid within the interval

$$\frac{1}{\sqrt{eB}} \ll s_0 \le s \le \infty, \tag{61}$$

where the lower bound $s_0$ depends on the external magnetic field—it should be larger than the Larmour radius $L_B = (eB)^{-1/2}$ and tend to zero together with it, as the magnetic field tends to infinity. The stronger the field, the closer to the origin $s = 0$ the interval of validity of this equation extends. If the magnetic field is not sufficiently strong, the lower bound $s_0$ falls beyond the region where the solution is mostly concentrated and the limiting form of the Bethe-Salpeter equation becomes inefficient, since it only relates to the asymptotic (large $s$) region, while the rest of the $s$ axis is served by a more complicated initial Bethe-Salpeter equation, not reducible to the two-dimensional form there. This is how the strength of the magnetic field participates—note that the coefficients of Eq. (60) do not contain it.

Solutions of (60) behave near the singular point $s = 0$ like $s^\sigma$, where

$$\sigma = \pm 2\sqrt{-\frac{\alpha}{\pi}}. \tag{62}$$

The fall-down onto the center [13] occurs if $\alpha > \alpha_{\rm cr} = 0$, i.e., for arbitrary small attraction, the genuine value $\alpha = 1/137$ included. This differs crucially from the case of zero magnetic field where $\alpha_{\rm cr} = \pi/8$ [14]. This difference is a purely geometrical consequence [20] of the dimensional reduction of the Minkowski space from (1,3) to (1,1).

In discussing the physical consequences of the falling to the center, we appeal to the approach recently developed by one of the present authors as applied to the Schrödinger equation with singular potential [21] and to the Dirac equation in the supercritical Coulomb field [22]. Within this approach the singular center looks like a black hole. The solutions of the differential equation that oscillate near the singularity point are treated as free particles emitted and absorbed by the singularity. This treatment becomes natural after the differential equation is written as the generalized eigenvalue problem with respect to the coupling constant. Its solutions make a (rigged) Hilbert space and are subject to orthonormality relations with a singular measure. This singularity makes it possible for the oscillating solutions to be normalized to $\delta$ functions, as free particle wave functions should be. The nontrivial, singular measure that appears in the definition of the scalar product of quantum states in the Hilbert space of quantum mechanics introduces the geometry of a black hole of nongravitational origin and the idea of horizon. The deviation from the standard quantum theory manifests itself in this approach only when particles are so close to one another that the mutual Coulomb field they are subjected to falls beyond the range, where the standard theory may be referred to as firmly established [22].

Following this theory we shall be using $s_0$ as the lower edge of the normalization box [21,22]. For doing this, it is necessary that $s_0$ be much smaller than the electron Compton length, $s_0 \ll m^{-1} \simeq 3.9 \times 10^{-11}$ cm, the only dimensional parameter in Eq. (60). In this case the asymptotic regime of small distances is achieved and the region $s < s_0$ beyond the normalization volume is left behind the event horizon and may not affect the problem. In this way the interval where the two-dimensional equations (47), (52), (57), and (59) and hence (60) are not valid and the space-time for charged particles remains four dimensional is excluded.

Alternatively, we might treat $s_0$ as the cutoff parameter. In this case we have had to extend Eq. (60) continuously to the region $0 \le s \le s_0$, simultaneously replacing the singularity $s^{-2}$ in it by a model function of $s$, nonsingular or less singular in the origin, say, a constant $s_0^{-2}$. In this approach the results are dependent on the choice of the model function which is intended to substitute for the lack of a treatable equation in that region. Besides, the limit $s_0 \to 0$ does not exist. The latter fact implies that the approach should become invalid for sufficiently small $s_0$, i.e., large $B$. Nevertheless, we shall also test the consequences of this approach later in this section to make sure that in our special problem the result is essentially unaffected.

### B. Maximum magnetic field

With the substitution $\Phi(s) = \Psi(s)/\sqrt{s}$ Eq. (60) acquires the standard form of a Schrödinger equation,

$$-\frac{d^2\Psi(s)}{ds^2} + \frac{-4\frac{\alpha}{\pi} - \frac{1}{4}}{s^2}\Psi(s) + m^2\Psi(s) = 0. \tag{63}$$

Equation (63) is valid in the interval

$$s_0 \le s \le \infty, \qquad s_0 \gg L_B = (eB)^{-1/2}. \tag{64}$$

Treating the applicability boundary $s_0$ of this equation as the lower edge of the normalization box, as discussed above, $s_0 \ll m^{-1}$, we impose the standing-wave boundary condition,

$$\Psi(s_0) = 0, \tag{65}$$

on the solution of (63)

$$\Psi(s) = \sqrt{s}\mathcal{K}_\nu(ms), \qquad \nu = \mathrm{i}2\sqrt{\alpha/\pi} \simeq 0.096\mathrm{i} \tag{66}$$

which decreases as $s \to \infty$. It behaves near the singular point $s = 0$ as

$$\left(\frac{s}{2}\right)^{1+\nu}\frac{1}{\Gamma(1+\nu)} - \left(\frac{s}{2}\right)^{1-\nu}\frac{1}{\Gamma(1-\nu)}. \tag{67}$$

Here the Euler $\Gamma$ functions appear. Starting with a certain small value of the argument $ms$, the McDonald function with imaginary index $\mathcal{K}_\nu(ms)$ (66) oscillates, as $s \to 0$, passing the zero value an infinite amount of times. Therefore, if $s_0$ is sufficiently small the standing-wave boundary condition (65), prescribed by the theory of Refs. [21,22], can be definitely satisfied. Keeping to the genuine value of the coupling constant $\alpha = 1/137$ ($\nu = 0.096\mathrm{i}$) one may ask, what is the largest possible value $s_0^{\max}$ of $s_0$, for which the boundary problem (63) and (65) can be solved? By demanding, in accordance with the validity condition (61) of Eqs. (60) and (63), that the value of $s_0^{\max}$ should exceed the Larmour radius,

$$s_0^{\max} \gg (eB)^{-1/2} \quad \text{or} \quad B \gg \frac{1}{e(s_0^{\max})^2}, \tag{68}$$

one establishes how large the magnetic field should be in order that the boundary problem might have a solution, in other words, that the point $P_0 = \mathbf{P} = 0$ might belong to the spectrum of bound states of the Bethe-Salpeter equation in its initial form (3).

One can use the asymptotic form of the McDonald function near zero to see that the boundary condition (65) is satisfied provided that

$$\left(\frac{ms_0}{2}\right)^{2\nu} = \frac{\Gamma(1+\nu)}{\Gamma^*(1-\nu)} \tag{69}$$

or

$$\nu \ln\frac{ms_0}{2} = \mathrm{i}\arg\Gamma(\nu+1) - \mathrm{i}\pi n, \qquad n = 0, \pm1, \pm2, \ldots \tag{70}$$

Since $|\nu|$ is small we may exploit the approximation $\Gamma(1+\nu) \simeq 1 - \nu C_\mathrm{E}$, where $C_\mathrm{E} = 0.577$ is the Euler constant, to get

$$\ln\left(\frac{ms_0}{2}\right) = -\frac{n}{2}\sqrt{\frac{\pi^3}{\alpha}} - C_\mathrm{E}, \qquad n = 1, 2, \ldots \tag{71}$$

We have expelled the nonpositive integers $n$ from here, since they would lead to the roots for $ms_0$ of the order of or larger than unity in contradiction to the adopted condition $s_0 \ll m^{-1}$. For such values, Eq. (67) is not valid. It may be checked that there are no other zeros of the McDonald function, apart from (71). The maximum value for $s_0$ is provided by $n = 1$. We finally get

$$\ln\left(\frac{ms_0^{\max}}{2}\right) = -\frac{1}{2}\sqrt{\frac{\pi^3}{\alpha}} - C_\mathrm{E} \quad \text{or}$$

$$s_0^{\max} = \frac{2}{m}\exp\left\{-\frac{1}{2}\sqrt{\frac{\pi^3}{\alpha}} - C_\mathrm{E}\right\} \simeq 10^{-14}\frac{1}{m}. \tag{72}$$

This is 14 orders of magnitude smaller than the Compton length $m^{-1} = 3.9 \times 10^{-11}$ cm and is about $10^{-25}$ cm. Now, in accordance with (68), if the magnetic field exceeds the maximum value of

$$B_{\max} = \frac{m^2}{4e}\exp\left\{\frac{\pi^{3/2}}{\sqrt{\alpha}} + 2C_\mathrm{E}\right\} \simeq 1.6 \times 10^{28} B_0, \tag{73}$$

the positronium ground state with the center-of-mass 4-momentum equal to zero appears. Here $B_0 = m^2/e \simeq 1.22 \times 10^{13}$ Heaviside-Lorentz units are the Schwinger critical field, or $B_0 = m^2c^3/e\hbar \simeq 4.4 \times 10^{13}$ G. The value of $B_{\max}$ is $\sim 10^{42}$ G, that is, a few orders of magnitude smaller than the highest magnetic field in the vicinity of superconductive cosmic strings [7]. Excited positronium states with $P_1 = 0$, $n > 1$ may also reach the spectral point $P_0 = P_3 = 0$, but this occurs for magnetic fields, tens of orders of magnitude larger than (73)—to be found in the same way from (71) with $n = 2, 3, \ldots$.

The ultrarelativistic state $P_\mu = 0$ has the internal structure of what was called a *confined* state in [21,22], i.e. the one whose wave function behaves as a standing-wave combination of free particles incoming from behind the lower edge of the normalization box and then totally reflected back to this edge. It decreases as $\exp(-ms)$ at large distances like the wave function of a bound state. The effective "Bohr radius," i.e. the value of $s$ that provides the maximum to the wave function (66), is $s_{\max} = 0.17\ m^{-1}$ (this fact is established by numerical analysis). This is certainly much less than the standard Bohr radius $a_0 = (\alpha m)^{-1}$. Taken at the level of 1/2 of its maximum value, the wave function is concentrated within the limits $0.006\ m^{-1} < s < 1.1\ m^{-1}$. But the effective region occupied by the confined state is still much closer to $s = 0$. The point is that the probability density of the confined state is the wave function squared *weighted with the measure* $s^{-2}\mathrm{d}s$ *singular in the origin* [21,22] and is hence concentrated near the edge of the normalization box $s_0 \simeq 10^{-25}$ cm, and not in the vicinity of the maximum of the wave function. The electric fields at such distances inside the positronium atom are about $10^{43}$ V/cm. Certainly, there is no evidence that the standard quantum theory should be valid under such conditions. This remark gives us the freedom of applying the theory presented in Refs. [21,22].

A relation like (73) between a fermion mass and the magnetic field is present in [23]. There, however, a different problem is studied and, correspondingly, a different meaning is attributed to that relation: it expresses the mass acquired dynamically by a primarily massless fermion in terms of the magnetic field applied to it. The mass generation is described by the homogeneous Bethe-Salpeter equation, whose solution is understood [23,24] as the wave function of the Goldstone boson corresponding to the spontaneous breaking of the chiral symmetry inherent in the massless QED. It is claimed, moreover, that the resulting relation between the magnetic field and the acquired mass is independent of the choice of the gauge for the photon propagator. The equations of Ref. [23] may well be exploited, formally, in our problem of the compensation of the positronium rest mass by the mass defect in a magnetic field, too, and the resulting expression may be used for determining the corresponding magnetic field, provided that the electron mass $m$ is substituted for the acquired mass $m_{\rm dyn}$ of [23]. There is, however, an important discrepancy in numerical coefficients in the characteristic exponential between (73) and the corresponding formula in [23]: the latter contains $\exp\{\pi^{3/2}/(2\alpha)^{1/2}\}$ in place of $\exp\{\pi^{3/2}/\alpha^{1/2} + 2C_{\rm E}\}$ in (73) and its direct use would lead to a more favorable estimate of the maximum value of the magnetic field, $2.6 \times 10^{19} B_0$, than (73). Although the basic mechanisms, the dimensional reduction and falling to the center, acting here and in [23], are essentially the same, the procedures are very much different, and the origin of the discrepancy remains unclear. Later, in [25], the authors revised their relation in favor of a different approximation. Supposedly, the revised relation may also be of use in the problem of the maximum magnetic field dealt with here.

It is interesting to compare the value (73) with the analogous value, obtained earlier by the present authors (see p. 393 of Ref. [10]) by extrapolating the nonrelativistic result for the positronium binding energy in a magnetic field to the extreme relativistic region:

$$B_{\max}|_{\rm nonrel} = \frac{\alpha^2 m^2}{e} \exp\left\{\frac{2\sqrt{2}}{\alpha}\right\} \simeq 10^{164} B_0. \tag{74}$$

Such is the magnetic field that makes the binding energy of the ground state equal to $-2m$. (This is worth comparing with the magnetic field, estimated [26] as $\alpha^2 \exp(2/\alpha) B_0$, that makes the mass defect of the nonrelativistic hydrogen atom comparable with the electron rest mass). We see that the relativistically enhanced attraction has resulted in a drastically lower value of the maximum magnetic field. Note the difference in the character of the essential non-analyticity with respect to the coupling constant: it is $\exp(\pi\sqrt{\pi}/\sqrt{\alpha})$ in (73) and $\exp(2\sqrt{2}/\alpha)$ in (74). Another effect of relativistic enhancement is that within the semi-relativistic treatment of the Bethe-Salpeter equation [9–11], as well as within the one using the Schrödinger equation [1], only the lowest level could acquire unlimited negative energy with the growth of the magnetic field, whereas according to (71) in our fully relativistic treatment all excited levels with $n > 1$ are subjected to the falling to the center and can reach, in turn, the point $P_\parallel = 0$.

Let us see now how the result (73) is altered if the cutoff procedure of Ref. [13] is used. Consider Eq. (63) in the domain $s_0 < s < \infty$, but replace it with another equation,

$$-\frac{{\rm d}^2\Psi_0(s)}{{\rm d}s^2} - \frac{\frac{4\alpha}{\pi} + \frac{1}{4}}{s_0^2}\Psi_0(s) + m^2\Psi_0(s) = 0 \tag{75}$$

in the domain $0 < s < s_0$. The singular potential is replaced by a constant near the origin in (75). Demand, in place of (65), that $\Psi_0(0) = 0$, $[\Psi_0'(s_0)/\Psi_0(s_0)) = (\Psi'(s_0)/\Psi(s_0))]$. Then, the result (73) should be multiplied by the factor

$$\exp\left\{-\frac{2}{\sqrt{\frac{4\alpha}{\pi} + \frac{1}{4}}\cot(\frac{4\alpha}{\pi} + \frac{1}{4}) - \frac{1}{2}}\right\}. \tag{76}$$

This expression may be approximately taken at the value $\alpha = 0$. Thus, the result (73) is only modified by a factor of $\exp(-4/3) \simeq 0.25$. Generally, the estimate of the limiting magnetic field (73) is practically nonsensitive to the way of the cutoff, in other words, to any solution of the initial equation inside the region $0 < s < s_0$, where the magnetic field does not dominate over the mutual attraction force between the electron and positron. This fact takes place because the term $(\pi^{3/2}/\sqrt{\alpha}) \simeq 65$, singular in $\alpha$, is prevailing in (73), the details of the behavior of the wave function close to the origin $s = 0$ being unessential against its background.

### C. Radiative corrections

#### *1. Vacuum polarization*

We should answer the question of whether the effects of vacuum polarization in a strong magnetic field may or may not screen the interaction between the electron and positron in such a way as to prevent the falling to the center in the positronium atom. It is clear *a priori* that, no matter how strong the magnetic field is, the ultraviolet singularity dominates over its influence in the photon propagator, if the interval sufficiently close to the light cone is involved. Therefore, there is a competition between the magnetic field and this characteristic interval, which is, in our problem, the Larmour radius that itself depends on the magnetic field. We have to consider the outcome of this competition.

To include the effect of the vacuum polarization we should use the photon propagator in a magnetic field, whose influence is realized via the vacuum polarization radiative corrections, instead of its free form (24) used above. The photon propagator in a constant and homogeneous magnetic field has the following approximation-independent structure [27–31]:

$$D_{ij}(x) = \frac{1}{(2\pi)^4}\int \exp({\rm i}kx) D_{ij}(k){\rm d}^4k, \qquad i, j = 0, 1, 2, 3, \tag{77}$$

$$D_{ij}(k) = \sum_{a=1}^{4} D_a(k) \frac{b_i^{(a)} b_j^{(a)}}{(b^{(a)})^2},$$
$$D_a(k) = \begin{cases} -[k^2 + \kappa_a(k)]^{-1}, & a = 1, 2, 3, \\ \text{arbitrary}, & a = 4. \end{cases} \quad (78)$$

Here $b^{(a)}$ and $\kappa^a$ are four eigenvectors and four eigenvalues of the polarization operator $\Pi_{ij}$,

$$\Pi_i{}^j b_j^{(a)} = \kappa_a(k) b_i^{(a)}. \quad (79)$$

The eigenvectors are known in the final form:

$$b_i^{(1)} = (F^2k)_i k^2 - k_i(kF^2k), \qquad b_i^{(2)} = (\tilde{F}k)_i,$$
$$b_i^{(3)} = (Fk)_i, \qquad b_i^{(4)} = k_i, \quad (80)$$

where $F$, $\tilde{F}$, and $F^2$ are the external electromagnetic field tensor, its dual, and its tensor squared, respectively, contracted with the photon 4-momentum $k$. On the contrary, the eigenvalues $\kappa_{1,2,3}(k)$ are generally unknown—subject to approximate calculations—scalar functions of two Lorentz-invariant combinations of the momentum and the field, which, in the special frame, where the external electromagnetic field is given by (7), are $k_0^2 - k_3^2$ and $k_1^2 + k_2^2 \equiv k_\perp^2$. The eigenvalue $\kappa_4$ is equal to zero as a trivial consequence of the transversity $\Pi_i{}^j k_j = 0$ of the polarization operator. The eigenvectors (80) with $a = 1, 2, 3$ are 4-potentials of the three photon modes, while the dispersion laws of the corresponding electromagnetic eigenwaves are obtained by equalizing the denominators of $D_a(k)$ in (78) with zero. In the special frame, the eigenvectors (80), up to normalizations, are

$$b_i^{(1)} = k^2 \begin{pmatrix} 0 \\ k_1 \\ k_2 \\ 0 \end{pmatrix}_i + k_\perp^2 \begin{pmatrix} k_0 \\ k_1 \\ k_2 \\ k_3 \end{pmatrix}_i,$$
$$b_i^{(2)} = \begin{pmatrix} k_3 \\ 0 \\ 0 \\ k_0 \end{pmatrix}_i, \qquad b_i^{(3)} = \begin{pmatrix} 0 \\ k_2 \\ -k_1 \\ 0 \end{pmatrix}_i. \quad (81)$$

When calculated [27,28] within the one-loop approximation of the Furry picture (i.e. using exact Dirac propagators in the external magnetic field without radiative corrections) these eigenvalues have the following asymptotic behavior [29] (see also [28,30–32]) for large fields $eB \gg m^2$, $eB \gg |k_3^2 - k_0^2|$:

$$\kappa_1(k_0^2 - k_3^2, k_\perp^2) = -\frac{\alpha k^2}{3\pi}\left(\ln\frac{B}{B_0} - C - 1.21\right), \quad (82)$$

$$\kappa_2(k_0^2 - k_3^2, k_\perp^2) = \frac{\alpha B m^2 (k_0^2 - k_3^2)}{\pi B_0} \exp\left(-\frac{k_\perp^2}{2m^2}\frac{B_0}{B}\right)$$
$$\times \int_{-1}^{1} \frac{(1-\eta^2)\mathrm{d}\eta}{4m^2 - (k_0^2 - k_3^2)(1-\eta^2)}, \quad (83)$$

$$\kappa_3(k_0^2 - k_3^2, k_\perp^2) = -\frac{\alpha k^2}{3\pi}\left(\ln\frac{B}{B_0} - C\right) - \frac{\alpha}{3\pi}[0.21 k_\perp^2$$
$$- 1.21(k_0^2 - k_3^2)]. \quad (84)$$

$C = 0.577$ is the Euler constant. Equations (82) and (84) are accurate up to terms decreasing with $B$ like $(B_{\rm cr}/B) \times \ln(B/B_{\rm cr})$ and faster. Equation (83) is accurate up to terms logarithmically growing with $B$. In $\kappa_{1,3}$ we also took the limit $k_\perp^2 \ll (B/B_{\rm cr})m^2$, which is not the case for $\kappa_2$, wherein the factor $\exp(-k_\perp^2 B_0/2m^2 B)$ is kept different from unity. Although the components $\kappa_{1,2,3}$ contain the growing logarithm $\alpha \ln(B/B_0)$, the latter is yet small for the values of the magnetic field of the order of $B_{\max}$ (73). This is not the case for the linearly growing part of (83). The full asymptotic expansion of the polarization tensor for large magnetic fields—however, in the static limit and without spatial dispersion—is given in [33].

Let us inspect the contributions of the photon propagator (78) into the equation that should appear in place of (47). To match the diagonal form (24) corresponding to the Feynman gauge, we fix the gauge arbitrariness by choosing

$$D_4(k) = -[k^2 + \kappa_1(k)]^{-1}. \quad (85)$$

In the isotropic case where no magnetic field is present, all three nontrivial eigenvalues are the same, $\kappa_a(k) = \kappa(k)$, $a = 1, 2, 3$. Then, with the choice (85) in (78) the photon propagator in this limit becomes diagonal,

$$D_{ij}(k) = -\frac{1}{k^2 + \kappa(k)} \sum_{a=1}^{4} \frac{b_i^{(a)} b_j^{(a)}}{(b^{(a)})^2} = -\frac{g_{ij}}{k^2 + \kappa(k)}, \quad (86)$$

since the eigenvectors (79) or (81) make an orthogonal basis irrespective of whether the magnetic field is present or not.

In spite of the presence [15,29] of a term, linearly growing with the field in (83), the component $D_2$ does contribute in the limit of high fields into the right-hand side of an equation to replace (57), because the ultraviolet singularity at the distance of the Larmour radius from the light cone dominates. To see this note that the right-hand side of the analog of (47) should get the contribution from $D_2$:

$$\frac{1}{(2\pi)^4} \int \frac{[k_3\gamma_0 - k_0\gamma_3]_{\lambda\lambda^{\rm e}} [k_3\gamma_0 - k_0\gamma_3]_{\mu\lambda^{\rm p}}}{(k_0^2 - k_3^2)}$$
$$\times \frac{\exp[\mathrm{i}(kx)]}{k^2 + \kappa_2} \mathrm{d}^4 k. \quad (87)$$

After this is contracted with the unit matrix we get, for the corresponding contribution into the right-hand side of the equation to be written in place of Eq. (57), the expression

$$-\frac{1}{(2\pi)^4}\int \frac{\exp[i(kx)]}{k^2+\kappa_2}d^4k. \tag{88}$$

Once the Hermite functions (17) restrict $x_\perp$ in integrals like (29) and (32) to the region inside the Larmour radius, the region $k_\perp^2 \gg L_B^{-2} = m^2 B/B_0$ in the integral (88) is important. There, however, $k_2$ disappears due to the exponential factor in (83) and we are left with the contribution, the same as the one coming from the free photon propagator. Moreover, as the light-cone singularity is formed exclusively due to integration over near-infinite values of all four photon momentum components, Eq. (88) behaves like $1/x^2$, the same as (24), near the light cone $x^2 = 0$.

We need, however, to also estimate the contribution immediately close to this singularity. To this end, let us disregard the spatial dispersion of the dielectric constant in the transverse plane, i.e. take $\kappa_2$ at the value $k_\perp = 0$ in (88). By doing so we essentially underestimate the contribution of the mode-2 photon as a carrier of the electromagnetic interaction into the attraction force between the electron and positron near the light cone, because we keep the term linearly growing with the field in the denominator for large $k_\perp$, where it, in fact, disappears. This approximation does not affect the light-cone singularity, which remains $1/x^2$, but makes the screening correction to the singular part larger than it is. We shall see, nevertheless, that, even within this approximation, with the screening overestimated, the effect of the latter is small. Once our working domain is restricted to the intervals $z^2 - t^2$ much closer to the light cone than the Compton length, we may confine ourselves to the condition $|k_0^2 - k_3^2| \gg m^2$ in the integral (88). Then $\kappa_2$ (83) should be taken in (88) as

$$\kappa_2 = -\frac{2\alpha B m^2}{\pi B_0}. \tag{89}$$

Then (88) becomes the well-known expression for the free propagator of a massive particle with the mass squared $M^2 = -\kappa_2$. To avoid a possible misunderstanding, we emphasize that this mass should not be referred to as an effective photon mass [34]. In the adiabatic approximation of Sec. II C the dimensional reduction yields again the prescription to disregard the dependence on $x_\perp$ in it by setting $x_\perp = 0$, $x^2 = -s^2$. Then the contribution from (88) is

$$\frac{iM}{4\pi^2 s}\mathcal{K}_1(Ms) \simeq \frac{i}{4\pi^2 s^2}\left(1 - \frac{s^2M^2}{2}\left|\ln\frac{Ms}{2}\right|\right). \tag{90}$$

Here $\mathcal{K}_1$ is the McDonald function of order 1, and we have specified its asymptotic behavior near the point $s^2 = z^2 - t^2 = +0$. According to (72) near the lower edge of the normalization volume and with the magnetic field (73) the quantity $sM = (2\alpha/\pi)^{1/2}$ is 0.068, and hence the second term inside the brackets in (90) is only $-7.8\times 10^{-3}$. Therefore the screening effect, although overestimated, is still negligible, the contribution of $D_2$ making one-half of the full contribution of the free photon propagator considered above. The one half originates from the absence of the factor 2 that appeared above when $[\gamma_i]_{\lambda\lambda^e}[\gamma_i]_{\mu\lambda^p}$ in (47) was later contracted with the unity $I$ to lead to (57): $\sum_{i=0,3} g_{ii}\gamma_i\gamma_i = 2$. The other half comes from the contribution of $D_1$ and $D_4$.

The quantity $D_3$ contains only $k_\perp$ components that give rise to $[k^i\gamma_i]_{\lambda\lambda^e}[k^j\gamma_j]_{\mu\lambda^p}$, $i, j = 1, 2$, in an equation to appear in place of (32), and consequently contribute only to the nondiagonal in the Landau quantum numbers part of the Bethe-Salpeter equation like (40), that does not survive the limit of a high magnetic field. On the contrary, the contributions of $D_1$ and $D_4$ do go to the diagonal part. This occurs because these contain the components $k_0$ and $k_3$ carrying the matrices $\gamma_0$ and $\gamma_3$ that may lead to the term diagonal in Landau quantum numbers, as explained when passing from (39) to (46) and (47). It follows from (78), (81), and (85) that the common contribution from $D_1$ and $D_4$ in the (0,3) subspace is determined by the expression

$$\frac{(k_\perp^2)^2 k_i k_j}{(b^{(1)})^2} + \frac{k_i k_j}{k^2} = \frac{k_i k_j}{k_0^2 - k_3^2}, \qquad i, j = 0, 3. \tag{91}$$

Then the counterpart of (87) reads

$$-\frac{1}{(2\pi)^4}\int \frac{[k_0\gamma_0 - k_3\gamma_3]_{\lambda\lambda^e}[k_0\gamma_0 - k_3\gamma_3]_{\mu\lambda^p}}{(k_0^2 - k_3^2)} \times \frac{\exp[i(kx)]}{k^2+\kappa_1}d^4k, \tag{92}$$

and the counterpart of (88) becomes [the spatial dispersion across the magnetic field, i.e. the dependence of $\kappa_2$ upon $k_\perp^2$, being disregarded already in writing Eq. (82)]

$$-\frac{1}{\mathcal{A}(2\pi)^4}\int \frac{\exp[i(kx)]}{k^2}d^4k = \frac{1}{\mathcal{A} i 4\pi^2 x^2}, \tag{93}$$

where

$$\mathcal{A} = 1 - \frac{\alpha}{3\pi}\left(\ln\frac{B}{B_0} - C - 1.21\right) \tag{94}$$

in view of (82). For the fields as large as $B = B_{\max}$ (73) the number $\mathcal{A}$ is very close to unity: $\mathcal{A} = 1 - 0.04$. (Its difference from unity is the measure of the antiscreening effect of the running coupling constant $\alpha/\mathcal{A}$ for a large magnetic field due to the lack of asymptotic freedom in pure quantum electrodynamics).

We conclude that the vacuum polarization does not essentially affect the falling to the center or hence the estimate of the maximum magnetic field. This contradicts the prescription that $\alpha$ should be replaced by $\alpha/2$ in the expression for the latter. Such a prescription would result if we applied the corresponding conclusion from Ref. [23] to the problem under consideration. The point is that in

Ref. [23] the contribution of $D_2$ is completely disregarded for the reason that the term (89), linearly growing with the magnetic field, is in the denominator of $D_2$. We saw above that this cannot be done: $D_2$ essentially contributes to the falling-to-the-center asymptotic regime of $s \gg 10^{-11}m^{-1}$, where the probability to find the system is concentrated.

Gathering the results of the present consideration together, we conclude that the effect of the vacuum polarization leads, in the approximation where the spatial dispersion in the orthogonal direction is neglected, to the replacement of Eq. (47) by the following two-dimensional Bethe-Salpeter equation for a high magnetic field limit including effects of the vacuum polarization,

$$\begin{aligned}[\mathrm{i}\hat{\partial}_{\parallel}{}^{\mathrm{e}} - m]_{\lambda\beta}[\mathrm{i}\hat{\partial}_{\parallel}{}^{\mathrm{p}} - m]_{\mu\nu}[\chi_{0,p_1^{\mathrm{e}};0,p_1^{\mathrm{p}}}(x_{\parallel}^{\mathrm{e}}, x_{\parallel}^{\mathrm{p}})]_{\beta\nu} &= \frac{\alpha}{\pi}\Bigg\{\frac{[\mathrm{i}\hat{\partial}_{\parallel}]_{\lambda\beta}[\mathrm{i}\hat{\partial}_{\parallel}]_{\mu\nu}}{\mathcal{A}\Box_2}\frac{1}{z^2 + \frac{P_1^2}{(eB)^2} - t^2} \\ &+ \frac{[\gamma_0\mathrm{i}\partial_z + \gamma_3\mathrm{i}\partial_t]_{\lambda\beta}[\gamma_0\mathrm{i}\partial_z + \gamma_3\mathrm{i}\partial_t]_{\mu\nu}}{\Box_2} \\ &\times \frac{M\mathcal{K}_1(M(z^2 + \frac{P_1^2}{(eB)^2} - t^2)^{1/2})}{(z^2 + \frac{P_1^2}{(eB)^2} - t^2)^{1/2}}\Bigg\}[\chi_{0,p_1^{\mathrm{e}};0,p_1^{\mathrm{p}}}(x_{\parallel}^{\mathrm{e}}, x_{\parallel}^{\mathrm{p}})]_{\beta\nu}.\end{aligned} \tag{95}$$

Here the action of the derivatives over $t$ and $z$ does not extend beyond the braces, $\mathrm{i}\hat{\partial}_{\parallel} = \mathrm{i}\gamma_0\partial_t + \mathrm{i}\gamma_3\partial_z$, and $\Box_2 = \partial_z^2 - \partial_t^2$. Remember that $t = x_0^{\mathrm{e}} - x_0^{\mathrm{p}}$ and $z = x_3^{\mathrm{e}} - x_3^{\mathrm{p}}$. The equation that follows from (95) for the singlet component in place of (57) is

$$(-\Box_2 + m^2)\Phi(t,z) = \frac{2\alpha}{\pi}\Bigg\{\frac{1}{\mathcal{A}(z^2 + \frac{P_1^2}{(eB)^2} - t^2)} + \frac{M\mathcal{K}_1(M(z^2 + \frac{P_1^2}{(eB)^2} - t^2)^{1/2})}{(z^2 + \frac{P_1^2}{(eB)^2} - t^2)^{1/2}}\Bigg\}\Phi(t,z). \tag{96}$$

Finally, the Bessel equation (60) for the (1,1) rotationally invariant solution now becomes

$$-\frac{\mathrm{d}^2\Phi}{\mathrm{d}s^2} - \frac{1}{s}\frac{\mathrm{d}\Phi}{\mathrm{d}s} + m^2\Phi = \frac{2\alpha}{\pi s}\left(\frac{1}{s} + M\mathcal{K}_1(Ms)\right)\Phi. \tag{97}$$

We neglected the difference of $\mathcal{A}$ from unity.

### 2. Mass corrections

Mass radiative corrections should be taken into account by inserting the mass operator into the Dirac differential operators in the left-hand sides of the Bethe-Salpeter equation (3) or (47). We shall estimate now whether this may affect the above conclusions concerning the positronium mass compensation by the mass defect.

In a strong magnetic field the one-loop calculation of the electron mass operator leads to the so-called double-logarithm mass correction growing with the field $B$ as [35]

$$\tilde{m} = m\left(1 + \frac{\alpha}{4\pi}\ln^2\frac{B}{B_0}\right). \tag{98}$$

For $B \simeq B_{\max}$ the corrected mass is $\tilde{m} = 3.45m$. This implies that the mass annihilation due to the falling to the center is opposed by the radiative corrections and requires a field somewhat larger than (73). To determine its value, substitute $\tilde{m}$ (98) for $m$ and $L_B = (eB)^{-1/2}$ for $s_0$ into Eq. (71) with $n = 1$. The resulting equation for the maximum magnetic field, modified by the mass radiative corrections, $B_{\mathrm{corr}}$,

$$\left(1 + \frac{\alpha}{4\pi}\ln^2\frac{B_{\mathrm{corr}}}{B_0}\right)^2 = 4\frac{B_{\mathrm{corr}}}{B_0}\exp\left(-\sqrt{\frac{\pi^3}{\alpha}} + C_{\mathrm{E}}\right), \tag{99}$$

has the numerical solution $B_{\mathrm{corr}} \simeq 13B_{\max}$.

When going beyond the one-loop approximation by summing the rainbow diagrams, two different expressions for $\tilde{m}$ were obtained by different authors. Reference [36] reports

$$\tilde{m} = m\exp\left(\frac{\alpha}{4\pi}\ln^2\frac{B}{B_0}\right). \tag{100}$$

The use of this formula gives rise to an increase of the maximum value by 2 orders of magnitude, $B_{\mathrm{corr}} = 3.5 \times 10^2 B_{\max}$, whereas the use of the result of Ref. [37],

$$\tilde{m} = \frac{m}{\cos(\sqrt{\frac{\alpha}{2\pi}}\ln\frac{B}{B_0})}, \tag{101}$$

would leave the maximum value practically unchanged: $B_{\mathrm{corr}} = 1.5B_{\max}$. Finally, if the vacuum polarization is taken into account while summing the leading contributions to the large-field asymptotic behavior of the mass operator, the following result [38],

$$\tilde{m} = \frac{m}{1 - \frac{\alpha}{2\pi}(\ln\frac{\pi}{\alpha} - C_{\mathrm{E}})\ln\frac{B}{B_0}}, \tag{102}$$

is obtained, from where the double logarithm is absent due to the effect of the term (89) in the photon propagator when substituted into electron-photon loops. The use of (102) would result in $B_{\mathrm{corr}} = 3B_{\max}$.

Anyway, we see that the mass correction, increasing the maximum value $B_{\max}$ by at the most 2 orders of magnitude, is not essential, bearing in mind the huge values (73) of the latter. Moreover, based on the most recent results concerning the mass correction [38], we conclude that the latter do not affect the value of the hypercritical field obtained above (73) practically at all.

## IV. SUMMARY AND DISCUSSION

In this paper we have considered the system of two charged relativistic particles—especially the electron and positron—in interaction with each other, when placed in a strong constant and homogeneous magnetic field **B**. The Bethe-Salpeter equation in the ladder approximation in the Feynman gauge is used without exploiting any nonrelativistic assumption. We have derived the maximum two-dimensional form of the Bethe-Salpeter equation, when the magnetic field tends to infinity, with the help of expansion over the complete set of Ritus matrix eigenfunctions [17]. The latter accumulate the spatial and spinor dependence on the transverse-to-the-field degree of freedom. The Fourier-Ritus transform of the Bethe-Salpeter amplitude obeys an infinite chain of coupled differential equations that decouple in the limit of large $B$, so that we are left with one closed equation for the amplitude component with the Landau quantum numbers of the electron and positron both equal to zero, while the components with other values of Landau quantum numbers vanish in this limit. The resulting equation is a differential equation with respect to two variables that are the differences of the particle coordinates: along the time $t = x_0^{\rm e} - x_0^{\rm p}$ and along the magnetic field $z = x_3^{\rm e} - x_3^{\rm p}$. It contains only two Dirac matrices, $\gamma_0$ and $\gamma_3$, and can be alternatively written using $2 \times 2$ Pauli matrices. By introducing different masses, the resulting two-dimensional equation may be easily modified to also cover the case of a one-electron atom in a strong magnetic field and/or other pairs of charged particles.

It is worth noting that the two-dimensionality holds only with respect to the degrees of freedom of charged particles, while the photons remain four dimensional in the sense that the singularity of the photon propagator is determined by the inverse d'Alembertian operator in the four-dimensional, and not two-dimensional, Minkowski space. (Otherwise it would be weaker).

We have made sure that in the case under consideration the critical value of the electromagnetic coupling constant is zero, $\alpha_{\rm cr} = 0$, i.e., the falling to the center caused by the ultraviolet singularity of the photon propagator as a carrier of the interaction is present already for its genuine value $\alpha = 1/137$, in contrast to the no-magnetic-field case, where $\alpha_{\rm cr} > 1/137$. If the magnetic field is large, but finite, the dimensional reduction holds everywhere except a small neighborhood of the singular point $s = 0$, wherein the mutual interaction between the particles dominates over their interaction with the magnetic field. The dimensionality of the space-time in this neighborhood remains to be 4, and its size is determined by the Larmour radius $L_B = (eB)^{-1/2}$, which is zero in the limit $B = \infty$. The Larmour radius supplies the singular problem with a regularizing length. The larger the magnetic field, the smaller the regularizing length, and the deeper the level.

We have found the maximum magnetic field that provides the full compensation of the positronium rest mass by the binding energy, and the wave function of the corresponding state as a solution to the Bethe-Salpeter equation. This state is described in terms of the theory of the falling to the center, developed in [21,22], as a "confined" state, different from the usual bound state. The appeal to this theory is necessitated by the fact that the falling to the center draws the electron and positron so close together that the mutual field is so large that the standard treatment may become inadequate. The maximum value is estimated to be unaffected by the radiative corrections modifying the mass and polarization operators.

In spite of the huge value, expected to be present, perhaps, only in superconducting cosmic strings [7], the magnetic field magnitude obtained may be important for setting the limits of applicability of QED or presenting the maximum value of the magnetic field admissible within pure QED. The point is that, at this field, the energy gap separating the electron-positron system from the vacuum disappears. If the maximum magnetic field is exceeded, the restructuring of the vacuum should take place. The vacuum restructuring is typical of other problems—with or without the magnetic field—where the falling to the center takes place: the supercharged nucleus [39,40] and a moderately charged nucleus with a strong magnetic field [41]. This issue is, in a preliminary way, discussed in the two adjacent papers [20,42]. The formal mechanisms that realize the magnetic field instability and may lead to prevention of its further growth via the decay of the confined state found here require a further study and will be considered elsewhere.

## ACKNOWLEDGMENTS

The authors are grateful to V. Gusynin, V. Miransky, and N. Mikheev for valuable discussions. This work was supported by the Russian Foundation for Basic Research (Project No. 05-02-17217) and the President-of-Russia Programme (LSS-4401.2006.2), as well as by the Israel Science Foundation of the Israel Academy of Sciences and Humanities.

[1] R. Loudon, Am. J. Phys. **27**, 649 (1959); R. J. Elliot and R. Loudon, J. Phys. Chem. Solids **8**, 382 (1959); **15**, 196 (1960).
[2] M. A. Ruderman, Phys. Rev. Lett. **27**, 1306 (1971); D. Lai, Rev. Mod. Phys. **73**, 629 (2001).
[3] P. A. Sturrock, Astrophys. J. **164**, 529 (1971); M. A. Ruderman and P. G. Sutherland, Astrophys. J. **196**, 51 (1975); D. Eichler, M. Livio, T. Piran, and D. Schramm,

Nature (London) **340**, 126 (1989); T.R. White and A.P. Lightman, Astrophys. J. **340**, 1024 (1989); V.V. Usov and D.B. Melrose, Aust. J. Phys. **48**, 571 (1995); C. Thompson and R.C. Duncan, Mon. Not. R. Astron. Soc. **275**, 255 (1995); V.V. Usov, Phys. Rev. Lett. **80**, 230 (1998); M.G. Baring and A.K. Harding, Astrophys. J. **547**, 929 (2001).

[4] R.N. Manchester, G.B. Hobbs, A. Teoh, and M. Hobbs, Astron. J. **129**, 1993 (2005).

[5] F.C. Michel, *Theory of Neutron Star Magnetospheres* (University of Chicago Press, Chicago, 1991).

[6] A.E. Shabad and V.V. Usov, Astrophys. Space Sci. **117**, 309 (1985); H. Herold, H. Ruder, and G. Wunner, Phys. Rev. Lett. **54**, 1452 (1985); V.V. Usov and D.B. Melrose, Astrophys. J. **464**, 306 (1996).

[7] E. Witten, Nucl. Phys. **B249**, 557 (1985); E.M. Chudnovsky, G.B. Field, D.N. Spergel, and A. Vilenkin, Phys. Rev. D **34**, 944 (1986).

[8] V. Berezinsky, B. Hnatyk, and A. Vilenkin, Phys. Rev. D **64**, 043004 (2001).

[9] L.B. Leinson and V.N. Oraevskii, Sov. J. Nucl. Phys. **42**, 245 (1985); Phys. Lett. B **165**, 422 (1985).

[10] A.E. Shabad and V.V. Usov, Astrophys. Space Sci. **128**, 377 (1986).

[11] L.B. Leinson and A. Pérez, J. High Energy Phys. 11 (2000) 039.

[12] A.E. Shabad and V.V. Usov, Phys. Rev. Lett. **96**, 180401 (2006).

[13] L.D. Landau and E.M. Lifshits, *Quantum Mechanics* (Pergamon Press, Oxford, 1991).

[14] In this point the situation is different from the no-magnetic-field case, where the falling to the center holds for nonphysical values of the fine structure constant, exceeding a positive critical value, $\alpha_{\rm cr} \lesssim 1$, and gives rise to the Goldstein solution [J.S. Goldstein, Phys. Rev. **91**, 1516 (1953)] of the Bethe-Salpeter equations, which also corresponds to the shrinking of the energy gap between electrons and positrons. See the review articles [24,43] and our recent publication [20].

[15] Yu.M. Loskutov and V.V. Skobelev, Phys. Lett. A **56**, 151 (1976).

[16] S.S. Schweber, *An Introduction to Relativistic Quantum Field Theory* (Row, Peterson and Co., Evanston, IL and Elmsford, NY, 1961).

[17] V.I. Ritus, Zh. Eksp. Teor. Fiz. **75**, 1560 (1978) [Sov. Phys. JETP **48**, 778 (1978)]; Trudy Fiz. Inst. im. P. N. Lebedeva, Akad. Nauk. SSSR **168**, 52 (1986). Our definition of the matrix eigenfunctions differs from that of Ritus in that the longitudinal degrees of freedom are not included and the factor $\exp(\mathrm{i}p_1x_1)$ is separated.

[18] Henceforth, if the superscript e or p is omitted, the corresponding equations relate both to electrons and positrons.

[19] D. Lai and E.E. Salpeter, Phys. Rev. A **52**, 2611 (1995).

[20] A.E. Shabad and V.V. Usov, hep-th/0512236.

[21] A.E. Shabad, hep-th/0403177.

[22] A.E. Shabad, J. Phys. A **38**, 7419 (2005).

[23] V.P. Gusynin, V.A. Miransky, and I.A. Shovkovy, Phys. Rev. D **52**, 4747 (1995); Nucl. Phys. **B462**, 249 (1996); C.N. Leung, Y.J. Ng, and A.W. Ackley, Phys. Rev. D **54**, 4181 (1996).

[24] P.I. Fomin, V.P. Gusynin, V.A. Miransky, and Yu.A. Sitenko, Riv. Nuovo Cimento **6**, numero 5 (1983).

[25] V.P. Gusynin, V.A. Miransky, and I.A. Shovkovy, Phys. Rev. Lett. **83**, 1291 (1999); Nucl. Phys.**B563**, 361 (1999); C.N. Leung and S.-Y. Wang, hep-ph/0510066.

[26] R.C. Duncan, astro-ph/0002442.

[27] I.A. Batalin and A.E. Shabad, Zh. Eksp. Teor. Fiz. **60**, 894 (1971) [Sov. Phys. JETP **33**, 483 (1971)]; A.E. Shabad, Ann. Phys. (N.Y.) **90**, 166 (1975).

[28] A.E. Shabad, *Polarization of the Vacuum and a Quantum Relativistic Gas in an External Field* (Nova Science Publishers, New York, 1991); Trudy Fiz. Inst. im. P. N. Lebedeva, Akad. Nauk. SSSR **192**, 5 (1988).

[29] A.E. Shabad, Kratkie Soobshchenia po Fizike, Sov. Phys. Lebedev Inst. Reps. **3**, 13 (1976).

[30] W. Dittrich and M. Reuter, *Effective Lagrangians in Quantum Electrodynamics* (Springer-Verlag, Berlin, 1985).

[31] A. Kuznetsov and N. Mikheev, *Electroweak Processes in External Electromagnetic Fields* (Springer, New York, Berlin, Heidelberg, 2004).

[32] A.E. Shabad, Zh. Eksp. Teor. Fiz. **125**, 210 (2004) [JETP **98**, 186 (2004)]. Note the difference in the signs in front of $k^2$ due to a different metric convention used here.

[33] J.S. Heyl and L. Hernquist, J. Phys. A **30**, 6485 (1997).

[34] The mass of the photon defined as its rest energy is equal to zero, corresponding to the fact that the point $k_0 = k_3 = k_1 = k_2 = 0$ always belongs to a solution of the dispersion equations $k^2 + \kappa_a(k) = 0$ due to gauge invariance. The "mass" $M$ appears in the ultraviolet, and not the infrared, regime. Massive solutions to the dispersion equations may also exist, but these are not photons. See a discussion of these issues in [32].

[35] B. Jancovici, Phys. Rev. **187**, 2275 (1969); R.G. Newton, Phys. Rev. **96**, 523 (1954); Phys. Rev. D **3**, 626 (1971); V.N. Baier, V.M. Katkov, and V.M. Strakhovenko, Zh. Eksp. Teor. Fiz. **67**, 67 (1974); Yad. Fiz. **24**, 379 (1976).

[36] Yu.M. Loskutov and V.V. Skobelev, Teor. Mat. Fiz. **48**, 44 (1981).

[37] V.P. Gusynin and A.V. Smilga, Phys. Lett. B **450**, 267 (1999).

[38] A.V. Kuznetsov, N.V. Mikheev, and M.V. Osipov, Mod. Phys. Lett. A **17**, 231 (2002).

[39] Ya.B. Zel'dovich and V.S. Popov, Usp. Fiz. Nauk **105**, 403 (1971) [Sov. Phys. Usp. **14**, 673 (1972)].

[40] W. Greiner and J. Reinhardt, *Quantum Electrodynamics* (Springer-Verlag, Berlin, 1992).

[41] V.N. Oraevskii, A.I. Rez, and V.B. Semikoz, Zh. Eksp. Teor. Fiz. **72**, 821 (1977) [Sov. Phys. JETP **45**, 428 (1977)].

[42] A.E. Shabad and V.V. Usov, astro-ph/0601542.

[43] N. Setô, Prog. Theor. Phys. Suppl. **95**, 25 (1988).