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Non-Abelian semilocal strings in $\mathcal{N} = 2$ supersymmetric QCD

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We consider a benchmark bulk theory in four dimensions: $\mathcal{N} = 2$ supersymmetric QCD with the gauge group U(N) and N_f flavors of fundamental matter hypermultiplets (quarks). The nature of the Bogomol'nyi-Prasad-Sommerfield (BPS) strings in this benchmark theory crucially depends on N_f. If $N_f \geq N$ and all quark masses are equal, it supports non-Abelian BPS strings which have internal (orientational) moduli. If $N_f > N$ these strings become semilocal, developing additional moduli ρ related to (unlimited) variations of their transverse size. Using the U(2) gauge group with $N_f = 3$, 4 as an example, we derive an effective low-energy theory on the (two-dimensional) string world sheet. Our derivation is field theoretic, direct and explicit: we first analyze the Bogomol'nyi equations for string-geometry solitons, suggest an *ansatz*, and solve it at large ρ. Then we use this solution to obtain the world-sheet theory. In the semiclassical limit our result confirms the Hanany-Tong conjecture, which rests on brane-based arguments, that the world-sheet theory is an $\mathcal{N} = 2$ supersymmetric U(1) gauge theory with N positively and $N_e = N_f - N$ negatively charged matter multiplets and the Fayet-Iliopoulos term determined by the four-dimensional coupling constant. We conclude that the Higgs branch of this model is not lifted by quantum effects. As a result, such strings cannot confine. Our analysis of infrared effects, not seen in the Hanany-Tong consideration, shows that, in fact, the derivative expansion can make sense only provided that the theory under consideration is regularized in the infrared, e.g. by the quark mass differences. The world-sheet action discussed in this paper becomes a *bona fide* low-energy effective action only if $\Delta m_{AB} \neq 0$.

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I. INTRODUCTION

The recent discovery in certain supersymmetric gauge theories of Bogomol'nyi-Prasad-Sommerfield (BPS)-saturated solitons that can be interpreted as *non-Abelian strings* [1–4] has led to a number of exciting developments [5–18]: from confined monopoles to non-Abelian boojums, from enhanced supersymmetry on the world sheet to possible applications in cosmic strings and beyond. The above-mentioned non-Abelian strings are characterized by non-Abelian moduli and present a generalization of Z_N strings [19–25] which, in turn, generalize the famous Abrikosov-Nielsen-Olesen (ANO) strings [26].

Other topological defects with stringy geometry—-sigma-model lumps—have been known for decades. For instance, instantons in two-dimensional CP($N - 1$) models, lifted to four dimensions, provide probably the most clear-cut example of such lumps. The topological defects that interpolate between the ANO strings and lumps are called *semilocal strings* (for a review see [27]). While nontrivial topology behind the ANO strings is related to $\pi_1(\mathrm{U}(1))$, the sigma-model lumps are supported by $\pi_2(\mathcal{T})$ where $\mathcal{T}$ is the target space of the sigma model at hand. Unlike the ANO strings whose size in the transverse plane is fixed, that of the semilocal string is a modulus. Both the ANO strings and lumps can be studied in a unified manner in the framework of gauged linear sigma models with a judiciously chosen Higgs potential. The special potentials which are required here are due to the Fayet-Iliopoulos (FI) terms [28]. In an appropriate limit, the $\mathcal{N} = 2$ supersymmetric gauge theories with the Fayet-Iliopoulos term develop Higgs branches. In the low-energy limit, effective theories on the Higgs branches become nonlinear sigma models whose target spaces have hyper-Kähler geometry.

In view of the recent developments, it is natural to raise the question of *non-Abelian semilocal* strings, in particular, how they emerge as BPS-saturated solitons in $\mathcal{N} = 2$ supersymmetric QCD which was previously shown to support non-Abelian local strings. This question was first addressed in [1] (see also [4]) where it was argued, on the basis of a brane-based analysis, that the effective low-energy theory on the world sheet of such string is given by a particular two-dimensional sigma model with a non-compact target space presenting an example of certain special manifolds called *toric varieties*. For an illuminating discussion see [29].

Needless to say, it is highly desirable to verify the Hanany-Tong conjecture by a straightforward derivation of the world-sheet theory for the non-Abelian semilocal string within the field-theoretic framework, starting from $\mathcal{N} = 2$ SQCD in the bulk. Here we carry out this derivation in a certain limit, and demonstrate that in this limit the result of our direct field-theoretic calculation coincides with the Hanany-Tong formula. *En route*, we clarify subtle aspects associated with the infrared (IR) regularization of the zero modes. These aspects, crucial for maintaining the BPS nature of the solution, were only mentioned in passing in [1].

We explain why, in spite of the fact that the (nonvanishing) tension of the semilocal strings under consideration is exactly determined by the central charge of the underlying theory, the semilocal strings do not lead to linear confinement in the conventional sense of this word.

The paper is organized as follows. In Sec. II we present our bulk model: $\mathcal{N} = 2$ supersymmetric QCD with the gauge group U(2) and N_f flavors of fundamental matter fields. In Sec. III we review Abelian semilocal string solutions. Section IV demonstrates that their formation leads to deconfinement. In Sec. V we find BPS solutions for non-Abelian semilocal strings. Section VI is devoted to the effective theory on the world sheet of the non-Abelian semilocal string. In Sec. VII we consider this theory in the semiclassical limit, and in Sec. VIII compare this theory to the one conjectured in [1,4]. Quantum effects in the world-sheet theory are discussed in Sec. IX. Section X presents our conclusions.

II. THE BULK MODEL

The local non-Abelian strings[1] were discovered in $\mathcal{N} = 2$ supersymmetric QCD with the gauge group $SU(N) \times U(1)$ and N flavors of the matter fields. Then, if the mass terms for all matter fields are the same, the theory possesses a global flavor $SU(N)$ symmetry, and the symmetry breaking pattern is

$$SU(N)_{\rm gauge} \times SU(N)_{\rm flavor} \to SU(N)_{\rm diagonal}.$$

To get semilocal non-Abelian strings, all we have to do is extend the matter sector of this theory. Namely, we must introduce N_e extra flavors, so that the total number of flavors $N_f = N + N_e$. Below, we will briefly summarize the main features of our basic model, limiting ourselves to $N = 2$, for simplicity. Generalization to $N > 2$ is straightforward.

Thus, we will consider $\mathcal{N} = 2$ supersymmetric QCD with the $SU(2) \times U(1)$ gauge group and N_f flavors of fundamental hypermultiplets, which we call "quarks." If $N_f = 4$ (i.e. $N_e = 2$), the β function of the theory vanishes, while the further increase of N_f leads to the loss of asymptotic freedom. Thus, we will limit ourselves to $N_e = 1$ and 2 (i.e. $N_f = 3$ and 4).

Our theory is perturbed by the FI term of the U(1) gauge factor with the FI parameter ξ. This parameter sets the scale of massive states in the theory, as well as the scale of the string tension. Indeed, the $(\frac{1}{2}, \frac{1}{2})$ central charge of the theory is

$$\{Q^I_\alpha, \bar{Q}^J_{\dot\alpha}\} = \delta^{IJ} 2(P_{\alpha\dot\alpha} + Z_{\alpha\dot\alpha}), \qquad I, J = 1, 2,$$
$$Z_\mu = \xi \int d^3x \varepsilon_{0\mu\nu\rho} \partial^\nu A^\rho, \tag{1}$$

where A^ρ is the U(1) gauge field. Thus, the tension of the minimal BPS string is

$$T = 2\pi\xi.$$

The field content of $SU(2) \times U(1)$ $\mathcal{N} = 2$ SQCD with N_f flavors is as follows. The $\mathcal{N} = 2$ vector multiplet consists of the U(1) gauge fields A_μ, SU(2) gauge field A^a_μ, (here $a = 1, 2, 3$), their Weyl fermion superpartners $(\lambda^1_\alpha, \lambda^2_\alpha)$ and $(\lambda^{1a}_\alpha, \lambda^{2a}_\alpha)$, and complex scalar fields a and a^a, the latter in the adjoint of SU(2). The spinorial index of λ's runs over $\alpha = 1, 2$. In this sector, the global $SU(2)_R$ symmetry inherent to the model at hand manifests itself through rotations $\lambda^1 \leftrightarrow \lambda^2$.

The quark multiplets of the $SU(2) \times U(1)$ theory consist of the complex scalar fields q^{kA} and $\tilde{q}_{Ak}$ (squarks) and the Weyl fermions ψ^{kA} and $\tilde{\psi}_{Ak}$, all in the fundamental representation of the SU(2) gauge group. Here $k = 1, 2$ is the color index while A is the flavor index,

$$A = 1, \ldots, N_f, \qquad N_f = 3 \quad \text{or} \quad 4.$$

Note that the scalars q^{kA} and $\bar{\tilde{q}}^{kA} \equiv \overline{\tilde{q}_{Ak}}$ form a doublet under the action of the global $SU(2)_R$ group.

The bosonic part of our $SU(2) \times U(1)$ theory (in Euclidean space) has the form

$$S = \int d^4x \bigg[\frac{1}{4g_2^2} (F^a_{\mu\nu})^2 + \frac{1}{4g_1^2} (F_{\mu\nu})^2 + \frac{1}{g_2^2} |D_\mu a^a|^2 + \frac{1}{g_1^2} |\partial_\mu a|^2 + |\nabla_\mu q^A|^2 + |\nabla_\mu \bar{\tilde{q}}^A|^2 + V(q^A, \tilde{q}_A, a^a, a) \bigg]. \tag{2}$$

Here D_μ is the covariant derivative in the adjoint representation of SU(2), while

$$\nabla_\mu = \partial_\mu - \frac{i}{2} A_\mu - i A^a_\mu \frac{\tau^a}{2}, \tag{3}$$

where we suppress the color SU(2) indices, and τ^a are the SU(2) Pauli matrices. The coupling constants g_1 and g_2 correspond to the U(1) and SU(2) sectors, respectively. With our conventions, the U(1) charges of the fundamental matter fields are $\pm 1/2$. The potential $V(q^A, \tilde{q}_A, a^a, a)$ in the Lagrangian (2) is a sum of D and F terms,

[1]According to the generally accepted—albeit rather confusing—terminology, local as opposed to semilocal strings are those whose transverse size is fixed. In this sense the ANO string is local.

$$
\begin{aligned}
V(q^A, \tilde{q}_A, a^a, a) = {} & \frac{g_2^2}{2}\left(\frac{1}{g_2^2}\varepsilon^{abc}\bar{a}^b a^c + \bar{q}_A\frac{\tau^a}{2}q^A - \tilde{q}_A\frac{\tau^a}{2}\bar{\tilde{q}}^A\right)^2 \\
& + \frac{g_1^2}{8}(\bar{q}_A q^A - \tilde{q}_A\bar{\tilde{q}}^A - 2\xi)^2 \\
& + \frac{g_2^2}{2}|\tilde{q}_A\tau^a q^A|^2 + \frac{g_1^2}{2}|\tilde{q}_A q^A|^2 \\
& + \frac{1}{2}\sum_{A=1}^{N_f}\left\{\left|\left(a + \sqrt{2}m_A + \tau^a a^a\right)q^A\right|^2\right. \\
& \left. + \left|\left(a + \sqrt{2}m_A + \tau^a a^a\right)\bar{\tilde{q}}_A\right|^2\right\}, \qquad (4)
\end{aligned}
$$

where the sum over the repeated flavor indices A is implied. For the time being, we keep all N_f mass terms m_A distinct.

The first and second lines represent D terms, the third line represents the F_a terms, while the fourth and the fifth lines represent the squark F terms. Note that the FI term does not break $\mathcal{N}=2$ supersymmetry [30,31].

The Fayet-Iliopoulos term triggers the spontaneous breaking of the gauge symmetry, forcing the squark fields to develop vacuum expectation values (VEV's). If all quark mass terms are different, there are $N_f(N_f-1)/2$ isolated vacua in which a pair of quark flavors develop VEV's [6]. We denote these vacua as AB vacua, where A and B are the quark flavors which develop VEV's.

Consider, say, the 12 vacuum. Up to gauge rotations, the VEV's of the squark fields can be chosen as

$$
\begin{aligned}
&\langle q^{kA}\rangle = \sqrt{\xi}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \qquad \langle\bar{\tilde{q}}^{kA}\rangle = 0, \\
&\langle q_e^{kB}\rangle = \langle\bar{\tilde{q}}_e^{kB}\rangle = 0, \qquad k = 1, 2, \\
&A = 1, 2, \qquad B = 3, \ldots, N_f,
\end{aligned} \qquad (5)
$$

where we arrange the squark fields of the first two flavors in a 2×2 matrix q, while q_e denotes extra quark flavors (the subscript e is for extra). The VEV's of the adjoint fields are given by

$$
\langle a^3\rangle = -\frac{m_1 - m_2}{\sqrt{2}}, \qquad \langle a\rangle = -\frac{m_1 + m_2}{\sqrt{2}}. \qquad (6)
$$

Consider first the case $m_1 = m_2$. The color-flavor locked form of the quark VEV's in Eq. (5) and the absence of VEV's of the adjoint scalar a^a in Eq. (6) results in the fact that, while the theory is fully Higgsed, a diagonal $SU(2)_{C+F}$ survives as a global symmetry. This symmetry involves a global gauge transformation together with a flavor rotation of the first two flavors. Say, for quark fields, it acts as

$$
q \to UqU^{-1}, \qquad q_e \to Uq_e, \qquad (7)
$$

where the global gauge rotation acts from the left while the flavor rotation acts from the right. It is clear that the vacuum (5) is invariant under this transformation. This invariance, a particular case of the Bardakci-Halpern mechanism [32], leads to the emergence [2] of orientational zero modes of the Z_2 strings in the model (2).

If $N_f = 3$, the SU(2) part of the gauge group is asymptotically free, implying generation of a dynamical scale Λ. In the infrared, if descent to Λ was uninterrupted, the gauge coupling g_2^2 would explode at this scale. Moreover, strong coupling effects in the SU(2) subsector at the scale Λ would break the SU(2) subgroup through the Seiberg-Witten mechanism [33]. Since we want to stay at weak coupling, we assume that $\sqrt{\xi} \gg \Lambda$, so that the running of the SU(2) coupling is frozen by the squark condensation at a small value,

$$
\frac{8\pi^2}{g_2^2} = \ln\frac{\sqrt{\xi}}{\Lambda} + \cdots \gg 1. \qquad (8)
$$

If $N_f = 4$, the SU(2) sector of the theory is conformally invariant, and hence the coupling g_2 does not run. In this case we also assume that

$$
\frac{8\pi^2}{g_2^2} \gg 1. \qquad (9)
$$

Now let us discuss the mass spectrum in the theory (2). Since both U(1) and SU(2) gauge groups are broken by squark condensation, all gauge bosons become massive. From (2) we get for the U(1) gauge boson

$$
m_\gamma = g_1\sqrt{\xi} \qquad (10)
$$

while three gauge bosons of the SU(2) group acquire the same mass,

$$
m_W = g_2\sqrt{\xi}. \qquad (11)
$$

It is not difficult to see from (4) that the adjoint fields a and a^a as well as the components of the quark matrix q acquire the same masses as the corresponding gauge bosons. Altogether we have one long $\mathcal{N}=2$ multiplet (eight bosonic + eight fermionic states) with the mass (10) and three long $\mathcal{N}=2$ multiplets with the mass (11). If the extra quark masses are different from $m_{1,2}$, the extra quark flavors acquire masses determined by the mass differences $\Delta m_{AB} = m_A - m_B$. The extra flavors become massless in the limit $\Delta m_{AB} \to 0$, which we will consider momentarily.

If all quark mass terms are equal, then the $N_f(N_f-1)/2$ isolated vacua we had in the case of unequal mass terms coalesce; a Higgs branch develops from the common root whose location on the Coulomb branch is given by Eq. (6) with $m_1 = m_2$. The dimension of this branch is $8N_e$; see [25,34]. The Higgs branch is noncompact and has a hyper-Kähler geometry [34,35]. At a generic point on the Higgs branch the BPS-saturated string solutions do not exist [36]; strings become non-BPS if we move along noncompact directions [37]. However, the Higgs branch has a compact base manifold defined by the condition

$$\tilde{q}_{Ak} = 0, \qquad A = 1, \ldots, N_f. \tag{12}$$

The dimension of this manifold is $4N_e$, half as much as the total dimension of the Higgs branch. The real dimension of the base manifold is 4 for $N_f = 3$ and 8 for $N_f = 4$. The BPS-saturated string solutions exist on the base manifold of the Higgs branch; therefore, the vacua belonging to the base manifold are our prime focus.

The base of the Higgs branch can be generated by flavor rotations of the 12-vacuum (5). For $N_f = 3$, the flavor rotations generate the manifold

$$\frac{SU(3)}{SU(2)_{C+F} \times U(1)} \tag{13}$$

where $SU(2)_{C+F}$ is a global unbroken color-flavor rotation which involves the first two flavors, while the U(1) factor stands for the unbroken U(1) flavor rotation of the third flavor. Dimension of this quotient is 4, indeed. For $N_f = 4$, the base of the Higgs branch is isomorphic to

$$\frac{SU(4)}{SU(2)_{C+F} \times U(2)} \tag{14}$$

where the U(2) factor stands for flavor rotations of the third and fourth flavors left unbroken by (5). Dimension of this quotient is 8, as was expected.

III. ABELIAN SEMILOCAL STRINGS

The flux tube (string) solutions on the Higgs branches (which are typical for multiflavor theories) usually are not conventional ANO strings, but, rather, semilocal strings (see [27] for a review). Here we give a brief introduction to semilocal strings in a simplified nonsupersymmetric environment in the U(1) model.

As was mentioned, the semilocal string interpolates between the ANO string and the two-dimensional sigma-model instanton lifted to four dimensions (this is referred to as the lump). The semilocal string possesses an additional zero mode associated with the string's transverse size ρ. At $\rho \to 0$ we have the ANO string while at $\rho \to \infty$ it becomes a lump. At nonzero $\rho \neq 0$ the profile functions of the semilocal string fall off at infinity as inverse powers of the distance, instead of the exponential falloff characteristic of ANO strings at $\rho = 0$. This leads to a dramatic physical effect—semilocal strings, in contradistinction to the ANO strings, do not support linear confinement (see below).

The simplest model where the semilocal strings appear is the Abelian Higgs model with *two* complex flavors,

$$S_{\rm AH} = \int d^4x \left\{ \frac{1}{4g^2} F_{\mu\nu}^2 + |\nabla_\mu q^A|^2 + \frac{g^2}{8}(|q^A|^2 - \xi)^2 \right\}. \tag{15}$$

Here $A = 1, 2$ is the flavor index. The model contains only bosonic fields; it is not supersymmetric. The scalar potential in Eq. (15) is inspired by supersymmetric models with the Fayet-Iliopoulos term [28]. The covariant derivative is defined as

$$\nabla_\mu = \partial_\mu - \frac{i}{2} A_\mu,$$

so that the electric charge of both quarks is 1/2.

If $\xi > 0$, the scalar fields develop VEV's breaking the U(1) gauge group. The photon field gets Higgsed, and gets a mass, together with one real scalar. For the particular choice of the quartic coupling presented in Eq. (15) this scalar has the same mass as the photon, since our toy model (15) is a bosonic reduction of an $\mathcal{N} = 1$ supersymmetric theory in which the vortices are BPS saturated. Two other scalars remain massless.

The topological reason for the existence of the ANO vortices is that $\pi_1[U(1)] = Z$. On the other hand, we can go to the low-energy limit in (15) assuming that $m_\gamma \to \infty$ and integrating out the massive photon and the real massive scalar field. This will lead us to a four-dimensional sigma model on the vacuum manifold

$$|q^1|^2 + |q^2|^2 = \xi.$$

This vacuum manifold has dimension $4 - 1 - 1 = 2$, where we subtract one real condition and one gauge phase. (One can always choose the gauge in which, say, q^1 is real.) The target space of the sigma model represents the two-dimensional sphere S_2. Thus, the low-energy limit of the theory (15) is the O(3) sigma model. Now recall that

$$\pi_2[S_2] = \pi_1[U(1)] = Z.$$

This is the topological reason for the existence of instantons in the two-dimensional O(3) sigma model. Lifted to four dimensions they become stringlike objects (lumps).

So, now the question is, what is the relation between the ANO flux tubes of scalar QED (15) and the lumps of the O(3) sigma model? It is clear that the model (15) supports the ANO strings. Say, if we put the second flavor field $q^2 = 0$, this model reduces to the standard framework for the critical ANO strings.

However, it turns out (see [38]) that the ANO solution in the model at hand has a zero mode associated with exciting the second flavor. This zero mode is parametrized by a complex parameter ρ where $|\rho|$ plays the role of the transverse size of the string while the phase of ρ describes a U(1) rotation angle in O(3). To see that this zero mode indeed occurs let us examine the solution. To this end we will modify the standard parametrization [26] for the ANO string, including the second flavor,

$$q^1(x) = \phi(r)e^{i\alpha}, \qquad q^2(x) = \chi(r),$$
$$A_i(x) = -2\epsilon_{ij}\frac{x_j}{r^2}[1 - f(r)], \qquad i, j = 1, 2, \tag{16}$$

where r and α are polar coordinates in the perpendicular (1,2) plane. We assume that the string is aligned along the

x_3 axis. Note that the second flavor does *not* wind at infinity. Therefore, its boundary condition at infinity is $\chi(\infty) = 0$, while at $r = 0$ the function χ need not vanish. The boundary conditions for other profile functions are

$$\phi(0) = 0, \qquad f(0) = 1, \qquad \phi(\infty) = \sqrt{\xi}, \qquad f(\infty) = 0. \tag{17}$$

These boundary conditions ensure that $|q^A|^2 \to \xi$ at infinity, while the string carries one unit of the magnetic flux and has a finite tension. The first-order Bogomol'nyi equations [39] for the profile functions take the form

$$\begin{aligned} r\frac{\mathrm{d}}{\mathrm{d}r}\phi(r) - f(r)\phi(r) &= 0, \\ r\frac{\mathrm{d}}{\mathrm{d}r}\chi(r) - (f(r) - 1)\chi(r) &= 0, \\ -\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}f(r) + \frac{g^2}{4}\{\phi^2(r) + \chi^2(r) - \xi\} &= 0. \end{aligned} \tag{18}$$

The ANO string solution implies that $\chi = 0$. In fact, the second equation in (18) can be solved in the general form,

$$\chi = \frac{\rho}{r}\phi, \tag{19}$$

expressing χ in terms of ϕ and an arbitrary complex parameter ρ. If we set $\rho = 0$, the second flavor profile function indeed vanishes. However, at $\rho \neq 0$ it does not.

The solution to Eqs. (18) at $\rho \neq 0$ is very different [38,40] from that for the ANO string. It has a long-range power falloff at infinity for all profile functions. In particular, in the limit of a very large transverse size of the string, $\rho \gg 1/g\sqrt{\xi}$, the solution has the form

$$\phi(r) = \sqrt{\xi}\frac{r}{\sqrt{r^2 + |\rho|^2}}, \qquad \chi(r) = \sqrt{\xi}\frac{\rho}{\sqrt{r^2 + |\rho|^2}}, \qquad f = \frac{|\rho|^2}{r^2 + |\rho|^2}. \tag{20}$$

This solution certainly has the same tension as the ANO string,

$$T = 2\pi\xi. \tag{21}$$

Equation (20) is valid at distances $r \gg 1/g\sqrt{\xi}$. Examining Eq. (20) we see that the scalar fields in this solution lie on the vacuum manifold $|q^A|^2 = \xi$ at any r as long as these expressions are valid. That is not the case for the ANO string. Inside the ANO string the scalar fields tend to zero; they approach the vacuum point only at $r \to \infty$.

The fact that $|q^A|^2 = \xi$ at any r means that we can relate the solution (20) to the O(3) sigma-model lump. To this end we use the standard relation between the O(3) and CP(1) model variables,

$$\frac{1}{\xi}\bar{q}_A(\tau_3)^A_B q^B = \frac{1 - |w|^2}{1 + |w|^2}, \qquad \frac{1}{\xi}\bar{q}_A(\tau_1)^A_B q^B = 2\frac{\mathrm{Re}w}{1 + |w|^2}, \qquad \frac{1}{\xi}\bar{q}_A(\tau_2)^A_B q^B = 2\frac{\mathrm{Im}w}{1 + |w|^2}, \tag{22}$$

where $\tau_{1,2,3}$ are flavor Pauli matrices. With this substitution, the low-energy limit of the action (15) reduces to that of the following O(3) sigma model:

$$S_{\mathrm{eff}} = \xi\int d^4x\frac{|\partial_\mu w|^2}{(1 + |w|^2)^2}, \tag{23}$$

with ξ playing the role of the coupling constant. In this model the standard lump solution centered at the origin takes the form

$$w_{\mathrm{lump}} = \frac{\rho}{x_1 + ix_2}, \tag{24}$$

where the complex modulus ρ is associated with the lump's size. Reexpressing this solution in terms of the quark fields through (22) we recover the solution (20). This is a direct and transparent demonstration of the fact that the semilocal string in the limit of large ρ is described by the lump solution of the O(3) sigma model.

IV. SEMILOCAL STRINGS AND CONFINEMENT

The semilocal strings discussed in Sec. III are BPS saturated. As was mentioned, their tension $T = 2\pi\xi$ irrespective of the value of ρ. At first sight it might seem that they must support linear confinement of monopoles, much in the same way as the ANO strings. The transverse size of the ANO string is $\sim 1/g\sqrt{\xi}$; if the string length $L \gg g\sqrt{\xi}$, the energy of this configuration is

$$V(L) = TL. \tag{25}$$

This linear potential ensures confinement of monopoles. Needless to say, if $L \ll g\sqrt{\xi}$, there is no linear potential.

For semilocal strings, the transverse size is a modulus. However, the adequate formulation of the problem is as follows. Assume we have a monopole-antimonopole pair separated by a distance L. Then the string to which the (anti)monopoles are attached has length L. If L is finite, the collective coordinate ρ loses its moduli status. At small ρ a slightly negative mode develops, since it is energetically favorable to increase ρ. This instability in ρ will be regulated by the string length parameter L itself. In other words, the transverse size of the finite-length semilocal string will be stabilized at $\rho \sim L$.

Clearly, the problem becomes three dimensional. The monopole flux is not trapped now inside a narrow flux tube. Instead, it is freely spread over a large three-dimensional volume of size $\sim L^3$. This produces a Coulomb-type potential between the probe monopole and antimonopole,

$$V(L) \sim 1/L, \tag{26}$$

up to possible logarithms. The energy of this configuration is lower than the one of the stringy configuration (25); therefore, it is energetically favored. The semilocal string increases its size with L and effectively disintegrates, which leads to a Coulomb-type interaction. It should be added that lattice studies confirm [41] that the semilocal string thickness tends to increase upon small perturbations. The formation of semilocal strings on the Higgs branches, replacing the ANO strings existing when we deal with isolated vacua, leads to a dramatic physical effect—deconfinement.

The above argument—that for the finite-length semilocal string it is energetically favorable to increase its transverse size up to $\rho \sim L$, producing a Coulomb-type interaction—is admittedly heuristic. At the moment, we cannot give a decisive proof based on analytic considerations, since the finite-length semilocal string is not BPS saturated. Apparently, dedicated numerical studies are needed. We believe that a tachyonic instability in the ρ mode (at finite fixed L) can be detected in this way. Such an investigation presents a subject of an independent research project.

Below we turn to non-Abelian semilocal strings in the theory (2) with $N_f = 3, 4$ to find out whether or not the size modulus ρ is lifted in quantum theory after taking into account a strong coupling of the corresponding zero mode to interacting orientational zero modes.

V. NON-ABELIAN SEMILOCAL STRINGS

If the above material can be viewed, in a sense, as an extended introduction, we now turn to construction and analysis of the semilocal non-Abelian strings in earnest.

Local non-Abelian BPS-saturated strings were found in $\mathcal{N} = 2$ QCD with the gauge group SU(N) × U(1) in [1–4]. As was mentioned, their crucial feature is the occurrence of orientational zero modes associated with rotation of the magnetic flux inside the SU(N) group. The key ingredient in the construction of non-Abelian strings is the presence of an unbroken global non-Abelian color-flavor subgroup [$SU(2)_{C+F}$ in the $N = 2$ case considered here]; see Eq. (7). This symmetry is broken, however, on the string solution. The Goldstone modes associated with the above breaking become orientational zero modes of the non-Abelian string.

For isolated vacua in $\mathcal{N} = 2$ QCD with the gauge group SU(N) × U(1) and $N_f = N$, the non-Abelian string solutions were explicitly found in [2]. It was done in two steps. First, a Z_N string solution was obtained. Then, rotations from $SU(N)_{C+F}$ were applied to this solution, producing a family of solutions parametrized by the orientational moduli. Now, we will generalize the procedure of [2] to cover the case of the semilocal strings in the theory (2). We will consider $N = 2$ and $N_e = 1$ and 2.

We will focus on string solutions on the base of the Higgs branch defined by the condition (12); hence, we assume that $\tilde{q} = 0$ on the string solution. For the ANO strings, the adjoint fields a and a^a played no role in the solution provided that $m_1 = m_2$. Assuming that the same is true for the semilocal strings we can simplify our theory by dropping the adjoint field (in addition to $\tilde{q}$) from the action (2).

Then the Bogomol'nyi completion [39] of the action leads to the following first-order equations:

$$F_3^{*a} + \frac{g_2^2}{2}(\bar{q}_A \tau^a q^A) = 0, \qquad a = 1, 2, 3;$$
$$F_3^* + \frac{g_1^2}{2}(|q^A|^2 - 2\xi) = 0; \qquad (\nabla_1 + i\nabla_2)q^A = 0, \tag{27}$$

where

$$F_m^* = \tfrac{1}{2}\varepsilon_{mnk}F_{nk}, \qquad m, n, k = 1, 2, 3 \tag{28}$$

(see Ref. [2]).

The minimal or elementary Z_2 string emerges when the first flavor has the unit winding number while the second flavor does not wind at all, to be referred to as the (1,0) string. Extra flavors have vanishing VEV's and cannot wind; see Eq. (5). Needless to say, there is another Z_2 string solution in which the second flavor has the unit winding number while the first flavor does not wind. This is called the (0,1) string. Together, they form a set of two Z_2 strings.

The conventional Abelian string forces both flavors to have the unit winding number; therefore, it must be viewed as the (1,1) string [2,3]. Its magnetic flux and tension are twice as large as those of the Z_2 strings. Consider for definiteness the (1,0) string. To find an appropriate solution to first-order equations (27) we modify the *ansatz* (16) as follows (see also [2]):

$$q(x) = \begin{pmatrix} e^{i\alpha}\phi_1(r) & 0 \\ 0 & \phi_2(r) \end{pmatrix}, \quad q_e(x) = \begin{pmatrix} \chi_1(r) & 0 \\ 0 & \chi_2(r) \end{pmatrix},$$
$$A_i^3(x) = -\varepsilon_{ij}\frac{x_j}{r^2}(1 - f_3(r)), \quad i, j = 1, 2, \tag{29}$$
$$A_i(x) = -\varepsilon_{ij}\frac{x_j}{r^2}(1 - f(r)),$$

where the real profile functions ϕ_1, ϕ_2 and complex functions χ_1, χ_2 for the scalar fields, as well as f_3, f for the gauge fields, depend only on r. The above *ansatz* refers to $N_f = 4$. The case $N_f = 3$ can be readily obtained from (29) by truncating the 2 × 2 matrix for q_e, replacing it by a two-component column with the entries

$$\begin{pmatrix} \chi_1 \\ 0 \end{pmatrix}.$$

Applying this *ansatz*, one can rearrange the first-order equations (27) in the form

$$\begin{gathered} r\frac{d}{dr}\phi_1(r) - \frac{1}{2}(f(r) + f_3(r))\phi_1(r) = 0, \qquad r\frac{d}{dr}\phi_2(r) - \frac{1}{2}(f(r) - f_3(r))\phi_2(r) = 0, \\ r\frac{d}{dr}\chi_1(r) - \frac{1}{2}(f(r) + f_3(r) - 2)\chi_1(r) = 0, \qquad r\frac{d}{dr}\chi_2(r) - \frac{1}{2}(f(r) - f_3(r))\chi_2(r) = 0, \\ -\frac{1}{r}\frac{d}{dr}f(r) + \frac{g_1^2}{2}[(\phi_1(r))^2 + (\phi_2(r))^2 + |\chi_1(r)|^2 + |\chi_2(r)|^2 - 2\xi] = 0, \\ -\frac{1}{r}\frac{d}{dr}f_3(r) + \frac{g_2^2}{2}[(\phi_1(r))^2 - (\phi_2(r))^2 + |\chi_1(r)|^2 - |\chi_2(r)|^2] = 0. \end{gathered} \tag{30}$$

Again, we hasten to add that these equations are written down for $N_f = 4$. If $N_f = 3$, one should put $\chi_2 = 0$ in Eq. (30).

Next, we need to specify the boundary conditions which would determine the profile functions in these equations. It is not difficult to see that one must require

$$f_3(0) = 1, \qquad f(0) = 1; \qquad f_3(\infty) = 0, \qquad f(\infty) = 0 \tag{31}$$

for the gauge fields, while the boundary conditions for the squark fields are

$$\begin{gathered} \phi_1(\infty) = \sqrt{\xi}, \qquad \phi_2(\infty) = \sqrt{\xi}, \qquad \phi_1(0) = 0 \\ \chi_1(\infty) = 0, \qquad \chi_2(\infty) = 0. \end{gathered} \tag{32}$$

Note that since the field ϕ_2 does not wind, it need not vanish at the origin, and it does not.

As in the Abelian case, the equations for χ's can be solved in the general form,

$$\chi_1 \sim \frac{1}{r}\phi_1, \qquad \chi_2 \sim \frac{1}{r}\phi_2. \tag{33}$$

Now let us consider the solutions to the first-order equations assuming the size ρ of the semilocal string to be very large,

$$|\rho| \gg \frac{1}{m_\gamma}, \ \frac{1}{m_W}, \tag{34}$$

where the masses of the gauge bosons are given in Eqs. (10) and (11) and we assume that $m_\gamma \sim m_W$. For the case $N_f = 3$ we have

$$q(x) = \begin{pmatrix} e^{i\alpha}\phi(r) & 0 \\ 0 & \sqrt{\xi} \end{pmatrix}, \qquad q_e(x) = \begin{pmatrix} \rho \\ 0 \end{pmatrix}\frac{\phi(r)}{r}, \qquad f_3 = f, \tag{35}$$

where the profile functions $\phi(r)$ and $f(r)$ are presented in Eq. (20). The modulus ρ is the complexified size of the lump. We see that the second flavor essentially plays no role in the solution since it is equal to its VEV everywhere, $|q^2|^2 \equiv \xi$. The first and the third flavors vary along the base of the Higgs branch,

$$|q^1|^2 + |q^2|^2 + |q^3|^2 = 2\xi \quad \text{at all} \quad r. \tag{36}$$

As in the Abelian case, this solution is, in fact, a lump of the low-energy four-dimensional sigma model on the base of the Higgs branch. Clearly, we can interchange the first and the second flavors, simultaneously flipping the sign of f_3, to get the other Z_2 string solution, namely, the (0,1) string.

If we have four flavors the solution takes the form

$$q(x) = \begin{pmatrix} e^{i\alpha}\phi(r) & 0 \\ 0 & \sqrt{\xi} \end{pmatrix}, \quad q_e(x) = \begin{pmatrix} \rho_1 & \rho_2 \\ 0 & 0 \end{pmatrix}\frac{\phi(r)}{r}, \quad f_3 = f, \tag{37}$$

where we use the possibility of arbitrary U(2) flavor rotations for the third and fourth flavors, which ensures the parametrization of the solution by two complex numbers ρ_1 and ρ_2. In this case, the size of the string ρ entering the profile functions (20) is given by

$$|\rho|^2 \equiv |\rho_1|^2 + |\rho_2|^2. \tag{38}$$

The elementary Z_2 strings give rise to the non-Abelian strings provided the condition $m_1 = m_2$ is satisfied [1–4]. Orientational moduli are generated corresponding to spontaneous breaking of the "flat" vacuum $SU(2)_{C+F}$ symmetry on the solutions (35) and (37). The color-flavor locked $SU(2)_{C+F}$ is broken down to U(1). This implies two orientational moduli [$2(N-1)$ for the bulk theory with the $SU(N) \times U(1)$ gauge group].

To obtain the semilocal non-Abelian string solution from the Z_2 string (35) and (37) we apply the diagonal color-flavor rotation (7), preserving the vacuum (5). To this end it is convenient to pass to the singular gauge where the scalar fields have no winding at infinity. They are aligned, while the string magnetic flux is saturated near the origin. In this gauge we have for the gauge fields

$$A_i^a(x) = S^a\varepsilon_{ij}\frac{x_j}{r^2}f_3(r), \qquad A_i(x) = \varepsilon_{ij}\frac{x_j}{r^2}f(r), \tag{39}$$

where S^a is a moduli vector defined as

$$S^a\tau^a = U\tau^3U^{-1}, \qquad a = 1, 2, 3, \qquad \sum_{a=1}^{3} S^aS^a = 1, \tag{40}$$

and U is a matrix from $SU(2)_{C+F}$. For $N_f = 3$, the quark fields have the form

$$q(x) = U\begin{pmatrix} \phi(r) & 0 \\ 0 & \sqrt{\xi} \end{pmatrix} U^{-1}, \qquad q_e(x) = U\begin{pmatrix} \rho \\ 0 \end{pmatrix} \frac{\phi(r)}{r}, \tag{41}$$

while for $N_f = 4$ we get

$$q(x) = U\begin{pmatrix} \phi(r) & 0 \\ 0 & \sqrt{\xi} \end{pmatrix} U^{-1}, \qquad q_e(x) = U\begin{pmatrix} \rho_1 & \rho_2 \\ 0 & 0 \end{pmatrix} \frac{\phi(r)}{r}. \tag{42}$$

VI. EFFECTIVE THEORY ON THE STRING WORLD SHEET

The fact that a proper *ansatz* for non-Abelian semilocal strings can be found and solved (in an explicit analytic form at $|\rho| \gg m_{\gamma,W}^{-1}$) is tantalizing by itself. This is not the end of the story, however. Our next task is to derive the world-sheet theory of moduli. In this section we will address this issue. To this end, we will promote the string moduli parameters to 2D fields on the string world sheet, assuming adiabatic dependence on the world-sheet coordinates. As usual, the translational moduli decouple; we will ignore them hereafter. We will focus on internal dynamics of the string at hand. For local non-Abelian strings occurring in the isolated vacua [such strings are obtained if $N_f = 2$ in the action (2)], the internal moduli are given by the orientation vector S^a. The low-energy world-sheet theory governing these orientational moduli is CP(1), with the action[2]

$$S_{N_f=2}^{(1+1)} = \frac{\beta}{2} \int dtdz (\partial_k S^a)^2, \tag{43}$$

where $k = 0, 3$, while the two-dimensional coupling constant β is related to the four-dimensional coupling as

$$\beta = \frac{2\pi}{g_2^2}, \tag{44}$$

at the scale $\sqrt{\xi}$ which determines the string's transverse size [2,3].

Now, we introduce one or two extra flavors, $N_f = 3, 4$, which triggers the conversion of the non-Abelian local string into a semilocal string. In addition to the orientational moduli S^a, the semilocal string acquires the size moduli ρ_i; see Eqs. (39), (41), and (42). Below we study interplay between the orientational and size moduli and derive an effective world-sheet theory—first, for the case of equal quark masses $\Delta m_{AB} = 0$ and, later, introducing small mass differences. The latter are crucial for infrared regularization.

[2]In this and many subsequent expressions for the world-sheet action we omit the fermion part.

A. The equal mass case

Assume that the orientational and size collective coordinates S^a and ρ_i are slow-varying functions of the string world-sheet coordinates x_k where $k = 0, 3$. Then these moduli become fields of a $(1+1)$-dimensional sigma model on the string world sheet. Since they parametrize the impact of the string zero modes, no potential term emerges. We must derive only the kinetic term.

To obtain the kinetic term, we substitute our solution, which depends on the moduli S^a and ρ_i in the action (2) assuming that the moduli fields acquire a dependence on the coordinates x_k via $S^a(x_k)$ and $\rho_i(x_k)$. As in the case of local non-Abelian strings [2,3], we will have to modify our string solution, extending our *ansatz* to include the $k = 0, 3$ components of the SU(2) gauge field,

$$A_k = -\varepsilon^{abc} S^b \partial_k S^c \omega(r), \qquad k = 0, 3, \tag{45}$$

where a new profile function $\omega(r)$ is introduced.

The function $\omega(r)$ in Eq. (45) is determined through a minimization procedure which generates ω's own equation of motion. Note that ω must satisfy the boundary conditions

$$\omega(\infty) = 0, \qquad \omega(0) = 1, \tag{46}$$

which ensure finiteness of the contribution to the action due to the gauge kinetic term $\mathrm{Tr}F_{ki}^2$.

Let us start from $N_f = 3$. Symmetry arguments forbid mixed kinetic terms involving both derivatives of S^a and derivatives of ρ. Hence, we can proceed in two steps. First, assume that S^a has an adiabatic dependence on the world-sheet coordinates while ρ is constant. This will give us a part of two-dimensional action containing the kinetic term for S^a. Then, we will assume, instead, that only the field ρ is x_k dependent, ignoring the x_k dependence of S^a. This will give rise to the kinetic term for ρ.

Substituting the string solution (39), (41), and (45) in the action (2) and ignoring the x_k dependence of ρ, we get the CP(1) model (43) with the coupling constant $\beta = \beta_S$ where now β_S is given by the following normalizing integral:

$$\beta_S = \frac{2\pi}{g_2^2} m_W \int_0^\infty rdr \left\{ \omega^2 + (1-\omega)\left(\frac{\phi}{\xi} - 1\right)^2 + (1-2\omega)\frac{|\rho|^2}{2r^2}\left(\frac{\phi}{\xi}\right)^2 \right\}. \tag{47}$$

Here we used the condition (34)—our semilocal string solution (39) and (41) [or (42)] was obtained in the limit of the large string size, $|\rho| \gg m_{\gamma,W}^{-1}$.

We will continue to heavily rely on the condition (34) in our studies of the effective theory on the string world sheet. In the opposite case of $\rho \lesssim 1/m_W$, the string reduces to a local non-Abelian string, which is well understood [1–4].

To determine the profile function $\omega(r)$, the functional (47) must be minimized with respect to ω. Varying (47) with respect to ω, one readily obtains

$$\omega = 1 - \frac{\phi}{\xi}. \tag{48}$$

This solution automatically satisfies the boundary conditions (46).

Substituting this solution back into the expression for the sigma-model coupling constant (47) one gets

$$\beta_S = \frac{2\pi}{g_2^2} m_W^2 |\rho|^2 \frac{1}{2} \ln\frac{L}{|\rho|}. \tag{49}$$

The integral over r in (47) is logarithmically divergent in the infrared. To regularize this divergence we introduce an infrared cutoff L in (49). Since this element is very important, we pause here to discuss it in more detail. The logarithmic divergence is due to long-range tails of the semilocal string which fall off as powers of r rather than exponentially. The fact that the ρ zero modes of semilocal strings [CP$(N-1)$ instantons] are logarithmically non-normalizable was noted long ago [40,42,43].

The problem is ill defined unless a physical IR regularization is provided. One possibility is to replace an infinite-length string by that of a finite length L. This will also regulate the spread of the string in the transverse plane [44]. However, at the same time, the problem loses its two-dimensional geometry and becomes essentially three dimensional. BPS saturation is also lost. In the logarithmic approximation, when $\ln|\rho|$ is considered to be a large number, while nonlogarithmic terms are neglected, technically the IR regularization by a finite length of the string remains a viable option.

A more convenient IR regularization, which maintains the BPS nature of the solution, can be provided by a small mass difference $\Delta m_{AB} \neq 0$; see Sec. VI B. In this case, $\ln L/|\rho|$ must be replaced by $\ln(|\rho| m_{AB})^{-1}$.

Now let us switch on the x_k dependence of ρ, assuming that the S^a moduli are constant (x_k independent). In this case, the gauge potentials (45) vanish. Substituting (41) in the action (2) we readily obtain

$$\frac{2\pi}{g_2^2} m_W^2 \int dt dz |\partial_k \rho|^2 \ln\frac{L}{|\rho|}. \tag{50}$$

This expression is valid with logarithmic accuracy, i.e. under the assumption that the logarithm is large and nonlogarithmic terms can be neglected. We will consistently exploit this approximation throughout the paper.

Now, assembling both parts of the action, the orientational and ρ moduli field kinetic terms (49) and (50), we finally get an effective low-energy theory on the world sheet of the semilocal non-Abelian string. Namely, with logarithmic accuracy,

$$S^{(1+1)}_{N_f=3} = \frac{2\pi}{g_2^2} m_W^2 \int dt dz \left\{ \frac{1}{4} |\rho|^2 (\partial_k S^a)^2 + |\partial_k \rho|^2 \right\} \ln\frac{L}{|\rho|}. \tag{51}$$

The two-dimensional theory (51) contains four real degrees of freedom: two orientational moduli S^a and a complex field ρ related to the size of the semilocal string. Note that both kinetic terms are proportional to the infrared logarithm. We see that the coupling constant of the CP(1) part which describes dynamics of orientational modes is now determined by $m_W^2 |\rho|^2 \ln|\rho|$. The geometry of the target space is $\mathcal{C}^2 \times S_2$, where the radius of S_2 is given by the above-mentioned value and depends on the position in the complex plane ρ.

Repeating the same procedure with the string solution (42) in the theory with four flavors we get practically the same action,

$$S^{(1+1)}_{N_f=4} = \frac{2\pi}{g_2^2} m_W^2 \int dt dz \left\{ \frac{1}{4} |\rho|^2 (\partial_k S^a)^2 + |\partial_k \rho_i|^2 \right\} \ln\frac{L}{|\rho|}, \tag{52}$$

with the obvious replacement $|\partial_k \rho|^2 \to |\partial_k \rho_1|^2 + |\partial_k \rho_2|^2$. Now, in addition to two independent fields S^a, we have four (real) fields ρ_i, $i = 1, 2$, while the size of the string is given by (38). Equations (51) and (52) describe the low-energy limit of the world-sheet theory—they represent a two-derivative truncation in the derivative expansion. The zero-mode interaction contains higher derivatives too. The derivative expansion runs in powers of $|\rho|\partial_k$ implying that the effective sigma models (51) and (52) are applicable at scales below the inverse string thickness $1/|\rho|$ which, thus, plays the role of an ultraviolet (UV) cutoff for the world-sheet theories (51) and (52). This is another reason why ρ has to be regularized in the infrared.

B. Unequal masses

To get a deeper insight into physics of the world-sheet theory for semilocal strings, let us explicitly introduce small mass differences for quark flavors in the bulk theory. Generally speaking, this will lift all internal moduli, introducing a shallow potential for the moduli fields in the world-sheet theory. We will assume, however, that

$$\Delta m_{AB} \ll m_W. \tag{53}$$

The smallness of Δm_{AB} ensures that the effective description in terms of a two-dimensional sigma model is still valid.

To warm up, let us start from the case of the local non-Abelian string, $N_f = 2$, considered previously. In this case the mass difference Δm_{12} breaks the global $SU(2)_{C+F}$ symmetry down to U(1), generating a VEV of the adjoint field; see (6). Thus, orientational moduli are lifted. The corresponding world-sheet theory [3,4], the CP(1) model with twisted mass [45], still possesses $\mathcal{N} = 2$. The action is

$$S^{(1+1)}_{N_f=2} = \beta \int dt dz \left\{ \frac{1}{2} (\partial_k S^a)^2 + \frac{|\Delta m_{12}|^2}{2} (1 - S_3^2) \right\}. \tag{54}$$

It is clearly seen that the moduli fields S^a are no longer massless (even at the classical level). The theory (54) has two vacua $S^a = (0, 0, \pm 1)$ which correspond to the (1,0) and (0,1) Z_2 strings.

Now we can consider semilocal non-Abelian strings, starting from $N_f = 3$. Let us first introduce a mass difference Δm_{13}, while keeping $m_1 = m_2$. Substituting the string solution (39) and (41) in the action, along with the vacuum values (6) for the adjoint fields, we see that the term in the fourth line of Eq. (4) gives a nonvanishing contribution for $A = 3$. A straightforward calculation yields

$$S^{(1+1)}_{N_f=3} = \frac{2\pi}{g_2^2} m_W^2 \int dt dz \bigg\{ \frac{1}{4} |\rho|^2 (\partial_k S^a)^2 + |\partial_k \rho|^2 + |m_1 - m_3|^2 |\rho|^2 \bigg\} \ln(|\Delta m_{13}||\rho|)^{-1}. \tag{55}$$

As was expected, the ρ zero modes are lifted. Another effect seen in (55) is that Δm_{13}^{-1} does indeed assume the role of the infrared cutoff L. The reason is quite evident: if $\Delta m_{13} \neq 0$, the Higgs branch of the bulk theory degenerates down to isolated vacua, we no longer have massless fields in the bulk. Therefore, at very large r, i.e. $r \gg 1/\Delta m_{13}$, the profile functions (20) in our solution modify to acquire an exponential falloff $\sim \exp(-|\Delta m_{13}| r)$. This exponential tail cuts off the logarithmic r integral resulting in (55).

Strictly speaking, if $\Delta m_{13} \neq 0$, semilocal strings cease to exist as exact solutions. The vacuum of the theory (55) is at $\rho = 0$ where the string under consideration becomes local (and our analytic solution is inapplicable). We keep a very small Δm_{13} in what follows, much smaller than any other physical parameter of dimension of mass, in order to cut off the infrared logarithmic divergences in the world-sheet theory [cf. Eq. (55)]. Δm_{13} is kept in the argument of the logarithms, but powers of Δm_{13} will be neglected. As was mentioned, $\Delta m_{13} \neq 0$ does not spoil the BPS nature of the string.

Now let us take into account $\Delta m_{12} \neq 0$, assuming $\Delta m_{13} \ll \Delta m_{12}$. Much in the same way as in the case of the local strings [3], we have to modify our solution (39) and (41), including in the *ansatz* an expression for the adjoint field. Following [3],

$$a^a = \frac{m_1 - m_2}{\sqrt{2}} [\delta^{a3} b + S^a S^3 (1 - b)], \tag{56}$$

where $b(r)$ is a new profile function subject to the boundary conditions

$$b(0) = 0, \qquad b(\infty) = 1. \tag{57}$$

Next, we substitute the string solution (39), (41), and (56) in the action. Calculation goes along the same lines as in [3], therefore it is appropriate to skip details. As in the case of the local strings, the minimization procedure yields

$$b(r) = 1 - \omega(r) = \frac{\phi(r)}{\xi}. \tag{58}$$

Finally, we obtain

$$S^{(1+1)}_{N_f=3} = \frac{2\pi}{g_2^2} m_W^2 \int dt dz \bigg\{ \frac{1}{4} |\rho|^2 (\partial_k S^a)^2 + |\partial_k \rho|^2 + \frac{|\Delta m_{12}|^2}{2} |\rho|^2 (1 - S_3) \bigg\} \ln(\Delta |m_{13}||\rho|)^{-1}. \tag{59}$$

The orientational moduli are lifted by $\Delta m_{12} \neq 0$. Classically, the vacuum manifold of this theory consists of a single branch,

$$S_3 = 1, \qquad \rho \text{ arbitrary}. \tag{60}$$

From the bulk point of view, the above vacuum is interpreted as the (1,0) semilocal Z_2 string. What about the (0,1) string?

As a matter of fact, the Z_2 symmetry is explicitly broken in (59), in contradistinction with the $N_f = 2$ case. There is no semilocal (0,1) string under the above choice of parameters. Changing the orientation vector S^a from $S_3 = 1$ to $S_3 = -1$ implies that the second rather than the first quark flavor winds at infinity; see (29). However, since $m_2 \neq m_3$, this is impossible, and no semilocal string of this type develops. Of course, we still have the local (0,1) string. It corresponds to $S_3 = -1$ and $\rho = 0$. It is not seen in the large ρ approximation.

Expanding the potential in Eq. (59) around a point on the vacuum manifold we see that two fields, S_1 and S_2, have masses $(m_1 - m_2)$ while ρ remains massless. The (real) dimension of the branch is 2,

$$\dim H^{N_f=3}_{(1,0)} = 2. \tag{61}$$

It is not difficult to generalize this analysis to the case $N_f = 4$. We assume $m_{12} \neq 0$ while very small mass differences $\Delta m_{13} = \Delta m_{24}$ are kept only in the argument of the logarithm, for the purpose of the IR regularization, $\Delta m_{13} \ll \Delta m_{12}$. Substituting the string solution (39), (42), and (56) in the action, we get

$$S^{(1+1)}_{N_f=4} = \frac{2\pi}{g_2^2} m_W^2 \int dt dz \bigg\{ \frac{1}{4} |\rho|^2 (\partial_k S^a)^2 + |\partial_k \rho_i|^2 + \frac{|m_1 - m_2|^2}{2} |\rho_1|^2 (1 - S_3) + \frac{|m_1 - m_2|^2}{2} |\rho_2|^2 (1 + S_3) \bigg\} \ln \frac{1}{|m_1 - m_3||\rho|}. \tag{62}$$

This theory classically has two vacuum branches located at

$$S_3 = 1, \qquad \rho_2 = 0, \qquad \rho_1 \text{ arbitrary}, \tag{63}$$

and

$$S_3 = -1, \qquad \rho_1 = 0, \qquad \rho_2 \text{ arbitrary}. \tag{64}$$

On the first branch we obtain four (real) states with mass $(m_1 - m_2)$, namely, S_1, S_2 plus the complex field ρ_2, while ρ_1 is massless. This branch has dimension

$$\dim H_{(1,0)}^{N_f=4} = 2. \tag{65}$$

It corresponds to the (1,0) semilocal string.

On the second branch we have four massive states too, with mass $(m_1 - m_2)$, namely, S_1, S_2 plus the complex field ρ_1, while ρ_2 is massless. The dimension of this branch is

$$\dim H_{(0,1)}^{N_f=4} = 2. \tag{66}$$

It corresponds to the (0,1) semilocal string.

Concluding this section, we would like to emphasize again that the effective theories (59) and (60) on the world sheet of the semilocal string were derived in the approximation of large but not too large values of ρ,

$$\frac{1}{m_W} \ll |\rho| \ll \frac{1}{|m_1 - m_3|}. \tag{67}$$

We commented on the first inequality more than once above. The second inequality ensures that the infrared logarithm in (59) and (62) is a large parameter. Please remember that the theories (59) and (62) are derived with logarithmic accuracy.

VII. SEMICLASSICAL LIMIT

The general expressions for $S_{N_f=3}^{(1+1)}$ and $S_{N_f=4}^{(1+1)}$ obtained above can be further simplified in the semiclassical limit, using a number of approximations. Let us reiterate these approximations.

(i) We will work in the window (67). In this window, $\ln(\Delta m_{13}|\rho|)^{-1}$ is large so that the logarithmic approximation—neglecting nonlogarithmic terms compared to logarithmic—can be consistently applied.

(ii) We will work only with the quadratic terms in the derivative expansion.

(iii) We will keep only the leading terms in the expansion in $(m_W|\rho|)^{-1}$.

(iv) We will assume that $\Delta m_{12} \gg \Delta m_{13}$, Δm_{24}. Moreover, the parameters Δm_{13}, Δm_{24} are kept in the arguments of logarithms but are neglected elsewhere. For brevity we will introduce the notation

$$\Delta m \equiv \Delta m_{12}. \tag{68}$$

Needless to say, $\Delta m_{12} \ll m_{\gamma,W}$ so that all fields in the bulk are very heavy compared to the masses on the string world sheet. An extra assumption regarding Δm_{12} needed at $N_f = 3$ will be specified below.

The subsequent derivations are quite straightforward, albeit somewhat cumbersome and involve a few rescalings/redefinitions. Algebraic manipulations to be presented below should not overshadow a simple statement that, at the very end, we obtain in the semiclassical limit the theory of free complex fields, two fields for $N_f = 3$ and three fields for $N_f = 4$.

First, we introduce a new variable z, replacing the ρ moduli,

$$z_i = \rho_i \left[2\pi\xi \ln\frac{1}{|m_1 - m_3||\rho|} \right]^{1/2}. \tag{69}$$

In terms of these new variables z_i, applying the logarithmic approximation we rewrite the world-sheet theories as

$N_f = 3$:

$$S_{N_f=3}^{(1+1)} = \int d^2x \left\{ \frac{1}{4}|z|^2(\partial_k S^a)^2 + |\partial_k z|^2 + \frac{|\Delta m_{12}|^2}{2}|z|^2(1 - S_3) \right\};$$

$N_f = 4$:

$$S_{N_f=4}^{(1+1)} = \int d^2x \left\{ \frac{1}{4}|z|^2(\partial_k S^a)^2 + |\partial_k z_i|^2 + \frac{|\Delta m_{12}|^2}{2}[|z_1|^2(1 - S_3) + |z_2|^2(1 + S_3)] \right\}. \tag{70}$$

Here $|z|^2 \equiv \sum_i |z_i|^2$. In terms of z, the window (67) becomes

$$\frac{2\pi}{g_2^2} \ll |z|^2 \ll \frac{\xi}{|m_1 - m_3|^2}. \tag{71}$$

As we will see later, the effective world-sheet theory for the semilocal string at $N_f = 3$ is asymptotically free and generates its own dynamical scale Λ_σ. In this section we will assume that

$$\Delta m \equiv \Delta m_{12} \gg \Lambda_\sigma. \tag{72}$$

This ensures the weak coupling regime in the world-sheet theory since, under (72), its coupling constant is frozen at the scale Δm_{12}. The world-sheet theory in the case $N_f = 4$ turns out to be conformal so the limitation (72) does not apply.

As usual, we begin with $N_f = 3$. The O(3) sigma model is known to be equivalent to the CP(1) model (for a review see e.g. [46]). The CP(1) model is a U(1) gauge theory of the complex charged doublet n^l where $l = 1, 2$, subject to the condition $|n^l|^2 = 1$. The relation between S^a and n^l is as follows:

$$S^a = \bar{n}_p (\tau^a)_l^p n^l. \tag{73}$$

In terms of n^l the O(3) sigma-model action (43) takes the form

$$S_{N_f=2}^{(1+1)} = 2\beta \int d^2x |\nabla_k n^l|^2, \tag{74}$$

where $\nabla_k = \partial_k - iA_k$. The gauge field A_k enters the action with no kinetic term and can be eliminated, which would lead us back to (43). We will trade the fields S^a in (70) for n^l. It is convenient to parametrize z through its modulus and phase,

$$z = |z| \exp(i\gamma), \qquad 0 \leq \gamma < 2\pi, \tag{75}$$

and introduce new variables

$$\varphi^l = |z| n^l, \qquad \eta \equiv \varphi^{l=1} = |z| n^1, \qquad \chi \equiv \varphi^{l=2} = |z| n^2. \tag{76}$$

Then Eq. (70) implies

$$S^{(1+1)}_{N_f=3} = \int d^2x \{ |\nabla_k \varphi^l|^2 + |\varphi^l|^2 (\partial_k \gamma)^2 + |\Delta m|^2 |\chi|^2 \}. \tag{77}$$

Let us have a closer look at this theory near its Higgs branch. It is immediately seen that, since the field χ is massive, its quantum fluctuations are small compared to the value of the massless η. Indeed, while $\chi \sim 1$, at the same time, $\eta^2 \approx |\varphi^l|^2 = |z|^2$ is huge due to (71). This means that we can parametrize η as

$$\eta = |z| \exp(i\delta) \tag{78}$$

where δ is a phase, $0 \le \delta < 2\pi$. It is easy to see that, under the circumstances, we get (to the leading order in $1/|z|$)

$$A_k = (\partial_k \delta). \tag{79}$$

Substituting this expression back in (77) we arrive at

$$S^{(1+1)}_{N_f=3} = \int d^2x \{ |\partial_k z|^2 + |\partial_k \tilde{\chi}|^2 + |\Delta m|^2 |\tilde{\chi}|^2 \}, \tag{80}$$

where

$$\tilde{\chi} = \chi \exp(i\delta). \tag{81}$$

The field η totally disappears. Its role was delegated to other terms. Its absolute value (equal to $|z|$) enters the kinetic term for the complex field z, while its phase δ is gauged away.

Thus, in the semiclassical limit we managed to reduce the world-sheet theory at $N_f = 3$ to a free theory of one massive complex field $\tilde{\chi}$ and one massless complex field z. This is obviously the bosonic sector of an $\mathcal{N} = 2$ sigma model with the twisted mass for the $\tilde{\chi}$ field. The massless field develops a huge VEV $|z|$ on the two-dimensional Higgs branch of the theory. This VEV corresponds to a large size $|\rho|$ of the semilocal string.

It is clear that the flat metric in (80) must have corrections running in powers of $2\pi/(g_2^2|z|)$ which, however, must preserve its Kähler nature. We do not see them in our approximation.

Now let us follow the same road to complete our derivation of the world-sheet theory in the $N_f = 4$ bulk model. Again, our starting point is Eq. (70). Rewriting it as a U(1) gauge theory we get

$$S^{(1+1)}_{N_f=4} = \int d^2x \{ |\nabla_k \varphi^l|^2 + |\varphi^l|^2 |\partial_k u_i|^2 + |\Delta m|^2 |u_1|^2 |\chi|^2 + |\Delta m|^2 |u_2|^2 |\eta|^2 \}, \tag{82}$$

where instead of the angle γ we now introduce a complex doublet u_i via

$$z_i = |z| u_i, \qquad |u_i|^2 = 1, \tag{83}$$

while φ^l's are defined as in (76). As was discussed in Sec. VI, this theory has two Higgs branches. Now they are located at $\chi = 0$, $u_2 = 0$ and $\eta = 0$, $u_1 = 0$, respectively.

Consider the first Higgs branch. The field χ is massive; its fluctuations are of order 1. On the contrary, the field η develops a large VEV, $|\eta| = |z|$. Using the parametrization (78) and eliminating the gauge field by virtue of Eq. (79), we get (to the leading order in $1/|z|$)

$$S^{(1+1)}_{N_f=4} = \int d^2x \{ |\partial_k z_i|^2 + |\partial_k \tilde{\chi}|^2 + |\Delta m|^2 (|\tilde{\chi}|^2 + |z_2|^2) \}. \tag{84}$$

The field η again disappears in the same sense as in the $N_f = 3$ case.

The model (84) is a free theory of two massive fields χ and z_2 and one massless field z_1 parametrizing the two-dimensional Higgs branch. As in the $N_f = 3$ case, corrections to the flat metric run in powers of $2\pi/(g_2^2|z|)$. The second Higgs branch of the theory (82) has the same free field description, with the interchange

$$\{\tilde{\chi}, z_2\} \leftrightarrow \{\tilde{\eta}, z_1\}. \tag{85}$$

VIII. COMPARISON WITH THE HANANY-TONG FORMULA

As was mentioned in Sec. I, non-Abelian semilocal strings were analyzed previously [1,4] within a complementary approach based on D branes. The advantage of this approach is that it is not limited to the semiclassical approximation. Its disadvantage is a rather indirect relation to field theory. To make contact with field theory it is highly instructive to compare our field-theoretic results with those obtained by Hanany and Tong. They conjectured that the effective theory on the world sheet of the non-Abelian semilocal string is given by the strong coupling limit ($e^2 \to \infty$) of a two-dimensional U(1) gauge theory which, in the case of $\mathrm{SU}(2)_{\mathrm{color}}$ under consideration, has the form

$$\begin{aligned} S = \int d^2x \Big\{ & |\nabla_k \varphi^l|^2 + |\tilde{\nabla}_k z_i|^2 + \frac{1}{4e^2} F_{kl}^2 + \frac{1}{e^2} |\partial_k \sigma|^2 \\ & + 2\Big|\sigma - \frac{m_l}{\sqrt{2}}\Big|^2 |\varphi^l|^2 + 2\Big|\sigma - \frac{m_{i+2}}{\sqrt{2}}\Big|^2 |z_i|^2 \\ & + \frac{e^2}{2} \Big(|\varphi^l|^2 - |z_i|^2 - \frac{2\pi}{g_2^2} \Big)^2 \Big\}, \end{aligned} \tag{86}$$

$$l = 1, 2, \qquad i = 1, \ldots, N_e, \qquad \tilde{\nabla}_k = \partial_k + iA_k.$$

With respect to the U(1) gauge field, the fields φ^l and z_i have charges $+1$ and -1, respectively. If only charge $+1$ fields were present, in the limit $e^2 \to \infty$ we would get a conventional twisted-mass deformed CP($N - 1$) model. The charge -1 fields z_i convert the target space of the corresponding sigma model into a toric variety. In the theory with $N_f = 3$, we have one complex field $z_1 \equiv z$, with negative charge, while in the case $N_f = 4$, we have two negatively charged complex fields, z_1 and z_2.

The action (86) is a bosonic part of a supersymmetric U(1) gauge theory with four supercharges, which corresponds to extended $\mathcal{N} = 2$ supersymmetry in two dimensions. In particular, the last term in (86) is a D term while the complex scalar σ is an $\mathcal{N} = 2$ superpartner of the photon. The field content above fits our expectations since the string we work with is $1/2$ BPS and, therefore, preserves four supercharges on its world sheet (half of supersymmetry in the bulk theory).

There is a rather convincing field-theoretic argument in favor of the Hanany-Tong conjecture. Consider the bulk theory (2) with $N_f = 2$ at the singular point (6) on the Coulomb branch, and take the limit $\xi \to 0$ [the point (6) becomes an isolated vacuum once we switch on the FI parameter ξ]. As was shown in [47], the BPS spectrum of dyons on the Coulomb branch of the 4D bulk theory (2) identically coincides with the BPS spectrum in the 2D twisted-mass deformed CP(1) model (54). The reason for this coincidence was revealed in [3,4] (see also Sec. 9 in [48]).

Consider a monopole of the SU(2) sector of the bulk theory at $\xi = 0$. This is the 't Hooft-Polyakov monopole [49] with mass given by the classical formula $\Delta m_{12}/g_2^2$. Quantum corrections to this result are determined by the exact Seiberg-Witten solution [35] of the bulk theory. If we now switch on the FI parameter $\xi \neq 0$, the quarks condense [see Eq. (5)], triggering formation of flux tubes and confinement of monopoles. In fact, the magnetic flux of the SU(2) monopole, 4π, exactly matches the difference of the magnetic fluxes of two elementary Z_2 strings, (1,0) and (0,1); see Eq. (29). The confined monopole is represented by a junction of these two Z_2 strings. In the CP(1) world-sheet theory (54) the confined monopole is seen as a kink interpolating between two vacua ($S_3 = \pm 1$) of this theory [3–5]. Although the 't Hooft-Polyakov monopole on the Coulomb branch looks very different from the string junction of the theory in the Higgs phase, amazingly, their masses are the same [3,4]. This is due to the fact that the mass of BPS states (the string junction is a $1/4$-BPS state) cannot depend on ξ because ξ is a nonholomorphic parameter. Since the confined monopole emerges as a kink of the world-sheet theory, the Seiberg-Witten formula for its mass should coincide with the exact result for the kink mass in the two-dimensional $\mathcal{N} = 2$ twisted-mass deformed CP(1) model found in [47]. Thus, we arrive at the statement of coincidence of the BPS spectra in both theories.[3]

We expect this correspondence to be generalizable to theories with $N_f > N$. The 2D theory (86) was studied in [50] (at generic $N_f > N$) where the BPS spectrum was shown to agree with the spectrum of the U(N) four-dimensional QCD with N_f flavors.

The coupling constant in (86) is classically identified with the coupling $2\pi/g_2^2$ of the bulk theory in much the same way as in the $N_f = 2$ case; see (44). Moreover, the one-loop coefficient of the β function equals $2N - N_f$ for both theories. This leads to identification of their coupling constants in quantum theory and identification of their scales, $\Lambda_\sigma = \Lambda$ at $N_f = 3$ [see (8) and (9)] and conformality at $N_f = 4$. The coincidence of the BPS spectra makes the theory (86) a promising candidate for the effective theory on the world sheet of non-Abelian semilocal strings. The D term in (86) determines the Higgs branch of this theory,

$$|\varphi^l|^2 - |z_i|^2 = \frac{2\pi}{g_2^2}. \tag{87}$$

Now comes the main point of our comparison. We will show that, in the limit

$$|z| \gg 2\pi/g_2^2$$

[see Eq. (71)], the metric on this Higgs branch becomes flat, and the Hanany-Tong theory (86) reduces to our results quoted in Eqs. (80) and (84) for $N_f = 3$ and $N_f = 4$, respectively.

Let us start from the case $N_f = 3$. In Eq. (86) we put $m_1 = m_3$ and $\Delta m = m_1 - m_2$. The Higgs branch of (86) is located at

$$\sigma = \frac{m_1}{\sqrt{2}}, \qquad \varphi^{l=2} \equiv \chi = 0, \tag{88}$$

while $\varphi^{l=1} \equiv \eta$ and z are determined by the condition (87),

$$|\eta|^2 - |z|^2 = \frac{2\pi}{g_2^2}. \tag{89}$$

These fields have four real components subject to one constraint (89). This gives $4 - 1 - 1 = 2$ for the dimension of the Higgs branch, where we subtract, in addition to the constraint (89), one U(1) gauge phase. This coincides with the dimension quoted in Eq. (61).

Near this Higgs branch, the field χ is massive, with mass Δm, and, hence, it does not develop large fluctuations. Therefore, we neglect $\chi \sim 1$ compared to $|z| \gg 1$, as we did in Sec. VII. Eliminating the field σ by virtue of its equation of motion yields

$$\sigma = \frac{m_1}{\sqrt{2}}[1 + O(1/|z|)].$$

Substituting this in (86) we get the mass term for χ identical to that in Eq. (80).

Now, let us ignore $2\pi/g_2^2$ in the right-hand side of Eq. (87), along with the contribution of χ in its left-hand

[3]In fact, Dorey [47] deals with the SU(N) theory at the root of the baryonic Higgs branch defined by the condition $\sum_A m_A = 0$. However, one can check that the BPS spectra of massive states in these two 4D theories are the same upon identification of m_A of the SU(N) theory with $m_A - \frac{1}{N}\sum_A m_A$ of the U(N) theory. Note that there is no Higgs branch in the vacuum (5) and (6) with $N_f = 2$ in the U(N) bulk theory, and all states in the bulk are massive.

side, which is legitimate since $|z| \gg 2\pi/g_2^2$. This gives $|\eta| = |z|$. The mean value of phases,

$$\frac{1}{2}(\arg\eta + \arg z),$$

can be gauged away by an appropriate choice of gauge in Eq. (86). At the same time, the relative phase

$$\arg\eta - \arg z$$

can be combined with the modulus $|z|$ to form a complex field with the flat metric. Thus, we arrive at the theory identical to (80) up to corrections in powers of $2\pi/(g_2^2|z|)$.

Next, let us consider the case $N_f = 4$. We put $m_1 = m_3$, $m_2 = m_4$, and $\Delta m = m_1 - m_2$. Now the theory (86) has two Highs branches located at

$$\sigma = \frac{m_1}{\sqrt{2}}, \qquad \chi = z_2 = 0, \tag{90}$$

and

$$\sigma = \frac{m_2}{\sqrt{2}}, \qquad \eta = z_1 = 0, \tag{91}$$

exactly as in our theory (82). Consider the first Higgs branch. Near this Higgs branch the fields χ and z_2 are massive, with mass Δm. Ignoring these fields in comparison with large VEV's of the fields

$$|\eta| = |z_1| \gg 2\pi/g_2^2$$

and eliminating the field σ and the gauge field A_k, we arrive at the theory (84), to the leading order in the parameter $2\pi/(g_2^2|z|)$.

Summarizing, we confirm the Hanany-Tong conjecture (86) by explicit field-theoretic calculation of the action of the world-sheet theory of the semilocal non-Abelian strings in the limit of large string size. Our derivation clearly shows that the limits of applicability of the derivative expansion are set by ρ^{-1}, which can be stabilized by the quark mass differences. This feature is not seen in the analysis of Hanany and Tong. It would be extremely interesting to check whether the theory (86) correctly reproduces corrections to the flat metric in powers of the parameter $2\pi/(g_2^2|z|)$. General arguments in favor of the theory (86) summarized at the beginning of this section (see [1,4]) indicate that this is plausible. However, even if the matching of the power expansions is demonstrated, the theory (86) definitely cannot be the exact answer for a low-energy theory on the string world sheet of the semilocal string. It misses corrections

$$O\left(\left[\ln\frac{\sqrt{\xi}}{|z||m_1 - m_3|}\right]^{-1}\right) \tag{92}$$

suppressed by a large infrared logarithm.

As we increase $|\rho|$, corrections in powers of $2\pi/(g_2^2|z|)$ to the flat metric of the world-sheet theory become exceedingly smaller. However, if we take $|\rho|$ too large, the logarithmic corrections to the metric of the type (92) become important. As we have already mentioned, both types of corrections are small inside the window (67).

IX. QUANTUM REGIME

In this section we will consider the theory (86) at $N_f = 3$ in the quantum regime

$$\Delta m \ll \Lambda_\sigma, \tag{93}$$

eventually taking the limit of equal quark masses which converts quasimoduli into genuine moduli. The theory (86) is studied in [50]; here we briefly review some results obtained in this paper and translate them in terms of the semilocal strings in four dimensions. As we already mentioned, the theory (86) is asymptotically free with the first (and the only) coefficient of the β function equal to $2N - N_f = 1$. The theory runs towards the strong coupling in the infrared and develop its own scale $\Lambda_\sigma = \Lambda$. At nonvanishing Δm_{12} the orientational zero modes S^a are lifted while the size moduli ρ remain massless. They correspond to motion along the two-dimensional Higgs branch of the theory (60).

At $\Delta m_{12} \to 0$ the color-flavor $SU(2)_{C+F}$ symmetry is restored in the bulk theory. *Classically*, we would expect "spontaneous symmetry breaking" on the string world sheet: we would expect the vector S^a to point in some particular direction and two orientational modes to become massless Goldstone modes. This does not happen in two dimensions. Quantum effects restore the $SU(2)_{C+F}$ symmetry on the world sheet and the orientational moduli never become massless. In fact, the dimension of the Higgs branch of the world-sheet theory remains 2 [50] at $\Delta m_{12} \to 0$. Orientational moduli n^l acquire mass of order of Λ much in the same way as in the CP(1) model. This means that, although the size ρ of the semilocal string can have arbitrary values, the orientational vector S^a does not have any particular direction. It is smeared all over. The string is in a highly quantum non-Abelian regime at $\Delta m_{12} \to 0$. This is in one-to-one correspondence with the case of local non-Abelian strings [2,3] occurring in the theory with $N_f = N$.

The quantum regime in the theory (86) with $N_f = 4$ is quite different. It is conformal; no dynamical scale develops. The quasiclassical analysis of Sec. VI can be extended to include the limit $\Delta m_{12} \to 0$, provided the coupling constant g_2 is small. In particular, we see that the Higgs branch of the theory gets enhanced in the limit $\Delta m_{12} \to 0$. In fact, two two-dimensional Higgs branches (63) and (64) fuse to become a connected six-dimensional vacuum manifold,

$$\dim H^{N_f = 4} = 6. \tag{94}$$

It corresponds to all four size moduli fields plus two orientational moduli fields becoming massless. This indicates that the string is in the "classical non-Abelian regime,'' namely, the orientation vector S^a points in some particular direction. The $SU(2)_{C+F}$ group is not restored on the world sheet by quantum effects. This regime does not occur for local non-Abelian strings in quantum theory [2,3].

In conclusion, we stress that in both cases, $N_f = 3$ and $N_f = 4$, the size zero moduli ρ_i of the semilocal non-Abelian string are *not* lifted by interactions with the orientational moduli in the quantum regime. This means that taking account of quantum effects does not change the fact that the size of the semilocal non-Abelian string is arbitrary, provided all mass differences are switched off. As was discussed in Sec. IV this effectively leads to deconfinement.

Let us note that IR-conformal theories, such as the one in Eq. (86) with $N_f = 4$ and finite e^2, were studied in [51,52]. If $m_l = 0$ at $l = 1, 2, 3, 4$, and in the limit $2\pi/g_2^2 \to 0$, these theories were shown to develop both the Higgs and Coulomb branches with distinct values of the Virasoro central charge. Moreover, a tube metric for the field σ was shown to be generated at one loop, upon integrating out the matter fields. It is interpreted as a long tube connecting the Higgs and Coulomb branches.

X. CONCLUSIONS

In this paper we considered a benchmark bulk theory in four dimensions: $\mathcal{N} = 2$ supersymmetric QCD with the gauge group U(N) and N_f flavors of fundamental matter hypermultiplets (quarks). The nature of the BPS strings in this benchmark theory crucially depends on N_f. If $N_f \geq N$ and all quark masses are equal, it supports non-Abelian BPS strings which have internal (orientational) moduli associated with rotations of the color magnetic flux in the non-Abelian group SU(N). If $N_f > N$ these strings become semilocal, developing additional moduli related to (unlimited) variations of their transverse size.

Using the U(2) gauge group with $N_f = 3, 4$ flavors as an example, we derive an effective low-energy theory on the (two-dimensional) string world sheet. Our derivation is field theoretic; it is direct and explicit in the sense that we first analyze the Bogomol'nyi equations for string-geometry solitons, suggest an *ansatz*, and solve it at large ρ. Then we use this solution to obtained the world-sheet theory.

Our result considered in the semiclassical limit confirms the conjecture made previously by Hanany and Tong that this theory is $\mathcal{N} = 2$ supersymmetric U(1) gauge theory in two dimensions, with N positively and N_e negatively charged matter multiplets and the Fayet-Iliopoulos term determined by the four-dimensional coupling constant. We discuss physics of this model and conclude that its Higgs branch is not lifted by quantum effects. This means that the width of the string can freely grow. As a result, such strings cannot confine.

Our analysis of infrared effects shows that, in fact, the derivative expansion can make sense only when the theory under consideration is regularized, e.g. by the quark mass differences. The world-sheet action discussed in this paper becomes a *bona fide* low-energy effective action only if $\Delta m_{AB} \neq 0$.

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