Noncommutative field theory from twisted Fock space

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We construct a quantum field theory in noncommutative space time by twisting the algebra of quantum operators (especially, creation and annihilation operators) of the corresponding quantum field theory in commutative space time. The twisted Fock space and S-matrix consistent with this algebra have been constructed. The resultant S-matrix is consistent with that of Filk [Tomas Filk, Phys. Lett. B **376**, 53 (1996).]. We find from this formulation that the spin-statistics relation is not violated in the canonical noncommutative field theories.

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I. INTRODUCTION

There has been much interest in field theories [1–4] in noncommutative space time where coordinates satisfy the commutation relation,

$$[x^\mu, x^\nu] = i\theta^{\mu\nu}, \tag{1}$$

with $\theta^{\mu\nu}$ being an antisymmetric constant. By the Weyl-Moyal correspondence [5], a field theory in noncommutative space time satisfying Eq. (1) is transformed to a field theory in commutative space time in which product of two space time functions is defined by the Moyal product,

$$(\phi * \psi)(x) = e^{(i/2)\theta^{\mu\nu}\partial_{x^\mu}\partial_{y^\nu}}\phi(x)\cdot\psi(y)|_{x=y}. \tag{2}$$

Because of the existence of noncommutativity parameter, the theory is not Poincaré invariant. The properties such as causality and unitarity are suggested to be violated in the presence of space time noncommutativity (STNC) [6–10], while they remain satisfied in the space-space noncommutative (SSNC) case. There have been some attempts to cure these problems [11–15]. Those arguments have assumed that the state of particles is a representation of the Poincaré group while a canonical noncommutative field theory has symmetry of $SO(1,1)\times SO(2)$ which is a subgroup of the Poincaré group. Recently, Chaichian *et al.* [16], and Wess [17] have proposed a framework of making quantum theory in noncommutative space time invariant under the twisted Poincaré-Hopf algebra $\mathcal{U}_{\mathcal{F}}(\mathcal{P})$ using the proper twisting element $\mathcal{F}\in\mathcal{U}(\mathcal{P})\otimes\mathcal{U}(\mathcal{P})$, where $\mathcal{P}$ is Poincaré algebra and $\mathcal{U}(\mathcal{P})$ is its universal enveloping algebra. In light of their work, field theory in noncommutative space time can be regarded as field theory in commutative space time with twisted coproduct of Poincaré-generators. This justifies the use of the representation of Poincaré group in studying the noncommutative quantum field theory. There have been some attempts to apply this idea to the field theory with Θ-Poincaré symmetry [18], conformal symmetry [19,20], super conformal symmetry [21], Galilean symmetry [22], Galileo Schrödinger symmetry [23], translational symmetry of R^d [24], gauge symmetry [25,26] and diffeomorphic symmetry [27–29]. To construct a consistent quantization formalism of field theory we need also to twist the algebra of quantum field operators consistently. There have been some studies on this problem [30–32].

In this paper, we derive a twisted algebra of creation and annihilation operators $\{a, a^\dagger\}$ as a basis to construct the twisted Lorentz invariant quantum field theory. We propose a framework to construct a consistent quantum field theory with this deformed algebra. Though we focus on the quantization of scalar field theory in this article, we expect that the main ideas of this article can be applied to other field theories. In Sec. II, we introduce the way to twist Hopf algebras and target algebras, and briefly review Chaichian *et al.*'s work. In Sec. III, we introduce twisted algebra of quantum operators, and we propose a framework to construct twisted quantum field theory. We then apply the formulation to construct the Fock space and S-matrix of the scalar field theory in Sec. IV. Finally we discuss some physical implications of the formulation in Sec. V.

II. TWISTED HOPF ALGEBRA OF POINCARÉ GENERATORS

Let $\mathcal{U}(\mathcal{P})$ be a universal enveloping algebra of Poincaré Lie algebra $\mathcal{P}$ and $Y(=P_\rho, M_{\mu\nu})$ be its elements [16]. They satisfy Hopf algebra properties:

$$\Delta Y = Y\otimes 1 + 1\otimes Y, \qquad \epsilon(Y) = 0, \qquad S(Y) = -Y, \tag{3}$$

where Δ is coproduct, ϵ co-unit, and S antipode. The action of Y on the algebra of function space $\mathcal{A}$ satisfies the relation (hereafter we use Sweedler's notation $\Delta Y = \sum Y_{(1)}\otimes Y_{(2)}$)

$$Y\triangleright(\phi\cdot\psi) = \sum(Y_{(1)}\triangleright\phi)\cdot(Y_{(2)}\triangleright\psi), \tag{4}$$

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where ϕ, $\psi \in \mathcal{A}$, the symbol $\cdot$ is a multiplication in the algebra $\mathcal{A}$, and the symbol $\triangleright$ denotes the action of the Poincaré generators on the algebra $\mathcal{A}$ of the complex function space.

The representation of the action of Poincaré generators $(P_\rho, M_{\mu\nu})$ on the function space is given by

$$\begin{aligned} P_\rho \triangleright \phi(x) &= -i\partial_\rho \phi(x), \\ M_{\mu\nu} \triangleright \phi(x) &= -i(x_\mu \partial_\nu - x_\nu \partial_\mu)\phi(x). \end{aligned} \tag{5}$$

If we have some "twisting element" $\mathcal{F} \in \mathcal{U}(\mathcal{P}) \otimes \mathcal{U}(\mathcal{P})$, we can generate another Hopf algebra $\mathcal{U}_\mathcal{F}(\mathcal{P})$ by twisting $\mathcal{U}(\mathcal{P})$ with $\mathcal{F}$. This twisting element $\mathcal{F}$ must satisfy the so called "2-cocycle and co-unital condition" [33]:

$$\begin{aligned} (\mathcal{F} \otimes 1) \cdot (\Delta \otimes \mathrm{id})\mathcal{F} &= (1 \otimes \mathcal{F}) \cdot (\mathrm{id} \otimes \Delta)\mathcal{F}, \\ (\epsilon \otimes \mathrm{id})\mathcal{F} = 1 &= (\mathrm{id} \otimes \epsilon)\mathcal{F}. \end{aligned} \tag{6}$$

Since the Poincaré algebra $\mathcal{P}$ has a commutative subalgbra $\{P_\rho\}$, it is easy to construct a twisting element $\mathcal{F}$ from P_ρ's:

$$\mathcal{F} = \exp\left(\frac{i}{2}\theta^{\alpha\beta} P_\alpha \otimes P_\beta\right). \tag{7}$$

The new Hopf algebra generated from this twisting element is the same as the algebra part of the original Hopf algebra while it has different co-algebra structure. This means that the Lie algebra commutation relations have the same form and the representation of Poincaré generators remains unchanged.

The new coproduct $\Delta_\mathcal{F}$ has the form,

$$\Delta_\mathcal{F} Y = \mathcal{F} \cdot \Delta Y \cdot \mathcal{F}^{-1}, \tag{8}$$

with the same co-unit and antipode, $\epsilon_\mathcal{F} = \epsilon$, $S_\mathcal{F} = S$. Under the change of coproduct, the action of Y (Eq. (4)) does not transform covariantly in general. For the form of Eq. (4) to change covariantly, one has to twist the target algebra $\mathcal{A}$ properly. This consistent multiplication, $*$, of twisted algebra $\mathcal{A}_\mathcal{F}$ has the form

$$\phi * \psi = \cdot[\mathcal{F}^{-1} \triangleright (\phi \otimes \psi)]. \tag{9}$$

When ϕ, $\psi \in \mathcal{A}_\mathcal{F}$ are the functions of the same space time coordinate x^μ, the product $*$ becomes the well known Moyal product. Since $P_\alpha \to -i\partial_\alpha$ in this representation, the commutation relation between space time coordinates is deduced from this $*$-product:

$$\begin{aligned} x^\mu * x^\nu &= \cdot[e^{+(i/2)\theta^{\alpha\beta}\partial_\alpha \otimes \partial_\beta} \triangleright (x^\mu \otimes x^\nu)] \\ &= x^\mu \cdot x^\nu + \frac{i}{2}\theta^{\mu\nu}, \end{aligned} \tag{10}$$

which leads to the commutation relation

$$[x^\mu, x^\nu]_* = i\theta^{\mu\nu}. \tag{11}$$

The above arguments imply that with a twisting element satisfying 2-cocycle condition, one can construct a new algebra pair $\{\mathcal{U}_\mathcal{F}(\mathcal{P}), \mathcal{A}_\mathcal{F}\}$ from the original pair of algebras $\{\mathcal{U}(\mathcal{P}), \mathcal{A}\}$. Thus, one can think of a field theory belonging to a class $\{\mathcal{U}(\mathcal{P}), \mathcal{A}\}$ in noncommutative space time as a field theory belonging to a class $\{\mathcal{U}_\mathcal{F}(\mathcal{P}), \mathcal{A}_\mathcal{F}\}$ in commutative space time. The field theory that belongs to a class $\{\mathcal{U}_\mathcal{F}(\mathcal{P}), \mathcal{A}_\mathcal{F}\}$ has many advantages. An important feature of this class of theories is that the theory has the twist-deformed Poincaré symmetry. Moreover, the operators such as P^2 and W^2 ($W_\alpha = -\frac{1}{2}\epsilon_{\alpha\beta\gamma\delta}M^{\beta\gamma}P^\delta$) remain Casimir operators in the twisted algebra. So in this framework the symmetry of the theory is more transparent, and one can utilize the irreducible representation of the Poincaré algebra in studying the noncommutative field theory. This framework justifies the earlier studies in noncommuatative quantum field theory where the representation of the Poincaré algebra has been utilized [16].

III. TWISTED ALGEBRA OF CREATION AND ANNIHILATION OPERATORS

In the previous section, we summarized how the conventional field theory can be deformed to a field theory which has a twist-deformed Poincaré symmetry. Since the Poincaré generators act on physical Hilbert space also, it is natural to deform the algebra of operators on Hilbert space covariantly when we twist the Poincaré symmetry. Chaichian *et al.* [30] and Balachandran *et al.* [31] have studied this aspect of the noncommutative field theory and most recently Zahn has also investigated this aspect (he especially considered twisting the commutation relations of quantum operators) [32]. In this section, we construct a noncommutative field theory by twisting the algebra of quantum field operators in such a way to preserve the action of Poincaré group on the Fock space. In conventional field theory, if we create/annihilate n-particles of momenta $p_1, \ldots, p_n$ in a Lorentz frame, then it implies that we create/annihilate n-particles of momenta $\Lambda p_1, \ldots, \Lambda p_n$ in a Lorentz transformed frame. Since we want to preserve this relation in the twisted theory, the focus in this paper will mainly be on the relation between the action of Poincaré group and creation/annihilation operators.

A. The action of $\mathcal{U}(\mathcal{P})$ on algebra of operators Ω

Let Ω denote a vector space of selected operators whose domain and range are the physical Hilbert space $\mathcal{T}$. By defining the composite map of two operators as a multiplication (we denote it by the symbol $\circ$), Ω becomes an algebra if it is closed under this multiplcation $\circ$. In other words, for arbitrary $\Psi \in \mathcal{T}$, and for all $a, b \in \Omega$, the map

$$a \circ b\colon \mathcal{T} \to \mathcal{T} \qquad (a \circ b)\Psi = a(b(\Psi)), \tag{12}$$

defines a multiplication in Ω.

We denote the action of $Y \in \mathcal{U}(\mathcal{P})$ on a selected target algebra Ω as $\blacktriangleright$ to distinguish it from the action $\triangleright$ defined in the last section, and let $U(\Lambda, \epsilon)$ be a Poincaré transformation in the physical Hilbert space $\mathcal{T}$ (Λ *denotes a Lorentz transformation*, $x \to x' = \Lambda x$, *and* ϵ *denotes a translation*, $x \to x' = x + \epsilon$). From the relation,

$$U(\Lambda, \epsilon)(a\Psi) = U(\Lambda, \epsilon) \cdot a \cdot U^{-1}(\Lambda, \epsilon)(U(\Lambda, \epsilon)\Psi) = a_{(\Lambda,\epsilon)}\Psi_{(\Lambda,\epsilon)}, \quad (13)$$

where $\Psi \in \mathcal{T}$ and $a \in \Omega$, and from the definition of $\blacktriangleright$, $a_{(\Lambda,\epsilon)} = U(\Lambda, \epsilon)\blacktriangleright a$, we have $\delta_{(\Lambda,\epsilon)}a = -i\epsilon^c(Y_c\blacktriangleright a)$ where ϵ^c are infinitesimal parameters of Poincaré transformation and c denotes $\{[\mu\nu], \rho | \mu, \nu, \rho = 0, 1, 2, 3\}$. These relations give the form of the action $\blacktriangleright$:

$$(Y\blacktriangleright a)\Psi = Y\triangleright(a\Psi) - a(Y\triangleright\Psi), \quad (14a)$$

$$Y\blacktriangleright a \equiv [Y, a], \quad (14b)$$

where the commutator in Eq. (14b) is understood as in Eq. (14a). This operation satisfies the properties needed to be an action:

$$(Y \cdot Z)\blacktriangleright a = Y\blacktriangleright(Z\blacktriangleright a), \qquad \mathbb{1}\blacktriangleright a = a, \qquad Y\blacktriangleright\mathbb{1} = 0, \quad (15a)$$

$$Y\blacktriangleright(a \circ b) = \circ[\Delta Y\blacktriangleright(a \otimes b)] = \sum(Y_{(1)}\blacktriangleright a) \circ (Y_{(2)}\blacktriangleright b). \quad (15b)$$

Thus, we have to twist the quantum operators properly so as to preserve the relation Eq. (15).

B. Twisted algebra of creation and annihilation operators

As in Sec. II, the algebra of quantum operators Ω has to be deformed properly to make the form of Eq. (15b) covariant. Let this consistently twisted product of $\circ$ be denoted as $\star$. In order to distinguish this product from the Moyal product, we denote the twisted product of $\Omega_{\mathcal{F}}$ by $\star$ and the Moyal product of $\mathcal{A}_{\mathcal{F}}$ by $*$ throughout this paper. The consistent form of $\star$-product is,

$$(a \star b)\Psi = \circ[\mathcal{F}^{-1}\blacktriangleright(a \otimes b)]\Psi, \quad (16)$$

where $\mathcal{F}$ is the same twisting element of Eq. (7). The explicit form of the $\star$-product is expressed as,

$$\begin{aligned}(a \star b)\Psi &= \sum(\mathcal{F}^{-1}_{(1)}\blacktriangleright a) \circ ((\mathcal{F}^{-1}_{(2)}\blacktriangleright b)\Psi) \\ &= \sum_{k=0}^{\infty}\frac{1}{k!}\left(-\frac{i}{2}\right)^k\theta^{\alpha_1\beta_1}\cdots\theta^{\alpha_k\beta_k} \\ &\quad \cdot (P_{\alpha 1}\blacktriangleright(P_{\alpha_2}\blacktriangleright\cdots(P_{\alpha k}\blacktriangleright a)\cdots) \\ &\quad \circ \{(P_{\beta 1}\blacktriangleright(P_{\beta_2}\blacktriangleright\cdots(P_{\beta k}\blacktriangleright b)\cdots)\Psi\} \\ &= \sum_{k=0}^{\infty}\frac{1}{k!}\left(-\frac{i}{2}\right)^k\theta^{\alpha_1\beta_1}\cdots\theta^{\alpha_k\beta_k} \\ &\quad \cdot [P_{\alpha_1}, [P_{\alpha_2}, \cdots, [P_{\alpha_k}, a], \cdots] \\ &\quad \circ \{[P_{\beta_1}, [P_{\beta_2}, \cdots, [P_{\beta_k}, b], \cdots]\Psi\}. \end{aligned} \quad (17)$$

In conventional field theory, the scalar field operator is expressed as (see Section 14.2 of [34])

$$\hat{\phi}(x) = \int_p [\sigma_p(x) \cdot a(\bar{\sigma}_p) + \bar{\sigma}_p(x) \cdot a^\dagger(\sigma_p)], \qquad \int_p \equiv \int\frac{d^3p}{(2\pi)^3 2\omega_p}, \quad (18)$$

where σ_p denotes the positive frequency solution of the Klein-Gordon equation, and $a(\bar{\sigma})$ and $a^\dagger(\sigma)$ are the annihilation and the creation operators, respectively. In the above relation we regard the creation and the annihilation operators as basis operators which act on the physical Hilbert space and $\sigma_p(x)$ and $\bar{\sigma}_p(x)$ as coefficients. Thus the selected target algebra to be twisted is the algebra generated by $\{\mathbb{1}, a(\sigma_p), a^\dagger(\sigma_p), \forall p\}$. We abbreviate $a(\sigma_p)$ and $a^\dagger(\sigma_p)$ as a_p and $a_p^\dagger$, respectively, hereafter.

The action of P_α on this basis operators is represented as

$$P_\alpha\blacktriangleright a_q = -q_\alpha a_q, \qquad P_\alpha\blacktriangleright a_q^\dagger = +q_\alpha \cdot a_q^\dagger, \quad (19)$$

which will be denoted in a simplified notation as,

$$P_\alpha \Rightarrow \tilde{q}_\alpha = \begin{cases} +q_\alpha, & \text{for } a_q^\dagger; \\ -q_\alpha, & \text{for } a_q. \end{cases} \quad (20)$$

From Eq. (17), $\star$-product between two of the creation and/or the annihilation operators is expressed, in terms of the conventional multiplication, as

$$\begin{aligned} a_p \star a_q &= e^{-(i/2)p\wedge q}a_p \cdot a_q, \\ a_p^\dagger \star a_q^\dagger &= e^{-(i/2)p\wedge q}a_p^\dagger \cdot a_q^\dagger, \\ a_p \star a_q^\dagger &= e^{+(i/2)p\wedge q}a_p \cdot a_q^\dagger, \\ a_p^\dagger \star a_q &= e^{+(i/2)p\wedge q}a_p^\dagger \cdot a_q. \end{aligned} \quad (21)$$

This relation can be written in a compact form as

$$c_p \star c_q = e^{-(i/2)\tilde{p}\wedge\tilde{q}}c_p \cdot c_q, \qquad c_p = a_p \text{ and/or } a_p^\dagger. \quad (22)$$

In Appendix A we explicitly compute the product of n operators, a_p and/or $a_q^\dagger$, to be

$$c_{q_1} \star \cdots \star c_{q_n} = \mathcal{E}(\tilde{q}_1, \cdots, \tilde{q}_n)c_{q_1}\cdots c_{q_n}, \qquad \mathcal{E}(\tilde{q}_1, \cdots, \tilde{q}_n) = \exp\left(-\frac{i}{2}\sum_{i<j}^{n}\tilde{q}_i \wedge \tilde{q}_j\right), \quad (23)$$

where c_q represents a_q or $a_q^\dagger$ and $\tilde{q}$ is defined in Eq. (20). Thus the $\star$-products of n-number of creation or annihilation operators are just the conventional products of the corresponding operators multiplied by the phase factor $\mathcal{E}(\tilde{q}_1, \cdots, \tilde{q}_n)$. It is worth to note that this twisted algebra is associative as shown in Eq. (A2), and the complex conjugation is also compatible with this algebra:

$$(c_{q_1} \star \cdots \star c_{q_n})^\dagger = c_{q_n}^\dagger \star \cdots \star c_{q_1}^\dagger, \tag{24}$$

since $\bar{\mathcal{E}}(\tilde{q}_1, \cdots, \tilde{q}_n) = \mathcal{E}(-\tilde{q}_n, \cdots, -\tilde{q}_1)$.

In this construction, we note that the coefficient functions are not $\star$-producted:

$$\left(\sum a_i(x)c_i\right) \star \left(\sum b_j(x)c_j\right) \begin{cases} = \sum(a_i(x)\cdot b_j(x))(c_i \star c_j), \\ \neq \sum(a_i(x) * b_j(x))(c_i \star c_j). \end{cases} \tag{25}$$

Since $\star$-product is deduced by requiring covariance of the action of Poincaré algebra on Ω, it is enough to check this property of the $\star$-product by evaluating the action on the free scalar field case. Specifically, let U_Λ be a Lorentz transformation which acts only on the operator part, but not on the space time part, then

$$\begin{aligned}\hat{\phi}'(x) &= U_\Lambda \cdot \hat{\phi}(x) \cdot U_\Lambda^{-1} \\ &= \int_p [e^{ip\cdot x} \cdot U_\Lambda \cdot a_p \cdot U_\Lambda^{-1} + e^{-ip\cdot x} \cdot U_\Lambda \cdot a_p^\dagger \cdot U_\Lambda^{-1}] \\ &= \int_p [e^{ip\cdot x} \cdot a_{\Lambda p} + e^{-ip\cdot x} \cdot a_{\Lambda p}^\dagger] \\ &\equiv \int_p [e^{ip\cdot \Lambda x} \cdot a_p + e^{-ip\cdot \Lambda x} \cdot a_p^\dagger] = \hat{\phi}(\Lambda x),\end{aligned} \tag{26}$$

where the use has been made of the fact that $p \cdot x = \Lambda p \cdot \Lambda x$. This leads to

$$Y \blacktriangleright \hat{\phi}(x) \approx Y \triangleright \hat{\phi}(x). \tag{27}$$

This suggests other possibilities of twisting the algebra of quantum field operators. Since quantum field operators are functions of space time and are also operators in the Hilbert space, one may twist the quantum operator part as indicated in [35], and carried out in this paper, or twist the space time function part of the field operators as has been done by Chaichian *et al.* [30], and Zahn [32]. Or one may twist both the quantum operator part and the space time function part as has been done by Balachandran *et al.* [31], which will result different S-matrix due to the cancellation of the effects of the two twist operations. We have chosen in this paper to twist the algebra of quantum operator part of the field operators, which gives the same S-matrix as those of earlier studies [36], as will be shown in the next section.

Incidentally, our $\star$-product has the same expression as that of Ref. [30] when it is performed on the scalar field operators:

$$\begin{aligned}\hat{\phi}(x) \star \hat{\phi}(y) &= \int_{p,q} [e^{ipx}e^{iqy}(a_p \star a_q) + e^{ipx}e^{-iqy}(a_p \star a_q^\dagger) + e^{-ipx}e^{iqy}(a_p^\dagger \star a_q) + e^{-ipx}e^{-iqy}(a_p^\dagger \star a_q^\dagger)] \\ &= \int_{p,q} [e^{-(i/2)p\wedge q}e^{ipx}e^{iqy}(a_p \cdot a_q) + e^{+(i/2)p\wedge q}e^{ipx}e^{-iqy}(a_p \cdot a_q^\dagger) + e^{+(i/2)p\wedge q}e^{-ipx}e^{iqy}(a_p^\dagger \cdot a_q) \\ &\quad + e^{-(i/2)p\wedge q}e^{-ipx}e^{-iqy}(a_p^\dagger \cdot a_q^\dagger)] \\ &= e^{(i/2)\partial_x \wedge \partial_y}(\hat{\phi}(x) \cdot \hat{\phi}(y)) \equiv \hat{\phi}(x) \star \hat{\phi}(y)|_{\text{Chaichian et al.}}.\end{aligned} \tag{28}$$

This shows that our expression for the $\star$-product corresponds to the momentum space representation of Chaichian *et al.*'s $\star$-product for the free scalar field case.

IV. PHYSICAL FOCK SPACE AND *S*-MATRIX

In the previous section, we have constructed twisted algebra of quantum operators. The physical quantities such as S-matrix must also be written in twist covariant form. Since the physical quantities can be expressed as a sum of products of creation and annihilation operators in the conventional field theory, the physical quantities in the twisted theory must be expressed as the same quantities with the conventional product replaced by the $\star$-product. Thus we can consistently construct noncommutative field theory by twisting the conventional field theory.

A. Twisted Fock space

1. Commutative case

Let $\mathcal{H}$ denote a one particle Hilbert space of scalar field theory. Then the Fock space of this theory can be written as $T(\mathcal{H}) = C \oplus [\bigoplus_{n=0}^\infty \mathcal{H}_S^n]$, $\mathcal{H}_S^n \equiv \bigotimes_S^n \mathcal{H}$, where subscript '$S$' denotes symmetrization and $\bigotimes^n$ denotes n'th order tensor product $\mathcal{H} \otimes \cdots \otimes \mathcal{H}$. The action of creation and annihilation operators on the normalized n-particle state $|q_1, \cdots, q_n\rangle \in \mathcal{H}_S^n$, is expressed as

$$\begin{aligned} a_q^\dagger |q_1, \cdots, q_n\rangle &= |q, q_1, \cdots, q_n\rangle, \\ a_q |q_1, \cdots, q_n\rangle &= \sum_{k=1}^n \delta(q - q_k)|q_1, \cdots, q_{k-1}, q_{k+1}, \cdots q_n\rangle. \end{aligned} \tag{29}$$

They satisfy the fundamental commutation relations,

$$[a_p^\dagger, a_q^\dagger] = 0, \qquad [a_p, a_q] = 0, \qquad [a_p, a_q^\dagger] = \delta(p - q). \tag{30}$$

Any elements $|q_1, \cdots, q_n\rangle$ in the Fock space can be obtained by successive operations of creation operators to the vacuum state $|0\rangle$ ($a_q|0\rangle = 0$, for all momentum q),

$$|q_1, \cdots, q_n\rangle = a_{q_1}^\dagger \cdots a_{q_n}^\dagger |0\rangle. \tag{31}$$

2. Noncommutative case

In noncommutative space time, $|q_1, \cdots, q_n\rangle$ in Eq. (31) is mapped into

$$|q_1, \cdots, q_n\rangle_\star = a_{q_1}^\dagger \star a_{q_2}^\dagger \star \cdots \star a_{q_n}^\dagger |0\rangle, \tag{32}$$

which will be called twisted n-state rather than n-particle state. If the vacuum state is defined as $|0\rangle_\star = |0\rangle$, the twisted 1-state is the same as the one particle state in the conventional quantum field theory:

$$|q\rangle_\star = a_q^\dagger \star |0\rangle_\star = (a_q^\dagger \star \mathbb{1})|0\rangle = a_q^\dagger|0\rangle = |q\rangle. \tag{33}$$

This definition of twisted n-state seems very natural, but due to the noncommutativity of the $\star$-product, the state vector $|q_1, \cdots, q_n\rangle_\star$ is not symmetric under the permutation of $(q_1, q_2, \cdots, q_n)$. The explicit form of $|q_1, \cdots, q_n\rangle_\star$ is

$$\begin{aligned} |q_1, \cdots, q_n\rangle_\star &= \mathcal{E}(q_1, \cdots, q_n)|q_1, \cdots, q_n\rangle, \\ \mathcal{E}(q_1, \cdots, q_n) &= \exp\left(-\frac{i}{2}\sum_{i<j}^{n} q_i \wedge q_j\right). \end{aligned} \tag{34}$$

The state $|q_1, \cdots, q_n\rangle$ is symmetric under any permutation of $(q_1, \cdots, q_n)$, but the phase factor $\mathcal{E}(q_1, \cdots, q_n)$ is not symmetric in general for $n \geqq 2$. Since the phase factor has unit norm ($|\mathcal{E}| = 1$), $|q_1, \cdots, q_n\rangle_\star$ and $|q_1, \cdots, q_n\rangle$ are in the same ray of the physical Hilbert space.

Some properties of the phase factor $\mathcal{E}$ are listed in Appendix B. In this new algebra the creation and the annihilation operators do not satisfy the fundamental commutation relation Eq. (30), rather, they satisfy (the same form of relations appear in [31]),

$$\begin{aligned} a_p^\dagger \star a_q^\dagger &= e^{-ip\wedge q} \cdot a_q^\dagger \star a_p^\dagger, \\ a_p \star a_q &= e^{-ip\wedge q} \cdot a_q \star a_p, \\ a_p \star a_q^\dagger &= e^{+ip\wedge q} \cdot a_q^\dagger \star a_p + \delta(p - q). \end{aligned} \tag{35}$$

The action of creation and annihilation operators on the state $|q_1, \cdots, q_n\rangle_\star$ gives

$$\begin{aligned} a_q^\dagger \star |q_1, \cdots, q_n\rangle_\star &= |q, q_1, \cdots, q_n\rangle_\star \\ a_q \star |q_1, \cdots, q_n\rangle_\star &= \sum_{k=1}^{n} \delta(q - q_k) e^{iq\wedge(q_1+\cdots+q_{k-1})}|q_1, \ldots, \\ &\qquad q_{k-1}, q_{k+1}, \ldots, q_n\rangle_\star. \end{aligned} \tag{36}$$

From these twisted states $|q_1, \cdots, q_n\rangle_\star$, we define twisted Fock space as $T_{\mathcal{F}}(\mathcal{H}) = C \oplus [\bigoplus_{n=0}^{\infty} \mathcal{H}_{\mathcal{F}}^n]$, ($\mathcal{H}_{\mathcal{F}}^n \equiv \bigotimes^n \mathcal{H}$). Using the above twisting process, it is natural to define the total number operator as

$$N_{\mathcal{F}} = \sum_k a_k^\dagger \star a_k. \tag{37}$$

This number operator satisfies

$$[N_{\mathcal{F}}, a_q^\dagger]_\star = a_q^\dagger, \qquad [N_{\mathcal{F}}, a_q]_\star = -a_q. \tag{38}$$

The state $|q_1, \cdots, q_n\rangle_\star$ has the same eigenvalue for the number operator as that of the state $|q_1, \cdots, q_n\rangle$ in the conventional theory, i.e., the eigenvalue equation,

$$N|q_1, \cdots, q_n\rangle = n(q_1, \cdots, q_n) \cdot |q_1, \cdots, q_n\rangle,$$

leads to

$$\begin{aligned} N_{\mathcal{F}} \star |q_1, \cdots, q_n\rangle_\star &= \sum_k a_k^\dagger \star a_k \star a_{q_1}^\dagger \star \cdots \star a_{q_n}^\dagger |0\rangle \\ &= \sum_k \mathcal{E}(k, -k, q_1, \cdots, q_n) \cdot N_k \\ &\quad \cdot |q_1, \cdots, q_n\rangle \\ &= \mathcal{E}(q_1, \cdots, q_n) \cdot (N|q_1, \cdots, q_n\rangle) \\ &= n(q_1, \cdots, q_n) \cdot |q_1, \cdots, q_n\rangle_\star, \end{aligned} \tag{39}$$

where we have used the relation Eq. (B6).

The Hamiltonian for the free scalar field theory has the form,

$$H_{\mathcal{F}} = \sum_k \omega_k \cdot a_k^\dagger \star a_k, \quad \text{where } \omega_k = \sqrt{k^2 + m^2}. \tag{40}$$

Thus as in the number operator case, the state $|q_1, \cdots, q_n\rangle_\star$ has the same energy eigenvalues as the state $|q_1, \cdots, q_n\rangle$ for the free Hamiltonian of the commutative scalar field theory.

B. *S*-matrix

By properly twisting the algebra of the quantum operators we have the expression for the S-matrix in the noncommutative field theory:

$$\begin{aligned} S_\star &= T\exp\left(-i\int d^4x \mathcal{H}_I^\star(x)\right) \\ &= \sum_{k=0}^{\infty} \frac{(-i)^k}{k!} \int d^4x_1 \cdots d^4x_k T\{\mathcal{H}_I^\star(x_1) \star \cdots \\ &\quad \star \mathcal{H}_I^\star(x_k)\}, \end{aligned} \tag{41}$$

where $\mathcal{H}_I^\star$ is the interacting part of the Hamiltonian and T denotes the time ordering. Since we can not define interaction Hamiltonian $\mathcal{H}_I$ in space time noncommutative case in general, we assume the space-space noncommutative case only in this section. For the $g\phi^n(x)$ theory, the element of S-matrix can be calculated by using the properties of the phase factor $\mathcal{E}$, given in Appendix B. If we use the abbreviation

$$\phi_{\rm in}(x) = \int_q [\sigma_{-q}(x)a_q + \sigma_q(x)a_q^\dagger] \equiv \sum_{c_q=a_q,a_q^\dagger}\int_q c_q\cdot\sigma_{\tilde{q}}(x), \quad (42)$$

the interaction Hamiltonian can be written as

$$\mathcal{H}_I^\star(x) = g\int_{q_1}\cdots\int_{q_n}(c_{q_1}\star\cdots\star c_{q_n})\cdot\sigma_{\tilde{q}_1}(x)\cdots\sigma_{\tilde{q}_n}(x) \equiv g\sum_{c_Q}\int_Q c_Q^\star\cdot\sigma_{\tilde{Q}}(x), \quad (43)$$

where $\int_Q \equiv \int_{q_1}\cdots\int_{q_n}$, $c_Q^\star \equiv c_{q_1}\star\cdots\star c_{q_n}$, $\sigma_{\tilde{Q}}(x) \equiv \sigma_{\tilde{q}_1}(x)\cdots\sigma_{\tilde{q}_n}(x)$, $\sum_{c_Q} \equiv \sum_{c_{q_1}=a_{q_q},a_{q_1}^\dagger}\cdots\sum_{c_{q_n}=a_{q_n},a_{q_n}^\dagger}$.

The twisted S-matrix is then expressed as

$$\begin{aligned}
S_\star &= \sum_{k=0}^\infty\frac{(-i)^k}{k!}\int d^4x_1\cdots d^4x_k T\{\mathcal{H}_I^\star(x_1)\star\cdots\star\mathcal{H}_I^\star(x_k)\} \\
&= \sum_{k=0}^\infty(-i)^k\int d^4x_1\cdots d^4x_k\theta(x_1,\cdots,x_k)\mathcal{H}_I^\star(x_1)\star\cdots\star\mathcal{H}_I^\star(x_k) \\
&= \sum_{k=0}^\infty(-ig)^k\int_{Q_1}\cdots\int_{Q_k}\sum_{c_{Q_1}\cdots c_{Q_k}}c_{Q_1}^\star\star\cdots\star c_{Q_k}^\star\Big(\int d^4x_1\cdots d^4x_k\theta(x_1,\cdots,x_k)\sigma_{\tilde{Q}_1}(x_1)\cdots\sigma_{\tilde{Q}_k}(x_k)\Big) \\
&= \sum_{k=0}^\infty(-ig)^k\int_{Q_1}\cdots\int_{Q_k}\sum_{c_{Q_1}\cdots c_{Q_k}}\mathcal{E}(\tilde{Q}_1,\cdots,\tilde{Q}_k)c_{Q_1}\cdots c_{Q_k}\cdot\tilde{\Theta}(\tilde{Q}_1,\cdots,\tilde{Q}_k), \qquad (44)
\end{aligned}$$

where

$$\tilde{\Theta}(\tilde{Q}_1,\cdots,\tilde{Q}_k) = \int d^4x_1\cdots d^4x_k\theta(x_1,\cdots,x_k)\times\sigma_{\tilde{Q}_1}(x_1)\cdots\sigma_{\tilde{Q}_k}(x_k). \quad (45)$$

In the limit $\theta\to 0$, this S-matrix reduces to the one in the commutative case:

$$\begin{aligned}
S_\star \to S &= \sum_{k=0}^\infty(-ig)^k\int_{Q_1}\cdots\int_{Q_k}\sum_{c_{Q_1}\cdots c_{Q_k}}c_{Q_1}\cdots c_{Q_k}\cdot\tilde{\Theta}(\tilde{Q}_1,\cdots,\tilde{Q}_k) \\
&= \sum_{k=0}^\infty(-ig)^k\int_{Q_1}\cdots\int_{Q_k}\sum_{c_{Q_1}\cdots c_{Q_k}}S^k(\tilde{Q}_1,\cdots,\tilde{Q}_k), \qquad (46)
\end{aligned}$$

where $S^k(\tilde{Q}_1,\cdots,\tilde{Q}_k) = c_{Q_1}\cdots c_{Q_k}\cdot\tilde{\Theta}(\tilde{Q}_1,\cdots,\tilde{Q}_k)$ corresponds to the momentum space representation of k-th order term of the S-matrix in the conventional field theory. Equation (44) and (46) show the relation between the $S_\star$-matrix and the S-matrix of the corresponding commutative theory. For the S-matrix element ${}_\star\langle\beta|S_\star|\alpha\rangle_\star$, where $|\alpha\rangle_\star(|\beta\rangle_\star)$ denotes '*in*' twisted $n(m)$-state, we have

$$\begin{aligned}
{}_\star\langle\beta|S_\star|\alpha\rangle_\star &= \sum_{k=0}^\infty(-ig)^k\times\int_{Q_1}\cdots\int_{Q_k}\sum_{c_{Q_1}\cdots c_{Q_k}}\mathcal{E}(\tilde{\beta},\tilde{Q}_1,\cdots,\tilde{Q}_k,\tilde{\alpha})\times\langle\beta|c_{Q_1}\cdots c_{Q_k}|\alpha\rangle\cdot\tilde{\Theta}(\tilde{Q}_1,\cdots,\tilde{Q}_k) \\
&= \mathcal{E}(\tilde{\beta},\tilde{\alpha})\sum_{k=0}^\infty(-ig)^k\times\int_{Q_1}\cdots\int_{Q_k}\sum_{c_{Q_1}\cdots c_{Q_k}}\mathcal{E}(\tilde{Q}_1,\cdots,\tilde{Q}_k)\times\langle\beta|S^k(\tilde{Q}_1,\cdots,\tilde{Q}_k)|\alpha\rangle \\
(\because\tilde{\alpha}+\tilde{\beta}=0) &= \mathcal{E}(-\beta,\alpha)\langle\beta|S_\star|\alpha\rangle, \qquad (47a)
\end{aligned}$$

where the momenta $\tilde{\alpha}$, $\tilde{\beta}$ are related to those of $\alpha = |\alpha_1,\alpha_2,\cdots\rangle$, $\beta = |\beta_1,\beta_2,\cdots\rangle$ as shown in Fig. 1, and

$$\langle\beta|S_\star|\alpha\rangle = \sum_{k=0}^\infty(-ig)^k\int_{Q_1}\cdots\int_{Q_k}\sum_{c_{Q_1}\cdots c_{Q_k}}\mathcal{E}(\tilde{Q}_1,\cdots,\tilde{Q}_k)\times\langle\beta|S^k(\tilde{Q}_1,\cdots,\tilde{Q}_k)|\alpha\rangle. \quad (47b)$$

It can be easily shown that the result (47b) is the same as that of Filk [36]. Since the quantities $\langle\beta|c_{Q_1}\cdots c_{Q_k}|\alpha\rangle$ contain the energy-momentum conservation delta functions $\delta(\tilde{Q}_1)\cdots\delta(\tilde{Q}_k)$, $\mathcal{E}(\tilde{Q}_1,\cdots,\tilde{Q}_k)$ in Eq. (47b) can be written as $\mathcal{E}(\tilde{Q}_1)\cdots\mathcal{E}(\tilde{Q}_k)$, and each of these $\mathcal{E}(\tilde{Q}_i)$ gives the phase factor at each vertex in the Feynman diagram.

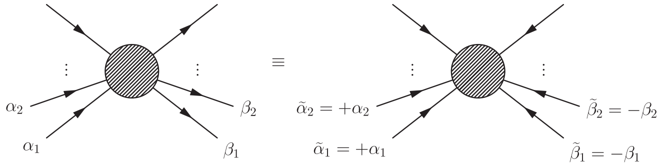

FIG. 1. Illustration of the notation for momenta $\tilde{\alpha}$ and $\tilde{\beta}$.

C. Statistics of indistinguishable particles

The unit matrices in the space of N-particle state can be expressed as

$$\mathbb{1}_D = \sum_\gamma |\gamma\rangle_D \langle\gamma|_D, \qquad \mathbb{1}_C = \frac{1}{N!}\sum_\gamma |\gamma\rangle\langle\gamma|, \qquad \mathbb{1}_\star = \frac{1}{N!}\sum_\gamma |\gamma\rangle_{\star\star}\langle\gamma| (\equiv \mathbb{1}_C), \tag{48}$$

where the subscript $C(D)$ denotes indistinguishable (distinguishable) state in the conventional field theory and $\star$ denotes the noncommutative case. We must be careful in the order of momenta in kets and bras because the twisted n-state is not symmetric under the permutation of the momenta. From Eq. (24) we find $|\mathbf{k_1}, \cdots, \mathbf{k_N}\rangle_\star^\dagger = {}_\star\langle \mathbf{k_N}, \cdots, \mathbf{k_1}|$, i.e., $|\gamma\rangle_\star \equiv |\mathbf{k_1}, \cdots, \mathbf{k_N}\rangle_\star$ and $\langle\gamma|_\star \equiv {}_\star\langle \mathbf{k_N}, \cdots, \mathbf{k_1}|$.

Let $\alpha(\beta)$ denote the free N-particle *in (out)* twisted $n(m)$-state, respectively, and $\mathcal{P}$ denotes arbitrary permutation. Then, for spatial dimensions $d \geq 3$, (See [37,38]), we have

$$\begin{aligned} {}_\star\langle\beta|\alpha\rangle_\star &= \mathcal{E}(-\beta, \alpha)\langle\beta|\alpha\rangle_C \\ &= \mathcal{E}(-\beta, \alpha)\sum_{\mathcal{P}} C_{\mathcal{P}}\langle\mathcal{P}(\beta)|\alpha\rangle_D, \end{aligned} \tag{49}$$

where $C_{\mathcal{P}}$ are complex constants. From the form of $\mathbb{1}_\star$ and $\mathbb{1}_D$ in Eq. (48) we finally obtain the relation,

$$\begin{aligned} {}_\star\langle\beta|\alpha\rangle_\star &= \frac{1}{N!}\sum_\gamma {}_\star\langle\beta|\gamma\rangle_{\star\star}\langle\gamma|\alpha\rangle_\star = \frac{1}{N!}\sum_\gamma \mathcal{E}(-\beta, \gamma)\mathcal{E}(-\gamma, \alpha)\sum_{\mathcal{P}', \mathcal{P}''} C_{\mathcal{P}'}C_{\mathcal{P}''}\langle\mathcal{P}'(\beta)|\gamma\rangle_D\langle\mathcal{P}''(\gamma)|\alpha\rangle_D \\ &= \mathcal{E}(-\beta, \alpha)\frac{1}{N!}\sum_{\mathcal{P}', \mathcal{P}''} C_{\mathcal{P}'}C_{\mathcal{P}''}\sum_\gamma\langle\mathcal{P}'(\beta)|\gamma\rangle_D\langle\mathcal{P}''(\gamma)|\alpha\rangle_D \\ &= \mathcal{E}(-\beta, \alpha)\frac{1}{N!}\sum_{\mathcal{P}', \mathcal{P}''} C_{\mathcal{P}'}C_{\mathcal{P}''}\sum_\gamma\langle\mathcal{P}''\mathcal{P}'(\beta)|\mathcal{P}''(\gamma)\rangle_D\langle\mathcal{P}''(\gamma)|\alpha\rangle_D = \mathcal{E}(-\beta, \alpha)\frac{1}{N!}\sum_{\mathcal{P}', \mathcal{P}''} C_{\mathcal{P}'}C_{\mathcal{P}''}\langle\mathcal{P}'\mathcal{P}''(\beta)|\alpha\rangle_D. \end{aligned} \tag{50}$$

Since $\mathcal{E}(-\beta, \alpha) \neq 0$, we have

$$\begin{aligned} \sum_{\mathcal{P}} C_{\mathcal{P}}\langle\mathcal{P}(\beta)|\alpha\rangle_D &= \frac{1}{N!}\sum_{\mathcal{P}', \mathcal{P}''} C_{\mathcal{P}'}C_{\mathcal{P}''}\langle\mathcal{P}''\mathcal{P}'(\beta)|\alpha\rangle_D \\ \Rightarrow C_{\mathcal{P}', \mathcal{P}''} &= C_{\mathcal{P}'} \cdot C_{\mathcal{P}''}, \end{aligned}$$

which is the one dimensional representation of the permutation group as we have in the conventional field theory case. Consequently, we have the same statistics for indistinguishable particles in the noncommutative field theories as in the corresponding commutative case [35,39].

V. SUMMARY AND DISCUSSIONS

We have constructed noncommutative quantum field theory by properly twisting the algebra of creation and annihilation operators.

As mentioned in Sec. IV the twisted n-state is not symmetric under the permutation of its momenta. If we permute its momenta the state changes by a phase factor $\mathcal{E}$ which has unit norm. Thus $|q_1, \cdots, q_n\rangle$ and $|q_1, \cdots, q_n\rangle_\star$ are in the same ray in Hilbert space, and as we have shown in Sec. IV the phase factor is always factorized out of the S-matrix element. Moreover, the states $|q_1, \cdots, q_n\rangle_\star$ and $|q_1, \cdots, q_n\rangle$ have the same eigenvalues for the corresponding number operators. Hence we have the same physics whether we use $|q_1, \cdots, q_n\rangle_\star$ or $|q_1, \cdots, q_n\rangle$ as a basis for *in*/*out* states (for example, twisted Lorentz transformation changes only the phase factor of the twisted n-states). Hence *we can define n-particle state as an equivalence class of this twisted n-states.*

We have shown that S-matrix elements differ by phase factors from the S-matrix elements of the conventional theory. In SSNC quantum field theory it gives the phase factor to every vertex in the Feynman diagram. This phase factor is the same as that in [36], thus justifying the results of Filk. The expression of S-matrix in this paper is manifestly twist Lorentz covariant, except that time ordering may break this symmetry since $\star$-commutator of two interacting Hamiltonians separated by spacelike distance does not vanish in general, i.e.,

$$[\mathcal{H}_I^\star(x), \mathcal{H}_I^\star(y)]_\star \neq 0, \quad \text{for } (x-y)^2 < 0. \tag{52}$$

This possible violation of locality is known to be inherited from the presence of space time noncommutativity. Some authors have argued that the micro causality is satisfied in the SSNC case, while it is violated in STNC case [40]. It appears that further studies are needed to have a consistent STNC field theory.

We have observed that the statistics of indistinguishable particles do not change in the properly twisted formulation of the noncommutative field theory. It reflects the cohomological properties of the phase factors, i.e. $\mathcal{E}(-\beta, \gamma)\mathcal{E}(-\gamma, \alpha) = \mathcal{E}(-\beta, \alpha)$.

In summary, by careful construction of quantum field theory using the twisted algebra applied to the quantum operator space, we have constructed the S-matrix of the interacting noncommutative scalar field theory, and the result is shown to be consistent with earlier ones [36]. We hope that this formulation can be generalized to the case of more general noncommutative field theories.

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APPENDIX A: EXPLICIT CALCULATIONS OF ⋆-PRODUCT OF CREATION AND ANNIHILATION OPERATORS

In this appendix, we derive the explicit form of ⋆-product of the creation and annihilation operators in terms of the conventional products.

(1) The ⋆-product of $(c_{p_1}\cdots c_{p_k})$ and $(c_{q_1}\cdots c_{q_l})$ is given by

$$\begin{aligned}(c_{p_1}\cdots c_{p_k})\star(c_{q_1}\cdots c_{q_l})\\ = e^{-(i/2)\tilde{P}\wedge\tilde{Q}}(c_{p_1}\cdots c_{p_k}c_{q_1}\cdots c_{q_l}),\end{aligned} \tag{A1}$$

where $\tilde{P} = \sum \tilde{p}_i$ and $\tilde{Q} = \sum \tilde{q}_j$.

Proof.

$$\begin{aligned}P_\alpha \blacktriangleright (c_{q_1}\cdots c_{q_l}) &= \cdot[\Delta^{(l)}P_\alpha \blacktriangleright (c_{q_1}\otimes\cdots\otimes c_{q_l})]\\ &= \cdot\sum_{i=1}^{l}[c_{q_1}\otimes\cdots\otimes(P_\alpha\blacktriangleright c_{q_i})\\ &\quad\otimes\cdots\otimes c_{q_l}]\\ &= \sum_i(\tilde{q}_i)_\alpha(c_{q_1}\cdots c_{q_i}\cdots c_{q_l})\\ &= \tilde{Q}_\alpha(c_{q_1}\cdots c_{q_l}),\end{aligned}$$

and since $(c_{p_1}\cdots c_{p_k}), (c_{q_1}\cdots c_{q_l}) \in \Omega$, we have

$$\begin{aligned}&(c_{p_1}\cdots c_{p_k})\star(c_{q_1}\cdots c_{q_l})\\ &\quad = \cdot\{e^{-(i/2)\theta^{\alpha\beta}P_\alpha\otimes P_\beta}\blacktriangleright[(c_{p_1}\cdots c_{p_k})\\ &\quad\quad\otimes(c_{q_1}\cdots c_{q_l})]\}\\ &\quad = e^{-(i/2)\theta^{\alpha\beta}\tilde{P}_\alpha\tilde{Q}_\beta}(c_{p_1}\cdots c_{p_k}c_{q_1}\cdots c_{q_l}).\end{aligned}$$

□

(2) For all natural number n, we have

$$\begin{aligned}c_{q_1}\star\cdots\star c_{q_n} &= \mathcal{E}(\tilde{q}_1,\cdots,\tilde{q}_n)c_{q_1}\cdots c_{q_n},\\ \mathcal{E}(\tilde{q}_1,\cdots,\tilde{q}_n) &= \exp\left(-\frac{i}{2}\sum_{i<j}^{n}\tilde{q}_i\wedge\tilde{q}_j\right).\end{aligned} \tag{A2}$$

Proof. (A2) can be shown by mathematical induction:

(i) It holds for $k = 1, 2$.

(ii) Suppose that this relation holds for $k, l \le 1, \ldots, n$ $(n \ge 2)$, then for $n + 1 \le N = k + l \le 2n$,

$$\begin{aligned}&(c_{q_1}\star\cdots\star c_{q_k})\star(c_{q_{k+1}}\star\cdots\star c_{q_N})\\ &\quad = \mathcal{E}(\tilde{q}_1,\cdots,\tilde{q}_k)\mathcal{E}(\tilde{q}_{k+1},\cdots,\tilde{q}_N)\\ &\quad\quad\cdot(c_{q_1}\cdots c_{q_k})\star(c_{q_{k+1}}\cdots c_{q_N})\\ &\quad = \mathcal{E}(\tilde{q}_1,\cdots,\tilde{q}_k)\mathcal{E}(\tilde{q}_{k+1},\cdots,\tilde{q}_N)\\ &\quad\quad\cdot e^{-(i/2)\tilde{Q}_1\wedge\tilde{Q}_2}(c_{q_1}\cdots c_{q_k})\cdot(c_{q_{k+1}}\cdots c_{q_N})\\ &\quad = \mathcal{E}(\tilde{q}_1,\cdots,\tilde{q}_k,\tilde{q}_{k+1},\cdots,\tilde{q}_N)\\ &\quad\quad\cdot(c_{q_1}\cdots c_{q_k}\cdot c_{q_{k+1}}\cdots c_{q_N})\\ &\quad = \mathcal{E}(\tilde{q}_1,\cdots,\tilde{q}_N)\cdot(c_{q_1}\cdots c_{q_N}),\end{aligned} \tag{A3}$$

where $Q_1 = q_1 + \cdots + q_k$, $Q_2 = q_{k+1} + \cdots + q_N$. Hence, (A2) is satisfied for all natural number n, and this equation also proves the associativity of ⋆-product. □

Using the above theorem we find the action of creation and annihilation operators on the twisted states to be,

$$\begin{aligned}a_q^\dagger\star|q_1,\cdots,q_n\rangle_\star &= \mathcal{E}(q,q_1,\ldots,q_n)\cdot|q,q_1,\cdots,q_n\rangle\\ &= |q,q_1,\cdots,q_n\rangle_\star,\end{aligned} \tag{A4a}$$

$$\begin{aligned}a_q\star|q_1,\cdots,q_n\rangle_\star &= \mathcal{E}(-q,q_1,\cdots,q_n)\cdot a_q a_{q_1}^\dagger\cdots a_{q_n}^\dagger|0\rangle\\ &= \mathcal{E}(-q,q_1,\cdots,q_n)\sum_{k=1}^{n}\delta(q-q_k)|q_1,\\ &\quad\ldots,q_{k-1},q_{k+1},\ldots,q_n\rangle\\ &= \sum_{k=1}^{n}\delta(q-q_k)\mathcal{E}(-q,q_1,\cdots,q_{k-1},q,\\ &\quad q_{k+1},\cdots,q_n)|q_1,\ldots,q_{k-1},q_{k+1},\ldots,q_n\rangle\\ &= \sum_{k=1}^{n}\delta(q-q_k)e^{iq\wedge(q_1+\cdots+q_{k-1})}|q_1,\ldots,\\ &\quad q_{k-1},q_{k+1},\ldots,q_n\rangle_\star.\end{aligned} \tag{A4b}$$

APPENDIX B: PROPERTIES OF PHASE FACTOR $\mathcal{E}(q_1,\cdots,q_n)$

In this appendix we summarize the useful properties of the phase factor:

$$\mathcal{E}(q_1, \cdots, q_n) = \exp\left(-\frac{i}{2}\sum_{i<j}^{n} q_i \wedge q_j\right),$$

where for $n = 1$, we define $\mathcal{E}(q) = 1$. From direct calculation we find

$$\mathcal{E}(k, -k) = 1, \qquad \mathcal{E}(q, k, -k) = 1, \tag{B1}$$

and since the phase factor is quadratic in q's, we have

$$\mathcal{E}(q_1, \cdots, q_n) = \mathcal{E}(-q_1, \cdots, -q_n). \tag{B2}$$

The phase factor is invariant under the cyclic permutations $\mathcal{P}$ of $(q_1, \ldots, q_n)$ if $\sum q_k = 0$, i.e.,

$$\mathcal{E}(q_1, \cdots, q_n) = \mathcal{E}(q_{\mathcal{P}1}, \cdots, q_{\mathcal{P}n}), \quad \text{for } \sum q_k = 0. \tag{B3}$$

It also has the following property:

$$\mathcal{E}(p_1, \cdots, p_n, q_1, \cdots, q_m) = e^{-(i/2)(p_1+\cdots+p_n)\wedge(q_1+\cdots+q_m)} \times \mathcal{E}(p_1, \cdots, p_n)\mathcal{E}(q_1, \cdots, q_m). \tag{B4}$$

If the sum of m-consecutive q's is zero, the phase factor is factorized into the product of two factors:

$$\mathcal{E}(q_1, \cdots, q_n) = \mathcal{E}(q_1, \ldots, q_k, q_{k+m+1}, \ldots, q_n) \cdot \mathcal{E}(q_{k+1}, \ldots, q_{k+m}),$$

$$\text{if } \sum_{i=k+1}^{k+m} q_i = 0.$$

This is a direct consequence of (B3) and (B4). For $m = 2$ case, we have

$$\mathcal{E}(q_1, \ldots, p, -p, \ldots, q_n) = \mathcal{E}(q_1, \cdots, q_n). \tag{B5}$$

From the above relations, we have

$$\bar{\mathcal{E}}(q_1, \cdots, q_n) = \mathcal{E}(q_n, \cdots, q_1) = \mathcal{E}(q_{\mathcal{P}n}, \cdots, q_{\mathcal{P}1}), \tag{B6}$$

where $\mathcal{P}$ is a cyclic permutation.

From these results we can derive the properties of $\mathcal{E}$ given in Ref. [36]:

$$\mathcal{E}(q_1, \ldots, q_{n_1}, p) \cdot \mathcal{E}(-p, q_{n_1+1}, \ldots, q_{n_2}) = \mathcal{E}(q_1, \ldots, q_{n_2}), \quad \text{for } q_1 + \cdots + q_{n_1} + p = 0, \tag{B7a}$$

$$\mathcal{E}(q_1, \ldots, q_{n_1}, p, q_{n_1+1}, \ldots, q_{n_2}, -p) = \mathcal{E}(q_1, \ldots, q_{n_2}), \quad \text{for } q_{n_1+1} + \cdots + q_{n_2} = 0. \tag{B7b}$$

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