Vacuum solutions of five dimensional Einstein equations generated by inverse scattering method. II. Production of the black ring solution

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We study vacuum solutions of five-dimensional Einstein equations generated by the inverse scattering method. We reproduce the black ring solution which was found by Emparan and Reall by taking the Euclidean Levi-Cività metric plus one-dimensional flat space as a seed. This transformation consists of two successive processes; the first step is to perform the three-solitonic transformation of the Euclidean Levi-Cività metric with one-dimensional flat space as a seed. The resulting metric is the Euclidean *C*-metric with extra one-dimensional flat space. The second is to perform the two-solitonic transformation by taking it as a new seed. Our result may serve as a stepping stone to find new exact solutions in higher dimensions.

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I. INTRODUCTION

In recent years, studies of black holes in higher dimensions have attracted much attention in the context of string theory and the brane world scenario. In fact, it has been predicted that higher-dimensional black holes would be produced in a future linear collider [1,2]. Such physical phenomena are expected not only to give us a piece of evidence for the existence of extra dimensions but also to help us to draw some information toward quantum gravity. Studies on the classical equilibrium state of black holes are important since it is extrapolated that we may detect the Hawking evaporation after the formation of stationary black holes in a collider.

Some classical studies on higher-dimensional black holes show that they have much more complicated and richer structure than four-dimensional ones. For instance, the topology of the event horizon in higher dimensions cannot be uniquely determined [3–5] in contrast to four-dimensional ones, which is restricted only to the two sphere under null energy condition [6,7]. In five dimensions, however, the possible geometric types of the horizon topology are either S^3 or $S^1 \times S^2$ [3], and in dimensions higher than five, more complicated [4,5].

As for the asymptotically flat, static solutions of higher-dimensional vacuum Einstein equations, the Schwarzschild-Tangherlini solution [8] is the unique solution [9], and moreover, it is stable against gravitational perturbations [10]. These are common properties as the counterpart in four dimensions. However, the situation radically changes for the stationary spacetime; for asymptotically flat, stationary, and axisymmetric solutions of the five-dimensional vacuum Einstein equations, Emparan and Reall discovered the black ring solution whose horizon topology is $S^1 \times S^2$ [11]. The black ring is rotating along the S^1 direction, which is necessary for the ring to keep a balance against its self-gravitational attractive force. The black ring is regular everywhere and has no closed timelike curves on and outside the event horizon. In addition to the black ring solution, the rotating black hole solution with S^3 horizon topology had been already found by Myers and Perry [12]. This means that, in the sense of four dimensions, uniqueness theorems of black hole do not hold in higher dimensions [13], although it has been shown that the five-dimensional Myers-Perry solution is unique if the topology is restricted to S^3 and the spacetime admits three commuting Killing vectors [14]. Furthermore, as several arguments suggest that the black ring solution is unstable [15–17], there might exist a new family of solutions with less symmetry as the final phase of instability [18]. These rich structures in higher-dimensional spacetimes have made it desirable to use techniques to find a new solution.

A new stationary and axisymmetric black ring solution with asymptotic flatness, which is rotating only in the two sphere direction, was found by Mishima and Iguchi [19,20] by using one of the solitonic solution-generating techniques [21]. Solitonic solution-generating methods are mainly classified into two types. One is called Bäcklund transformation [22,23], which is basically the technique to generate a new solution of the Ernst equation. The other is the inverse scattering technique, which Belinski and Zakharov [24] developed as an another type of soliton-generating technique. Both methods have produced vacuum solutions from a certain known vacuum solution and succeeded in generation of some four-dimensional exact solutions. As a matter of fact, the Kerr black hole solution, the multi-Kerr black hole solutions, and the Tomimastu-Sato solutions can be obtained from the Minkowski seed (other physically interesting solutions can also be obtained by these methods [25,26]). The latter technique is essentially based on the fact that the Einstein's second-order

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nonlinear partial differential equations can be replaced with a pair of first-order linear partial differential equations called Lax pair. It can be immediately generalized to D-dimensional vacuum spacetimes with $(D-2)$-commuting Killing vectors. The complete integrability in five-dimensional Einstein-Maxwell systems is discussed in Ref. [27]. As discussed in [28], the former cannot generate black hole/ring solutions with two angular momentum components due to its ansatz, the latter has an advantage to be able to produce such solutions apart from whether these solutions are regular or not, though this happens only in five dimensions. Therefore, there would be the possibility that new black hole/ring solutions can be found by using this method.

Recently, some higher-dimensional black hole/ring solutions have been generated by means of the inverse scattering method. As an infinite number of static solutions of the five-dimensional vacuum Einstein equations with axial symmetry, the five-dimensional Schwarzschild solution and the static black ring solution were reproduced [29], which gave the first example of the generation of a higher-dimensional asymptotically flat black hole solution by the inverse scattering method. The Myers-Perry solution with single and double angular momenta were regenerated from the Minkowski [28,30] and some technical seed [31], respectively. The black ring solutions with S^2 rotation were also generated by using this method from the Minkowski seed [28]. Albeit several articles have been devoted to construct the black ring solution with S^1 rotation by this method, it has not been successful so far. In this article, we reproduce the S^1-rotating black ring solution from the Levi-Cività solution via the inverse scattering method.

This article is organized as follows. In Sec. II, the outline of the inverse scattering technique is summarized. In Sec. III, we start from the Euclidean Levi-Cività metric plus one-dimensional flat space; the C-metric plus one-dimensional flat space is obtained as a three-soliton solution. By taking this solution as a new seed, the black ring solution is recaptured as a two-soliton solution. We will see this is indeed the case in Sec. IV. We will frame our conclusion in Sec. V.

II. PRELIMINARY

We will give a summary account of the inverse scattering method developed by Belinski and Zakharov [24], which is straightforwardly applied into five dimensions.

A. Five-dimensional n-soliton solutions

We consider the asymptotically flat, five-dimensional stationary and axisymmetric vacuum spacetime with three commuting Killing vector fields $V_{(i)}$ $(i = 1, 2, 3)$ following the argument in [32,33]. The commutativity of Killing vectors $[V_{(i)}, V_{(j)}] = 0$ enables us to find a coordinate system such that $V_{(i)} = \partial/\partial x^i (i = 1, 2, 3)$ and the metric is independent of the coordinates x^i, where $(\partial/\partial x^1)$ is the Killing vector field associated with time translation and $(\partial/\partial x^2)$, $(\partial/\partial x^3)$ denote the spacelike Killing vector fields with closed orbits. In the following section, we will put $x^1 = t$, $x^2 = \phi$, and $x^3 = \psi$. Here we invoke the theorem in [32,33].

In D-dimensional spacetime, let $V_{(i)}, i = 1, \ldots, D-2$ be $(D-2)$-commuting Killing vector fields such that

(1) $V_{(1)}^{[\mu_1} V_{(2)}^{\mu_2} \cdots V_{(D-2)}^{\mu_{D-2}]} D^\nu V_{(i)}^{\rho} = 0$ holds at least one point of the spacetime for a given $i = 1, \ldots, D-2$.

(2) $V_{(i)}^{\nu} R_\nu^{[\rho} V_{(1)}^{\mu_1} V_{(2)}^{\mu_2} \cdots V_{(D-2)}^{\mu_{D-2}]} = 0$ holds for all $i = 1, \ldots, D-2$.

Then the two-planes orthogonal to the Killing vector fields $V_{(i)}, i = 1, \ldots, D-2$ are integrable.

Thanks to the axisymmetry, condition (1) holds on the axis of rotation. Meanwhile, condition (2) is automatically satisfied as long as we restrict ourselves to the vacuum solutions of Einstein equations $R_{\mu\nu} = 0$, where μ, ν run over all spacetime indices. In such a spacetime, the metric can be written in the canonical form [32,33] as

$$ds^2 = f(d\rho^2 + dz^2) + g_{ij} dx^i dx^j, \tag{1}$$

where $f = f(\rho, z)$ and $g_{ij} = g_{ij}(\rho, z)$ are a function and an induced metric on the three-dimensional space, respectively. Both of them depend only on the coordinates ρ and z. Here it is the most convenient to choose the 3×3 matrix $g = (g)_{ij}$ as to satisfy the condition

$$\det g = -\rho^2. \tag{2}$$

This is compatible with the vacuum Einstein equation $g^{ij} R_{ij} = 0$, which reduces to $(\partial_\rho^2 + \partial_z^2)(-\det g)^{1/2} = 0$. It follows from $R_{ij} = 0$ that the matrix g satisfies the solitonic equation

$$(\rho g_{,\rho} g^{-1})_{,\rho} + (\rho g_{,z} g^{-1})_{,z} = 0. \tag{3}$$

From the other components of the Einstein equations $R_{\rho\rho} - R_{zz} = 0$ and $R_{\rho z} = 0$, we obtain the equations which determine the function $f(\rho, z)$ for a given solution of the solitonic equation (3)

$$(\ln f)_{,\rho} = -\frac{1}{\rho} + \frac{1}{4\rho} \mathrm{Tr}(U^2 - V^2), \tag{4}$$

$$(\ln f)_{,z} = \frac{1}{2\rho} \mathrm{Tr}(UV), \tag{5}$$

where the 3×3 matrices $U(\rho, z)$ and $V(\rho, z)$ are defined by

$$U := \rho g_{,\rho} g^{-1}, \qquad V := \rho g_{,z} g^{-1}. \tag{6}$$

The integrability condition with respect to f is automatically satisfied for the solution g of Eq. (3). Note also that $R_{\rho\rho} + R_{zz} = 0$ is consistent with the solution (3)–(5); i.e., this system is not overdetermined.

Although our immediate goal is to solve the differential equations (3), it cannot be generally solved due to its nonlinearity. In analogy with the soliton technique, we can find the Lax pair for the matrix equations (3). We consider Schrödinger-type equations for the 3×3 matrix $\Psi(\lambda, \rho, z)$ as in four dimensions [24,25]:

$$D_1\Psi = \frac{\rho V - \lambda U}{\lambda^2 + \rho^2}\Psi, \qquad D_2\Psi = \frac{\rho U + \lambda V}{\lambda^2 + \rho^2}\Psi, \tag{7}$$

where λ is a complex spectral parameter independent of ρ and z. The differential operators D_1 and D_2 are defined as

$$D_1 := \partial_z - \frac{2\lambda^2}{\lambda^2 + \rho^2}\partial_\lambda, \qquad D_2 := \partial_\rho + \frac{2\lambda\rho}{\lambda^2 + \rho^2}\partial_\lambda, \tag{8}$$

which can be shown to commute $[D_1, D_2] = 0$. Note that Eq. (8) is invariant under the transformation $\lambda \rightarrow -\rho^2/\lambda$. Then the compatibility condition $[D_1, D_2]\Psi = 0$ reduces to the Einstein equations (3) with

$$g(\rho, z) = \Psi(0, \rho, z). \tag{9}$$

It deserves note that the Einstein's second-order nonlinear partial differential equations (3) are reduced to a pair of first-order linear partial differential equations (7).

Let g_0, U_0, V_0, and Ψ_0 be particular solutions of Eq. (3) and (7). We shall call the known solution g_0 the seed solution. We are going to seek a new solution of the form

$$\Psi = \chi\Psi_0, \tag{10}$$

which leads the following equations that the dressing matrix $\chi(\lambda, \rho, z)$ must satisfy:

$$D_1\chi = \frac{\rho V - \lambda U}{\lambda^2 + \rho^2}\chi - \chi\frac{\rho V_0 - \lambda U_0}{\lambda^2 + \rho^2}, \qquad D_2\chi = \frac{\rho U + \lambda V}{\lambda^2 + \rho^2}\chi - \chi\frac{\rho U_0 + \lambda V_0}{\lambda^2 + \rho^2}. \tag{11}$$

In order for the solutions $g(\rho, z)$ to be real and symmetric, we impose the following conditions on the dressing matrix χ:

$$\bar{\chi}(\bar{\lambda}, \rho, z) = \chi(\lambda, \rho, z), \qquad \bar{\Psi}(\bar{\lambda}, \rho, z) = \Psi(\lambda, \rho, z) \tag{12}$$

and

$$g = \chi(-\rho^2/\lambda, \rho, z)g_0{}^T\chi(\lambda, \rho, z), \tag{13}$$

where $\bar{\chi}$ and ${}^T\chi$ denote complex conjugation and the transposition of χ. From Eqs. (10) and (13), the dressing matrix χ asymptotes to a unit matrix $\chi \rightarrow I$ as $\lambda \rightarrow \infty$.

The general n-soliton solutions for the matrix g are generated due to the presence of the simple poles of the dressing matrix on the complex λ-plane:

$$\chi = I + \sum_{k=1}^{n}\frac{S_k}{\lambda - \mu_k}, \tag{14}$$

where the matrices S_k and the position of the pole μ_k depend only on the variables ρ and z. Here and hereafter, the subscript k, l counts the number of solitons. It is the characteristic feature of solitons that the dressing matrix χ is represented as the meromorphic function on the complex λ-plane. Pole trajectories $\mu_k(\rho, z)$ are determined by the condition that the left-hand side of Eq. (11) have no poles of second order at $\lambda = \mu_k$, which yields following two differential equations for $\mu_k(\rho, z)$:

$$\mu_{k,z} = -\frac{2\mu_k^2}{\mu_k^2 + \rho^2}, \qquad \mu_{k,\rho} = \frac{2\rho\mu_k}{\mu_k^2 + \rho^2}, \tag{15}$$

which are expressed by the solutions of the following quadratic equations:

$$\mu_k^2 + 2(z - w_k)\mu_k - \rho^2 = 0, \tag{16}$$

where w_k are arbitrary constants. Solving Eq. (16), one can easily find

$$\mu_k = w_k - z \pm \sqrt{(z - w_k)^2 + \rho^2}. \tag{17}$$

Since the matrices S_k are degenerate at the poles $S_k\chi^{-1}(\mu_k) = 0$, which follows from the condition $\chi\chi^{-1} = I$ at $\lambda = \mu_k$, it is possible to write down the matrix elements of S_k in the form

$$(S_k)_{ij} = n_i^{(k)}m_j^{(k)}. \tag{18}$$

The fact that Eq. (11) has no residues at the poles $\lambda = \mu_k$ leads one to obtain the vectors $m_i^{(k)}$ as

$$m^{i(k)} = m_{0j}^{(k)}[\Psi_0^{-1}(\mu_k, \rho, z)]^{ji}, \tag{19}$$

where $m_{0i}^{(k)}$ are arbitrary constants. The vectors $n_i^{(k)}$, on the other hand, are determined by the condition that Eq. (13) is regular at $\lambda = \mu_k$ as

$$n_i^{(k)} = \sum_{l=1}^{n}\mu_k^{-1}(\Gamma^{-1})_{kl}L_i^{(l)}, \tag{20}$$

where the vectors $L_i^{(k)}$ and the symmetric matrix Γ_{kl} are given by

$$L_i^{(k)} = m^{j(k)}(g_0)_{ij}, \tag{21}$$

$$\Gamma_{kl} = \frac{m^{i(k)}(g_0)_{ij}m^{j(l)}}{\rho^2 + \mu_k\mu_l}, \tag{22}$$

respectively. Therefore one can now find from Eqs. (9), (10), and (14) that the matrix g becomes

$$g_{ij}^{(\mathrm{unphys})} = \Psi(0, \rho, z)_{ij} \tag{23}$$

$$= (g_0)_{ij} - \sum_{k,l=1}^{n}(\Gamma^{-1})_{kl}\mu_k^{-1}\mu_l^{-1}L_i^{(k)}L_j^{(l)}. \tag{24}$$

This metric does not necessarily meet the condition $\det g = -\rho^2$, which we have denoted $g^{(\mathrm{unphys})}$. In order to satisfy the gauge condition $\det g = -\rho^2$, the metric should be appropriately normalized. One example is to normalize all the metric components by the same weight as

$$g^{(\mathrm{phys})} = (-1)^{n/3}\rho^{-2n/3}\bigg(\prod_{k=1}^{n}\mu_k^{2/3}\bigg)g^{(\mathrm{unphys})}, \tag{25}$$

where $g^{(\mathrm{phys})}$ is the metric which fulfills the condition $\det g = -\rho^2$. Actually, the four-dimensional Kerr solution is obtained similarly by the overall normalization as Eq. (25). Substituting the physical metric solution $g^{(\mathrm{phys})}$ given by Eq. (25) into Eq. (4) and (5), we obtain a physical value of f as

$$f = C_0 f_0 \rho^{-n(n-1)/3}\det(\Gamma_{kl})\prod_{k=1}^{n}[\mu_k^{2(n+2)/3}(\mu_k^2+\rho^2)^{-1/3}] \cdot \prod_{k>l}^{n}(\mu_k-\mu_l)^{-4/3}, \tag{26}$$

where C_0 is an arbitrary constant, and f_0 is a value of f corresponding to the seed g_0.

B. Relation between two diagonal generating matrices

In this subsection, we study how the diagonal generating matrices are mutually related. Let $\Psi = \mathrm{diag}(\psi_1, \psi_2, \psi_3)$ and $\tilde{\Psi} = \mathrm{diag}(\tilde{\psi}_1, \tilde{\psi}_2, \tilde{\psi}_3)$ be generating matrices corresponding to the diagonal seeds $g = \mathrm{diag}(g_1, g_2, g_3)$ and $\tilde{g} = \mathrm{diag}(\tilde{g}_1, \tilde{g}_2, \tilde{g}_3)$ satisfying Eq. (3), respectively. If they are related as $\tilde{g}_i = \Omega_i g_i (i = 1, 2, 3)$ for $\Omega_i = \Omega_i(\rho, z)$, it follows from the Einstein equations (3) that the logarithm of Ω_i must be harmonic functions on the three-dimensional Euclidean space:

$$\triangle \ln\Omega_i := \bigg(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{\partial^2}{\partial z^2}\bigg)\ln\Omega_i = 0. \tag{27}$$

Hence, the generating matrix $\tilde{\Psi} = \mathrm{diag}(\tilde{\psi}_1, \tilde{\psi}_2, \tilde{\psi}_3)$ corresponding to $\tilde{g}$ satisfies

$$\begin{aligned} D_1(\ln\tilde{\psi}_i) &= \frac{\rho V_i - \lambda U_i}{\rho^2+\lambda^2} + \frac{\rho^2(\ln\Omega_i)_{,z} - \rho\lambda(\ln\Omega_i)_{,\rho}}{\rho^2+\lambda^2}, \\ D_2(\ln\tilde{\psi}_i) &= \frac{\rho U_i + \lambda V_i}{\rho^2+\lambda^2} + \frac{\rho^2(\ln\Omega_i)_{,\rho} + \rho\lambda(\ln\Omega_i)_{,z}}{\rho^2+\lambda^2}, \end{aligned} \tag{28}$$

where $U_i = \rho g_{i,\rho} g_i^{-1}$ and $V_i = \rho g_{i,z} g_i^{-1}$.

Since these equations are linear, we can easily find that the solutions of Eq. (28) are expressed in the form of $\tilde{\psi}_i = \hat{\psi}(\Omega_i)\psi_i$, where ψ_i is the solution of Eq. (7) for the metric g, and $\hat{\psi}(\Omega_i)$ is the solution of the following equations:

$$\begin{aligned} D_1(\ln\hat{\psi}(\Omega_i)) &= \frac{\rho^2(\ln\Omega_i)_{,z} - \rho\lambda(\ln\Omega_i)_{,\rho}}{\rho^2+\lambda^2}, \\ D_2(\ln\hat{\psi}(\Omega_i)) &= \frac{\rho^2(\ln\Omega_i)_{,\rho} + \rho\lambda(\ln\Omega_i)_{,z}}{\rho^2+\lambda^2}. \end{aligned} \tag{29}$$

If $\hat{\psi}(\Omega_i)$ and $\hat{\psi}(\omega_i)$ are solutions of Eq. (29), we can find that $\hat{\psi}(\Omega_i\omega_i)$ is also the solution of Eq. (29) and satisfies the relation $\hat{\psi}(\Omega_i\omega_i) = \hat{\psi}(\Omega_i)\hat{\psi}_i(\omega_i)$. We list the relevant solutions of Eq. (29) in our discussion:

$$\hat{\psi}(1) = 1, \qquad \hat{\psi}(\rho^2) = \rho^2 - 2z\lambda - \lambda^2 \tag{30}$$

$$\hat{\psi}(\lambda_1) = \lambda_1 - \lambda, \qquad \hat{\psi}(\lambda_2) = \lambda_2 + \lambda, \tag{31}$$

where λ_1 and λ_2 are given by

$$\lambda_1 := \sqrt{\rho^2 + (z-z_0)^2} - (z - z_0), \tag{32}$$

$$\lambda_2 := \sqrt{\rho^2 + (z-z_0)^2} + (z - z_0). \tag{33}$$

Here z_0 is an arbitrary constant.

III. GENERATION OF SEED SOLUTION FOR BLACK RING

As has been shown in [28], the black ring solution rotating S^2-direction [19] is obtained as a two-soliton solution by taking the five-dimensional Minkowski spacetime as a seed solution. In order to obtain the black ring solution with S^1-rotation, we start from the following metric:

$$ds^2 = -dt^2 + \rho^{1+d}d\psi^2 + \rho^{1-d}d\phi^2 + \rho^{(d^2-1)/2}(d\rho^2 + dz^2). \tag{34}$$

where d is a constant. The metric (34) is the direct product of the Euclidean Levi-Cività metric and one-dimensional flat space. Thus, it is a vacuum solution of five-dimensional Einstein equations. Performing the Wick rotation $\psi \to i\bar{t}$, the four-dimensional base space describes the Levi-Cività solution

$$ds^2 = -\rho^{1+d}d\bar{t}^2 + \rho^{1-d}d\phi^2 + \rho^{(d^2-1)/2}(d\rho^2 + dz^2). \tag{35}$$

We can find that the Minkowski and Rindler spacetime are recovered by setting $d = -1, 1$, respectively. $d = 0$ corresponds to one of Kinnersley's type D metrics [34]. From the Levi-Cività metric (35), we can obtain the Weyl class solutions in four dimensions by adding solitons [35].

For the diagonal seed solutions such as (34), diagonal solutions can be obtained by taking some of the parameters vanishing, e.g., $m_{01}^{(k)} = m_{02}^{(k)} = 0$. The assumption that the seed solution g_0 is of diagonal form quite simplifies analyses since the generating matrix Ψ_0 may be set to be diagonal $\Psi_0 = \mathrm{diag}(\psi_1, \psi_2, \psi_3)$, where ψ_i are functions which depend on λ, ρ, and z. If this is the case, the partial differential equations for the generating matrix Ψ_0 are decoupled into each component, which allow us to solve each ψ_i independently.

We should comment on the normalization of the metric a little more carefully. In the previous section, we have normalized all the metric components by the same weight as seen in Eq. (25). Nevertheless, we should note that there exists some freedom for the normalization when we construct a physical metric for the case in which the generated solution has at most one nondiagonal component. The main reason is that in this particular case the soliton equations (3) are decomposed into each component. As we will see, we adopt a distinct normalization for our solution because some components of the generated solution are identical to those of the seed (see Appendix A). Taking the case of $m_{01}^{(k)} = 0$, for example, the three-dimensional unphysical metric $g^{(\mathrm{unphys})}$ is decomposed into the 2 + 1 block matrix as

$$g^{(\mathrm{unphys})} = \begin{pmatrix} (g_0)_{11} & 0 \\ 0 & g_{AB}^{(\mathrm{unphys})} \end{pmatrix}, \tag{36}$$

where $g_{AB}^{(\mathrm{unphys})}$ $(A, B = 2, 3)$ is a 2×2 matrix dependent on the $2n$ parameters $m_{02}^{(k)}$, $m_{03}^{(k)}$ $(k = 1, \ldots, n)$; i.e., we add solitons exclusively to the block diagonal components $(g_0)_{AB}$. In the present case, we may choose a normalization which multiplies only $g_{AB}^{(\mathrm{unphys})}$ by the normalization factor in such a way that the metric satisfies the supplementary condition $\det g = -\rho^2$. Namely, we leave the component $(g_0)_{11}$ intact, i.e.,

$$g^{(\mathrm{phys})} = \begin{pmatrix} (g_0)_{11} & 0 \\ 0 & \left(\prod_{k=1}^{n} \frac{\mu_k}{\rho}\right) g_{AB}^{(\mathrm{unphys})} \end{pmatrix}. \tag{37}$$

We can easily show that, if a seed metric satisfies the condition $\det g_0 = -\rho^2$, the physical metric (37) also satisfies this condition. The general expression for the diagonal n-soliton solution under the normalization (37) is

$$g_{tt} = (g_0)_{tt}, \qquad g_{\psi\psi} = \left[\prod_{k=1}^{n}\left(\frac{\mu_k}{\rho}\right)\right](g_0)_{\psi\psi}, \qquad g_{\phi\phi} = -\frac{\rho^2}{g_{tt}g_{\psi\psi}}. \tag{38}$$

Here we focus attention on the three real-pole trajectories and the Levi-Cività metric with $d = 0$ is taken as a seed. As in [36], we choose the three parameters $w_k (k = 1, 2, 3)$ such that

$$w_1 = -\eta_1\sigma, \qquad w_2 = \eta_2\sigma, \qquad w_3 = \kappa\sigma, \tag{39}$$

where η_1, η_2, κ are constants satisfying

$$\kappa \geq 1, \quad -1 < \eta_1 < 1, \quad -1 < \eta_2 < 1, \quad \eta_1 + \eta_2 > 0. \tag{40}$$

The pole trajectories are given by

$$\mu_1^{\pm} = \pm R_{-\eta_1\sigma} - (z + \eta_1\sigma), \qquad \mu_2^{\pm} = \pm R_{\eta_2\sigma} - (z - \eta_2\sigma), \qquad \mu_3^{\pm} = \pm R_{\kappa\sigma} - (z - \kappa\sigma), \tag{41}$$

where we have set

$$R_d := \sqrt{\rho^2 + (z - d)^2}. \tag{42}$$

Note that $\mu_k^+ \mu_k^- = -\rho^2$ for each k. The resulting metric component is

$$g_{\psi\psi} = \frac{\mu_1^- \mu_2^+ \mu_3^-}{\rho^2} \tag{43}$$

$$= \frac{(R_{\kappa\sigma} + z - \kappa\sigma)(R_{\eta_2\sigma} - z + \eta_2\sigma)}{R_{-\eta_1\sigma} - z - \eta_1\sigma}. \tag{44}$$

The sign choice in Eq. (43) is for the sake of convenience. From Eq. (38), we have

$$g_{\phi\phi} = \frac{(R_{-\eta_1\sigma} - z - \eta_1\sigma)(R_{\kappa\sigma} - z + \kappa\sigma)}{R_{\eta_2\sigma} - z + \eta_2\sigma}, \tag{45}$$

from which we can obtain the rest of the metric components by the straightforward quadratures (4) and (5). Making the coordinate transformation in Appendix A, we can find the resulting three-soliton solution is the direct product spacetime of the Euclidean vacuum C-metric [37,38] and the one-dimensional flat space

$$ds^2 = -dt^2 + \frac{2\tilde{\kappa}^2}{(u - v)^2} \times \left[-G(u)d\psi^2 + G(v)d\phi^2 - \frac{du^2}{G(u)} + \frac{dv^2}{G(v)}\right], \tag{46}$$

where $G(\xi)$ is the structure function given by

$$G(\xi) = (1 - \xi^2)(1 + c\xi). \tag{47}$$

Since the metric (46) is the Euclidean vacuum C-metric plus one-dimensional flat space, it is obviously the vacuum solution of the five-dimensional Einstein equations. The C-metric describes a pair of point particles undergoing uniform acceleration [37]. $A = (\sqrt{2}\tilde{\kappa})^{-1}$ and $m = c\tilde{\kappa}/\sqrt{2}$ represent the acceleration and mass of the point particles, respectively.

In the next section, we use the solution

$$ds^2 = -dt^2 + g_2 d\phi^2 + g_3 d\psi^2 \tag{48}$$

as a seed to obtain the black ring solution, where

$$g_2 := \frac{(R_{-\eta_1\sigma} - z - \eta_1\sigma)(R_{\kappa\sigma} - z + \kappa\sigma)}{(R_{\eta_2\sigma} - z + \eta_2\sigma)}, \qquad g_3 := \frac{(R_{-\eta_1\sigma} + z + \eta_1\sigma)(R_{\eta_2\sigma} - z + \eta_2\sigma)}{(R_{\kappa\sigma} - z + \kappa\sigma)}. \tag{49}$$

The seed solution (48) satisfies the supplementary condition $\det g = -\rho^2$ and coincides with the seed obtained by Iguchi and Mishima to generate the black ring solution with the S^1-rotation [36]. For the $\kappa = \eta_2$ case, the seed (48) reduces to the Minkowski spacetime. Supposing $\Psi = \mathrm{diag}(\psi_1, \psi_2, \psi_2)$, we can find from the analysis in Sec II B that the generating matrix for the background metric (48) is expressed as

$$\psi_1[\lambda] = -1,$$
$$\psi_2[\lambda] = \frac{(R_{-\eta_1\sigma} - z - \eta_1\sigma - \lambda)(R_{\kappa\sigma} - z + \kappa\sigma - \lambda)}{(R_{\eta_2\sigma} - z + \eta_2\sigma - \lambda)},$$
$$\psi_3[\lambda] = \frac{(R_{-\eta_1\sigma} + z + \eta_1\sigma + \lambda)(R_{\eta_2\sigma} - z + \eta_2\sigma - \lambda)}{(R_{\kappa\sigma} - z + \kappa\sigma - \lambda)}, \tag{50}$$

where we have omitted the argument ρ and z in each ψ_i for simplicity, which is understood to represent $\psi_i[\lambda] = \psi_i(\lambda, \rho, z)$.

IV. PRODUCTION OF BLACK RING SOLUTION

In view of the fact that the $\psi - \psi$ component of the seed solution coincides that of the black ring of Emparan and Reall (see Appendix A), it is suitable to normalize the metric so that $(g_0)_{33}$ is unchanged:

$$g^{(\mathrm{phys})} = \begin{pmatrix} \left(\prod_{k=1}^{n} \frac{\mu_k}{\rho}\right) g_{AB}^{(\mathrm{unphys})} & 0 \\ 0 & (g_0)_{33} \end{pmatrix}, \tag{51}$$

where A, $B = 1$, 2. Here we consider the two-soliton solution. We choose the sign of plus in Eq. (17) and take the constants $w_1 = -w_2 = -\sigma$ without loss of generality. Under the special normalization (51), the two-soliton solution can be written in the following form:

$$g_{tt}^{(\mathrm{phys})} = -\frac{G_{tt}}{\mu_1\mu_2\Sigma}, \qquad g_{t\phi}^{(\mathrm{phys})} = -g_2\frac{(\rho^2 + \mu_1\mu_2)G_{t\phi}}{\mu_1\mu_2\Sigma},$$
$$g_{\phi\phi}^{(\mathrm{phys})} = -g_2\frac{G_{\phi\phi}}{\mu_1\mu_2\Sigma}, \tag{52}$$

$$g_{\psi\psi}^{(\mathrm{phys})} = g_3, \qquad g_{\phi\psi}^{(\mathrm{phys})} = g_{t\psi}^{(\mathrm{phys})} = 0, \tag{53}$$

where the functions G_{tt}, $G_{t\phi}$, $G_{\phi\phi}$ and Σ are defined as

$$\begin{aligned} G_{tt} = {}& -m_{01}^{(1)2}m_{01}^{(2)2}\psi_2[\mu_1]^2\psi_2[\mu_2]^2(\mu_1 - \mu_2)^2\rho^4 + m_{01}^{(1)2}m_{02}^{(2)2}g_2\mu_2^2(\rho^2 + \mu_1\mu_2)^2\psi_2[\mu_1]^2 \\ & + m_{01}^{(2)2}m_{02}^{(1)2}g_2\mu_1^2(\rho^2 + \mu_1\mu_2)^2\psi_2[\mu_2]^2 - m_{02}^{(1)2}m_{02}^{(2)2}g_2^2\mu_1^2\mu_2^2(\mu_1 - \mu_2)^2 \\ & - 2m_{01}^{(1)}m_{01}^{(2)}m_{02}^{(1)}m_{02}^{(2)}g_2\psi_2[\mu_1]\psi_2[\mu_2](\rho^2 + \mu_1^2)(\rho^2 + \mu_2^2)\mu_1\mu_2, \end{aligned} \tag{54}$$

$$\begin{aligned} G_{\phi\phi} = {}& m_{01}^{(1)2}m_{01}^{(2)2}\mu_1^2\mu_2^2(\mu_1 - \mu_2)^2\psi_2[\mu_1]^2\psi_2[\mu_2]^2 + m_{02}^{(1)2}m_{02}^{(2)2}g_2^2(\mu_1 - \mu_2)^2\rho^4 \\ & - m_{01}^{(1)2}m_{02}^{(2)2}g_2\mu_1^2\psi_2[\mu_1]^2(\rho^2 + \mu_1\mu_2)^2 - m_{01}^{(2)2}m_{02}^{(1)2}g_2\mu_2^2(g_2 - \mu_2)^2(\rho^2 + \mu_1\mu_2)^2 \\ & + 2m_{01}^{(1)}m_{01}^{(2)}m_{02}^{(1)}m_{02}^{(2)}g_2\mu_1\mu_2\psi_2[\mu_2]\psi_2[\mu_1](\rho^2 + \mu_1^2)(\rho^2 + \mu_2^2), \end{aligned} \tag{55}$$

$$\begin{aligned} G_{t\phi} = {}& m_{01}^{(1)}m_{01}^{(2)2}m_{02}^{(1)}\mu_2(\mu_1 - \mu_2)\psi_2[\mu_2]^2\psi_2[\mu_1](\rho^2 + \mu_1^2) + m_{01}^{(1)}m_{02}^{(1)}m_{02}^{(2)2}g_2\mu_2(\mu_2 - \mu_1)\psi_2[\mu_1](\rho^2 + \mu_1^2) \\ & + m_{01}^{(1)2}m_{01}^{(2)}m_{02}^{(2)}\mu_1(\mu_2 - \mu_1)\psi_2[\mu_1]^2\psi_2[\mu_2](\rho^2 + \mu_2^2) + m_{01}^{(2)}m_{02}^{(1)2}m_{02}^{(2)}\mu_1g_2\psi_2[\mu_2](\rho^2 + \mu_2^2)(\mu_1 - \mu_2), \end{aligned} \tag{56}$$

$$\begin{aligned} \Sigma = {}& m_{01}^{(1)2}m_{01}^{(2)2}\psi_2[\mu_1]^2\psi_2[\mu_2]^2(\mu_1 - \mu_2)^2\rho^2 + m_{02}^{(1)2}m_{02}^{(2)2}g_2^2(\mu_1 - \mu_2)^2\rho^2 + m_{01}^{(1)2}m_{02}^{(2)2}g_2\psi_2[\mu_1]^2(\rho^2 + \mu_1\mu_2)^2 \\ & + m_{02}^{(1)2}m_{01}^{(2)2}g_2\psi_2[\mu_2]^2(\rho^2 + \mu_1\mu_2)^2 - 2m_{01}^{(1)}m_{01}^{(2)}m_{02}^{(1)}m_{02}^{(2)}g_2\psi_2[\mu_1]\psi_2[\mu_2](\rho^2 + \mu_1^2)(\rho^2 + \mu_2^2), \end{aligned} \tag{57}$$

where the two functions g_2 and g_3 are given by Eq. (49).

In order for the metric to approach the Minkowski spacetime asymptotically, let us consider the coordinate transformation of the physical metric such that

$$t \to t' = t - C_1\phi, \qquad \phi \to \phi' = \phi, \tag{58}$$

where C_1 is a constant chosen to ensure the asymptotic flatness. We should note that the transformed metric also satisfies the supplementary condition $\det g = -\rho^2$. Under this transformation, the physical metric components become

$$\begin{aligned} & g_{tt}^{(\mathrm{phys})} \to g_{t't'}^{(\mathrm{phys})} = g_{tt}^{(\mathrm{phys})}, \\ & g_{t\phi}^{(\mathrm{phys})} \to g_{t'\phi'}^{(\mathrm{phys})} = g_{t\phi}^{(\mathrm{phys})} + C_1g_{tt}^{(\mathrm{phys})}, \\ & g_{\phi\phi}^{(\mathrm{phys})} \to g_{\phi'\phi'}^{(\mathrm{phys})} = g_{\phi\phi}^{(\mathrm{phys})} + 2C_1g_{t\phi}^{(\mathrm{phys})} + C_1^2g_{tt}^{(\mathrm{phys})}. \end{aligned} \tag{59}$$

If we choose the parameters such that

$$m_{01}^{(1)}m_{01}^{(2)} = \beta, \tag{60}$$

$$m_{01}^{(2)}m_{02}^{(1)} = \sigma^{1/2}(\kappa_2 - 1), \tag{61}$$

$$m_{01}^{(1)}m_{02}^{(2)} = -\sigma^{1/2}\alpha\beta(\kappa_1 + 1), \tag{62}$$

$$m_{02}^{(1)}m_{02}^{(2)} = -\sigma\alpha(\kappa_1 + 1)(\kappa_2 - 1), \tag{63}$$

$$C_1 = \frac{2\sigma^{1/2}\alpha}{1 + \alpha\beta}, \tag{64}$$

with

$$\begin{aligned} \kappa_1 + 1 &= \frac{(-\kappa + 1)(\eta_1 + 1)}{-\eta_2 + 1}, \\ \kappa_2 - 1 &= \frac{(\kappa + 1)(\eta_1 - 1)}{\eta_2 + 1}, \end{aligned} \tag{65}$$

and use the prolate spheroidal coordinate (x, y) defined by

$$\rho = \sigma\sqrt{(x^2-1)(1-y^2)}, \qquad z = \sigma xy, \tag{66}$$

we can confirm that the transformed metric coincides with the expression of a black ring solution in [36] with the aid of the equations in Appendix B:

$$\begin{aligned} ds^2 = &-\frac{A}{B}\left[dt - \left(2\sigma g_3^{-1/2}\frac{C}{A} + C_1\right)d\phi\right]^2 \\ &+ \frac{B}{A}g_3^{-1}\rho^2 d\phi^2 + g_3 d\psi^2 \\ &+ C_2\frac{x^2-y^2}{x^2-1}Be^{2\gamma'}g_3^{-1}\left(\frac{dx^2}{x^2-1} + \frac{dy^2}{1-y^2}\right), \end{aligned} \tag{67}$$

where

$$A := (x^2-1)(1+FG)^2 - (1-y^2)(F-G)^2, \tag{68}$$

$$B := [(x+1) + (x-1)FG]^2 + [(1+y)F + (1-y)G]^2, \tag{69}$$

$$\begin{aligned} C := &(x^2-1)(1+FG)[G - F - y(F+G)] \\ &+ (1-y^2)(G-F)[1 + FG + x(1-FG)], \end{aligned} \tag{70}$$

$$F := -\alpha\frac{\sigma^{3/2}(-\kappa+1)(\eta_1+1)}{-\eta_2+1}\frac{g_2^{1/2}(x+1)(1-y)}{\psi_2[\mu_2]\rho}, \tag{71}$$

$$G := -\beta\frac{(\eta_2+1)}{(\kappa+1)(\eta_1-1)\sigma^{3/2}}\frac{\rho\psi_2[\mu_1]}{g_2^{1/2}(x-1)(1-y)}, \tag{72}$$

$$\begin{aligned} \gamma' := &\tfrac{1}{4}[\ln(U_\sigma U_{-\sigma}^{-1}U_{\kappa\sigma}^2 U_{-\eta_1\sigma}^2 U_{\eta_2\sigma}^{-2}) \\ &- \ln(Y_{\sigma,\sigma}Y_{-\sigma,-\sigma}Y_{\kappa\sigma,\kappa\sigma}Y_{-\eta_1\sigma,-\eta_1\sigma}Y_{\eta_2\sigma\eta_2\sigma}Y_{\sigma,-\sigma}^{-2} \\ &\times Y_{\sigma,\kappa\sigma}Y_{-\sigma,\kappa\sigma}^{-1}Y_{\sigma,-\eta_1\sigma}Y_{\sigma,\eta_2\sigma}^{-1}Y_{-\sigma,-\eta_1\sigma}^{-1}Y_{-\sigma,\eta_2\sigma} \\ &\times Y_{\kappa\sigma,-\eta_1\sigma}^2 Y_{\kappa\sigma,\eta_2\sigma}^{-2}Y_{\eta_1\sigma,\eta_2\sigma}^{-2})], \end{aligned} \tag{73}$$

$$U_c := R_c + z - c, \tag{74}$$

$$Y_{c,d} := R_c R_d + (z-c)(z-d) + \rho^2, \tag{75}$$

$$C_2 := \frac{\sigma^2}{(1+\alpha\beta)^2}. \tag{76}$$

The constant C_2 is chosen to assure the asymptotic flatness of the solution. When $\kappa = \eta_2$, this solution reduces to the one obtained in [28], which is the S^2-rotating black ring solution. At first sight, the solution (67) includes six parameters σ, α, β, κ, η_1, and η_2, but one of which is a kinematical parameter so that one can freely choose the value of it. The solution is then specified by the remaining five parameters. This solution, however, has closed timelike curves in general. The condition for the absence of closed timelike curves restricts the value of α and β as [36]

$$\alpha = \sqrt{\frac{2(1-\eta_2)}{(\kappa-1)(1+\eta_1)}}, \qquad \beta = \sqrt{\frac{(\kappa+1)(1-\eta_1)}{2(1+\eta_2)}}. \tag{77}$$

These follow from the condition that the orbit $(\partial/\partial\phi)$ should close at $(x, y) = (1, \pm 1)$. The remaining free parameters are three. However, the above condition is not sufficient for the metric to be well-posed throughout the spacetime since the solution still has conical singularities. The lack of conical singularities, furthermore, imposes the following restriction on the rest of three parameters to

$$1 + \alpha\beta = \sqrt{\frac{\kappa+1}{\kappa-1}}\left(\frac{\kappa-\eta_2}{\kappa+\eta_1}\right)\left(1 + \alpha\beta\frac{\kappa-1}{\kappa+1}\right). \tag{78}$$

Consequently, the resultant metric is characterized by two parameters. As shown in Appendix A, under these physically reasonable conditions, this solution (67) accords with the one found by Emparan and Reall. Thus, the remaining two parameters are concerned with the mass and angular momentum, which specifies the black ring solution.

V. SUMMARY AND DISCUSSION

In this article, we studied the inverse scattering method for the asymptotically flat, stationary, and axisymmetric spacetimes of the five-dimensional vacuum Einstein equations. Our main result is the reproduction of the black ring solution found by Emparan and Reall via the inverse scattering technique. The steps of the procedure we have done is as follows: in the first place, we consider the direct product spacetime with one-dimensional timelike flat space and a four-dimensional base space. The four-dimensional base space is taken to be the Euclidean Levi-Cività solution with $d = 0$ (35), whence this metric is apparently the vacuum solution of the five-dimensional Einstein equations with cylindrical symmetry. Considering the fact that the static black ring solution belongs to the five-dimensional Weyl class solutions, it may be reasonable to take the Levi-Cività metric as a starting point since it describes the family of the cylindrically symmetric static solutions of vacuum Einstein equations, from which we can obtain the Weyl class solutions in four dimensions by adding solitons via the inverse scattering method. Subsequently, the Euclidean vacuum C-metric is generated as a three-soliton solution from the Levi-Cività background with $d = 0$ [35]. The appearance of the C-metric seems to be convincing since the original derivation of black ring solution has employed the Euclidean dilatonic C-metric. Lastly, the black ring solution with S^1-rotation is constructed as a two-soliton solution from the seed of the Euclidean Levi-Cività solution plus one-dimensional flat space. This solution coincides with the one obtained by Iguchi and Mishima [36], and encompasses a broader class than the one found by Emparan and Reall. But it generally has closed timelike curves and conical singularity in the vicinity of the event horizon. If we impose that the solution is free from causal violation, we finally obtain the black ring solution of Emparan and Reall (see Appendix A). The condition for the absence of conical singularity comes down to the completely regular black ring solution, which is characterized by the mass and angular momentum. Our consequence is the first example that clarifies the physical origin of the seed solution and indicates the possibility of finding new exact solutions in higher dimensions.

The point we would like to stress is the special normalization such as (37) and (51), namely, some of the components of the seed remain unchanged. Even the five-dimensional Schwarzschild solution is generated in this way [29]. We also comment that, if we want to obtain a nontrivial regular black holes/ring solution with nonvanishing two angular momentum components, the overall normalization (25) does not seem to yield regular spacetimes as two-soliton solutions unless the seed solution satisfies the condition $\det g = -\rho^2$. To obtain the metric with nonvanishing off-diagonal components, it may be ineluctable to adopt some technical seed in the sense that it does not satisfy the condition $\det g = -\rho^2$, which is to say, it is not the solution of vacuum Einstein equations although it is the solution of the Lax pair. In fact, the seed to generate the Myers-Perry black hole solution with two nonvanishing angular momentum components in [31] does not meet $\det g = -\rho^2$. Otherwise, it may be necessary to evaluate the multisoliton solutions.

It is also worth noting that the generated solution (67) reduces to the S^2-rotating black ring solution when $\kappa = \eta_2$ with $\eta_1 > 1$, because the Minkowski spacetime is recovered when $\kappa = \eta_2$ in the C-metric seed [see Eqs. (48) and (49)]. In consequence, our generation is the most general one rather than those heretofore discussed in that all the previous results are recaptured as special cases of our model; the S^2-rotating black ring solution [28], the Myers-Perry solution with one angular momentum [30], and of course the five-dimensional Schwarzschild solution [29] are realized.

Complete integrability of the Einstein equations highly resorts to the fact that the metric is written by the canonical form (1); on top of that we focus our attention exclusively on the solitonic solutions. In higher dimensions than five, even the Minkowski spacetime does not fall into this type due to the lack of Killing vectors meeting certain requirements. Accordingly, our discussions cannot be straightforwardly applicable into higher dimensions than five. In order to obtain such solutions, we might have to formulate the inverse scattering method in other coordinate systems rather than the canonical coordinate (1).

As mentioned in the Introduction, all asymptotically flat five-dimensional vacuum black hole solutions with a connected component of the event horizon which has already been discovered in this stage can be produced via the inverse scattering method. We expect to be able to construct new higher-dimensional exact solutions by this powerful tool although we have not yet found such solutions. We can speculate from the results in the present paper that an infinite class of higher-dimensional stationary and axisymmetric solutions of vacuum Einstein equations can be generated by means of the inverse scattering method by taking the higher-dimensional Weyl class solution as a seed. Besides that, it may also be of interest to include several matter fields appearing in string theory and supergravity. These prospective issues are worth investigating in the future.

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APPENDIX A: *C*-METRIC COORDINATES AND CANONICAL FORM

Define the C-metric coordinates (u, v) as

$$\rho := \frac{2\tilde{\kappa}^2\sqrt{-G(u)G(v)}}{(u-v)^2},\qquad \tilde{z} := z + \frac{\eta_1 - \eta_2}{2}\sigma = \frac{\tilde{\kappa}^2(1-uv)(2+cu+cv)}{(u-v)^2}, \tag{A1}$$

where $G(\xi)$ is a structure function given by

$$G(\xi) = (1-\xi^2)(1+c\xi). \tag{A2}$$

We have introduced new constants $c(0 < c < 1)$ and $\tilde{\kappa}$ defined as

$$c = \frac{\eta_1 + \eta_2}{2\kappa + \eta_1 - \eta_2}, \tag{A3}$$

$$\tilde{\kappa}^2 = \left(\kappa + \frac{\eta_1 - \eta_2}{2}\right)\sigma. \tag{A4}$$

The coordinates u and v lie in the ranges

$$-1 \le u \le 1, \qquad v \le -1. \tag{A5}$$

Defining a new constant b as

$$b = \frac{[\kappa + 1 + (\kappa - 1)\alpha\beta]^2 - (\kappa^2 - 1)(1+\alpha\beta)^2}{[\kappa + 1 + (\kappa - 1)\alpha\beta]^2 + (\kappa^2 - 1)(1+\alpha\beta)^2}, \tag{A6}$$

the black ring solution (67) with the conditions (77) and (78) is then transcribed as [11,32]

$$ds^2 = -\frac{F(v)}{F(u)}\left(dt - \mathcal{C}\kappa\frac{1+v}{F(v)}d\phi\right)^2 + \frac{2\tilde{\kappa}^2F(u)}{(u-v)^2}\times\left[-\frac{G(v)}{F(v)}d\phi^2 + \frac{G(u)}{F(u)}d\psi^2 + \frac{du^2}{G(u)} - \frac{dv^2}{G(v)}\right], \tag{A7}$$

where

$$F(\xi) = 1 + b\xi, \tag{A8}$$

$$\mathcal{C} = \sqrt{2b(b-c)\frac{1+b}{1-b}}. \tag{A9}$$

The parameters b and c take the value in the range

$$0 < c \leq b < 1. \tag{A10}$$

The ring (A7) generally has the conical singularity, which is cured by choosing the parameter b such that

$$b = \frac{2c}{1+c^2}. \tag{A11}$$

In the canonical coordinate ρ and z, the metric is rewritten as

$$g_{11} = -\frac{(1+b)(1-c)R_1 + (1-b)(1+c)R_2 - 2(b-c)R_3 - 2b(1-c^2)\tilde{\kappa}^2}{(1+b)(1-c)R_1 + (1-b)(1+c)R_2 - 2(b-c)R_3 + 2b(1-c^2)\tilde{\kappa}^2}, \tag{A12}$$

$$g_{12} = -\frac{2\mathcal{C}\tilde{\kappa}(1-c)[R_3 - R_1 + (1+c)\tilde{\kappa}^2]}{(1+b)(1-c)R_1 + (1-b)(1+c)R_2 - 2(b-c)R_3 + 2b(1-c^2)\tilde{\kappa}^2}, \tag{A13}$$

$$g_{22} = -\frac{\rho^2}{g_{11}g_{33}} + \frac{g_{12}^2}{g_{11}}, \tag{A14}$$

$$g_{33} = \frac{(R_3 + z - \tilde{\kappa}^2)(R_2 - z + c\tilde{\kappa}^2)}{R_1 - z - c\tilde{\kappa}^2}, \tag{A15}$$

$$f = [(1+b)(1-c)R_1 + (1-b)(1+c)R_2 - 2(b-c)R_3 + 2b(1-c^2)\tilde{\kappa}^2]\frac{(1-c)R_1 + (1+c)R_2 + 2cR_3}{8(1-c^2)^2R_1R_2R_3}, \tag{A16}$$

where we have defined

$$R_1 = \sqrt{\rho^2 + (\tilde{z} + c\tilde{\kappa}^2)^2}, \qquad R_2 = \sqrt{\rho^2 + (\tilde{z} - c\tilde{\kappa}^2)^2}, \qquad R_3 = \sqrt{\rho^2 + (\tilde{z} - \tilde{\kappa}^2)^2}. \tag{A17}$$

Observe that the g_{33} components of the black ring (A15) are the same as that of the seed (48).

APPENDIX B: ALTERNATIVE EXPRESSION FOR BLACK RING

We present the useful expression for the metric components of a general black ring solution. Substituting Eqs. (71) and (72) into (68)–(70), another expression of the metric components is derived as

$$g_{tt} = -\frac{\tilde{A}}{\tilde{B}}, \qquad g_{t\phi} = 2\sigma^{1/2}g_2\frac{\tilde{C}}{\tilde{B}} + C_1\frac{\tilde{A}}{\tilde{B}}. \tag{B1}$$

Here we have introduced new functions $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$ defined as

$$\begin{aligned}\tilde{A} = &-\beta^2\psi_2[\mu_1]^2\psi_2[\mu_2]^2(1+y)^2\\ &+ \sigma\alpha^2\beta^2(\kappa_1+1)^2g_2\psi_2[\mu_1]^2(x+1)^2\\ &+ \sigma(\kappa_2-1)^2g_2\psi_2[\mu_2]^2(x-1)^2\\ &- \sigma^2\alpha^2(\kappa_1+1)^2(\kappa_2-1)^2g_2^2(1-y)^2\\ &+ 2\sigma\alpha\beta(\kappa_1+1)(\kappa_2-1)g_2\psi_2[\mu_1]\psi_2[\mu_2](x^2-y^2),\end{aligned} \tag{B2}$$

$$\begin{aligned}\tilde{B} = &\beta^2\psi_2[\mu_1]^2\psi_2[\mu_2]^2(1-y)^2\\ &+ \sigma^2\alpha^2(\kappa_1+1)^2(\kappa_2-1)^2g_2^2(1-y^2)\\ &+ \sigma\alpha^2\beta^2(\kappa_1+1)^2\psi_2[\mu_1]^2g_2(x^2-1)\\ &+ \sigma(\kappa_2-1)^2\psi_2[\mu_2]^2g_2(x^2-1)\\ &+ 2\sigma\alpha\beta(\kappa_1+1)(\kappa_2-1)g_2\psi_2[\mu_1]\psi_2[\mu_2](x^2-y^2),\end{aligned} \tag{B3}$$

$$\begin{aligned}\tilde{C} = &-\beta(\kappa_2-1)\psi_2[\mu_1]\psi_2[\mu_2]^2(x+y)\\ &- \alpha\beta^2(\kappa_1+1)\psi_2[\mu_1]^2\psi_2[\mu_2](x-y)\\ &+ \sigma\alpha^2\beta(\kappa_2-1)(\kappa_1+1)^2g_2\psi_2[\mu_1](x+y)\\ &+ \sigma\alpha(\kappa_1+1)(\kappa_2-1)^2\psi_2[\mu_2](x-y),\end{aligned} \tag{B4}$$

where $\psi_2[\mu_1]$ and $\psi_2[\mu_2]$ are obtained by Eq. (50).

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