Higgs mechanism for gravity. II. Higher spin connections

Nicolas Boulanger
Université de Mons-Hainaut, Académie Wallonie-Bruxelles, Mécanique et Gravitation, Avenue du Champ de Mars 6, B-7000 Mons, Belgium

Ingo Kirsch
Jefferson Laboratory of Physics, Harvard University, Cambridge, Massachusetts 02138, USA
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We continue the work of [Phys. Rev. D **72**, 024001 (2005)] in which gravity is considered as the Goldstone realization of a spontaneously broken diffeomorphism group. We complete the discussion of the coset space Diff $(d, \mathbb{R})/SO(1, d-1)$ formed by the d-dimensional group of analytic diffeomorphisms and the Lorentz group. We find that this coset space is parametrized by coordinates, a metric, and an infinite tower of higher-spin or generalized connections. We then study effective actions for the corresponding symmetry breaking which gives mass to the higher spin connections. Our model predicts that gravity is modified at high energies by the exchange of massive higher spin particles.

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I. INTRODUCTION

Gauge field theories have a long and successful history in elementary particle physics. As it is generally known, the starting point for gauging is the experimental observation of a conserved charge which, via Noether's theorem, is related to a rigid symmetry. In particular, a conserved energy-momentum current corresponds to the invariance under global space-time translations. Since energy momentum is the source of gravity, one expects the gravitational interaction to emerge from gauging the global translational symmetry. Indeed, general relativity (GR) can be derived by gauging the translational group as was first conclusively shown in [1]. The gauge status of gravity remains however rather subtle, see e.g. [2–6] and many references therein.

In this paper we adopt the view that gauge theories of gravity describe only the low-energy effective, i.e. massless degrees of freedom of a more general gravitational theory featuring a spontaneously broken space-time symmetry. This view is based on a theorem in [7] which states that the gauge theory associated with a local group $G_{\rm loc}$ can be obtained by the nonlinear realization of the corresponding infinite-parameter group G with the Poincaré group H being the vacuum stability group. The Minkowski metric η_{ij} is assumed to be given from the beginning. Mechanisms for the selection of the signature of the metric have been proposed in [8,9].

Let us illustrate the theorem for the case in which $G_{\rm loc}$ is the local translational group $T(d)$ in d dimensions. By construction, the group $T(d)$ is locally isomorphic to the infinite-parameter group Diff $(d, \mathbb{R})$ of analytic diffeomorphisms [10]. Then, according to the theorem of [7], the (simplest) gauge theory of the translational group $G_{\rm loc} = T(d)$, general relativity, can be derived by nonlinearly realizing $G = \text{Diff}\,(d, \mathbb{R})$.

The key observation in the proof of this theorem is that the gauge potential of $G_{\rm loc}$, here the tetrad $e_i{}^j$, can be identified with a parameter of the coset space G/H. In fact, applying the nonlinear realization method [11] adapted to space-time groups [12–15], one finds that G/H is parametrized by the field $h^i{}_j$ and an infinite tower of fields ${}^{(s)}\omega^i{}_{j_1\ldots j_s k}$ ($s \geqslant 1$). The exponential of $h^i{}_j$, $e_j{}^i \equiv (e^h)^i{}_j$ transforms exactly as a tetrad [13]. The translational gauge potential thus arises as one of the Goldstone fields of a spontaneously broken diffeomorphism invariance.

This was first explicitly shown by Borisov and Ogievetsky [16]. These authors used the fact that the infinite-dimensional algebra of analytic diffeomorphisms can be represented as the closure of two finite-dimensional algebras [10]. This splitting, however, impeded the discussion of the remaining Goldstone fields ${}^{(s)}\omega$. It was pointed out later in [7] that the fields ${}^{(s)}\omega$ may acquire mass through a Higgs effect, leaving the tetrads as the only massless degrees of freedom. Since gauging translations only provides the tetrads but not the massive fields ${}^{(s)}\omega$, the gauge principle leads to the correct low-energy effective theory, at least as long as the masses of ${}^{(s)}\omega$ are high enough. However, if one takes the idea of a spontaneously broken diffeomorphism invariance seriously, the gravitational interaction will be modified at high energies by the exchange of massive fields ${}^{(s)}\omega$.

This paper is devoted to the study of these coset fields and the construction of a spontaneous symmetry-breaking mechanism [17] for the diffeomorphism group. (For brevity, we will refer to it as "Higgs mechanism," although this terminology is unfair, see Refs. [17].) To gain some insight into this Higgs mechanism, we first complete the nonlinear realization of Diff $(d, \mathbb{R})$ studied in [16,18,19] by providing the complete transformation laws for all coset fields of $\text{Diff}\,(d,\mathbb{R})/SO(1,d-1)$. We find that the fields ${}^{(s)}\omega$ naturally generalize the concept of a linear connection; the transformation law is inhomogeneous and contains the $s+1$th derivative of the diffeomorphism parameter $\varepsilon^i(x)$.

The nonlinear realization will also provide gauge transformations of the (linearized) form

$$ {}^{(1)}\omega'_{ijk} = {}^{(1)}\omega_{ijk} - \partial_k h_{ij}, \tag{1}$$

$$ {}^{(s)}\omega'^{i}{}_{j_1\ldots j_s k} = {}^{(s)}\omega^{i}{}_{j_1\ldots j_s k} - \partial_k {}^{(s-1)}\omega^{i}{}_{j_1\ldots j_s} \qquad (s \geqslant 2), \tag{2}$$

which show a mutual absorption of the Goldstone fields ${}^{(s)}\omega^{i}{}_{j_1\ldots j_s k}$ and h_{ij}. The generalized connection ${}^{(s)}\omega$ of level s eats the connection ${}^{(s-1)}\omega$ of level $s-1$, while the ordinary connection ${}^{(1)}\omega^{i}{}_{jk}$ absorbs the metric h_{ij}. This will give mass to each of the generalized connections ${}^{(s)}\omega$ ($s \geqslant 2$) and ${}^{(1)}\omega_{(ij)k}$. As the field with the lowest spin in the coset space, the metric remains massless.

In the second part of the paper we model the gravitational Higgs mechanism by concrete actions. We restrict here to find actions for the lowest two absorption processes given by Eqs. (1) and (2) for $s = 1, 2$. We consider these models as describing only a part of the full Higgs mechanism for the complete diffeomorphism group which, in full generality, appears to be quite complex.

The first model we present describes the breaking of the linear group $GL(d, \mathbb{R}) \subset \mathrm{Diff}\,(d, \mathbb{R})$ down to the Lorentz group $SO(1, d-1)$. Here we assume that the generalized connections ${}^{(s)}\omega$ with $s \geqslant 2$ have already been decoupled and we are left with a massless linear connection ${}^{(1)}\omega^{i}{}_{jk}$ of an effective affine space-time. The breaking of the tangential group $GL(d, \mathbb{R})$ will then be induced by the introduction of the metric as a Higgs field. This involves the absorption process (1) by which the symmetric part ${}^{(1)}\omega'_{(ij)k}$ of the connection acquires mass.

The model is largely based on that given in [19]. There are two essential improvements: (i) We explicitly show that the Higgs mechanism leads to a massive spin-3 field associated with the totally symmetric field ${}^{(1)}\omega'_{(ijk)}$. (ii) The field which was introduced in [19] as a kind of gravitational analog to the (so-called) Higgs particle plays now the role of an auxiliary field in the Singh-Hagen formulation [20] of the massive spin-3 field. We therefore do not predict a new Higgs particle.

The second model describes the absorption process (2) for $s = 2$ by means of which the field ${}^{(2)}\omega'^{i}{}_{j_1 j_2 k}$ becomes massive. The model is along the lines of the so-called "telescopic Higgs effect" (see [21] and references therein), more recently also known as "La Grande Bouffe" [22]. Here we aim at the more modest goal of constructing the Stückelberg Lagrangian for the massive field ${}^{(2)}\omega'^{i}{}_{j_1 j_2 k}$.

In the last part of the paper, we discuss a possible relation between the coset fields ${}^{(s)}\omega$ and higher spin connections as first introduced in the gauge formalism of higher spin fields in [23]. Note that a relation between the latter formalism and the nonlinear realization approach was recently pointed out in [24]. We also show that a space endowed with generalized connections satisfies the strong equivalence principle and is equivalent to a space-time with a velocity-dependent affine connection.

The paper is organized as follows. In Sec. II we study the coset space $\mathrm{Diff}\,(d, \mathbb{R})/SO(1, d-1)$ by means of the nonlinear realization approach [11]. We also discuss the double role of Goldstone fields in gravity as absorber fields and fields which get absorbed by other Goldstone fields. In Sec. III we construct Higgs models which lead to a ultraviolet modification of general relativity. In Sec. IV we discuss a possible link between the generalized connections ${}^{(s)}\omega$ and higher spin connections known from the literature. We also discuss the geometrical structure of a space-time equipped with generalized connections. We conclude in Sec. V with some final remarks and open questions.

II. NONLINEAR REALIZATION OF THE ANALYTIC DIFFEOMORPHISM GROUP

In this section we consider the (left) coset space $\mathrm{Diff}\,(d, \mathbb{R})/SO(1, d-1)$ formed by the d-dimensional group of analytic diffeomorphisms $\mathrm{Diff}\,(d, \mathbb{R})$ and its stabilizing Lorentz subgroup $SO(1, d-1)$. We show that the coset space is parametrized by a coordinate field, a metric and an infinite tower of generalized connections.

A. Review on the diffeomorphism algebra

We begin by briefly reviewing the algebra of analytic diffeomorphisms. The diffeomorphism algebra is generated by an infinite tower of generators $F_i^{(m)j_1\ldots j_{m+1}}$ ($m = -1, \ldots, \infty$) which are symmetric in the $m+1$ upper indices. The lowest generators are the translations $P_i \equiv F_i^{(-1)}$ and the generators of the linear group $L_i{}^j \equiv F_i^{(0)j}$. Generators $F_i^{(m)j_1\ldots j_{m+1}}$ with $m \geqslant 1$ generate nonlinear transformations.

The corresponding diffeomorphism algebra $\mathrm{diff}\,(d, \mathbb{R})$ is given by the commutation relations

$$[F_k^{(n)i_1\ldots i_{n+1}}, F_l^{(m)j_1\ldots j_{m+1}}] = i\sum_{a=1}^{m+1} \delta_k^{j_a} F_l^{(m+n)i_1\ldots i_{n+1} j_1\ldots \hat{j}_a \ldots j_{m+1}} - i\sum_{a=1}^{n+1} \delta_l^{i_a} F_k^{(m+n)i_1\ldots \hat{i}_a \ldots i_{n+1} j_1\ldots j_{m+1}}, \tag{3}$$

where indices with a hat are omitted. We easily identify the Lorentz (sub)algebra

$$[M_{ij}, M_{kl}] = i\eta_{j[k}M_{l]i} - i\eta_{i[k}M_{j]l}, \tag{4}$$

with Lorentz generators $M_{ij} \equiv L_{[i}{}^{k}\eta_{j]k} = F_{[i}^{(0)k}\eta_{j]k}$. We denote complete strength-one antisymmetrization on indices by using square brackets, while complete strength-one symmetrization is denoted by curved brackets. For example, $F^{(0)k}_{[i}\eta_{j]k} \equiv \frac{1}{2}(F^{(0)k}_{i}\eta_{jk} - F^{(0)k}_{j}\eta_{ik})$.

B. The coset space Diff $(d, \mathbb{R})/SO(1, d-1)$ and generalized connections

The coset space $G/H = \mathrm{Diff}\,(d, \mathbb{R})/SO(1, d-1)$ is parametrized by the fields ξ^i (d parameters), h_{ij} ($d(d+1)/2$ parameters), and an infinite set of fields ${}^{(s)}\omega^i{}_{j_1\ldots j_s k}$ ($s \geqslant 1$), each with

$$d\binom{d+s}{s+1} \tag{5}$$

components. These fields are associated with broken translations P_i, shear transformations and dilations $T_{ij} = L_{(ij)}$, and generators $F_i^{(s)j_1\ldots j_s k}$ ($s \geqslant 1$), respectively.

As was shown in [18,19] the parameters ξ^i transform as coordinates under the diffeomorphism group,

$$\delta\xi^i = \varepsilon^i(\xi), \tag{6}$$

with $\varepsilon^i(\xi)$ the parameters of Diff $(d, \mathbb{R})$. As explained in detail in [19], the breaking of translations makes the parameters of Diff $(d, \mathbb{R})$ dependent on the coordinates ξ^i and turns the coset parameters into space-time dependent fields. For a recent discussion on the tight link between the coset fields ξ^i and space-time coordinates, see [5].

The transformation behavior of the coset field $h_{ij}(\xi)$ has been known since the very first publications on nonlinear realizations of space-time groups [12,16]. It is usually given for the exponential

$$e_j{}^i \equiv (e^h)^i{}_j = \delta^i_j + h^i{}_j + h^i{}_k h^k{}_j/2 + \ldots \tag{7}$$

which transforms as a tetrad [16]. From now on we will be using Greek indices for the Minkowski metric $\eta_{\alpha\beta}$ and define a space-time metric, as usual, by

$$g_{ij} = e_i{}^\alpha e_j{}^\beta \eta_{\alpha\beta}. \tag{8}$$

Finally, the field $\omega^i{}_{jk}(\xi)$ associated with $F_i^{(1)jk}$ was shown to transform as a linear connection under the diffeomorphism group [18,19],

$$\delta\omega^i{}_{jk} = \frac{\partial\varepsilon^i}{\partial\xi^m}\omega^m{}_{jk} - 2\frac{\partial\varepsilon^m}{\partial\xi^{(j}}\omega^i{}_{k)m} + \frac{1}{2}\frac{\partial^2\varepsilon^i}{\partial\xi^j\partial\xi^k}. \tag{9}$$

Since $\omega^i{}_{jk}$ is symmetric in the indices j and k, there is no torsion.

So far not much attention has been paid to the fields ${}^{(s)}\omega^i{}_{j_1\ldots j_s k}$ associated with the nonlinear generators $F^{(s)}$ with $s > 1$. These fields are completely symmetric in the lower $s+1$ indices, which is ultimately a consequence of the assumed commutativity of the coordinates of $\mathbb{R}^d$ [19].

Using the general nonlinear realization technique [11–16], in App. A we compute the infinitesimal transformation law for the fields ${}^{(s)}\omega^i{}_{j_1\ldots j_s k}$. We obtain

$$\begin{aligned}\delta_\varepsilon {}^{(s)}\omega^i{}_{j_1\ldots j_s k} &= \varepsilon^i{}_m {}^{(s)}\omega^m{}_{j_1\ldots j_s k} - (s+1)\varepsilon^m{}_{(j_1} {}^{(s)}\omega^i{}_{j_2\ldots j_s k)m} \\ &\quad + \varepsilon^i{}_{j_1\ldots j_s k} + \mathcal{O}({}^{(s-1)}\omega),\end{aligned} \tag{10}$$

with

$$\varepsilon^i{}_{j_1\ldots j_s} = \frac{1}{s!}\frac{\partial^s\varepsilon^i}{\partial\xi^{j_1}\cdots\partial\xi^{j_s}}, \tag{11}$$

which generalizes Eq. (9) ($s=1$) to arbitrary values of s. The left-hand side (l.h.s.) of Eq. (10) is defined to be $\delta_\varepsilon {}^{(s)}\omega^i{}_{j_1\ldots j_s k} = {}^{(s)}\omega'^i{}_{j_1\ldots j_s k}(\xi') - {}^{(s)}\omega^i{}_{j_1\ldots j_s k}(\xi)$. We identify Eq. (10) with the transformation behavior of a *generalized connection*: The first line in Eq. (10) is the tensor part of the transformation, while the first term in the second line shows the inhomogeneity which contains the $s+1$th derivative of the diffeomorphism parameter $\varepsilon^i(\xi)$ [18]. The finite form of the transformation law is given by

$$\begin{aligned}{}^{(s)}\omega'^i{}_{j_1\ldots j_s k} &= \frac{\partial\xi'^i}{\partial\xi^m}\frac{\partial\xi^{l_1}}{\partial\xi'^{j_1}}\cdots\frac{\partial\xi^{l_s}}{\partial\xi'^{j_s}}\frac{\partial\xi^n}{\partial\xi'^k}{}^{(s)}\omega^m_{l_1\ldots l_s n} \\ &\quad - \frac{\partial\xi^{l_1}}{\partial\xi'^{j_1}}\cdots\frac{\partial\xi^{l_s}}{\partial\xi'^{j_s}}\frac{\partial\xi^n}{\partial\xi'^k}\frac{\partial^{s+1}\xi'^i}{\partial\xi^{l_1}\cdots\partial\xi^{l_s}\partial\xi^n} \\ &\quad + \mathcal{O}({}^{(s-1)}\omega).\end{aligned} \tag{12}$$

Upon substituting $\xi'^i = \xi^i + \varepsilon^i(\xi)$ with $\varepsilon^i(\xi)$ small into (12) and redefining ${}^{(s)}\omega \to -(s+1)!{}^{(s)}\omega$, we regain the infinitesimal transformation (10).

A new feature of the generalized connections is the occurrence of additional terms in the transformation law (10) which are summarized in $\mathcal{O}({}^{(s-1)}\omega)$. By using a convenient bracket notation, in App. A we give an algorithm to compute the complete transformation $\delta^{(s)}\omega$ including all terms in $\mathcal{O}({}^{(s-1)}\omega)$. In general, these terms contain connections of lower spin. For instance, the transformation law for the connection $\delta^{(2)}\omega^i{}_{j_1 j_2 k}$, Eq. (A13), contains the term

$$2\varepsilon^i{}_{l(j_1}\omega^l{}_{j_2 k)}. \tag{13}$$

This term involves the ordinary linear connection $\omega^i{}_{jk}$ which has one index less than ${}^{(2)}\omega^i{}_{j_1 j_2 k}$. Generalized connections mix and cannot be considered independently from each other.

In App. C we decompose the fields ${}^{(s)}\omega^i{}_{j_1\ldots j_s k}$ with respect to the general linear group and determine their spin content. We find that, unless further constraints are imposed, these fields describe several states of different spin, where the highest state possesses spin $s+2$. For instance, the highest component of a general linear connection $\omega^i{}_{jk}$ ($s=1$) has spin 3, see e.g. [3].[1] This leads us

[1]For a general linear connection $\omega^i{}_{jk}$, the totally symmetric component $\omega_{(ijk)}$ is nonvanishing. See [25] for recent works in metric-affine theory of gravity, where exact solutions are built that display a propagating spin-3 component of the linear connection.

TABLE I. Goldstone fields parametrizing the infinite-dimensional coset space Diff $(d, \mathbb{R})/SO(1, d-1)$.

Broken symmetry	Generators	Geometrical field
Translations	$P_i \equiv F_i^{(-1)}$	coordinates ξ^i
Shears/dilations	$T_{ij} \equiv F_{(ij)}^{(0)}$	metric g_{ij}
Nonlinear Transformations	$F_i^{(s)j_1\ldots j_s k}$ $(s \geqslant 1)$	gen. connections ${}^{(s)}\omega^i{}_{j_1\ldots j_s k}$

to the presumption that the generalized connections ${}^{(s)}\omega$ are related to higher spin connections. For the construction of actions, we will implicitly assume this relation. We will come back to the possible link with higher spin connections in Sec. IV A.

In Table I we summarize the parameters of the coset space $G/H = \mathrm{Diff}\,(d, \mathbb{R})/SO(1, d-1)$ and give their geometrical interpretation. We have shown that the fields ξ^i, h_{ij}, ${}^{(1)}\omega^i{}_{jk}$, ${}^{(2)}\omega^i{}_{jkl}$, etc. can be regarded as coordinates, metric, and an infinite tower of generalized connections, respectively. As we will see in Sec. III, most of these Goldstone bosons become massive though and decouple at low energies. Einstein gravity results as the appropriate low-energy effective theory of gravity.

As in [19] we define the space-time manifold $\mathcal{M}$ as that part of the coset space G/H which is spanned by the (global) translations. This part is parametrized by the coordinates ξ^i. If we wish to recover Einstein gravity at low energies, we need to have *local* Poincaré invariance in the tangent space of the manifold $\mathcal{M}$. Local translational invariance is ensured by the diffeomorphism invariance of the manifold $\mathcal{M}$, cf. Eq. (6).[2] Local Lorentz invariance is more subtle to see. Note that the vacuum stability group $H \subset \mathrm{Diff}\,(d, \mathbb{R})$ is just the *global* Lorentz group. However, in the present nonlinear realization the group H induces *local* Lorentz transformations: Recall that in the transformation law for coset elements $\sigma \in G/H$ [11,12],

$$g\sigma(\xi) = \sigma(\xi')h(\xi, g), \tag{14}$$

the elements $h \in H$ depend nonlinearly on $g \in G$ and the coset parameters ξ. Since global translations are broken, the group elements h depend, in particular, on the coordinates ξ^i, i.e. they are functions of ξ^i, $h = h(\xi^i, \ldots)$. We can thus perform an independent Lorentz transformation at each space-time point.

C. The total nonlinear connection

Let us now turn to the total nonlinear connection one-form Γ which can be expanded in the generators of $G = \mathrm{Diff}\,(d, \mathbb{R})$ as

$$\Gamma = i\vartheta^\alpha P_\alpha + i\sum_{s=1}^{\infty} {}^{(s)}\Gamma^\alpha{}_{\beta_1\ldots\beta_s} F_\alpha^{(s-1)\beta_1\ldots\beta_s}. \tag{15}$$

In order to find the coefficients ϑ^α and $\Gamma^\alpha{}_{\beta_1\ldots\beta_s}$ $(s \geqslant 1)$, we fix the stabilizing group to be $H = SO(1, d-1)$ such that an element σ of the coset space G/H is parametrized by

$$\sigma = e^{i\xi^m P_m} e^{ih^{ij}T_{ij}} e^{i\omega^i{}_{j_1 j_2} F_i^{(1)j_1 j_2}} \cdots e^{i{}^{(s)}\omega^i{}_{j_1\ldots j_{s+1}} F_i^{(s)j_1\ldots j_{s+1}}} \cdots. \tag{16}$$

The coefficients of the total nonlinear connection $\Gamma = \sigma^{-1}d\sigma$ are then given by the one-forms

$$\vartheta^\alpha = (e^{-1})_k{}^\alpha d\xi^k, \tag{17}$$

$${}^{(1)}\Gamma^\alpha{}_\beta = (e^{-1})_k{}^\alpha de^k{}_\beta - 2\omega^\alpha{}_{\beta\gamma}\vartheta^\gamma, \tag{18}$$

$$\begin{aligned}{}^{(2)}\Gamma^\alpha{}_{\beta_1\beta_2} = {}& d\omega^\alpha{}_{\beta_1\beta_2} - 3\omega^\alpha{}_{\beta_1\beta_2\gamma}\vartheta^\gamma - \omega^\delta{}_{\beta_1\beta_2}\omega^\alpha{}_{\delta\gamma}\vartheta^\gamma \\ & + 2\omega^\alpha{}_{\delta(\beta_1}\omega^\delta{}_{\beta_2)\gamma}\vartheta^\gamma + (e^{-1})_l{}^\alpha de^l{}_\gamma\omega^\gamma{}_{\beta_1\beta_2} \\ & - 2\omega^\alpha{}_{\gamma(\beta_1|}(e^{-1})_l{}^\gamma de^l{}_{|\beta_2)}\end{aligned} \tag{19}$$

and for $s \geqslant 3$ by

$$\begin{aligned}{}^{(s)}\Gamma^\alpha{}_{\beta_1\ldots\beta_s} = {}& d^{(s-1)}\omega^\alpha{}_{\beta_1\ldots\beta_s} - (s+1)^{(s)}\omega^\alpha{}_{\beta_1\ldots\beta_s\gamma}\vartheta^\gamma \\ & + \mathcal{O}({}^{(s-1)}\omega^2).\end{aligned} \tag{20}$$

Here we used Latin $(i, j, \ldots)$ and Greek $(\alpha, \beta, \ldots)$ letters for holonomic and anholonomic (frame) indices, respectively. $\mathcal{O}({}^{(s-1)}\omega^2)$ denotes terms of quadratic order and higher in ${}^{(s-1)}\omega, {}^{(s-2)}\omega, \ldots, {}^{(1)}\omega$. A general formula for the coefficients ${}^{(s)}\Gamma^\alpha{}_{\beta_1\ldots\beta_s}$ is given to all orders by Eq. (B2) in App. B. To evaluate Eq. (B2), one may use the bracket notation introduced in App. A.

As spelled out in [19], the coefficients ϑ^α and ${}^{(1)}\Gamma^\alpha{}_\beta$ can be interpreted as the coframe and linear connection. Linear connections can also be obtained by the gauging of the linear group as first proposed in [26] and elaborated on in metric-affine gravity [3], see also [27]. We observe that nonlinear realizations of Diff $(d, \mathbb{R})$ provide an alternative derivation of the linear connection.

Note that it is not possible in the nonlinear realization approach to single out a single generalized connection (or a finite number of such connections). Assume we break only a single generator $F^{(s)}$ which gives rise to a single connection ${}^{(s)}\omega$. Then, terms of higher order would be absent in Eqs. (10) and (20). However, the stabilizing subgroup H is not closed in this case, since e.g. the commutator $[F^{(s-1)}, F^{(1)}]$ ends on $F^{(s)}$. The nonlinear generators $F^{(s)}$ $(s \geqslant 1)$ can thus only be broken as a whole. This property is shared by the higher spin algebras, see e.g. [28] and references therein.

[2] G can be considered as a principal H-bundle over G/H, π: $G \to G/H$. Recall from [19], Sec. III A that we gain local translational invariance on $\mathcal{M}$ (i.e. on the base space G/H) at the expense of losing global translational invariance in the fiber.

D. Higgs phenomenon and double role of Goldstone fields in gravity

For the following it is useful to recall the Higgs phenomenon in elementary particle physics. For instance, in $U(1)$ gauge theory the gauge boson A_μ (spin 1) becomes massive due to the absorption of a Goldstone scalar ϕ. Usually this is achieved by the $U(1)$ gauge transformation

$$A'_\mu = A_\mu + \partial_\mu \phi \tag{21}$$

turning the Goldstone field ϕ into the longitudinal mode of the massive gauge boson A'_μ.

In the coset realization under consideration, the gravitational analog of Eq. (21) is given by the coefficients of the total nonlinear connection Γ. Equations (18)–(20) can be regarded as redefinitions of the generalized connections. There are basically two absorption processes:

(I) $s = 1$: The ordinary spin connection ${}^{(1)}\omega^\alpha{}_{\beta k}$ absorbs the degrees of freedom of the tetrad $e^i{}_\alpha$ as can be seen from Eq. (18). Since the tetrads are related to the shear and dilation parameters, this corresponds to the breaking of $GL(d, \mathbb{R})$ down to $SO(1, d-1)$.

(II) $s > 1$: The generalized connections ${}^{(s)}\omega$ eat the fields ${}^{(s-1)}\omega$ as described by Eq. (20). The connections ${}^{(s-1)}\omega$ parametrize the coset space $\mathrm{Diff}_0(d, \mathbb{R})/GL(d, \mathbb{R})$, where $\mathrm{Diff}_0(d, \mathbb{R})$ is the homogeneous part of the diffeomorphism group.

The absorption takes place in such a way that the fields ${}^{(s)}\Gamma^\alpha{}_{\beta_1\ldots\beta_s}$ $(s \geqslant 1)$ and ${}^{(1)}\Gamma_{(\alpha\beta)}$ $(\subset \Gamma_{G/H})$ turn into rank-$s+2$ tensors, while ${}^{(1)}\Gamma_{[\alpha\beta]}$ $(= \Gamma_H)$ remains a true connection. Recall that the coset part $\Gamma_{G/H}$ of the total connection transforms homogeneously under the diffeomorphism group, while Γ_H is a true connection.[3]

The coset fields ${}^{(s)}\omega$ play a fascinating double role at the absorption process as can be seen by linearizing Eq. (20):

$${}^{(s)}\Gamma^\alpha{}_{\beta_1\ldots\beta_s k} = \partial_k {}^{(s-1)}\omega^\alpha{}_{\beta_1\ldots\beta_s} - (s+1){}^{(s)}\omega^\alpha{}_{\beta_1\ldots\beta_s k}. \tag{22}$$

Here ${}^{(s-1)}\omega$ behaves as a genuine Goldstone field which gets absorbed by the field ${}^{(s)}\omega$. The same field plays however a different role on the next lower level. Considering Eq. (22) for $s-1$ instead of s, we see that ${}^{(s-1)}\omega$ itself absorbs ${}^{(s-2)}\omega$. In this aspect it resembles more the characteristic behavior of a gauge boson.

The fact that in gravity Goldstone bosons can also take over the role of absorber fields is related to the "inverse Higgs effect" [29], see also [30] for a recent review. Goldstone's theorem states that there is a *massless* mode for each broken symmetry. However, since some of the Goldstone bosons can become massive for spontaneously broken space-time groups [12], the theorem gives only an upper bound on the number of massless Goldstone modes.

The standard example is the spontaneous breaking of the conformal group $SO(4, 2)$ down to the Poincaré group ISO(1, 3) [12,30]. From the dimension of the coset space one would expect five massless Goldstone bosons, one corresponding to scale transformations and four corresponding to special conformal transformations. However, the special conformal parameter φ_μ becomes massive by the absorption of the dilaton ϕ as can be seen from the total nonlinear connection component along the dilations $\Gamma_D = \varphi'_\mu dx^\mu = (2\varphi_\mu - \partial_\mu \phi)dx^\mu$.

It is usually argued [29,30] that one can set the part $\Gamma_{G/H}$ of the total nonlinear connection Γ to zero, $\Gamma_{G/H} = 0$. This gives relations among the coset fields, which reduces the actual number of massless Goldstone fields. For instance, setting $\Gamma_D = 0$ in the above example implies that $2\varphi_\mu$ can be replaced by $\partial_\mu \phi$. Note however that $\Gamma_{G/H} = 0$ should be interpreted as an effective equation, since one ignores all massive Goldstone bosons (φ'_μ in the above example). This constraint is justified only at energies much below the mass of these Goldstone bosons.

In the realization considered in this paper, the relation $\Gamma_{G/H} = 0$ translates into ${}^{(s)}\Gamma = 0$ $(s > 1)$ and ${}^{(1)}\Gamma_{(\alpha\beta)} = 0$. The constraint ${}^{(s)}\Gamma = 0$ $(s > 1)$ relates all the generalized Goldstone connections by

$${}^{(s-1)}\omega_{[\alpha\beta_1]\ldots\beta_{s-1}k} = \frac{2}{s!}\partial_{\beta_2}\ldots\partial_{\beta_{s-1}}{}^{(1)}\omega_{[\alpha\beta_1]k}. \tag{23}$$

We have shown in [19] that by setting ${}^{(1)}\Gamma_{(\alpha\beta)} = 0$, the affine connection ${}^{(1)}\Gamma_{[\alpha\beta]k}$ becomes metric compatible, i.e. equivalent to the Christoffel connection. In this way all higher spin connections are given in terms of derivatives of the tetrad. We stress again that this is only true at low energies, where all higher spin fields are assumed to be decoupled.

III. HIGGS MECHANISM FOR GRAVITY

The spontaneous breaking of the diffeomorphism group down to the Lorentz group gives rise to an infinite tower of higher spin connections as well as to the metric. In this section we construct actions for some parts of the corresponding Higgs mechanism by which the higher spin connections get massive. Assuming their decoupling at low energies, general relativity results as the appropriate effective low-energy description of gravity.

A. The breaking of dilations and shear transformations

One part of the symmetry breaking of the diffeomorphism group is the spontaneous breaking of its linear subgroup $GL(d, \mathbb{R}) \subset \mathrm{Diff}(d, \mathbb{R})$ down to the Lorentz

[3]Note that, with G being $\mathrm{Diff}(d, \mathbb{R})$ and H being either $GL(d, \mathbb{R})$ or $SO(1, d-1)$, the commutator of a generator of G/H with a generator of H is a linear combination of generators of G/H (making G/H a reductive coset), which ensures that $\Gamma_{G/H}$ and Γ_H transform independently under G.

group $SO(1, d-1)$. In the following we propose a Higgs mechanism for this breaking which shows the occurrence of the metric as a Goldstone field in an affine space-time. The model is largely based on that of [19]. Previous Higgs models of the (special) linear group have been constructed in [31,32].

We begin by assuming that all higher spin connections have already been decoupled and we are left with a massless connection $\Gamma^i{}_{jk}$ of an effective affine space-time. This connection is considered as an independent dynamical variable and, in particular, does not depend on the existence of the metric. In this space-time the metric will be introduced as a Higgs field which breaks the linear group $GL(d, \mathbb{R})$ in the tangent space. Recall from Sec. II B that the breaking of shear and dilation invariance leads to the metric as a Goldstone field.

We construct the Higgs sector as follows. In analogy to the complex scalar Φ of $U(1)$ symmetry breaking, the breaking is induced by a (real) scalar field ϕ and a symmetric tensor φ_{ij}. Under the Lorentz group the tensor φ_{ij} decomposes into a scalar σ and a traceless symmetric tensor $\hat{h}_{ij}$ ($\mathbf{10} \rightarrow \mathbf{1} + \mathbf{9}$ in $d = 4$),

$$\varphi_{ij} = \hat{h}_{ij} + \frac{1}{d}\sigma\eta_{ij}, \qquad \eta^{ij}\hat{h}_{ij} \equiv 0. \tag{24}$$

The singlet σ has been introduced in analogy to the Higgs field in $U(1)$ symmetry breaking. The fields $h_{ij} = \hat{h}_{ij} + \frac{\phi}{d}\eta_{ij}$ are the $d(d+1)/2$ Goldstone fields parametrizing the coset space $GL(d, \mathbb{R})/SO(1, d-1)$. In fact, compared to the previous section, the coset fields h_{ij} associated with the linear generators T^{ij} have been rescaled and redefined so that they now possess mechanical dimension $m_P^{(d-2)/2}$. The quantity κh_{ij} has no dimension, if κ is the gravitational constant appearing in Einstein-Hilbert's action $S^{\rm EH}[g_{ij}] = \frac{2}{\kappa^2}\int d^dx\sqrt{-g}R$. In terms of the Planck mass m_P, we thus have $\kappa = m_P^{(2-d)/2}$. Similarly, the quantity $\kappa\sigma$ is dimensionless, and so are $\kappa\varphi_{ij}$ and $\kappa\phi$.

For later convenience, we also introduce another parametrization for φ_{ij}, which is the analog of the polar parametrization for the complex scalar field Φ in the $U(1)$ Higgs mechanism:

$$\varphi_{ij} = \frac{\sigma}{d}e^{-\kappa\phi}g_{ij}, \qquad g_{ij} := e^{\kappa\phi}(e^{\kappa\bar{h}})_{ij}. \tag{25}$$

In order for both parametrizations (24) and (25) to define one and the same field φ_{ij}, it is easy to see that the real, symmetric matrix $\bar{h}_{ij}$ must satisfy one constraint.[4] In other words, the matrix $\bar{h}$ possesses the same number of independent components as does $\hat{h}$, *viz.* $\frac{d(d+1)}{2} - 1$. The fields ϕ and $\bar{h}_{ij}$ may therefore as well parametrize the coset space $GL(d, \mathbb{R})/SO(1, d-1)$. The field ϕ parametrizes dilations while $\bar{h}_{ij}$ parametrizes $SL(d, \mathbb{R})/SO(1, d-1)$. We will use the metric g_{ij} and its inverse, denoted g^{ij}, to lower and raise the indices.

It is then convenient to define the *nonmetricity* tensor Q_{ijk} by

$$Q_{ijk} \equiv -D_k g_{ij} = -\partial_k g_{ij} + 2\Gamma^l{}_{k(i}g_{j)l}. \tag{26}$$

Note that this definition exactly reflects the symmetric part of the absorption equation (18) (identify ${}^{(1)}\Gamma_{(ij)k} \sim Q_{ijk}$ and $\omega^i{}_{jk} \sim \Gamma^i{}_{jk}$). Solving this for $\Gamma^i{}_{jk}$,

$$\Gamma^i{}_{jk} = \Gamma^{\{\}i}{}_{jk} + N^i{}_{jk}, \tag{27}$$

$$N^i{}_{jk} \equiv \tfrac{1}{2}(Q_{jk}{}^i + Q_k{}^i{}_j - Q^i{}_{jk}), \tag{28}$$

we observe that a general symmetric connection can be expressed in terms of the Christoffel connection $\Gamma^{\{\}i}{}_{jk}(g_{ij})$ and nonmetricity.

We can now write down a $GL(d, \mathbb{R})$ invariant action for the fields $\Gamma^i{}_{jk}(x)$, $\phi(x)$, $\varphi_{ij}(x)$ and the descendant $g_{ij}(\phi(x), \hat{h}_{ij}(x))$. However, it turns out to be more convenient to perform a change of variables by Eq. (27), $\Gamma^i{}_{jk}(x) \rightarrow Q_{ijk}(x)$, and to construct instead an action for $Q_{ijk}(x)$, $\phi(x)$, $\varphi_{ij}(x)$. Nonmetricity contains a totally symmetric part $\tilde{Q}_{ijk} \equiv Q_{(ijk)}$ which can be viewed as representing a massless spin-3 field, if no mass terms are introduced for Q. We therefore have to construct an action for a massive spin-2 field φ_{ij} (and a scalar ϕ) in the background of a massless spin-3 field $\tilde{Q}_{ijk}$.

We are not aware of any action that would consistently couple a massive spin-2 field to a massless spin-3 gauge field, but consistent nonlinear higher spin field equations have been constructed [33], that involve an infinite tower of higher spin gauge fields.

Our point of view in the present work is to postulate the existence of an action in which all higher spin gauge fields would consistently interact, and focus only at particular sectors of this action. Then, those subsectors need not be separately consistent but must obey the requirement that, in the free limit, they should reduce to a positive sum of Singh-Hagen [20] and/or Fronsdal [34] actions. Clearly, as we already mentioned, the mechanism we are presenting here must be seen as a very small part of a complete Higgs mechanism involving the infinite number of Goldstone fields of the coset space Diff $(d, \mathbb{R})/SO(1, d-1)$.

The action S we propose is given by

$$S = \int d^dx\sqrt{-g}[\kappa^p\mathcal{L}(\tilde{Q}_{ijk}) + 3\mathcal{L}(\phi, \varphi_{ij}) + \mathcal{L}_{\rm int}], \tag{29}$$

[4]Being real and symmetric, $\bar{h}_{ij}$ can be diagonalized by an orthogonal matrix O: $\bar{h} = ODO^{-1}$ where $D = \mathrm{diag}\,(\lambda_1, \ldots, \lambda_d)$. Now, the identification $\hat{h}_{ij} + \frac{1}{d}\sigma\eta_{ij} = \varphi_{ij} = \frac{\sigma}{d}(\exp[\bar{h}])_{ij}$ implies that $\hat{h}_{ij} = \frac{\sigma}{d}[\bar{h} + \frac{1}{2!}\bar{h}^2 + \frac{1}{3!}\bar{h}^3 + \ldots]_{ij}$. Finally, because $\eta^{ij}\hat{h}_{ij} \equiv 0$, one can easily see that the above equation yields $\prod_{n=1}^d \lambda_n = \ln d$, that is, one constraint on $\bar{h}$.

with ($\tilde{Q}_i \equiv \tilde{Q}_i{}^k{}_k$, $p = \frac{2(d-4)}{2-d}$)

$$\mathcal{L}(\tilde{Q}_{ijk}) = -\tfrac{1}{2}(D_i\tilde{Q}_{jkl})^2 + \tfrac{3}{2}(D^i\tilde{Q}_{ijk})^2 + \tfrac{3}{2}(D_i\tilde{Q}_j)^2 + 3(D^iD^j\tilde{Q}_{ijk})\tilde{Q}^k + \tfrac{3}{4}(D^i\tilde{Q}_i)^2, \tag{30}$$

$$\mathcal{L}(\phi, \varphi_{ij}) = \frac{1}{2}(D_i\varphi_{jk})^2 - \frac{1}{2}(D_i\varphi^k{}_k)^2 - D_i\varphi_{jk}D^k\varphi^{ij} + D^i\varphi^k{}_kD^j\varphi_{ij} + \frac{3d-d^2-2}{2d^2}(D_i\phi)^2 + D^i\phi D^j\varphi_{ij} - V(\phi, \varphi_{ij}), \tag{31}$$

$$\mathcal{L}_{\rm int} = -\varphi^{ij}\varphi_{ij}\tilde{Q}^{klm}\tilde{Q}_{klm} - \frac{3}{2d}(D^i\tilde{Q}_{ijk})\varphi^{jk}\varphi^l{}_l, \tag{32}$$

and the effective potential

$$V(\phi, \varphi_{ij}) = \frac{\lambda\kappa^p}{4}\left(\frac{\varphi^2}{\kappa^p} - M^2\right)^2 + \frac{m^2}{2}\phi^2 + \frac{\lambda'}{2\kappa^p}\varphi^2\phi^2,$$
$$\varphi^2 \equiv (\varphi_{ij}g^{ij})^2 - \varphi^{ij}\varphi_{ij}.$$

The kinetic terms in the Lagrangians $\mathcal{L}(\tilde{Q}_{ijk})$ and $\mathcal{L}(\phi, \varphi_{ij})$ are obtained from the Fronsdal Lagrangian for a massless spin-3 field $\tilde{Q}_{ijk}$ and the Fierz-Pauli Lagrangian for a massive spin-2 field φ_{ij}. The Lagrangian $\mathcal{L}(\phi, \varphi_{ij})$ contains also a kinetic term for the scalar ϕ and a symmetry-breaking potential $V(\phi, \varphi_{ij})$. The kinetic terms in $\mathcal{L}(\phi, \varphi_{ij})$ are invariant under the exchange of $\varphi^k{}_k$ and ϕ.

By construction, the action is invariant under the linear group. The linear connection is minimally coupled to the fields ϕ, φ_{ij}, and $\tilde{Q}_{ijk}$ via the covariant derivative

$$D_i\varphi_{jk} = \partial_i\varphi_{jk} - 2\Gamma^l{}_{i(j}\varphi_{k)l} = \nabla_i\varphi_{jk} - 2N^l{}_{i(j}\varphi_{k)l} \tag{33}$$

and similarly for $\tilde{Q}_{ijk}$. Here ∇_i is the covariant derivative constructed from the Christoffel connection $\Gamma^{\{\}i}{}_{jk}$ and $N^i{}_{jk}$ as in Eq. (28). $\mathcal{L}_{\rm int}$ contains some additional nonminimal interactions.

For brevity, we omitted kinetic terms for the field $\bar{Q}_{ijk} \equiv Q_{ijk} - \tilde{Q}_{ijk}$. $\bar{Q}_{ijk}$ enters the Lagrangian via the covariant derivatives and is required for linear invariance. Linear invariant actions for all components of the nonmetricity can be found in [3]. Note that $\bar{Q}_{ijk}$ is nonpropagating in $d = 4$ if massless.

The potential $V(\phi, \varphi_{ij})$ has a minimum at

$$v_\varphi^2 \equiv \langle\varphi^2\rangle = \kappa^pM^2 - \frac{\lambda'}{\lambda}\phi^2 \tag{34}$$

and is invariant in the Goldstone direction that parametrizes $SL(d,\mathbb{R})/SO(1,d-1)$. This can best be seen in the parametrization (25) in which V, upon rescaling $\sigma' = e^{-\kappa\phi}\sigma$, becomes identical to the potential of hybrid inflation [35]. Scale invariance is softly broken at energy scales of order of the parameter M and below (we assume $m^2 \ll M^2$).

As in hybrid inflation, we assume that the dilaton field ϕ is slow-rolling and large at the beginning of the breaking. As long as the dilaton ϕ is larger than the critical value $\phi_c^2 = \kappa^p\lambda M^2/\lambda'$, the field φ_{ij} is trapped at $\varphi_{ij} = 0$. The effective mass squared of φ_{ij},

$$m^2(\varphi_{ij}) = \frac{\lambda'}{\kappa^p}\phi^2 - \lambda M^2, \tag{35}$$

becomes negative as soon as the value of ϕ falls below ϕ_c, $\phi < \phi_c$, at which point the vacuum becomes metastable. Then the field φ_{ij} is not trapped at $\varphi_{ij} = 0$ anymore and rolls down the "waterfall" to its minimum $v_\varphi = \pm\kappa^{p/2}M$ at $\phi_0 = 0$. In this way the breaking of scale invariance triggers the spontaneous breaking of the special linear group $SL(d,\mathbb{R}) \subset GL(d,\mathbb{R})$ down to the Lorentz group.

Higgs phase and massive spin-3 fields

We now study the action (29) in the Higgs phase. Below Eq. (26), we identified the nonmetricity field Q_{ijk} with the component ${}^{(1)}\Gamma_{(ij)k}$ of the total nonlinear connection Γ. Since ${}^{(1)}\Gamma_{(ij)k} \subset \Gamma_{G/H}$, we expect Q_{ijk} to acquire mass during the symmetry breaking. We thus have to show that at the minimum of the potential the Lagrangian in (29) contains the Singh-Hagen Lagrangian for the massive spin-3 field $\tilde{Q}_{ijk}$.

Let us first verify that the Goldstone field $\hat{h}_{ij}$ becomes massless at the minimum of the potential. According to the general Higgs procedure, we have to expand ϕ and φ_{ij} around the absolute minimum of the potential at $\phi_0 = 0$ and $v_\varphi \equiv \sqrt{\langle\varphi^2\rangle} = \pm\kappa^{p/2}M$. We choose the parametrization

$$\varphi_{ij} = \left(\frac{v_\varphi}{\sqrt{d(d-1)}} + \frac{1}{d}\sigma\right)\eta_{ij} + \hat{h}_{ij}, \qquad \phi = \phi_0 + \tilde{\phi} = \tilde{\phi}, \tag{36}$$

where the normalization is chosen such that $\varphi^2 = v_\varphi^2 + \ldots$. Substituting this into the potential $V(\phi, \varphi_{ij})$, we observe that the Goldstone field $\hat{h}_{ij}$ is indeed massless, whereas the Higgs-like field σ obtains a positive mass,

$$m_\sigma^2 \sim \lambda M^2\frac{d-1}{d}. \tag{37}$$

It is much simpler to work in the unitary gauge

$$\varphi_{ij} = \left(\frac{v_\varphi}{\sqrt{d(d-1)}} + \frac{1}{d}\sigma\right)\eta_{ij}, \qquad \phi = 0, \tag{38}$$

in which the Goldstone bosons $\hat{h}_{ij}$ and ϕ are gauged away. This corresponds to the flat space limit $g_{ij} = \eta_{ij}$. In this gauge the Lagrangian in (29) reduces to

$$\mathcal{L} = \kappa^p\mathcal{L}_F(\tilde{Q}_{ijk}) + \frac{v_\varphi^2}{2}\tilde{Q}_{ijk}^2 - \frac{3}{2}v_\varphi^2(\tilde{Q}^l{}_{lj})^2 - \frac{9}{4}\lambda M^2\sigma^2 + \frac{9}{16}(\partial_i\sigma)^2 - \frac{3}{4}v_\varphi(\partial^i\tilde{Q}^l{}_{li})\sigma + \ldots, \tag{39}$$

where dots denote additional mixed terms. The Fronsdal Lagrangian $\mathcal{L}_F(\tilde{Q}_{ijk})$ in flat space follows directly from $\mathcal{L}(\tilde{Q}_{ijk})$ in (30), while the mass terms for $\tilde{Q}_{ijk}$ and its trace $\tilde{Q}^l{}_{lj}$ descend from the kinetic terms of the Goldstone bosons. If we choose $\lambda = 1$, then Eq. (39) is nothing but the Stückelberg Lagrangian for a massive spin-3 field $\tilde{Q}_{ijk}$ [36] which is equivalent to the spin-3 Singh-Hagen Lagrangian [20]. The mass of $\tilde{Q}_{ijk}$ is given by

$$m_Q^2 = \frac{1}{\kappa^p} v_\varphi^2 = M^2. \tag{40}$$

The vacuum expectation value $v_\varphi = \kappa^{p/2} M$ is a free parameter in the model and has to be determined by experiment. If we assume that our model is indeed related to hybrid inflation, we can make a rough estimation of the mass $m_Q = M$. It has been found [37] that the parameter M determined by the COBE normalization is roughly 10^{15}–10^{16} GeV.

The development of the field σ is quite exciting. Since we introduced σ as a Higgs-like field, we would have expected it to be an independent massive scalar, just like the Higgs particle in elementary particle physics. Instead, the field σ turned out to be the auxiliary scalar required in the Singh-Hagen Lagrangian for a massive spin-3 field. We thus do not have an additional Higgs particle.

In the general parametrization (36), there are additional terms involving the Goldstone metric. Let us assume that m_Q^2 is very high such that $\tilde{Q}_{ijk}$ and σ decouple at low energies. In this decoupling limit, the Lagrangian in (29) effectively reduces to the linearized Einstein-Hilbert Lagrangian $\mathcal{L}_F(h_{ij})$ with $h_{ij} = \hat{h}_{ij} + \frac{1}{d}\phi\eta_{ij}$. Gravity is thus effectively described by general relativity at the minimum of the potential. This shows explicitly that the condition $Q_{ijk} \equiv -D_k g_{ij} \sim {}^{(1)}\Gamma_{(ij)k} = 0$ imposed by the "inverse Higgs effect" is an effective equation.

B. Higgs mechanism for higher spin connections

In the previous section we proposed a full Higgs mechanism for the linear group including a symmetry-breaking potential. An essential part of the mechanism was the absorption process (I) of Sec. II D. A description of the breaking of the complete diffeomorphism group appears to be quite complex. We therefore aim at the more modest goal of modeling the absorption process (II) of Sec. II D for $s = 2$ without giving a symmetry-breaking potential. This is along the lines of [21,22,38] which discuss a "telescopic Higgs effect." The latter effect is briefly reviewed now.

1. Stückelberg formalism and the "telescopic Higgs effect"

At the basis of the Stückelberg formalism lies the well-known fact that $O(d-1)$, the little group for a massless particle in $d+1$ dimensions, is the same as the little group for a massive particle in a d-dimensional space-time. Consistent actions for massive particles can indeed be obtained by dimensional reduction of massless gauge-invariant actions. The dimensionally reduced action is itself invariant under a set of gauge invariances which display a "telescopic Higgs effect."

For example, in the previous section we recovered the Singh-Hagen Lagrangian for a massive spin-3 field as resulting from the expression, in the unitary gauge, of a Lagrangian containing an appropriate symmetry-breaking potential V. This Lagrangian can also be obtained starting from Fronsdal's gauge-invariant Lagrangian $\mathcal{L}_F(\phi_{MNP})$ for a massless spin-3 field ϕ_{MNP}, dimensionally reduced from $d+1$ down to d dimensions. Upon dimensional reduction, the field ϕ_{MNP} gives rise to the set of d-dimensional fields $\{\phi_{ijk}, \phi_{ij}, A_i, \psi\}$, entering a d-dimensional Stückelberg Lagrangian $\mathcal{L}_S(\phi_{ijk}, \phi_{ij}, A_i, \psi)$. However, not all of these fields will survive. The dimensionally reduced Lagrangian $\mathcal{L}_S(\phi_{ijk}, \phi_{ij}, A_i, \psi)$ inherits gauge invariances from $\mathcal{L}_F(\phi_{MNP})$ whose effect is to eliminate all the fields but ϕ_{ijk} and ϕ, the trace of ϕ_{ij}.

2. Higher spin connections in the Stückelberg formulation

We now apply the Stückelberg formalism to model the absorption process (II) of Sec. II D. We begin by splitting Eq. (22) into irreducible pieces under the linear group. The total symmetric and the "hook" parts are given by

$$\begin{aligned} {}^{(s)}\hat{\Gamma}_{(\alpha\beta_1\ldots\beta_s k)} &= {}^{(s-1)}\hat{\omega}_{(\alpha\beta_1\ldots\beta_s,k)} - (s+1){}^{(s)}\hat{\omega}_{(\alpha\beta_1\ldots\beta_s k)}, \\ {}^{(s)}\hat{\Gamma}_{[\alpha(\beta_1]\ldots\beta_s k)} &= {}^{(s-1)}\hat{\omega}_{[\alpha(\beta_1]\ldots\beta_s,k)} - (s+1){}^{(s)}\hat{\omega}_{[\alpha\beta_1]\ldots\beta_s k}, \end{aligned} \tag{41}$$

where in the second line we first symmetrize in $\beta_1, \ldots, \beta_s$, k and then antisymmetrize in α and β_1. A comma denotes a partial derivative, e.g. $\Phi_{,k} := \partial_k \Phi$.

We have restricted to (double) traceless fields, which is required for the construction of Fronsdal's Lagrangians. The hat on top of a field indicates its tracelessness in the anholonomic indices, e.g. ${}^{(s)}\hat{\omega}^\alpha{}_{\alpha\beta_2\ldots\beta_s k} = {}^{(s)}\hat{\omega}_\alpha{}^\beta{}_{\beta\beta_3\ldots\beta_s k} = 0$. This guarantees the double tracelessness of the field ${}^{(s)}\hat{\omega}_{\alpha\beta_1\ldots\beta_s k}$. Note that the fully anholonomic field

$${}^{(s-1)}\hat{\omega}_{\alpha\beta_1\ldots\beta_s} = {}^{(s-1)}\hat{\omega}_{\alpha\beta_1\ldots\beta_{s-1}k}\hat{e}^k{}_{\beta_s} \tag{42}$$

is traceless, rather than double traceless. In the nonlinear realization of Diff $(d, \mathbb{R})$ with the Lorentz group $SO(1, d-1)$ as stabilizing subgroup, it is indeed natural to decompose the generators $F_\alpha{}^{\beta_1\ldots\beta_2}$ of Diff $(d, \mathbb{R})$—as expressed in an anholonomic basis—with respect to irreducible representations of $SO(1, d-1)$. In other words, the generators $F_\alpha{}^{\beta_1\ldots\beta_2} \equiv e^i{}_\alpha F_i{}^{j_1\ldots j_s}(e^{-1})_{j_1}{}^{\beta_1}\ldots(e^{-1})_{j_s}{}^{\beta_s}$

can be decomposed into their traceless and pure-trace parts by using the Minkowski metric $\eta_{\alpha\beta}$.

It is convenient to define the fields

$$\phi_{\alpha\beta_1\ldots\beta_s k} \equiv {}^{(s)}\hat{\omega}_{(\alpha\beta_1\ldots\beta_s k)}, \quad \phi'_{\alpha\beta_1\ldots\beta_s k} \equiv {}^{(s)}\hat{\Gamma}_{(\alpha\beta_1\ldots\beta_s k)},$$
$$T_{\alpha\beta_1|\beta_2\ldots\beta_s k} \equiv {}^{(s)}\hat{\omega}_{[\alpha\beta_1]\ldots\beta_s k}, \quad T'_{\alpha\beta_1|\beta_2\ldots\beta_s k} \equiv {}^{(s)}\hat{\Gamma}_{[\alpha(\beta_1]\ldots\beta_s k)}, \tag{43}$$

which, by assumption (more details in Sec. IVA), satisfy the gauge transformation laws

$$\delta\phi_{\alpha\beta_1\ldots\beta_s} = s\partial_{(\alpha}\hat{\lambda}_{\beta_1\ldots\beta_s)}, \tag{44}$$

$$\delta T_{\alpha\beta_1|\beta_2\ldots\beta_s} = \partial_{[\alpha}\hat{k}_{\beta_1]|\beta_2\ldots\beta_s} - \frac{3(s-1)}{s+1}\partial_{([\beta_2}\hat{k}_{\alpha|\beta_1]\beta_3\ldots\beta_s)}, \tag{45}$$

where the last term in (45) is first antisymmetrized in the indices β_2, α, β_1 and then completely symmetrized in the indices $\beta_2, \beta_3, \ldots, \beta_s$ [39]. The gauge parameters $\hat{\lambda}$ and $\hat{k}$ satisfy $\hat{\lambda}^{\beta}{}_{\beta\beta_3\ldots\beta_s} = 0$ and $\hat{k}_{\beta_1|}{}^{\beta}{}_{\beta\beta_4\ldots\beta_s} = 0$. Using

$$\omega_{\alpha|\beta_1\ldots\beta_s} = \frac{2s}{s+1}T_{\alpha(\beta_1|\beta_2\ldots\beta_s)} + \phi_{\alpha\beta_1\ldots\beta_s}, \tag{46}$$

Eq. (41) can then be rewritten as

$$\phi'_{\alpha\beta_1\ldots\beta_s k} = \partial_{(k}\phi_{\alpha\beta_1\ldots\beta_s)} - (s+1)\phi_{\alpha\beta_1\ldots\beta_s k}, \tag{47}$$

$$\begin{aligned} T'_{\alpha\beta_1|\beta_2\ldots\beta_s k} = &-(s+1)T_{\alpha\beta_1|\beta_2\ldots\beta_s k} \\ &+ \frac{2s}{(s+1)^2}\partial_{[\beta_1}T_{\alpha](\beta_2|\beta_3\ldots\beta_s k)} \\ &+ \frac{s}{s+1}T_{\alpha\beta_1|\beta_2\ldots\beta_s,k} \\ &+ \frac{1}{s+1}\partial_{[\beta_1}\phi_{\alpha]\beta_2\ldots\beta_s k}. \end{aligned} \tag{48}$$

Equation (47) is the relevant gauge transformation involved in the Higgs mechanism for totally symmetric spin-$s+2$ fields $\phi_{\alpha\beta_1\ldots\beta_s k}$ (Young diagram $[s+2, 0]$), while Eq. (48) describes the Higgs effect for spin-$s+1$ fields $T_{\alpha\beta_1|\beta_2\ldots\beta_s k}$ in the hook representation $[s+1, 1]$.

For simplicity, we restrict to $s = 2$ in the following.

Let us compare Eqs. (47) and (48) with the fields arising in the Stückelberg formalism. Decomposing the massive representations $[4,0]_m$ and $[3,1]_m$ into massless representations, we obtain

$$\overbrace{[4,0]_m}^{\phi'_{\alpha\beta_1\beta_2 k}} \to \overbrace{[4,0]}^{\phi_{\alpha\beta_1\beta_2 k}} \oplus [3,0] \oplus [2,0] \oplus [1,0] \oplus [0,0],$$
$$\overbrace{[3,1]_m}^{T'_{\alpha\beta_1|\beta_2 k}} \to \overbrace{[3,1]}^{T_{\alpha\beta_1|\beta_2 k}} \oplus \overbrace{[2,1]}^{C_{\alpha\beta_1|\beta_2}} \oplus \overbrace{[2,0]}^{\phi_{\alpha\beta}} \oplus \overbrace{[1,1]}^{B_{\alpha\beta}} \oplus \overbrace{[1,0]}^{X_\alpha} \oplus \overbrace{[3,0]}^{S_{\alpha\beta_1\beta_2}}. \tag{49}$$

This shows that a massless representation [4, 0] has to absorb the massless representations [3, 0], [2, 0], [1, 0], [0, 0] to become massive. These representations descend themselves from the massive representation $[3,0]_m$ ($\phi_{\alpha\beta_1 k}$). Moreover, the massless representation [3, 1] must absorb the [2, 1], [2, 0], [1, 1], [1, 0], and [3, 0] to become massive. The first four of these representations come from the massive representation $[2,1]_m$ ($T_{\alpha\beta|k}$). The remaining representation [3, 0] originates from $\phi_{\alpha\beta_1 k}$, cf. with the last line in Eq. (48).

This agrees with the fact that, in a full Higgs mechanism including a symmetry-breaking potential, the Goldstone bosons $\phi_{\alpha\beta_1 k}$ and $T_{\alpha\beta|k}$ would be introduced inside a (tachyonic) Higgs field, i.e. as massive representations. At the minimum of the potential the Goldstone bosons condense and become massless. Note however that in the following we restrict to give the Stückelberg description of the massive representations which we consider as part of the full symmetry-breaking mechanism.

3. Stückelberg formulation of massive [3, 1] hook field

The Stückelberg formalism for the representation [4, 0] has been discussed in detail in [22].[5] We therefore need to construct only a field equation for the representation [3, 1] which describes the absorption (48) for $s = 2$.

Upon dimensional reduction $(x^i, y) \downarrow x^i$, the $(d+1)$-dimensional massless gauge field $T_{MN|PQ}(x, y)$ gives rise to the following d-dimensional gauge fields

$$\begin{aligned} T_{MN|PQ}(x,y) &= \frac{1}{\sqrt{2}}T_{MN|PQ}(x)e^{imy} + \text{c.c.}, \\ T_{ij|kl}(x,y) &= \frac{1}{\sqrt{2}}T_{ij|kl}(x)e^{imy} + \text{c.c.}, \\ T_{ij|ky}(x,y) &= \frac{i}{\sqrt{2}}C_{ij|k}(x)e^{imy} + \text{c.c.}, \\ T_{iy|jk}(x,y) &= \frac{i}{\sqrt{2}}\left[S_{ijk}(x) + \frac{2}{3}C_{i(j|k)}(x)\right]e^{imy} + \text{c.c.}, \\ T_{ij|yy}(x,y) &= \frac{1}{\sqrt{2}}B_{ij}(x)e^{imy} + \text{c.c.}, \\ T_{iy|yj}(x,y) &= \frac{1}{\sqrt{2}}\left[\phi_{ij}(x) + \frac{1}{2}B_{ij}(x)\right]e^{imy} + \text{c.c.}, \\ T_{iy|yy}(x,y) &= \frac{i}{\sqrt{2}}X_i(x)e^{imy} + \text{c.c.}. \end{aligned}$$

The descendant fields $T_{ij|kl}$, S_{ijk}, $C_{ij|k}$, B_{ij}, ϕ_{ij} and X_i are all real. The field $T_{MN|PQ}(x,y)$ has the following symmetries

$$T_{MN|PQ} = -T_{NM|PQ} = -T_{NM|QP}, \qquad T_{[MN|P]Q} \equiv 0,$$

while the descendant d-dimensional fields have the symmetries

[5]If fact, it is discussed for all totally symmetric fields $\phi_{i_1\ldots i_s}$ there.

$$C_{ij|k} = -C_{ji|k}, \qquad C_{[ij|k]} \equiv 0, \qquad S_{ijk} = S_{(ijk)},$$
$$B_{ij} = -B_{ji}, \qquad \phi_{ij} = \phi_{ji}.$$

The gauge transformations of the field $T_{MN|PQ}(x, y)$ are

$$\delta T_{MN|PQ} = \partial_{[M}\hat{K}_{N]|PQ} - \tfrac{3}{4}(\partial_{[P}\hat{K}_{M|N]Q} + \partial_{[Q}\hat{K}_{M|N]P}),$$

where the gauge parameter $\hat{K}_{N]|PQ}(x, y)$ possesses the following symmetries

$$\hat{K}_{M|NP} = \hat{K}_{M|PN}, \qquad 0 = \hat{K}_{M|N}{}^{N} \equiv \hat{K}_{M|\alpha}{}^{\alpha} + \hat{K}_{M|yy}.$$

The $(d+1)$-dimensional gauge parameter $\hat{K}_{M|NP}(x, y)$ generates the following d-dimensional gauge parameters upon dimensional reduction

$$\hat{K}_{M|NP}(x, y) = \frac{1}{\sqrt{2}}\hat{K}_{M|NP}(x)e^{imy} + \text{c.c.},$$
$$\hat{K}_{i|jk} = \frac{1}{\sqrt{2}}\left[\hat{k}_{i|jk} + \frac{\eta_{jk}}{d}a_i\right]e^{imy} + \text{c.c.},$$
$$\hat{K}_{i|jy} = \frac{i}{\sqrt{2}}t_{ij}e^{imy} + \text{c.c.}$$
$$\hat{K}_{y|ij} = \frac{i}{\sqrt{2}}\left[\hat{l}_{ij} + \frac{\eta_{ij}}{d}l\right]e^{imy} + \text{c.c.},$$
$$\hat{K}_{y|yi} = \frac{1}{\sqrt{2}}\epsilon_i e^{imy} + \text{c.c.}, \qquad \hat{K}_{y|yy} = \frac{-i}{\sqrt{2}}le^{imy} + \text{c.c.},$$

where the descendant gauge parameters $\hat{k}_{i|jk}$, a_i, t_{ij}, $\hat{l}_{ij}$, l, and ϵ_i are all real and

$$a_i = \hat{K}_{i|k}{}^{k}(x), \qquad l = \hat{K}_{y|k}{}^{k}(x).$$

They furthermore obey

$$\hat{k}_{i|jk} = \hat{k}_{i|kj}, \qquad \eta^{jk}\hat{k}_{i|jk} = 0 \qquad \hat{l}_{ij} = \hat{l}_{ji}, \qquad \eta^{ij}\hat{l}_{ij} = 0.$$

4. Gauge transformations and field redefinitions

The d-dimensional gauge transformations of the descendant fields read

$$\delta T_{ij|kl} = \partial_{[i}\hat{k}_{j]|kl} - \frac{3}{4}(\partial_{[k}\hat{k}_{i|j]l} + \partial_{[l}\hat{k}_{i|j]k}) + \frac{1}{2d}\eta_{kl}\partial_{[i}a_{j]} - \frac{1}{4d}(\eta_{jl}\partial_{[k}a_{i]} + \eta_{jk}\partial_{[l}a_{i]} + \eta_{il}\partial_{[j}a_{k]} + \eta_{ik}\partial_{[j}a_{l]}),$$
$$\delta C_{ij|k} = \frac{3}{4}(\partial_{[i}t_{j]k} - \partial_{[k}t_{ij]}) - \frac{m}{4}\left[\hat{k}_{[i|j]k} + \frac{1}{d}\eta_{k[j}a_{i]}\right] + \frac{1}{4}\partial_{[i}\hat{l}_{j]k} + \frac{1}{4d}\eta_{k[j}\partial_{i]}l,$$
$$\delta S_{ijk} = \frac{1}{2}\partial_{(i}\hat{l}_{jk)} + \frac{1}{2d}\eta_{(ij}\partial_{k)}l - \frac{m}{2}\left(\hat{k}_{(i|jk)} + \frac{1}{d}\eta_{(ij}a_{k)}\right),$$
$$\delta B_{ij} = \tfrac{1}{2}(\partial_{[i}\epsilon_{j]} - \partial_{[i}a_{j]} + mt_{[ij]}),$$
$$\delta\phi_{ij} = \tfrac{1}{2}(\partial_{(i}\epsilon_{j)} + mt_{(ij)}), \qquad \delta X_i = \tfrac{1}{2}(ma_i - \partial_i l).$$

The field X_i drops out of the action by doing the following field redefinitions

$$\phi_{ij} \to \phi'_{ij} = \phi_{ij}, \tag{50}$$

$$B_{ij} \to B'_{ij} = B_{ij} + \frac{1}{m}\partial_{[i}X_{j]}, \tag{51}$$

$$S_{ijk} \to S'_{ijk} = S_{ijk} + \frac{1}{D}\eta_{(ij}X_{k)}, \tag{52}$$

$$C_{ij|k} \to C'_{ij|k} = C_{ij|k} + \frac{1}{2D}\eta_{k[j}X_{i]}, \tag{53}$$

$$T_{ij|kl} \to T'_{ij|kl} = T_{ij|kl} + \frac{3}{2mD}[\eta_{(jk}\partial_{l)}X_i + \eta_{i(j}\partial_k X_{l)} - \eta_{(ik}\partial_{l)}X_j - \eta_{j(i}\partial_k X_{l)}]. \tag{54}$$

The redefined fields transform as

$$\delta T'_{ij|kl} = \partial_{[i}\hat{k}_{j]|kl} - \tfrac{3}{4}(\partial_{[k}\hat{k}_{i|j]l} + \partial_{[l}\hat{k}_{i|j]k}), \tag{55}$$

$$\delta C'_{ij|k} = \frac{3}{4}(\partial_{[i}t_{j]k} - \partial_{[k}t_{ij]}) - \frac{m}{4}\hat{k}_{[i|j]k} + \frac{1}{4}\partial_{[i}\hat{l}_{j]k}, \tag{56}$$

$$\delta S'_{ijk} = \frac{1}{2}\partial_{(i}\hat{l}_{jk)} - \frac{m}{2}\hat{k}_{(i|jk)},$$
$$\delta B'_{ij} = \frac{1}{2}(\partial_{[i}\epsilon_{j]} + mt_{[i|j]}), \qquad \delta\phi'_{ij} = \frac{1}{2}(\partial_{(i}\epsilon_{j)} + mt_{(i|j)}). \tag{57}$$

Performing the field redefinitions (50)–(54) is equivalent to going in the gauge

$$a_i = \frac{1}{m}\partial_i l, \tag{58}$$

whose effect is to eliminate X_i from the action. Of course we must have $m \neq 0$. There is no redefined field varying with respect to the gauge parameters a_i and l.

The next gauge-fixing condition that we choose is $(m \neq 0)$

$$t_{ij} = -\frac{1}{m}\partial_i\epsilon_j \tag{59}$$

which is equivalent to gauging B_{ij} and ϕ_{ij} away. Note that (55)–(57) are unaffected by this gauge-fixing condition. In terms of field redefinition, the gauge (59) translates as

$$T'_{ij|kl} \to T''_{ij|kl} = T'_{ij|kl},$$
$$C'_{ij|k} \to C''_{ij|k} = C'_{ij|k} - \frac{3}{2m}\partial_{[i}\phi'_{j]k} - \frac{1}{2m}(\partial_k B'_{ji} + \partial_{[i}B'_{j]k}),$$
$$S'_{ijk} \to S''_{ijk} = S'_{ijk}.$$

As a result, the fields B' and ϕ' disappear from the action and we have

$$\delta T''_{ij|kl} = \partial_{[i}\hat{k}_{j]|kl} - \tfrac{3}{4}(\partial_{[k}\hat{k}_{i|j]l} + \partial_{[l}\hat{k}_{i|j]k}),$$
$$\delta C''_{ij|k} = \tfrac{1}{4}(\partial_{[i}\hat{\lambda}_{j]k} - m\hat{k}_{[i|j]k}),$$
$$\delta S''_{ijk} = \tfrac{1}{2}(\partial_{(i}\hat{\lambda}_{jk)} - m\hat{k}_{(i|jk)}).$$

Obviously, the next gauge condition we impose is ($m \neq 0$)

$$\hat{k}_{i|jk} = \frac{1}{m}\partial_i\hat{l}_{jk}, \tag{60}$$

which enables us to eliminate the (jk)-*traceless* part of $P_{i|jk} := S''_{ijk} + \frac{8}{3}C''_{i(j|k)}$ since it can be seen that

$$\delta P_{i|jk} = \tfrac{1}{2}(\partial_i\hat{l}_{jk} - m\hat{k}_{i|jk}).$$

In other words, we can gauge away S''_{ijk} and $C''_{ij|k}$ (the two independent components of $P_{i|jk}$) *except for the trace* $\eta^{jk}P_{ij|k}$ which will remain in the action, playing the role of an auxiliary vector field V_i that we need for the action of a massive [3, 1] field. At the level of the action, the gauge (60) translates as the field redefinition

$$T''_{ij|kl} \to H_{ij|kl} = T''_{ij|kl} + \frac{2}{m}\left[\partial_{[i}P_{j]|kl} - \frac{3}{4}(\partial_{[k}P_{i|j]l} + \partial_{[l}P_{i|j]k})\right].$$

This equation expresses the absorption (48). Then, at the end of all these field redefinitions which are the translation of the gauge-fixing conditions (58)–(60), all the fields but $H_{ij|kl}$ and $V_i := S''_i + \frac{8}{3}C''_i$ remain in the action. The field $H_{ij|kl}$ does not transform anymore, it has become a massive field. The field $V_i := S''_i + \frac{8}{3}C''_i$ does not transform either, it is an auxiliary field, as we show explicitly in the following.

5. *Field equations*

The field equations for a massless [3, 1] irreducible hook field $T_{MN|RS}$ in dimension $(d+1)$ are [39,40] (see also [41] in different symmetry conventions)

$$F_{MN|AB} = 0, \tag{61}$$

where $F_{MN|AB}$ is the kinetic tensor

$$F_{MN|AB} = \partial_R\partial^R T_{MN|AB} + 2\partial^R\partial_{[M}T_{N]R|AB} - 2\partial^R T_{MN|R(A,B)} + 4\partial_{[M}T^R{}_{N]|R(A,B)} + \partial_A\partial_B T_{MN|}{}^R{}_R.$$

As before, a comma $\Phi_{B,A}$ denotes a partial derivative $\partial_A\Phi_B$. Partially decomposing the field equation according to $x^M = (x^i, y)$ gives

$$\begin{aligned}0 = {} & \Box T_{MN|AB} + \partial_y\partial_y T_{MN|AB} - \partial_M\partial^k T_{kN|AB} \\ & - \partial_M\partial_y T_{yN|AB} + \partial_N\partial^k T_{kM|AB} + \partial_N\partial_y T_{yM|AB} \\ & - \partial_A\partial^k T_{MN|kB} - \partial_A\partial_y T_{MN|yB} - \partial_B\partial^k T_{MN|kA} \\ & - \partial_B\partial_y T_{MN|yA} + \partial_A\partial_M T^k{}_{N|kB} + \partial_A\partial_M T_{yN|yB} \\ & - \partial_A\partial_N T^k{}_{M|kB} - \partial_A\partial_N T_{yM|yB} + \partial_B\partial_M T^k{}_{N|kA} \\ & + \partial_B\partial_M T_{yN|yA} - \partial_B\partial_N T^k{}_{M|kA} - \partial_B\partial_N T_{yM|yA} \\ & + \partial_A\partial_B T^k_{MN|k} + \partial_A\partial_B T_{MN|yy}.\end{aligned}$$

We can now decompose the above expression where the indices $MN|AB$ take the values $ij|ab$, $iy|ab$, $ij|yy$, $iy|yb$, and $iy|yy$, respectively. We find

$$\begin{aligned}0 = {} & F_{ij|ab} - m^2T_{ij|ab} - 2m\partial_{[i}S_{j]ab} - \tfrac{4}{3}m\partial_{[i}C_{j](a|b)} \\ & + \partial_a\partial_b B_{ij} + 2mC_{ij|(a,b)} - 4\partial_{[i}\phi_{j](a,b)} - 2\partial_{[i}B_{j](a,b)},\end{aligned}$$
$$\begin{aligned}0 = {} & \Box(S_{iab} + \tfrac{2}{3}C_{i(a|b)}) - \partial_i\partial^k(S_{kab} + \tfrac{2}{3}C_{k(a|b)}) \\ & - \partial^k\partial_a(S_{bik} + \tfrac{2}{3}C_{i(k|b)}) - \partial^k\partial_b(S_{aik} + \tfrac{2}{3}C_{i(k|a)}) \\ & + \partial_a\partial_b(S^k_{ik} + \tfrac{2}{3}C^k_i) + m\partial^k T_{ki|ab} - 2mT^k{}_{i|k(a,b)} \\ & + \partial_a\partial_b X_i,\end{aligned}$$
$$\begin{aligned}0 = {} & \Box(\phi_{ib} + \tfrac{1}{2}B_{ib}) - \partial_b\partial^k(\phi_{ib} + \tfrac{1}{2}B_{ib}) + \partial_b\partial_i\phi^k{}_k \\ & + m\partial^k S_{kib} - 2m\partial_{(i}S_{b)k}{}^k - m\partial_b X_i \\ & - \partial_i\partial^k(\phi_{kb} + \tfrac{1}{2}B_{kb}) - \tfrac{4}{3}m\partial^k C_{ki|b} + \tfrac{1}{3}m\partial^k C_{ib|k} \\ & - \tfrac{1}{3}m\partial_i C^k{}_{b|k} - \tfrac{5}{3}m\partial_b C_i{}^k{}_{|k} + m^2T^k{}_{i|kb},\end{aligned}$$
$$\begin{aligned}0 = {} & \Box X_i - \partial_i\partial^k X_k - m^2X_i + 2m\partial^k B_{ki} - 2m\partial^k\phi_{ik} \\ & + 2m\partial_i\phi^k{}_k - m^2(S_{ik}{}^k + \tfrac{8}{3}C_i{}^k{}_{|k}),\end{aligned}$$

where $F_{ij|ab}$ is the kinetic tensor for the field $T_{ij|ab}$.

We now perform all the field redefinitions given in the previous section. The above field equations read

$$0 = F_{ij|ab}(H) - m^2H_{ij|ab} - \frac{2}{m}\partial_a\partial_b\partial_{[i}(S''{}_{j]k}{}^k + \tfrac{8}{3}C''{}_{j]k|}{}^k),$$
$$\begin{aligned}0 = {} & \partial_a\partial_b(S''{}_{ik}{}^k + \tfrac{8}{3}C''{}_{ik|}{}^k) + m(\partial^k H_{ki|ab} - \partial_a H_{ki|b}{}^k \\ & - \partial_b H_{ki|a}{}^k),\end{aligned}$$
$$0 = m^2H_{ki|b}{}^k - m\partial_b(S''{}_{ik}{}^k + \tfrac{8}{3}C''{}_{ik|}{}^k),$$
$$0 = m^2[S''{}_{ik}{}^k + \tfrac{8}{3}C''{}_{ik|}{}^k].$$

All together, these field equations imply

$$(\Box - m^2)H_{ij|kl} = 0, \qquad \partial^i H_{ij|kl} = 0, \qquad \eta^{jk}H_{ij|kl} = 0,$$

which are the field equations for a massive d-dimensional [3, 1] hook field. We thus derived the correct field redefinitions which express the absorption phenomenon by which a massless [3, 1] hook field becomes massive.

IV. GEOMETRICAL INTERPRETATION OF HIGHER SPIN CONNECTIONS

In Sec. II we found an infinite tower of generalized connections ${}^{(s)}\omega$ parametrizing the coset space $\mathrm{Diff}_0(d, \mathbb{R})/GL(d, \mathbb{R})$ associated with nonlinear coordinate transformations. In Sec. IV A we will relate them with higher spin connections known from the frame formalism of higher spin fields [23]. In particular, we derive some gauge invariance principle for the generalized connections which leads to a geometrical interpretation of higher spin connections. In Secs. IV B and IV C we then study the geometrical structure of a space-time equipped with higher spin connections.

A. Gauge transformations of higher spin connections

In the frame formalism for higher spin gauge fields in Minkowski space [23], Lorentz-like connections $\omega_k{}^{\alpha|\beta_1\ldots\beta_{S-1}}$ are given in terms of framelike fields $e_k{}^{\beta_1\ldots\beta_{S-1}}$. These fields are symmetric in the indices $\beta_1, \ldots, \beta_{S-1}$ and satisfy the relations

$$\omega_{k|}{}^{\beta}{}_{|\beta\beta_2\ldots\beta_{S-1}} = 0, \qquad \omega_{k|\alpha|}{}^{\beta}{}_{\beta\beta_3\ldots\beta_{S-1}} = 0,$$
$$\omega_{k|(\alpha|\beta_1\ldots\beta_{S-1})} = 0, \qquad e_k{}^{\beta}{}_{\beta}{}^{\beta_3\ldots\beta_{S-1}} = 0.$$

The higher spin connections and tetrads are invariant under the gauge transformations

$$\begin{aligned}\delta\omega_{k|\alpha|\beta_1\ldots\beta_{S-1}} &= \partial_k a_{\alpha|\beta_1\ldots\beta_{S-1}} + \Sigma_{k|\alpha|\beta_1\ldots\beta_{S-1}},\\ \delta e_{k|\beta_1\ldots\beta_{S-1}} &= \partial_k\lambda_{\beta_1\ldots\beta_{S-1}} + a_{k|\beta_1\ldots\beta_{S-1}},\end{aligned} \qquad (62)$$

where the gauge parameters a, Σ, and λ are traceless, completely symmetric in the indices $(\beta_1\ldots\beta_{S-1})$ and possess the following supplementary symmetry properties:

$$a_{(\alpha|\beta_1\ldots\beta_{S-1})} = 0 = \Sigma_{k|(\alpha|\beta_1\ldots\beta_{S-1})},$$
$$\Sigma_{k|\alpha|\beta_1\ldots\beta_{S-1}} = \Sigma_{\alpha|k|\beta_1\ldots\beta_{S-1}}.$$

Of course, similar gauge transformation formulas are also present in the metriclike formulation of higher spin gauge fields [34] (see also [42]) and are crucial for the construction of consistent higher spin theories.

Though nonlinear realizations are different from gauging, the group action on the coset fields is very similar to a gauge transformation [12]. We may exploit this similarity to derive a gauge transformation for the generalized connections ${}^{(s)}\omega$ which is basically given by Eq. (62).

We begin by rewriting the defining equations of the nonlinear connection one-forms ${}^{(s)}\Gamma$. For simplicity, we consider once again the linearized version of Eq. (20):

$$\delta^{(s)}\omega^{\alpha}{}_{\beta_1\ldots\beta_s k} = \partial_k{}^{(s-1)}\omega^{\alpha}{}_{\beta_1\ldots\beta_s}, \qquad (s > 1) \qquad (63)$$

where the variation $\delta^{(s)}\omega^{\alpha}{}_{\beta_1\ldots\beta_s k}$ has been defined by

$$\delta^{(s)}\omega^{\alpha}{}_{\beta_1\ldots\beta_s k} \equiv {}^{(s)}\Gamma^{\alpha}{}_{\beta_1\ldots\beta_s k} - (-s-1){}^{(s)}\omega^{\alpha}{}_{\beta_1\ldots\beta_s k}$$

and where one takes the traceless projection of this equation in the anholonomic indices, as we did in Sec. III B 2. For the interpretation of Eq. (63) as the gauge transformation of the coset field ${}^{(s)}\omega$, we have to consider the field ${}^{(s-1)}\omega^{\alpha}{}_{\beta_1\ldots\beta_s}$ as the gauge parameter of ${}^{(s)}\omega$. Indeed, if we define $a_{[\alpha|\beta_1]\beta_2\ldots\beta_s} := {}^{(s-1)}\omega_{[\alpha\beta_1]\beta_2\ldots\beta_s}$ and $\omega_{k|[\alpha|\beta_1]\beta_2\ldots\beta_s} := {}^{(s)}\omega_{[\alpha\beta_1]\beta_2\ldots\beta_s k}$, then Eq. (63) antisymmetrized in $(\alpha\beta_1)$ is equivalent to the transformation (62) in the manifestly antisymmetric conventions.[6]

It is crucial to observe here that a certain coset field ${}^{(s)}\omega$ plays simultaneously the role of a gauge field as well as that of a gauge parameter: On the one hand, the field ${}^{(s-1)}\omega^{\alpha}{}_{\beta_1\ldots\beta_s}$ acts as the gauge parameter of the connection ${}^{(s)}\omega^{\alpha}{}_{\beta_1\ldots\beta_s k}$. On the other hand, on the next higher level, ${}^{(s)}\omega^{\alpha}{}_{\beta_1\ldots\beta_s\beta_{s+1}}$ has to be interpreted itself as the gauge parameter of ${}^{(s+1)}\omega^{\alpha}{}_{\beta_1\ldots\beta_{s+1}k}$. We have already encountered this double role in Sec. II D, where we interpreted Eq. (20) as an absorption equation.

Why do we expect Eq. (20) to reproduce the gauge transformations of higher spin connections? In Sec. II D we interpreted Eq. (20) as an absorption equation for Goldstone bosons. In the standard Higgs mechanism of elementary particle physics the absorption of a Goldstone boson by a gauge field is identical to a gauge transformation in which the gauge parameter is identified with the Goldstone boson. It is thus natural to regard Eq. (20) as a kind of gravitational gauge transformation.

Continuing the analogy to gauging even further, we may ask which global symmetry is made local by the generalized connections. Note that the fields ${}^{(s)}\omega^{\alpha}{}_{\beta_1\ldots\beta_s k}$ are the components of the connection one-forms $\Gamma^{\alpha}{}_{\beta_1\ldots\beta_s}$ associated with the generators $F_{\alpha}^{(s-1)\beta_1\ldots\beta_s}$ $(s \geqslant 1)$. In this sense, the global symmetries generated by $F_{\alpha}^{(s-1)\beta_1\ldots\beta_s}$ are "gauged" by ${}^{(s)}\omega^{\alpha}{}_{\beta_1\ldots\beta_s k}$. For $s = 1$ this implies that the ordinary connection $\omega^{\alpha}{}_{\beta k}$ is the gauge potential of the linear group.

[6] Since ${}^{(s)}\omega^{\alpha}{}_{\beta_1\ldots\beta_s k}$ is already completely symmetric in the indices $(\beta_1\ldots\beta_s k)$, there is no further parameter $\Sigma_{k|\alpha|\beta_1\ldots\beta_s}$ on the right-hand side (r.h.s.) of Eq. (63).

B. The strong equivalence principle

Gravity in a space-time equipped with generalized connections obeys the strong equivalence principle (SEP). The SEP states that gravitational interactions can be gauged away by an appropriate coordinate transformation. To see this, we prove that at each point P there exists a coordinate system such that

$$ {}^{(s)}\omega^{i}{}_{j_1\ldots j_s k}|_P = 0 $$

for all $s \geqslant 1$.

Let us choose P as the point of origin $x^i = 0$ (choose gauge $\xi^i = x^i$) and perform the coordinate transformations

$$ x^i \to x'^i = x^i + \frac{1}{(s+1)!}\varepsilon^{i}{}_{j_1\ldots j_s k} x^{j_1}\cdots x^{j_s}x^k. \tag{64} $$

Substituting this into the transformation law (12), we obtain[7]

$$ {}^{(s)}\omega'^{i}{}_{j_1\ldots j_s k}|_P = {}^{(s)}\omega^{i}{}_{j_1\ldots j_s k}|_P - \varepsilon^{i}{}_{j_1\ldots j_s k}. $$

If we choose the parameters $\varepsilon^{i}{}_{j_1\ldots j_s k} = {}^{(s)}\omega^{i}{}_{j_1\ldots j_s k}|_P$, we get ${}^{(s)}\omega'^{i}{}_{j_1\ldots j_s k}|_P = 0$ and, from this, ${}^{(s)}\Gamma^{i}{}_{j_1\ldots j_s k}|_P = 0$ for all $s \geqslant 1$. It is thus possible to find a coordinate system at a point P in which there is no gravitational force on a point particle, i.e. $\ddot{x}^i|_P = 0$ (SEP). All higher spin connections have been gauged away.

C. Velocity-dependent affine connection

There exists an interesting alternative view of a space-time endowed with higher spin connections. This view is based on a geometrical object called *N-connection* (N for nonlinear). The concept of an N-connection $N^{i}{}_{j}(x, \dot{x})$ was first introduced by É. Cartan [43] in his work on Finsler spaces, see [44] for a modern review. The N-connection is related to a velocity-dependent affine connection $\gamma^{i}{}_{jk}(x, \dot{x})$ by

$$ N^{i}{}_{j}(x, \dot{x}) = \frac{1}{2}\frac{\partial}{\partial \dot{x}^j}(\gamma^{i}{}_{nk}(x, \dot{x})\dot{x}^n\dot{x}^k). \tag{65} $$

The affine connection $\gamma^{i}{}_{jk}(x, \dot{x})$ can now be defined in terms of the higher spin connections ${}^{(s)}\Gamma$,

$$ \gamma^{i}{}_{nk}(x, \dot{x}) \equiv \sum_{s=1}^{\infty} {}^{(s)}\Gamma^{i}{}_{nj_2\ldots j_s k}\dot{x}^{j_2}\cdots \dot{x}^{j_s} \tag{66} $$

which transforms as required:

$$ \delta\gamma^{i}{}_{nk}(x, \dot{x}) = \varepsilon^{i}{}_{m}\gamma^{m}{}_{nk} - 2\varepsilon^{m}{}_{(n}\gamma^{i}{}_{k)m} - \varepsilon^{i}{}_{,nk}. \tag{67} $$

The inhomogeneity $\varepsilon^{i}{}_{,nk}$ follows from the variation $\delta^{(1)}\Gamma$, while the terms with $s > 1$ on the r.h.s. of (66) transform as (use $\delta\dot{x}^i = \dot{\varepsilon}^i = \varepsilon^{i}{}_{m}\dot{x}^m$)

$$ \delta({}^{(s)}\Gamma^{i}{}_{nj_2\ldots j_s k}\dot{x}^{j_2}\cdots\dot{x}^{j_s}) = (\varepsilon^{i}{}_{m}{}^{(s)}\Gamma^{m}{}_{nj_2\ldots j_s k} - 2\varepsilon^{m}{}_{(n|}{}^{(s)}\Gamma^{i}{}_{j_2\ldots j_s|k)m})\dot{x}^{j_2}\cdots\dot{x}^{j_s}, \tag{68} $$

where only the indices n and k are symmetrized. Here terms involving the variations $\delta\dot{x}^i$ have cancelled $s-1$ terms in the tensor transformation of ${}^{(s)}\Gamma$ ($s > 1$).

Physically, Eq. (66) means that a space-time equipped with higher spin connections is equivalent to a space-time with a velocity-dependent affine connection $\gamma^{i}{}_{jk}(x, \dot{x})$. The gravitational force on a test particle thus depends not only on the location of the particle, but also on its velocity similar as in a Finsler space. However, since $\gamma^{i}{}_{jk}$ is not derived from any metric structure, this space-time is more general than a Finsler space.

D. Matter currents

We have not yet discussed the matter currents associated with the generalized connections. Here, we restrict ourselves to a few comments. A thorough discussion of the matter currents is beyond the scope of this paper.

Consider a general matter Lagrangian $\mathcal{L} = \mathcal{L}(\Psi, d\Psi, \vartheta^\alpha, d\vartheta^\alpha, {}^{(s)}\Gamma, d^{(s)}\Gamma)$ which includes a matter field Ψ, the coframe ϑ^α, and the generalized connections ${}^{(s)}\Gamma^{\alpha}{}_{\beta_1\ldots\beta_{s+1}}$ ($s \geqslant 0$) as given by Eqs. (17)–(20). We may then define the $d-1$-form currents

$$ \Sigma_\alpha := \frac{\delta\mathcal{L}}{\delta\vartheta^\alpha}, \tag{69} $$

$$ \Delta_{\alpha}^{(s)\beta_1\ldots\beta_{s+1}} := \frac{\delta\mathcal{L}}{\delta^{(s)}\Gamma^{\alpha}{}_{\beta_1\ldots\beta_{s+1}}} \qquad (s \geqslant 0). \tag{70} $$

Here Σ_α is the canonical energy-momentum current and $\Delta_{\alpha}^{(s)\beta_1\ldots\beta_{s+1}}$ denotes currents which we will call *hypermomentum currents* of degree s. The currents $\Delta_{\alpha}^{(s)\beta_1\ldots\beta_{s+1}}$ generalize the hypermomentum current $\Delta_{\alpha}^{(0)\beta}$ known from the metric-affine theory of gravity [3]. Hypermomentum is the sum of the spin current $\tau_{\alpha\beta} = \Delta^{(0)}_{[\alpha\beta]}$ and the shear and dilation current $\Delta^{(0)}_{(\alpha\beta)}$.

The components of $\Sigma_\alpha = \Sigma_{k\alpha}dx^k$ and $\Delta_{\alpha}^{(s)\beta_1\ldots\beta_{s+1}} = \Delta_{k\alpha}^{(s)\beta_1\ldots\beta_{s+1}}dx^k$ may be used to define the generators of the diffeomorphism algebra. In fact, integrating the components $\Sigma_{0\alpha}$ and $\Delta_{0\alpha}^{(s)\beta_1\ldots\beta_{s+1}}$ over a $d-1$-dimensional spacelike hypersurface, we recover (gauge $\xi^i = x^i$)

$$ P_\alpha = \int d^{d-1}x\Sigma_{0\alpha}, \tag{71} $$

[7] We perform the coordinate transformation (64) first for ${}^{(1)}\omega$, then for ${}^{(2)}\omega$, etc. In this way the term $\mathcal{O}({}^{(s-1)}\omega)$ in (12) is absent, since we have already set all lower spin connections to zero.

$$F_{\alpha}^{(s)\beta_1\ldots\beta_{s+1}} = \int d^{d-1}x \Delta_{0\alpha}^{(s)\beta_1\ldots\beta_{s+1}} \tag{72}$$

which, by construction, satisfy the algebra (3).

Which are the matter fields carrying these currents? Representations of the Poincaré group carry only energy momentum and spin. In order to have also sources for hypermomentum, we would have to construct field equations for representations of the double covering of $\overline{GL}(d, \mathbb{R})$ or the diffeomorphism group $\overline{\mathrm{Diff}}\,(d, \mathbb{R})$. We briefly commented on this in [19], Sec. IV A, see also [3] and references therein.

V. CONCLUSIONS

In this paper we discussed the higher spin Goldstone fields ${}^{(s)}\omega^i{}_{j_1\ldots j_s k}$ of the spontaneous breaking of the group of analytic diffeomorphisms and its relevance for gravity. It is quite a challenge to construct a Higgs mechanism for the complete diffeomorphism group.

As a partial realization, we provided a Higgs mechanism for the breaking of its linear subgroup down to the Lorentz group. Our model predicts that gravity is modified at high energies by the exchange of a massive spin-3 field. This field was identified as the totally symmetric part of the nonmetricity field Q_{ijk}. In [19] we suggested the name "triton" for the corresponding particle. The range of this additional spin-3 force is of order of the Compton wavelength $\lambda_c = h/m_Q c$ and appears to be extremely short-ranged. The mass m_Q of nonmetricity enters our model as a free parameter and has to be measured experimentally. Under the assumption that our model is related to hybrid inflation, we estimated m_Q to be at 10^{15}–10^{16} GeV.

Of course, one expects [7] all higher spin fields ${}^{(s)}\omega$ to become massive due to a similar Higgs effect. To gain insight into the complexity of the Higgs effect, we therefore modeled also the absorption process for ${}^{(2)}\omega^i{}_{j_1 j_2 k}$ adopting the Stückelberg formalism.

From the nonlinear realization discussion, it is clear that the complete symmetry breaking of the diffeomorphism group should provide a massless graviton and an infinite tower of massive higher spin particles. This particle spectrum reminds to that of string theory, but with the difference that here the fields acquire mass by a Higgs mechanism. It would be exciting to find a constraint in a generalization of our Higgs model to higher spin fields which constraints the corresponding particles to lie on Regge trajectories.

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APPENDIX A: THE TRANSFORMATION BEHAVIOR OF THE COSET FIELDS ${}^{(s)}\boldsymbol{\omega}$

In this appendix we compute the transformation behavior of the coset fields ${}^{(s)}\omega^i{}_{j_1\ldots j_s k}$ associated with the broken generators $F_i^{(s)j_1\ldots j_s k}$ of the diffeomorphism group. The computation is analog to that in [19], App. A, where it was performed for the special case $s = 1$.

For simplicity, we only consider fields with holonomic indices and restrict on the coset space $G/H = \mathrm{Diff}\,(d, \mathbb{R})/GL(d, \mathbb{R})$. For the coset element $\sigma \in G/H$, the group elements $g \in G$ and $h \in H$, we choose the parametrizations

$$\sigma(\xi, \omega) = e^{i\xi\cdot P} e^{i\overset{(1)}{\omega}\cdot F^{(1)}} e^{i\overset{(2)}{\omega}\cdot F^{(2)}} \cdots, \tag{A1}$$

$$g(\epsilon) \approx 1 + i\epsilon \cdot P + i\epsilon \cdot F^{(0)} + i\epsilon \cdot F^{(1)} + \ldots, \tag{A2}$$

$$h(\alpha) \approx 1 + i\alpha \cdot M, \qquad \alpha = \alpha(\epsilon; \xi, \omega), \tag{A3}$$

where

$$\begin{aligned} \epsilon \cdot P = \epsilon^i P_i, \qquad \epsilon \cdot F^{(0)} = \epsilon^i{}_j F_i^{(0)j}, \\ \epsilon \cdot F^{(1)} = \epsilon^i{}_{j_1 j_2} F_i^{(1)j_1 j_2}, \text{ etc.} \end{aligned} \tag{A4}$$

In order to obtain the transformation behavior $\delta^{(s)}\omega^i{}_{j_1\ldots j_s k}$, we substitute the above parametrizations into the nonlinear transformation law for elements σ of G/H given by [11,12]

$$g(\epsilon)\sigma(\xi, \omega) = \sigma(\xi', \omega')h(\epsilon, \xi, \omega). \tag{A5}$$

Solving for $h(\epsilon, \xi, \omega)$, we get

$$\begin{aligned} 1 + i\alpha \cdot F^{(0)} = {}& \cdots e^{-i\overset{(2)}{\omega}\cdot F^{(2)}} e^{-i\overset{(1)}{\omega}\cdot F^{(1)}} (1 + i\overset{(0)}{\varepsilon} \cdot F^{(0)} \\ & + i\overset{(1)}{\varepsilon} \cdot F^{(1)} + \ldots) e^{i\overset{(1)}{\omega}\cdot F^{(1)}} e^{i\overset{(2)}{\omega}\cdot F^{(2)}} \cdots \\ & + \sum_{n=1}^{\infty} \cdots e^{-i\overset{(n+1)}{\omega}\cdot F^{(n+1)}} e^{-i\overset{(n)}{\omega}\cdot F^{(n)}} \\ & \times \left(i \sum_{i_n=0}^{\infty} \frac{(-1)^{i_n+1}}{(i_n+1)!} (\overset{(n)}{\omega})^{i_n} [\delta \overset{(n)}{\omega}] \cdot F^{(n i_n + n)} \right) \\ & \times e^{i\overset{(n)}{\omega}\cdot F^{(n)}} e^{i\overset{(n+1)}{\omega}\cdot F^{(n+1)}} \cdots, \end{aligned} \tag{A6}$$

where

$$\overset{(n)}{\varepsilon} \cdot F^{(n)} = \frac{1}{(n+1)!} \frac{\partial^{n+1} \varepsilon^i(\xi)}{\partial \xi^{j_1} \ldots \partial \xi^{j_n} \partial \xi^k} F_i^{(n) j_1 \ldots j_n k}, \qquad \varepsilon^i(\xi) \equiv \epsilon^i + \epsilon^i_j \xi^j + \epsilon^i_{j_1 j_2} \xi^{j_1} \xi^{j_2} + \ldots = \delta \xi^i. \tag{A7}$$

Note that we have already performed the multiplication with $e^{\pm i \xi \cdot P}$. As shown in detail in [19], App. A1, this promotes the parameters of $g(\epsilon)$ to space-time dependent fields: $\epsilon \rightarrow \varepsilon(\xi)$.

For the computation of (A6), it turns out to be convenient to introduce the following bracket notation: For any two tensors $T^{(n)} \rightsquigarrow T^i{}_{j_1 \ldots j_n k}$ and $U^{(q)} \rightsquigarrow U^i{}_{j_1 \ldots j_q k}$ of type $(1, n+1)$ and $(1, q+1)$, completely symmetric in their covariant indices, we have the following bracket which gives a tensor of type $(1, n+q+1)$, completely symmetric in its covariant indices as well

$$[,]: (T^{(n)}, U^{(q)}) \rightarrow [T^{(n)}, U^{(q)}]^{(n+q)},$$
$$[T^{(n)}, U^{(q)}]^i{}_{j_1 \ldots j_{n+q} k} = (n+1) T^i{}_{l(j_1 \ldots j_n} U^l{}_{j_{n+1} \ldots j_{n+q} k)} - (q+1) U^i{}_{l(j_1 \ldots j_q} T^l{}_{j_{q+1} \ldots j_{n+q} k)}. \tag{A8}$$

If we further define the notation

$$\overset{(n)}{\omega}[\overset{(p)}{\varepsilon}] := [\overset{(p)}{\varepsilon}, \overset{(n)}{\omega}],$$
$$(\overset{(n)}{\omega})^2[\overset{(p)}{\varepsilon}] := [[\overset{(p)}{\varepsilon}, \overset{(n)}{\omega}], \overset{(n)}{\omega}],$$
$$(\overset{(n)}{\omega})^3[\overset{(p)}{\varepsilon}] := [[[\overset{(p)}{\varepsilon}, \overset{(n)}{\omega}], \overset{(n)}{\omega}], \overset{(n)}{\omega}],$$
$$\vdots$$

and

$$\overset{(n_1)}{\omega} \cdots \overset{(n_s)}{\omega}[\overset{(p)}{\varepsilon}] := [\ldots[\overset{(p)}{\varepsilon}, \overset{(n_s)}{\omega}], \overset{(n_{s-1})}{\omega}], \ldots], \overset{(n_1)}{\omega}], \tag{A9}$$

then, for example,

$$e^{-i\overset{(r)}{\omega} \cdot F^{(r)}} (1 + i \overset{(0)}{\varepsilon} \cdot F^{(0)} + i \overset{(1)}{\varepsilon} \cdot F^{(1)} + \ldots) e^{i\overset{(r)}{\omega} \cdot F^{(r)}} = 1 + i \sum_{s=1}^{\infty} \left(\sum_{k=0}^{s-1} \frac{1}{k!} (\overset{(r)}{\omega})^k [\overset{(s-k-1)}{\varepsilon}] \right) \cdot F^{(s-1)}. \tag{A10}$$

Here we used the Baker-Campbell-Hausdorff formula in the form

$$e^{-B} A e^{B} = A + [A, B] + \frac{1}{2!}[[A, B], B] + \ldots \tag{A11}$$

for two operators A and B.

We have now an algorithm to write down $\delta \overset{(s)}{\omega}$ in a closed form. Comparing successively the coefficients of $F^{(1)}$, $F^{(2)}$, $F^{(3)}$, etc. in Eq. (A6), we obtain

$$\delta \overset{(1)}{\omega} = \overset{(1)}{\varepsilon} + \overset{(1)}{\omega}[\overset{(0)}{\varepsilon}], \tag{A12}$$

$$\delta \overset{(2)}{\omega} = \overset{(2)}{\varepsilon} + \overset{(2)}{\omega}[\overset{(0)}{\varepsilon}] + \tfrac{1}{2} \overset{(1)}{\omega}[\overset{(1)}{\varepsilon}], \tag{A13}$$

$$\delta \overset{(3)}{\omega} = \overset{(3)}{\varepsilon} + \overset{(3)}{\omega}[\overset{(0)}{\varepsilon}] + \overset{(1)}{\omega}[\overset{(2)}{\varepsilon}] - \frac{1}{3!} \overset{(1)}{\omega} \overset{(1)}{\omega}[\overset{(1)}{\varepsilon}], \text{ etc.} \tag{A14}$$

For general $\delta \overset{(s)}{\omega}$, we therefore get

$$\delta \overset{(s)}{\omega} = \overset{(s)}{\varepsilon} + \overset{(s)}{\omega}[\overset{(0)}{\varepsilon}] + \ldots, \tag{A15}$$

which is identical to Eq. (10).

APPENDIX B: THE TOTAL NONLINEAR CONNECTION

In this appendix we give a compact expression for the total nonlinear connection $\Gamma = \sigma^{-1} d\sigma$. The coset element $\sigma \in \mathrm{Diff}\,(d, \mathbb{R})/SO(1, d-1)$ will be parametrized as in Eq. (16). After a short computation, we get

$$\Gamma = \sum_{n=-1}^{\infty} \prod_{m=n+1}^{\infty} e^{-i^{(m)}\omega \cdot F^{(m)}} (e^{-i^{(n)}\omega \cdot F^{(n)}} d e^{i^{(n)}\omega \cdot F^{(n)}}) e^{i^{(m)}\omega \cdot F^{(m)}}, \tag{B1}$$

where

$$e^{-i^{(n)}\omega \cdot F^{(n)}} d e^{i^{(n)}\omega \cdot F^{(n)}} = \sum_{i_n=0}^{\infty} \frac{i(^{(n)}\omega)^{i_n}[d^{(n)}\omega]}{(i_n+1)!} \cdot F^{(n i_n + n)}.$$

Here we defined $^{(-1)}\omega \equiv \xi$ and $^{(0)}\omega \equiv h$, where h is the shear coset parameter corresponding to $GL(d, \mathbb{R})/SO(1, d-1)$. It is understood that the exponentials have to be written in ascending (descending) order on the right (left) of the central factor $e^{-\cdots} d e^{\cdots}$.

Using the bracket notation of App. A, we find for the one-forms $\Gamma^i{}_{j_1 \ldots j_s} = (\Gamma|_{F^{(s-1)}})^i{}_{j_1 \ldots j_s}$ the compact expression

$$\Gamma|_{F^{(s-1)}} = \sum_{n=-1}^{s-1} \sum_{i_n, \ldots, i_s=0}^{s} (^{(s)}\omega)^{i_s} \cdots (^{(n)}\omega)^{i_n}[d^{(n)}\omega] \times \frac{\delta(i_n, \ldots, i_s, n)}{(i_n+1)! i_{n+1}! \ldots i_s!} \delta_{i_{-1}, 0}, \tag{B2}$$

where

$$\delta(i_n, i_{n+1}, \ldots, i_s, n) = 1, \tag{B3}$$

if $n + n i_n + (n+1) i_{n+1} + \ldots + s i_s = s - 1$, zero otherwise.

To linear order this can be expanded as

$$\Gamma|_{F^{(s-1)}} = d^{(s-1)}\omega + {}^{(s)}\omega[d\xi] + \mathcal{O}(\omega^2), \tag{B4}$$

where the first two terms correspond to $n = s-1$, $i_n = 0$ and $n = -1$, $i_1 = \ldots = i_{s-1} = 0$, $i_s = 1$, respectively.

The first five coefficients are

$$\Gamma|_{F^{(-1)}} = \vartheta \equiv e^h[d\xi] = (1 + h + \tfrac{1}{2}h^2 + \ldots)[d\xi], \tag{B5}$$

$$\Gamma|_{F^{(0)}} = e^{-1} de + {}^{(1)}\omega[\vartheta], \tag{B6}$$

$$\Gamma|_{F^{(1)}} = d^{(1)}\omega + \left({}^{(2)}\omega + \frac{1}{2!}({}^{(1)}\omega)^2\right)[\vartheta] + {}^{(1)}\omega[e^{-1}de], \tag{B7}$$

$$\Gamma|_{F^{(2)}} = d^{(2)}\omega + \left({}^{(3)}\omega + {}^{(2)}\omega{}^{(1)}\omega + \frac{1}{3!}({}^{(1)}\omega)^3\right)[\vartheta] + \left({}^{(2)}\omega + \frac{1}{2!}({}^{(1)}\omega)^2\right)[e^{-1}de] + \frac{1}{2!}{}^{(1)}\omega[d^{(1)}\omega], \tag{B8}$$

$$\Gamma|_{F^{(3)}} = d^{(3)}\omega + \left({}^{(4)}\omega + {}^{(3)}\omega{}^{(1)}\omega + \frac{1}{2!}({}^{(2)}\omega)^2 + \frac{1}{4!}({}^{(1)}\omega)^4\right)[\vartheta] + \left({}^{(3)}\omega + {}^{(2)}\omega{}^{(1)}\omega + \frac{1}{3!}({}^{(1)}\omega)^3\right) \times [e^{-1}de] + \left({}^{(2)}\omega + \frac{1}{3!}({}^{(1)}\omega)^2\right)[d^{(1)}\omega], \tag{B9}$$

with

$$e^{-1}de = \sum_{i_0=0}^{\infty}\frac{1}{(i_0+1)!}h^{i_0}[dh]. \tag{B10}$$

If we apply the rule for the bracket in Eq. (A8), we find Eqs. (17)–(20).

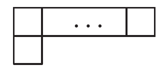
The transformation law for the one-forms $\Gamma^i{}_{j_1\ldots j_s} = (\Gamma|_{F^{(s-1)}})^i{}_{j_1\ldots j_s k}d\xi^k$ follows from those for ${}^{(s)}\omega$ given in App. A. Since $\Gamma_{G/H}$ transforms as a tensor, we expect

$$\delta\Gamma|_{F^{(s-1)}} = (\Gamma|_{F^{(s-1)}})[\overset{(0)}{\varepsilon}] = [\overset{(0)}{\varepsilon}, \Gamma|_{F^{(s-1)}}] \tag{B11}$$

for $s \geqslant 2$. We explicitly checked this for $\Gamma|_{F^{(1)}}$ using Eqs. (A12), (A13), and (B7).

APPENDIX C: DECOMPOSITION OF HIGHER SPIN CONNECTIONS

Upon lowering the upper index i, the higher spin connection $\omega^i{}_{j_1\ldots j_s k}$ can be decomposed under $GL(d,\mathbb{R})$ into a totally symmetric part corresponding to the Young tableau $[s+2,0]$ and a part corresponding to $[s+1,1]$:

	$GL(d,\mathbb{R})$	dimension
${}^{(s)}\omega_{(ij_1\ldots j_s k)}$		$\frac{(d+s+1)!}{(d-1)!(s+2)!}$,
${}^{(s)}\omega_{[ij_1]\ldots j_s k}$		$\frac{(d+s)!(s+1)}{(d-2)!(s+2)!}$.

In total, the higher spin connection ${}^{(s)}\omega$ has

$$d\binom{d+s}{s+1} \tag{C1}$$

off shell components.

Let us consider the case in which ${}^{(s)}\omega$ is massless. Then, in order to apply the Fronsdal description for these fields, we have to split ${}^{(s)}\omega$ into double-traceless fields. For instance, ${}^{(s)}\omega_{(ij_1\ldots j_s k)}$ is equivalent to the sum of double-traceless fields ${}^{(s)}\hat{\hat{\omega}}_{(ij_1\ldots j_s k)}$, ${}^{(s)}\hat{\hat{\omega}}_{(ij_1\ldots j_{s-4} k)}$, ${}^{(s)}\hat{\hat{\omega}}_{(ij_1\ldots j_{s-8} k)}$, etc.

The number of on shell degrees of freedom are given by the same Young diagram, now labeling an $O(d-2)$ irreducible representation. In $d=4$, the fields ${}^{(s)}\hat{\hat{\omega}}_{(ij_1\ldots j_{s-n} k)}$ $(n = 0, 4, 8, \ldots)$, have spin $s-n+2$ and 2 on shell degrees of freedom, while the fields ${}^{(s)}\hat{\hat{\omega}}_{[ij_1]\ldots j_{s-n} k}$ $(n = 0, 4, 8, \ldots)$ are *nondynamical*. The hook representations have vanishing on shell degrees of freedom, since the dimension of the same Young tableau under the little group $O(2)$ is zero.

[1] Y. M. Cho, Phys. Rev. D **14**, 2521 (1976).
[2] C. Fronsdal, J. Geom. Phys. **7**, 305 (1990).
[3] F. W. Hehl, J. D. McCrea, E. W. Mielke, and Y. Neeman, Phys. Rep. **258**, 1 (1995).
[4] M. Blagojevic, *Gravitation and Gauge Symmetries* (Institute of Physics and Physical Society, Bristol, UK, 2002), p. 522.
[5] R. Tresguerres, Phys. Rev. D **66**, 064025 (2002).
[6] G. Sardanashvily, gr-qc/0512115.
[7] E. A. Ivanov and V. I. Ogievetsky, Lett. Math. Phys. **1**, 309 (1976); Pis'ma Zh. Eksp. Teor. Fiz. **23**, 661 (1976) [JETP Lett. **23**, 606 (1976)].
[8] Y. Itin and F. W. Hehl, Ann. Phys. (N.Y.) **312**, 60 (2004).
[9] M. Leclerc, Ann. Phys. (N.Y.) **321**, 708 (2006).
[10] V. I. Ogievetsky, Lett. Nuovo Cimento Soc. Ital. Fis. **8**, 988 (1973).
[11] S. R. Coleman, J. Wess, and B. Zumino, Phys. Rev. **177**, 2239 (1969); C. G. Callan, S. R. Coleman, J. Wess, and B. Zumino, Phys. Rev. **177**, 2247 (1969).
[12] A. Salam and J. Strathdee, Phys. Rev. **184**, 1750 (1969); **184**, 1760 (1969).
[13] C. J. Isham, A. Salam, and J. A. Strathdee, Ann. Phys. (N.Y.) **62**, 98 (1971).
[14] D. V. Volkov and V. P. Akulov, Pis'ma Zh. Eksp. Teor. Fiz. **16**, 621 (1972); D. V. Volkov, Fiz. Elem. Chastits At. Yadra **4**, 3 (1973) [Sov. J. Part. Nucl. **4**, 1 (1973)].
[15] V. Ogievetsky, *10th Winter School of Theor. Phys.* (Karpach, Poland, 1973).
[16] A. B. Borisov and V. I. Ogievetsky, Teor. Mat. Fiz. **21**, 329 (1974) [Theor. Math. Phys. (Engl. Transl.) **21**, 1179 (1974)].
[17] F. Englert and R. Brout, Phys. Rev. Lett. **13**, 321 (1964); P. W. Higgs, Phys. Lett. **12**, 132 (1964); Phys. Rev. Lett. **13**, 508 (1964); G. S. Guralnik, C. R. Hagen, and T. W. B.

Kibble, Phys. Rev. Lett. **13**, 585 (1964).
[18] A. Pashnev, hep-th/9704203.
[19] I. Kirsch, Phys. Rev. D **72**, 024001 (2005).
[20] L. P. S. Singh and C. R. Hagen, Phys. Rev. D **9**, 898 (1974).
[21] C. S. Aulakh, I. G. Koh, and S. Ouvry, Phys. Lett. B **173**, 284 (1986).
[22] M. Bianchi, P. J. Heslop, and F. Riccioni, J. High Energy Phys. 08 (2005) 088.
[23] M. A. Vasiliev, Yad. Fiz. **32**, 855 (1980) [Sov. J. Nucl. Phys. **32**, 439 (1980)].
[24] M. A. Vasiliev, Int. J. Geom. Meth. Mod. Phys. **3**, 37 (2006).
[25] C. Heinicke, P. Baekler, and F. W. Hehl, Phys. Rev. D **72**, 025012 (2005); P. Baekler and F. W. Hehl, gr-qc/0601063; Y. N. Obukhov, Phys. Rev. D **73**, 024025 (2006).
[26] C. N. Yang, Phys. Rev. Lett. **33**, 445 (1974).
[27] R. Jackiw, physics/0403109.
[28] E. S. Fradkin and M. A. Vasiliev, Phys. Lett. B **189**, 89 (1987); Nucl. Phys. **B291**, 141 (1987); M. A. Vasiliev, Nucl. Phys. **B616**, 106 (2001); **B652**, 407(E) (2003); K. B. Alkalaev and M. A. Vasiliev, Nucl. Phys. **B655**, 57 (2003).
[29] E. A. Ivanov and V. I. Ogievetsky, Teor. Mat. Fiz. **25**, 164 (1975).
[30] I. Low and A. V. Manohar, Phys. Rev. Lett. **88**, 101602 (2002).
[31] Y. Ne'eman and Dj. Šijački, Phys. Lett. B **200**, 489 (1988).
[32] I. Kirsch and Dj. Šijački, Classical Quantum Gravity **19**, 3157 (2002).
[33] M. A. Vasiliev, Phys. Lett. B **567**, 139 (2003); for a review, see X. Bekaert, S. Cnockaert, C. Iazeolla, and M. A. Vasiliev, Proceedings of the first Solvay Workshop on Higher Spin Fields, Brussels, 2004 (unpublished).
[34] C. Fronsdal, Phys. Rev. D **18**, 3624 (1978); J. Fang and C. Fronsdal, Phys. Rev. D **18**, 3630 (1978).
[35] A. D. Linde, Phys. Rev. D **49**, 748 (1994).
[36] F. Riccioni, Laurea thesis, available on: http://people.roma2.infn.it/~stringhe/activities.htm.
[37] T. Watari and T. Yanagida, Phys. Lett. B **589**, 71 (2004).
[38] C. Aragone, S. Deser, and Z. Yang, Ann. Phys. (N.Y.) **179**, 76 (1987).
[39] N. Boulanger, S. Cnockaert, and M. Henneaux, J. High Energy Phys. 06 (2003) 060.
[40] X. Bekaert and N. Boulanger, Phys. Lett. B **561**, 183 (2003).
[41] K. S. Chung, C. W. Han, J. K. Kim, and I. G. Koh, Phys. Rev. D **37**, 1079 (1988).
[42] B. de Wit and D. Z. Freedman, Phys. Rev. D **21**, 358 (1980).
[43] É. Cartan, *Les Espaces de Finsler* (Hermann, Paris, 1935).
[44] S. Vacaru, P. Stavrinos, E. Gaburov, and D. Gonta, gr-qc/0508023.