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Volume of black holes

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We propose a simple definition of volume for stationary spacetimes. The proposed volume is constant in time, independent of the choice of stationary time slicing, and applies even in the absence of a globally timelike Killing vector. We then consider whether it is possible to construct spacetimes that have finite horizon area but infinite volume, by letting the radius go to infinity while making discrete identifications to preserve the horizon area. We show that, in three or four dimensions, no such solutions exist that are not inconsistent in some way. This may constrain the statistical interpretation of the Bekenstein-Hawking entropy.

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I. INTRODUCTION

Time and space are often regarded as being interchanged across a black hole horizon; the interior of a Schwarzschild black hole, for example, can usefully be thought of as a collapsing universe. Moreover, what one means by the volume of space depends on how spacetime is split into space and time; spatial volume is not slicing invariant. Hence, at first sight, it does not seem to make any sense to talk about the volume of a black hole.

This is unsatisfying because one of the most celebrated facts about quantum gravity is that the entropy is vastly reduced from what it would have been in quantum field theory. The Bekenstein-Hawking entropy equals area$/4l_{\rm p}^2$, which, it is proclaimed, is numerically much less than volume$/l_{\rm p}^3$. This begs the question: what volume?

Now, thermodynamic notions typically call for thermal equilibrium, and, geometrically, "equilibrium" means that the spacetime possesses a symmetry under time translation i.e. there exists a timelike Killing vector. Suppose we have such a timelike Killing vector. Can one determine a volume in this more restricted setting?

Remarkably, the answer is yes. In this paper, we will show that, if the spacetime admits a somewhere-timelike Killing vector then it is possible to define a meaningful notion of volume, even in the absence of a globally timelike Killing vector. The volume that we define is not only constant in time, but also independent of the choice of stationary time slice (with the one proviso that the asymptotic form of the metric also be preserved).

Armed with a working definition of volume, an interesting next question is: are there families of spacetimes whose horizons have bounded area but whose volume can be arbitrarily large? For example, one might try to send the mass of a black hole to infinity, while simultaneously making discrete identifications on the spacetime to preserve the horizon area. Were such a construction to exist, it would be more than a curious fact: as we will argue, it would suggest that the Bekenstein-Hawking entropy counts only those Hilbert states that "live" near the horizon—as opposed to all the states in the black hole's Hilbert space. However, we will be able to show that, at least in three or four spacetime dimensions, no such families of spacetimes exist. We interpret this as evidence that the Bekenstein-Hawking entropy might not be independent of the interior of the horizon.

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II. THE VOLUME OF A BLACK HOLE

For illustration, we will have in mind nonrotating black holes; the final formula, though, requires only stationarity and applies equally to rotating black holes. Consider then an Einstein space with a horizon and a line element of the form

$$ds^2 = -\alpha(r)dt_s^2 + \frac{dr^2}{\alpha(r)} + r^2 d\Sigma_{D-2}^2(\vec{x}), \qquad (1)$$

where $d\Sigma_{D-2}^2$ can be taken to be the line element of a maximally symmetric $D-2$-dimensional space. For instance, α could be

$$\alpha(r) = \frac{2\Lambda}{(D-1)(D-2)} r^2 + \eta - \frac{2M}{r^{D-3}}. \qquad (2)$$

One could also consider adding charge. When $\Lambda \geq 0$, η is $+1$ but, in anti-de Sitter space (AdS), η can also be 0 or -1, corresponding to black holes with flat or negatively curved horizons. The horizon is at r_+ where r_+ is the largest root of $\alpha(r) = 0$. The time coordinate, t_s, is static time; the metric is invariant not only under $t_s \to t_s + c$, but also under $t_s \to -t_s$. However, the coordinate breaks down at the horizon, as evidenced by the divergence of g_{rr} and g^{tt}. To continue through the horizon, one defines a new coordinate, t; static time is then expressed as $t_s(t, r, \vec{x})$. Then ∂_t is a Killing vector if and only if the transformation takes the form

$$t_s = \lambda t + f(r, \vec{x}). \qquad (3)$$

Here λ is a constant, which we can take to be positive to

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preserve the orientation of time. In fact, our definition of volume will require that λ be restricted to 1:

$$t_s \equiv t + f(r, \vec{x}). \tag{4}$$

When $\eta \neq 0$, λ can be set to 1 by demanding a fixed asymptotic form of the metric, or by fixing the asymptotic normalization of the Killing vector. There remains an enormous class of time slicings, since each choice of the almost arbitrary function $f(r, \vec{x})$ defines a different time slice. To reduce clutter, take $f(r, \vec{x})$ to be a function only of r. The $t - r$ part of the line element is now

$$ds^2 = -\alpha(r)dt^2 - 2\alpha(r)f' dt dr + dr^2\left(\frac{1}{\alpha(r)} - \alpha(r)f'^2\right). \tag{5}$$

By choosing f so that f' is real and such that g_{rr} stays positive and finite, one obtains a stationary slicing that extends through the horizon. Note that, although ∂_t may become spacelike across the horizon, the normal to a surface of constant t is—thanks to an off-diagonal term—everywhere timelike; such surfaces constitute bona fide spatial sections. We would like to define an invariant measure on these sections. First, note that, if we were to take the volume to be the proper volume of a hypersurface of constant t, we would not get an invariant volume because g_{rr} manifestly depends on the choice of time slice through its dependence on f. Indeed, by considering a slicing that is nearly lightlike, one can arrange for the proper three-volume to be as close to zero as one wants.

Instead, observe that the determinant of the spacetime metric has no dependence on the time slicing: f' drops out and λ has been set to 1. This suggests the following definition of spatial volume. Consider the differential *spacetime* volume

$$dV_D(t) = \int_t^{t+dt} dt' \int dr \int d^{D-2}x\sqrt{-g_{(D)}}. \tag{6}$$

While the combination $d^D x\sqrt{-g_{(D)}}$ is slicing invariant (in fact, coordinate invariant), dV_D is not, because the limits on the integral are defined in terms of a time coordinate. However, if the time coordinate is of the form (4)—that is, if ∂_t is a Killing vector—then the integrand is time independent with time appearing in dV_D only through the multiplicative factor, dt.

We therefore propose that

$$V_{\text{space}} \equiv \frac{dV_D}{dt} = \int d^{D-1}x\sqrt{-g_{(D)}}. \tag{7}$$

In other words, if, rather than using $\sqrt{g_{(D-1)}}$ as the measure, one uses $\sqrt{-g_{(D)}}$ instead, then two things happen. First, the volume stays constant in time for all choices of Killing time since the integrand is time independent. Second, the integral is now invariant under stationary time slices.

Here is why. Imagine the spacetime integral, (6), as a Riemann sum of little strips, each of coordinate length dt, lined up side by side from $r = 0$ to $r = r_+$. According to (4), a particular constant-time slice merely shifts these strips up or down along the orbit of the Killing vector in an f-dependent manner. But the metric is unchanged under such shifts. Hence, the spacetime integral is invariant even though different time slicings correspond to integration over different spacetime regions. After dividing out by dt, we therefore obtain an invariant spatial volume. Indeed, when λ is fixed to 1, it is the unique invariant volume. Nor is the construction affected by the nature—timelike, spacelike, or null—of the Killing vector.

Actually, this is also the notion of volume that appears in thermodynamics. To see this, write the partition function as

$$Z = \exp(-F\beta) = \exp\left(-\int d^D x\sqrt{-g_{(D)}}\mathcal{L}\right). \tag{8}$$

Now, in the region where thermodynamics applies, the inverse temperature, β, is the period of a complexified time coordinate, τ. Notice: there is no $\sqrt{-g_{\tau\tau}}$ factor in β. Suppose the field is constant in τ. Then the free energy is $F = \int d^{D-1}x\sqrt{-g_{(D)}}\mathcal{L}$. For an extensive system, the free energy is proportional to the volume. We see that this is not inconsistent with regarding $\int d^{D-1}x\sqrt{-g_{(D)}}$ as the volume.

Let us now evaluate the volume for some simple spacetimes. For a four-dimensional spherically symmetric black hole [1], we find that the volume takes a satisfyingly familiar form:

$$V_{\text{spherical hole}} = \tfrac{4}{3}\pi r_+^3. \tag{9}$$

This result is in accord with what an observer in the time-independent region might consider to be the black hole volume. Indeed, it is amusing that this is precisely the proper three-volume of flat Euclidean space. A slicing in which the constant-time hypersurfaces are flat is given by Painlevé coordinates, a coordinate system that has already proven its utility in tunneling calculations [2,3]. However, in general, a smooth flat slice that extends to the singularity need not exist.

Finally, it should be clear that the above arguments relied only on stationarity so the same volume formula applies to rotating black holes. In four dimensions, one finds

$$V_{\text{Kerr hole}} = \tfrac{4}{3}\pi r_+(r_+^2 + a^2). \tag{10}$$

III. FINITE AREA BUT INFINITE VOLUME?

Now that we have a notion of the volume of a black hole, we might ask whether there are families of spacetimes with bounded area but unbounded volume. The motivation for searching for such families is this. Several of the statistical interpretations of black hole entropy refer only to the outside of the horizon; they ignore the hole's interior. If this perspective is correct, it should not matter what the volume is; there should be no statistical mechanical obstruction to finding finite entropy solutions with arbitrarily large volume. The existence of such families would then serve as evidence for the "outside" view of gravitational entropy [4].

As an elementary example, consider D-dimensional Rindler space, with $D > 2$. In Cartesian coordinates, an observer moving with constant acceleration in the positive X_1 direction has a future Rindler horizon described by the light-sheet $T = X_1$. The light-sheet has infinite extent in the X_i directions, for $i = 2 \ldots D - 1$, so the horizon has infinite area. The volume of the spacetime behind the Rindler horizon is also intuitively infinite. If we now make a toroidal compactification of all the transverse directions,

$$X_i \sim X_i + L_i, \qquad i = 2 \ldots D - 1, \tag{11}$$

the horizon area becomes finite: $A = \prod_i L_i$. (The compactification does not imply a dimensional reduction; the L_i could be chosen to be enormous compared with the D-dimensional Planck length.) However, because X_1 is not identified, spatial sections behind the horizon are noncompact and intuitively have infinite volume. Thus this would appear to be an example of a spacetime with a horizon of finite area and infinite volume. However, it has been shown that Rindler space with all but one spatial direction compactified is inconsistent [5] with the finiteness of the entropy.

To find other spacetimes with this property, we note that Rindler space is the infinite mass limit of a nonextremal black hole. Thus, in general, we would like to take a spacetime with a horizon and send the radius of the horizon to infinity, while making discrete identifications to keep the area finite as the radius diverges. More precisely, we would like to quotient by groups obeying:

(i) The group must be a subgroup of the isometry group of the spatial section of the horizon. This is necessary so that the quotient space has a well-defined metric.

(ii) The group must act freely on the spacetime: no fixed points. Otherwise, we would introduce singularities. However, we may allow a fixed point to occur at a "point" that is already singular since singular points are not formally part of the manifold.

(iii) The fundamental domain must not have any noncontractible loops, or 1-cycles, whose length vanishes during the process of simultaneously blowing up the horizon radius and quotienting by the groups. This is because, if there were vanishing 1-cycles, the gravity description could not be trusted; winding modes of closed strings winding around the cycle would become lighter than momentum modes. This is a restrictive requirement. It implies that the identifications have to act democratically in all dimensions along the horizon. Otherwise, the directions in which they do act would be forced to become vanishingly small to preserve the area as the radius grows.

A. Spherical horizons

Consider first spacetimes whose horizons, when sliced using stationary time, are spheres. These have the isometry group $O(D - 1)$. We need a family of discrete subgroups of arbitrarily high order so that, by quotienting with groups of ever larger order, we can keep the area bounded even as the radius diverges. For $D > 3$, there are two infinite families of discrete subgroups of $O(D - 1)$: the cyclic and the dihedral groups. The cyclic groups, C_n, have order n and are isomorphic to Z_n. They act by identifying points in the azimuthal direction: $\phi \sim \phi + 2\pi/n$. C_n does not act freely because, for example, it leaves the poles of the two-sphere fixed, in violation of requirement (ii). The dihedral groups, D_n, have order $2n$, and are isomorphic to $Z_2 \rtimes Z_n$. They are non-Abelian and act freely. However, both C_n and D_n essentially act mainly along the azimuth. The fundamental domain, after modding out by D_n, can be regarded as a wedge extending down from the pole to the equator, much like a segment of an orange. As the radius of the sphere becomes ever greater, the width of the segment must vanish to preserve the area, thus violating requirement (iii).

In three dimensions, stationary sections of the horizon are just circles. So here we need subgroups of $O(2)$. Obviously, we can mod out by Z_n. There are two spacetimes with horizons in three dimensions: the Bañados-Teitelboim-Zanelli (BTZ) black hole and three-dimensional de Sitter space. For de Sitter space modding out by Z_n in the angular direction results in a conical singularity at $r = 0$. (After appropriate relabelings, this can be regarded as a Schwarzschild-de Sitter space, without an identification.)

This leaves the BTZ black hole. The horizon is a circle with circumference

$$C = 2\pi l \sqrt{8GM}, \tag{12}$$

where M is the mass parameter. Now let $M \rightarrow sM$, define $n = \sqrt{[s]}$, and make the identification $\phi \sim \phi + \frac{2\pi}{n}$. Then, as $s \rightarrow \infty$, $C \rightarrow C$. Thus, M can be made arbitrarily large without causing the circumference to diverge. However, there is a problem here. Although discrete identifications can be performed, there is no invariant volume because, since $\eta = 0$ in (2), there is no way to fix λ to 1; time can be rescaled.

We have shown that there are no spherically symmetric spacetimes with finite area and infinite volume. We could also have tried quotienting nonspherically symmetric spacetimes such as the Kerr black hole or Taub-NUT space. However, their isometry groups are just proper subgroups of those of a sphere; hence they also do not yield finite area and infinite volume quotients.

B. Flat horizons

Flat horizons exist in Rindler space, which we have already rejected, as well as in AdS. The AdS black brane solutions have $\Lambda < 0$ and $\eta = 0$ in (2). The isometry group of stationary sections of the horizon is $E(D-2)$ i.e. $ISO(D-1)$. The lattice groups are discrete subgroups with no fixed points. Thus, we can make a toroidal identification on the horizon: $X_i \sim X_i + L_i$. It is easy to see that this satisfies all the requisite properties. After identification, the topology of the stationary slices is now T^{D-2}. But, since $\eta = 0$, λ cannot be set to 1. So, as with the BTZ black hole, there is no invariant volume.

C. Hyperbolic horizons

AdS also has solutions with hyperbolic horizons. These have $\Lambda < 0$ and $\eta = -1$. A stationary section of the horizon is a hyperbolic space, H^{D-2}, a noncompact Riemannian manifold with constant negative curvature (i.e. "Euclidean" anti-de Sitter space).

Consider $D = 4$. Hawking's uniqueness theorem [6] on horizon topology does not apply to AdS black holes; indeed, H^2 has infinitely many topologically inequivalent compactifications [7]. One might hope that some of these might possess finite area and infinite volume horizons. However, the global Gauss-Bonnet theorem says that the integral of the Ricci scalar is related to the Euler characteristic, χ, of the horizon:

$$\frac{1}{4\pi}\int R dA = \chi \Rightarrow A = 4\pi(g-1)r_+^2 \geq 4\pi r_+^2, \qquad (13)$$

where g is the genus, and $\chi = 2 - 2g$. Thus, we see that, irrespective of the compactification, the area becomes infinite as the radius goes to infinity.

In conclusion, there are no families of spacetimes with bounded horizon area but unbounded volume, at least in three or four dimensions. It would be interesting to see whether this no-go theorem can be extended to higher dimensions. Two loopholes in higher dimensions are that asymptotically flat black geometries can have horizons with more complicated topologies such as $S^1 \times S^2$ [8], and that higher-dimensional hyperbolic horizons are not subject to the Gauss-Bonnet theorem.

IV. DISCUSSION

We have defined a slicing-invariant volume and shown that one cannot construct families of spacetimes that have horizons of bounded area but unbounded volume. It is intriguing that, in each case where this might have worked, something went wrong: either there was a cycle of vanishing length (Schwarzschild black holes), or there was a conical singularity (de Sitter space), or the definition of volume became ambiguous (AdS branes and BTZ), or there was a conflict with symmetries (Rindler space), or the area itself diverged (hyperbolic horizons). Perhaps there is a deeper reason why such a construction may be impossible.

One deeper reason might be a conflict with holography. The statistical interpretation of gravitational entropy remains contentious [4,9–12]. One school of thought holds that the entropy enumerates all possible gravitational degrees of freedom within the volume enclosed by the area. That is, it counts the total number of states in the quantum gravity Hilbert space. An alternative view is that entropy counts only entangled states [9]. (Among other possibilities, the entropy might also enumerate horizon states [11], or configurations of the black hole's thermal atmosphere [12].) The fact that the entropy scales as the area has two very different implications from these two perspectives. The first implies that quantum gravity is highly nonlocal, with far fewer degrees of freedom than a local quantum field theory would have had. In contrast, the second implies that quantum gravity is local, locality being precisely the reason that the field deep inside the hole is not entangled with the field outside. Which of these two interpretations is correct is still in dispute because we are unable to count the quantum gravity states directly. For example, in string theory, the counting of microstates [13] is typically done in a dual picture for which the string coupling is weak, leaving the gravity picture of the states unclear (though recent work attempts a more direct approach [14]).

Now, entanglement entropy is indifferent to the volume of space. Indeed, its most appealing feature is that the entropy-area relation appears quite naturally. From the perspective of entanglement entropy—or indeed of any of the interpretations that do not refer to the interior region—there appears to be no reason why spacetimes with finite area could not have unbounded volume. In fact, their existence would have been evidence against the alternate interpretation; that the total number of Hilbert states in an infinite volume might be finite seems hard to believe. However, the fact that no such families of spacetimes exist suggests (if indeed the underlying reason is holographic) that volume and entropy may not be independent. The Bekenstein-Hawking entropy might really be counting all the Hilbert states of quantum gravity.

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