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Classical string in curved backgrounds

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The Mathisson-Papapetrou method is originally used for derivation of the particle world line equation from the covariant conservation of its stress-energy tensor. We generalize this method to extended objects, such as a string. Without specifying the type of matter the string is made of, we obtain both the equations of motion and boundary conditions of the string. The world sheet equations turn out to be more general than the familiar minimal surface equations. In particular, they depend on the internal structure of the string. The relevant cases are classified by examining canonical forms of the effective 2-dimensional stress-energy tensor. The case of homogeneously distributed matter with the tension that equals its mass density is shown to define the familiar Nambu-Goto dynamics. The other three cases include physically relevant massive and massless strings, and unphysical tachyonic strings.

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I. INTRODUCTION

The original motivation for introducing strings in particle physics came from the analysis of meson resonances. As it appears, the known resonances, characterized by the angular momentum J and the mass M, follow the pattern $J = \alpha M^2 + \text{const}$, where α is a universal constant. These are called Regge trajectories.

To explain Regge trajectories, the meson resonances are viewed as excited 2-quark bound states. It has been shown then that a relativistic rotating string with light quarks attached to its ends indeed reproduces the above pattern. The string is characterized by the tension alone, and has no other structure. It was realized later that realistic field configurations with such properties really exist. Such is, for example, the flux tube solution of Ref. [1].

In what follows, we shall not be concerned with particular field-theoretical models that accommodate flux tubes, or any other linelike configurations of fields. We shall merely assume that such kink configurations exist, and try to draw from it as much information as possible. In particular, we want to obtain the world sheet equations of motion.

Our motivation for considering stringy shaped matter in curved backgrounds is twofold. First, as we have already explained, realistic strings (like flux tubes) are really believed to exist, and to be relevant for the description of hadronic matter. Second, the basic Nambu-Goto string action [2,3] is in literature often modified to include interaction with additional background fields. Apart from the target-space metric, the antisymmetric tensor field $B_{\mu\nu}(x)$ and the dilaton field $\Phi(x)$ are considered [4–7]. While the spacetime metric has obvious geometric interpretation, the background fields $B_{\mu\nu}(x)$ and $\Phi(x)$ do not. The attempts have been made in literature to interpret $B_{\mu\nu}$ and Φ as originating from the background torsion and nonmetricity, respectively [8–12]. It seems to us that string dynamics in target-spaces of general geometry is worth considering.

Basically, we are interested in the influence of the target-space torsion on the string dynamics. Our idea is to consider a field-theoretical model that naturally includes torsion (like Poincaré gauge theory of gravity) and find the equations of motion of a stringy shaped material object. Hopefully, the effective action of Refs. [4–7] would be recovered, and the real geometric nature of the background fields $B_{\mu\nu}(x)$ and $\Phi(x)$ found.

In this paper, we shall restrict our considerations to the simpler case of purely Riemannian spacetime. Thus, the geometry is given in terms of the metric tensor alone, and the dynamics is governed by the Einstein's equations

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}. \tag{1}$$

The stress-energy tensor of matter fields is symmetric, $T^{\mu\nu} = T^{\nu\mu}$, and covariantly conserved, $\nabla_\nu T^{\mu\nu} = 0$. In Riemannian spacetimes, the connection in the definition of the covariant derivative $\nabla_\nu v^\mu = \partial_\nu v^\mu + \Gamma^\mu{}_{\rho\nu} v^\rho$ is the Levi-Civita connection. As a consequence, the stress-energy covariant conservation law $\nabla_\nu T^{\mu\nu} = 0$ is rewritten in the form

$$\partial_\nu(\sqrt{-g}T^{\mu\nu}) + \Gamma^\mu{}_{\rho\nu}\sqrt{-g}T^{\rho\nu} = 0. \tag{2}$$

This equation will be the starting point in our analysis of motion of extending objects in curved spacetimes. For practical purposes, we shall consider extended objects with the attributes of test bodies. This way, their influence on the background geometry becomes negligible.

The method we use is a straightforward generalization of the Mathisson-Papapetrou method for pointlike matter [13,14]. It boils down to the analysis of the covariantly conserved stress-energy tensor of matter fields, without specifying their nature. The basic assumption used is the existence of a stringlike localized kink solution in a curved background. Then, the world sheet effective equations of

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motion are obtained in the approximation of an infinitely thin string.

Although the case under consideration is a simple one (torsionless, Riemannian geometry), the resulting world sheet equations turn out to depend on the internal structure of the string. This dependence enters the equations of motion through the effective 2-dimensional stress-energy tensor of the string. Classifying possible canonical forms of the stress-energy 2-tensor, we shall discover a specific distribution of matter characterized by the tension alone. Its world sheet dynamics is fully determined by the target-space geometry, and coincides with that of the known Nambu-Goto string [2,3]. Different matter distributions will contribute to different world sheet equations, and different boundary conditions. In particular, if the string mass is localized in a point, the world sheet turns into a conventional geodesic line.

The layout of the paper is as follows. In Sec. II, the point particle is considered as a demonstration of our method and conventions. The known result is reproduced, but the emphasis is put on the fact that the mass parameter transforms as a 1-dimensional stress-energy tensor. In Sec. III, the effective world sheet equations are derived from the covariant conservation law of the stress-energy tensor of matter fields. Instead of the mass parameter in the point particle case, here, the effective 2-dimensional stress-energy tensor m^{ab} appears to characterize the internal structure of the string. The world sheet equations imply the covariant conservation of m^{ab} with respect to the induced 2-dimensional world sheet metric γ_{ab}. If the string is open, the world sheet equations also include some boundary conditions. Section IV is devoted to the analysis of possible canonical forms of m^{ab}, and to some examples. It is shown how the assumption of homogeneously distributed matter with the tension that equals its mass density leads to the known Nambu-Goto string. As an example, the Nielsen-Olesen vortex line solution of a Higgs type scalar electrodynamics is briefly examined [1]. In Sec. V we give our final remarks. In particular, we emphasize that the results we have obtained are easily generalized to hold for any p-brane in an external Riemannian spacetime.

Our conventions are as follows. Greek indices from the middle of the alphabet, $\mu, \nu, \ldots$, are the target-space indices, and run over $0, 1, \ldots, D-1$. Greek indices from the beginning of the alphabet, $\alpha, \beta, \ldots$, refer to the spatial section of the target-space, and run over $1, 2, \ldots, D-1$. Latin indices $a, b, \ldots$ are the world sheet indices and run over 0, 1. The target-space and world sheet coordinates are denoted by x^μ and ξ^a, respectively. The target-space and world sheet metric tensors are denoted by $g_{\mu\nu}(x)$ and $\gamma_{ab}(\xi)$, respectively. The signature convention is defined by $\eta_{\mu\nu} = \mathrm{diag}(1, -1, \ldots, -1)$, $\eta_{ab} = \mathrm{diag}(1, -1)$. Target-space indices $\mu, \nu, \ldots$ are lowered and raised by the target-space metric $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$, while world sheet indices $a, b, \ldots$ are lowered and raised by the world sheet metric γ_{ab} and its inverse γ^{ab}. Throughout the paper, we shall restrict our considerations to $D = 4$, but we note that all the results remain valid for arbitrary spacetime dimension.

II. PARTICLE DYNAMICS

We begin with the treatment of a point particle in a curved background spacetime. The problem was studied in the early days of relativity by Einstein, Infeld, Hoffman, Mathisson, Papapetrou, and others [13–18]. Here, we formalize the calculations, and adjust the algorithm for the case of a string in the next section.

A. Stress-energy tensor

We need a general form of the stress-energy tensor, suitable for the description of a point particle.

Let us introduce a timelike curve $x^\mu = z^\mu(\tau)$ in spacetime, with τ an arbitrary parameter. We shall consider spacetimes of topology $\Sigma \times R$, where R stands for time, and Σ is an arbitrary 3-manifold representing spatial sections. Both, the spacetime and the curve are supposed to be *nondegenerate* and *complete*. In simple terms, only infinite curves are considered. This way, the unphysical matter distributions, such as instantons, are excluded.

In general, the spacetime coordinates x^μ, and the world line parameter τ are arbitrary. Still, we shall partially fix this freedom to make the exposition more transparent. First, the coordinates x^μ are chosen in accordance with the demand that equal-time surfaces are spacelike. As a consequence, our curve intersects these surfaces only once, and the function $z^0(\tau)$ becomes invertible. Second, the time coordinate $x^0 \equiv t$ is chosen to parametrize the curve. The choice $\tau = t$, or equivalently, $z^0(\tau) \equiv \tau$, puts the constraint $u^0 = 1$ to the form of the tangent vector $u^\mu \equiv dz^\mu/d\tau$. Otherwise, it is an arbitrary timelike vector satisfying $g_{\mu\nu}u^\mu u^\nu \equiv u^2 > 0$.

Next, we expand $\sqrt{-g}T^{\mu\nu}(x)$ into the δ function series around the point $\vec{x} = \vec{z}(t)$, while treating the time coordinate as a parameter. Using the formula (A1) (case $d = 3$), we have:

$$\sqrt{-g}T^{\mu\nu}(t, \vec{x}) = \sqrt{\gamma}b^{\mu\nu}(t)\delta^{(3)}(\vec{x} - \vec{z}(t)) + \sqrt{\gamma}b^{\mu\nu\alpha}(t)\partial_\alpha\delta^{(3)}(\vec{x} - \vec{z}(t)) + \cdots.$$

Here, γ is the induced metric on the curve. It is defined by $ds^2 = g_{\mu\nu}(z)dz^\mu dz^\nu \equiv \gamma(\tau)d\tau^2$, and is introduced for later convenience. The coefficients in the expansion are given by the formula (A2):

$$\sqrt{\gamma}b^{\mu\nu} = \int d^3\vec{x}\,\sqrt{-g}\,T^{\mu\nu},$$

$$\sqrt{\gamma}b^{\mu\nu\alpha} = -\int d^3\vec{x}(x^\alpha - z^\alpha)\sqrt{-g}T^{\mu\nu},$$

etc. Note that the coefficients in the expansion equal the Papapetrou moments $M^{\mu\nu}$, $M^{\alpha\mu\nu}$ etc., as defined in [14].

Now, we introduce the basic assumption about matter. It is localized around the line $z^\mu(\tau)$, i.e. the *stress-energy tensor drops exponentially to zero as we move away from the line*. Of course, this assumption means that field equations allow solutions of such a type. Field theories that have such properties are known to exist, but we shall not be interested in details of particular models.

As a consequence of this assumption, each coefficient $b^{\mu\nu\alpha_1\ldots\alpha_n}$ gets smaller as n gets larger. In the lowest approximation (the so called *single-pole* approximation), all b's except the first are neglected, and we end up with

$$\sqrt{-g}T^{\mu\nu}(t,\vec{x}) = \sqrt{\gamma}b^{\mu\nu}(t)\delta^{(3)}(\vec{x}-\vec{z}(t)). \tag{3}$$

This equation is not covariant with respect to the target-space coordinate transformations. To cast it into a covariant form, we add an extra δ function and an extra integration. Thus, we obtain

$$\sqrt{-g}T^{\mu\nu}(x) = \int d\tau \sqrt{\gamma}b^{\mu\nu}(\tau)\delta^{(4)}(x-z(\tau)), \tag{4}$$

which reduces to (3) in the gauge $z^0(\tau)=\tau$. This form of stress-energy tensor is covariant with respect to both spacetime coordinate transformations and world line reparametrizations. From the known transformation properties of $T^{\mu\nu}$, we infer the transformation properties of $b^{\mu\nu}$. It is a tensor with respect to general coordinate transformations, and scalar with respect to world line reparametrizations. Equation (4) describes matter localized around the line $z^\mu(\tau)$, and in this form, we shall use it to solve the Eq. (2).

B. Equations of motion

We look for the solution of the Eq. (2) in the form (4), where $b^{\mu\nu}(\tau)$ and $z^\mu(\tau)$ are the unknown functions to be determined. Thus, we obtain

$$\int d\tau\sqrt{\gamma}[b^{\mu\nu}\partial_\nu\delta^{(4)}(x-z) + b^{\rho\nu}\Gamma^\mu{}_{\rho\nu}\delta^{(4)}(x-z)] = 0.$$

Viewed as the δ function expansion, this equation is decoupled into a pair of equations determining z^μ and $b^{\mu\nu}$. To see this, we multiply the equation with an arbitrary function $f(x)$ of *compact support*, and integrate over the spacetime. The compact support of $f(x)$ allows switching the order of integrations, which results in

$$\int d\tau\sqrt{\gamma}\left[-b^{\mu\nu}\frac{\partial f(z)}{\partial z^\nu} + b^{\rho\nu}\Gamma^\mu{}_{\rho\nu}(z)f(z)\right] = 0. \tag{5}$$

The scalar field $f(x)$ is arbitrary, but the longitudinal component of the gradient $f_{,\nu}\equiv \partial f(z)/\partial z^\nu$ is not independent of $f(z)$. So, we decompose the gradient $f_{,\nu}$ into the parallel and orthogonal components:

$$f_{,\nu} = f^{\perp}_{\nu} + f^{\|}u_\nu.$$

By definition, $f^\perp_\nu u^\nu = 0$, and the coefficient $f^\|$ is obtained from the identity $df/d\tau \equiv f_{,\nu}u^\nu$. Thus,

$$f_{,\nu} = f^\perp_\nu + \frac{1}{\gamma}\frac{df}{d\tau}u_\nu.$$

Now, $f(z(\tau))$ and $f^\perp_\nu(z(\tau))$ are independent and arbitrary. We arrange (5) as follows:

$$\int d\tau\left\{\sqrt{\gamma}b^{\mu\nu}f^\perp_\nu - \left[\frac{d}{d\tau}\left(\frac{1}{\sqrt{\gamma}}u_\nu b^{\mu\nu}\right) + \sqrt{\gamma}b^{\rho\nu}\Gamma^\mu{}_{\rho\nu}\right]f + \frac{d}{d\tau}\left(\frac{1}{\sqrt{\gamma}}u_\nu b^{\mu\nu}f\right)\right\} = 0. \tag{6}$$

Each of the three terms in the above equation must separately vanish.

The third term gives no contribution to the world line equations. Indeed, the world line is assumed to be infinite, and the function f to have compact support. As a consequence, the corresponding boundary integral vanishes.

The second term gives the world line equations in the form

$$\frac{d}{d\tau}\left(\frac{1}{\sqrt{\gamma}}u_\nu b^{\mu\nu}\right) + \sqrt{\gamma}b^{\rho\nu}\Gamma^\mu{}_{\rho\nu} = 0. \tag{7}$$

Viewed as an equation for $z^\mu(\tau)$, it contains the undetermined coefficients $b^{\mu\nu}(\tau)$.

In the first term, we decompose $b^{\mu\nu}$ into the parallel and orthogonal components with respect to the second index:

$$b^{\mu\nu} = b^{\mu\nu}_\perp + b^\mu u^\nu,$$

where $b^{\mu\nu}_\perp u_\nu \equiv 0$. Then, vanishing of the term $b^{\mu\nu}f^\perp_\nu$ for every $f^\perp_\nu$ implies $b^{\mu\nu}_\perp = 0$, and consequently,

$$b^{\mu\nu} = b^\mu u^\nu.$$

As $b^{\mu\nu}$ is a symmetric tensor, $b^\mu u^\nu$ must equal $b^\nu u^\mu$, so that $b^\mu \propto u^\mu$. Therefore,

$$b^{\mu\nu} = mu^\mu u^\nu, \tag{8}$$

where $m(\tau)$ is an arbitrary coefficient. We see that, up to a multiplicative term, $b^{\mu\nu}$ is fully determined by $z^\mu(\tau)$.

We can now substitute (8) into (7) and obtain

$$\frac{d}{d\tau}(\sqrt{\gamma}mu^\mu) + \sqrt{\gamma}m\Gamma^\mu{}_{\rho\nu}u^\rho u^\nu = 0. \tag{9}$$

This is our final, covariant world line equation. It contains the undetermined $m(\tau)$, but this coefficient is constrained by the very same equation. Indeed, the projection of (9) on the tangent vector u^μ can straightforwardly be brought to the simple form

$$\frac{d}{d\tau}(m\gamma) = 0. \tag{10}$$

We see that $m\gamma$ is a constant of motion, and consequently, it can easily be eliminated from the world line equations. In fact, using the proper distance s to parametrize the curve (which is equivalent to fixing the gauge $\gamma = 1$), we get

$m = \text{const}$, and restore the standard geodesic equation

$$\frac{d^2 z^\mu}{ds^2} + \Gamma^\mu{}_{\rho\nu} \frac{dz^\rho}{ds} \frac{dz^\nu}{ds} = 0.$$

C. Discussion

The world line equations we have obtained are manifestly covariant with respect to both general coordinate transformations and world line reparametrizations. The quantities $b^{\mu\nu}$, u^μ, and m, besides being spacetime tensors (second rank tensor, vector, and scalar, respectively), are also tensors with respect to the reparametrizations $\tau' = \tau'(\tau)$ (scalar, vector, and second rank tensor, respectively). In particular,

$$m'(\tau') = \frac{d\tau'}{d\tau} \frac{d\tau'}{d\tau} m(\tau).$$

Thus, m transforms as a second rank contravariant tensor with respect to world line reparametrizations. This gives us the idea that m can be viewed as an effective one-dimensional stress-energy tensor of the pointlike matter. In support of this interpretation, note that the earlier established conservation of $m\gamma$, as given by (10), can be rewritten as

$$\nabla_\tau m = 0, \tag{11}$$

where ∇_τ stands for the one-dimensional, Riemannian covariant derivative ($\nabla_\tau v \equiv \partial_\tau v + \Gamma v$, where Γ is one-dimensional Levi-Civita connection). Thus, our coefficient m can really be viewed as an effective, *covariantly conserved* one-dimensional energy-momentum tensor. In this respect, m should be considered the particle mass.

The results of this section can be summarized as follows. The stress-energy conservation Eq. (2) is applied to a linelike distribution of matter. In the lowest approximation, such matter distribution is covariantly described by (4), with $z^\mu(\tau)$ and $b^{\mu\nu}(\tau)$ the unknown functions. We found that (a) the world line, parametrized by the proper distance s, satisfies the geodesic equation

$$\frac{d^2 z^\mu}{ds^2} + \Gamma^\mu{}_{\rho\nu} \frac{dz^\rho}{ds} \frac{dz^\nu}{ds} = 0,$$

and (b) the stress-energy tensor takes the form

$$\sqrt{-g} T^{\mu\nu}(x) = m \int ds u^\mu u^\nu \delta^{(4)}(x - z(s)),$$

with m a constant interpreted as the particle mass.

The above results are obtained in the lowest approximation in the δ function expansion. If, however, the second term (*pole-dipole* approximation), or higher order terms were included, the world line equation would depend on the internal structure of the particle. In particular, the particle angular momentum would couple to the spacetime curvature, giving deviations from the geodesic trajectory. The analysis of the higher order particle moments has extensively been done in the literature (see, for example [14]). Here, we just prepare the setting for the study of string dynamics in the next section.

III. STRING DYNAMICS

The calculations presented in the previous section are well known, and there are papers [19,20] that generalize the procedure to include torsion, and explore modifications that it brings to the theory. However, this research has been focused on the particle case, and we want to address the problem of finding equations of motion of an extended object, such as a string. In this section, we generalize the Papapetrou method to linelike matter, and present the results.

A. Stress-energy tensor

As in the particle case, we begin with the stress-energy covariant conservation law in the form (2).

In contrast to the particle, the string is an extended, one-dimensional object whose trajectory is not a world line, but rather a two-dimensional world sheet $\mathcal{M}$. Let us introduce a two-dimensional surface $x^\mu = z^\mu(\xi^a)$ in spacetime, where ξ^0 and ξ^1 are the surface coordinates. We shall assume that the surface is *everywhere regular*, and the coordinates ξ^a well defined. As in the particle case, we shall consider only time-infinite string trajectories. This means that every spatial section of the spacetime has nonempty intersection with the world sheet. As for the intersection itself, it is supposed to be of finite length. Thus, only closed, or finite open strings are considered. In the conventional parametrization, $\xi^0 \equiv \tau$ goes from minus to plus infinity, while $\xi^1 \equiv \sigma$ takes values in the interval $[0, \pi]$. In this parametrization, the world sheet boundary is defined by the coordinate lines $\sigma = 0$ and $\sigma = \pi$.

In what follows, we shall frequently use the notion of the world sheet coordinate vectors

$$u_a^\mu \equiv \frac{\partial z^\mu}{\partial \xi^a},$$

and the world sheet induced metric tensor

$$\gamma_{ab} = g_{\mu\nu} u_a^\mu u_b^\nu.$$

If the world sheet is regular, and the coordinates ξ^a are well defined, the two tangent vectors u_0^μ and u_1^μ are linearly independent. The induced metric is assumed to be nondegenerate, $\det(\gamma_{ab}) \neq 0$, and of Minkowski signature $(+, -)$. With this assumption, each point on the world sheet accommodates a timelike tangent vector. This is how the notion of the timelike curve is generalized to the two-dimensional case. In what follows, we shall also discuss the situations in which this assumption is violated on the world sheet boundary.

We shall restrict our considerations to 4-dimensional spacetimes. As before, we expand the stress-energy tensor into a δ function series around the world sheet. The procedure is basically the same as in the particle case, the only difference being in the use of $\delta^{(2)}$ instead of $\delta^{(3)}$ functions. In the single-pole approximation, we drop all the terms in the expansion except the leading one. In this approximation, the stress-energy tensor contains no δ-function derivatives. Similar to the particle case, the covariantization is achieved by employing two more δ functions, and two more integrations. Thus, we obtain a covariant expression for the stress-energy tensor:

$$\sqrt{-g}T^{\mu\nu}(x) = \int d^2\xi \sqrt{-\gamma} b^{\mu\nu}(\xi)\delta^{(4)}(x - z(\xi)). \quad (12)$$

The coefficients $b^{\mu\nu}$ transform covariantly with respect to both, target-space and world sheet reparametrizations.

B. Equations of motion

Using the ansatz (12) in the Eqs. (2) yields

$$\int d^2\xi \sqrt{-\gamma}[b^{\mu\nu}\partial_\nu \delta^{(4)}(x - z) + b^{\rho\nu}\Gamma^\mu{}_{\rho\nu}\delta^{(4)}(x - z)] = 0.$$

The left-hand side of this equation is almost in the form of the δ-function series. To make use of this, we multiply the equation with an arbitrary function $f(x)$ of *compact support*, and integrate over the spacetime. The compact support of $f(x)$ allows switching the order of integrations. Using partial integration in the first term, we get

$$\int d^2\xi \sqrt{-\gamma}\left[-b^{\mu\nu}\frac{\partial f(z)}{\partial z^\nu} + b^{\rho\nu}\Gamma^\mu{}_{\rho\nu}(z) f(z)\right] = 0. \quad (13)$$

The scalar field $f(x)$ is arbitrary, but the projection of the gradient $f_{,\nu} \equiv \partial f(z)/\partial z^\nu$ on the world sheet is not independent of $f(z)$. So, we decompose the gradient $f_{,\nu}$ into the parallel and orthogonal components:

$$f_{,\nu} = f^\perp_\nu + f^\parallel_a u^a_\nu.$$

By definition, $f^\perp_\nu u^\nu_a = 0$, and the coefficients $f^\parallel_a$ can be expressed through $\partial f/\partial\xi^a \equiv f_{,\nu}u^\nu_a$. Thus, we have:

$$f_{,\nu} = f^\perp_\nu + \frac{\partial f}{\partial \xi^a}u^a_\nu.$$

Now, f and $f^\perp_\nu$ are mutually independent on the world sheet. We arrange (13) as follows:

$$\int_{\mathcal{M}} d^2\xi\left\{ \sqrt{-\gamma}b^{\mu\nu}f^\perp_\nu - \left[\frac{\partial}{\partial\xi^a}(\sqrt{-\gamma}b^{\mu\nu}u^a_\nu) + \sqrt{-\gamma}b^{\rho\nu}\Gamma^\mu{}_{\rho\nu}\right] f + \frac{\partial}{\partial\xi^a}(\sqrt{-\gamma}b^{\mu\nu}u^a_\nu f)\right\} = 0. \quad (14)$$

Owing to the arbitrariness and mutual independence of f and $f^\perp_\nu$, each of the three terms in the integrand of the above equation must separately vanish.

The third term is a two-divergence, and by Stokes theorem, reduces to a line integral over the boundary $\partial\mathcal{M}$. It must vanish for any choice of the function f evaluated on the boundary, and therefore, implies the boundary condition

$$\sqrt{-\gamma}b^{\mu\nu}u^a_\nu n_a|_{\partial\mathcal{M}} = 0. \quad (15)$$

Here, n_a is the outward directed normal to the boundary $\partial\mathcal{M}$. If the boundary line $\xi^a = \xi^a(\varsigma)$ is parametrized by some parameter ς, the normal will take the form

$$n_a = \varepsilon_{ab}\frac{d\xi^b}{d\varsigma}, \quad (16)$$

where ε_{ab} is the antisymmetric Levi-Civita tensor. The boundary conditions (15) do not appear if the string is closed. In that case, $\partial\mathcal{M} = \varnothing$, and the third term of Eq. (14) identically vanishes.

The vanishing of the second term in (14) yields the world sheet equation

$$\frac{\partial}{\partial\xi^a}(\sqrt{-\gamma}b^{\mu\nu}u^a_\nu) + \sqrt{-\gamma}b^{\rho\nu}\Gamma^\mu{}_{\rho\nu} = 0. \quad (17)$$

It generalizes the particle world line Eq. (7). As an equation for $z^\mu(\xi)$, it should be supplemented by appropriate constraints on the unknown coefficients $b^{\mu\nu}$.

Finally, consider the first term. First, we split $b^{\mu\nu}$ into the parallel and orthogonal components with respect to the second index:

$$b^{\mu\nu} = b^{\mu\nu}_\perp + b^{\mu a}u^\nu_a,$$

where $b^{\mu\nu}_\perp u_{a\nu} \equiv 0$. Vanishing of the term $b^{\mu\nu}f^\perp_\nu$ for every $f^\perp_\nu$ implies $b^{\mu\nu}_\perp = 0$, and we are left with

$$b^{\mu\nu} = b^{\mu a}u^\nu_a.$$

Again, we make use of the symmetry of the stress-energy tensor, and obtain

$$b^{\mu a}u^\nu_a = b^{\nu a}u^\mu_a.$$

This means that $b^{\mu a}$ is a linear combination of vectors u^μ_a, and consequently,

$$b^{\mu\nu} = m^{ab}u^\mu_a u^\nu_b. \quad (18)$$

Here, $m^{ab}(\xi)$ are arbitrary coefficients. They transform as scalars with respect to spacetime diffeomorphisms, and as components of a contravariant symmetric second rank tensor with respect to the world sheet reparametrizations. Apart from arbitrariness in m^{ab}, $b^{\mu\nu}$ is fully determined by u^μ_a, which are, in turn, fully determined by $z^\mu(\xi)$. The boundary conditions (15) now read

$$\sqrt{-\gamma}m^{ab}n_b u^\mu_a|_{\partial\mathcal{M}} = 0, \quad (19)$$

while the world sheet equation (17) takes the form

$$\partial_a(\sqrt{-\gamma}m^{ab}u^\mu_b) + \sqrt{-\gamma}m^{ab}u^\rho_a u^\nu_b\Gamma^\mu{}_{\rho\nu} = 0. \quad (20)$$

The world sheet equations can be written in a manifestly covariant way. To this end, we make use of the *total covariant derivative* ∇_a, which acts on both, spacetime and world sheet indices:

$$\nabla_b v^{\mu a} = \partial_b v^{\mu a} + \Gamma^\mu{}_{\nu\rho} u^\rho_b v^{\nu a} + \Gamma^a{}_{cb} v^{\mu c}.$$

Here, $\Gamma^a{}_{cb}$ is the induced connection on the world sheet. In the absence of torsion, it is defined in terms of γ_{ab} via the known Christoffel formula. With this definition, the metricity condition is satisfied for both metric tensors,

$$\nabla_a \gamma_{bc} = 0, \qquad \nabla_a g_{\mu\nu} = 0,$$

and (20) is rewritten as

$$\nabla_a (m^{ab} u^\mu_b) = 0. \tag{21}$$

[The equivalent covariantization in the particle case could be achieved by employing the one-dimensional induced connection $\Gamma = (1/2\gamma)(d\gamma/d\tau)$.] Viewed as an equation for the string trajectory, this equation contains the unknown coefficients m^{ab}. It can be shown, however, that m^{ab} are not fully arbitrary. Instead, they are constrained by the same Eq. (21). To see this, we project (21) on u^c_μ, and obtain

$$\nabla_a m^{ac} + m^{ab} u^c_\mu \nabla_a u^\mu_b = 0. \tag{22}$$

The second term is shown to identically vanish, and we end up with

$$\nabla_a m^{ac} = 0. \tag{23}$$

Thus, m^{ab} is a covariantly conserved, symmetric world sheet tensor. As such, it is seen as the effective two-dimensional stress-energy tensor of the string.

C. Discussion

The Eq. (19) represents a valid form of the boundary conditions, irrespective of the world sheet parametrization used. Even the situations in which the coordinates ξ^a are not well defined on the boundary are included. If everywhere regular coordinates ξ^a are used, the coordinate vectors u^μ_a are linearly independent, and the boundary conditions (19) reduce to

$$\sqrt{-\gamma} m^{ab} n_b|_{\partial\mathcal{M}} = 0. \tag{24}$$

This form of boundary conditions can further be simplified by employing the standard parametrization $\xi^0 = \tau$, $\xi^1 = \sigma$. In these coordinates, the boundary is defined by $\sigma = 0$, $\sigma = \pi$, the n_0 component of the normal (16) vanishes, and the boundary conditions take the simple form

$$\sqrt{-\gamma} m^{a1}|_{\sigma=0,\pi} = 0. \tag{25}$$

Although the metric γ_{ab} is assumed nondegenerate in the interior of the world sheet, we shall retain the term $\sqrt{-\gamma}$ in the above formula to allow violations of this assumption on the boundary. This way, we prevent losing some important solutions of the world sheet equations. In particular, the known Nambu-Goto dynamics belongs to this class of solutions.

The boundary conditions obtained in this section are naturally associated with the familiar Neumann boundary conditions of the conventional string theory. This is a consequence of the fact that only "freely falling" strings in an external gravitational field are considered. In the standard variational approach, they are obtained by allowing free variation of the string boundary. The alternative choice is to use Dirichlet boundary conditions, which are defined by imposing additional constraints on the variation of the string boundary. Precisely, the string ends are attached to an external p-brane, which (partially or fully) fixes their trajectories.

In our approach, this situation is unsatisfactory, as the interaction of the string with the p-brane violates the covariant conservation of the stress-energy tensor at the string ends. We could, of course, impose these constraints by hand, but the natural way to incorporate Dirichlet boundary conditions within our approach is to consider the p-brane and the attached string as a single object moving in an external gravitational field. Although such complex matter configurations are interesting by themselves, we do not study them in the present work. Instead, we consider a string that interacts only with the spacetime geometry. This means that the string ends have nothing else to interact with, which in turn explains why the derivation of the equations of motion yields precisely the Neumann boundary conditions.

The results of this section can be summarized as follows. We considered the stress-energy conservation Eq. (2), and looked for a solution describing a stringlike distribution of matter. In the lowest approximation (an infinitely thin string with no structure in the transverse direction), the stress-energy tensor has the form (12), with $z^\mu(\xi)$ and $b^{\mu\nu}(\xi)$ the unknown functions. We have found that (a) the string dynamics obeys the equation

$$\nabla_a (m^{ab} u^\mu_b) = 0,$$

and (b) the world sheet stress-energy tensor m^{ab} is covariantly conserved

$$\nabla_a m^{ab} = 0.$$

Further, (c) the endpoints of an open string are subject to the boundary conditions

$$\sqrt{-\gamma} m^{a1}|_{\sigma=0,\pi} = 0,$$

and (d) the target-space stress-energy tensor in this approximation has the form

$$\sqrt{-g} T^{\mu\nu}(x) = \int d^2\xi \sqrt{-\gamma} m^{ab} u^\mu_a u^\nu_b \delta^{(4)}(x - z(\xi)).$$

As opposed to the particle case, the dynamics of a stringy shaped matter generally depends on its internal structure. Indeed, the two-dimensional stress-energy conservation $\nabla_a m^{ab} = 0$ has no unique solution. There is a variety of possibilities to choose m^{ab}, each leading to a different string dynamics.

Notice, however, that there exists a geometric solution analogous to the one-dimensional $m \propto \gamma^{-1}$. It has the form $m^{ab} \propto \gamma^{ab}$, and defines a string trajectory in full analogy with the geodesic line. In fact, this particular choice of m^{ab} yields the string dynamics familiar from the literature: the obtained world sheet equations and boundary conditions coincide with what we get by varying the standard Nambu-Goto action. In the next section, we shall classify possible canonical forms of m^{ab}, and explore their influence on the string dynamics. We shall also give some examples to illustrate the feasibility of stringlike solutions in ordinary field theories.

Let us note, in the end of this section, that the particle equations have the same form as those of a string. Indeed, if the indices $a, b, \ldots$ are restrained to take only one value, say 0, the world sheet equations will get the form of a geodesic equation. This is not a coincidence. In fact, it is possible to extend the whole discussion to a very general case of a p-brane moving in a D-dimensional curved spacetime. The equations of motion, boundary conditions, and the covariant conservation of the effective mass tensor m^{ab} are virtually the same, the only difference being in the dimensionality of the world sheet. The p-brane world sheet coordinate indices $a, b, \ldots$ take the values $0, 1, \ldots, p$, while spacetime indices $\mu, \nu, \ldots$ range through $0, 1, \ldots, D-1$.

IV. INTERNAL STRUCTURE OF THE STRING

As we have seen in the previous section, the world sheet equations depend on the type of matter the string is made of. To completely characterize the string trajectory, we need the type and distribution of its mass tensor m^{ab}. In this section, we shall classify possible canonical forms of m^{ab}, and provide some illustrative examples.

A. Canonical forms of the mass tensor

In this section, we shall analyze the eigenproblem of the two-dimensional mass tensor m^{ab}. The analogous 4-dimensional analysis has been done in [21], and the reduction to two dimensions is straightforward.

The eigenproblem of m^{ab} in a general world sheet with metric γ_{ab}, is defined by the equation

$$m^{ab} e_b = \lambda e^a,$$

where $e^a \equiv \gamma^{ab} e_b$. The existence of nonvanishing eigenvectors e^a is guaranteed by the condition $\det[m^{ab} - \lambda\gamma^{ab}] = 0$. It is rewritten as the quadratic equation

$$\lambda^2 - m^a{}_a \lambda + \gamma \det[m^{ab}] = 0,$$

with the discriminant

$$\Delta \equiv (m^a{}_a)^2 - 4\gamma \det[m^{ab}].$$

Because of the indefiniteness of the metric, three cases are possible: $\Delta > 0$, $\Delta = 0$, and $\Delta < 0$. The eigenvectors can be either timelike, spacelike or null. The mass tensor m^{ab} cannot always be diagonalized.

Let us analyze the behavior of m^{ab} in the vicinity of a point on the world sheet. We shall use such ξ^a coordinates which ensure $\gamma_{ab} = \eta_{ab}$, and $\Gamma^a{}_{bc} = 0$ in the chosen point. If we write m^{ab} in a matrix form as

$$m^{ab} = \begin{pmatrix} \rho & \pi \\ \pi & p \end{pmatrix},$$

we see that ρ represents energy density along the string, π is the energy flux, and $-p$ is the string tension. The components of the stress-energy tensor are subject to the physical condition that energy flux must not exceed the energy density: $\rho \geq |\pi|$. Otherwise, matter would travel faster than light [21]. This must be satisfied in every reference frame, which can be shown to imply the general conditions on the components of m^{ab}:

$$\rho + p \geq 2|\pi|, \qquad \rho \geq p. \tag{26}$$

Now, we proceed to examine the cases where diagonalization is, or is not, possible, and to give physical interpretation.

1. Case $\Delta > 0$

In this case, one can employ a Lorentz transformation that brings m^{ab} to a diagonal form:

$$m^{ab} = \begin{pmatrix} \lambda^{(1)} & 0 \\ 0 & -\lambda^{(2)} \end{pmatrix}, \qquad \lambda^{(1)} \neq \lambda^{(2)},$$

where $\lambda^{(1)}$ and $\lambda^{(2)}$ are the eigenvalues of m^{ab}. This means that there exists a *rest frame*, where the energy flux is zero, $\pi = 0$, and matter does not move. This is the case for the usual massive matter.

Conditions (26) are now rewritten as $\lambda^{(1)} \geq |\lambda^{(2)}|$, which means that the energy density ρ is always positive, and exceeds the absolute value of the tension. The string trajectory equations in the vicinity of the chosen world sheet point can further be simplified by using a local inertial frame in the target-space: $g_{\mu\nu} = \eta_{\mu\nu}$, $\Gamma^\mu{}_{\nu\rho} = 0$. Then, the world sheet equations $m^{ab}\nabla_a u^\mu_b = 0$ reduce to

$$\rho \frac{\partial^2 z^\mu}{\partial \tau^2} + p \frac{\partial^2 z^\mu}{\partial \sigma^2} = 0.$$

If the string tension is positive, $p < 0$, we may rewrite this as

$$\frac{\partial^2 z^\mu}{\partial \tau^2} - \omega^2 \frac{\partial^2 z^\mu}{\partial \sigma^2} = 0, \tag{27}$$

where $\omega = \sqrt{-p/\rho}$ is the wave speed of the familiar wave equation. The conditions (26) enforce $\rho > -p$, wherefrom $\omega < 1$. Thus, the speed of sound along the string is less than that of light, as expected for ordinary massive matter.

The world sheet metric γ_{ab} is assumed to be everywhere nondegenerate, including the boundary itself. In this case, the boundary conditions (24) reduce to

$$\lambda^{(1)} n^0|_{\partial\mathcal{M}} = \lambda^{(2)} n^1|_{\partial\mathcal{M}} = 0, \tag{28}$$

which means that at least one of the eigenvalues must vanish. The physical condition $\lambda^{(1)} \geq |\lambda^{(2)}|$ then singles out $\lambda^{(1)} \neq 0$, $\lambda^{(2)} = 0$, and consequently,

$$m^{ab}|_{\partial\mathcal{M}} = \begin{pmatrix} \rho & 0 \\ 0 & 0 \end{pmatrix},$$

with the interpretation that the tension $-p$ vanishes on the string ends. The conditions (28) also imply $n^0 = 0$, which means that the boundary coincides with a coordinate line $\sigma = \text{const}$. Thus, the form (25) of the boundary conditions could have been used, too.

2. Case $\Delta = 0$

In this case, there exists a boost that brings m^{ab} to the form

$$m^{ab} = \begin{pmatrix} \lambda + \mu & \mu \\ \mu & -\lambda + \mu \end{pmatrix}.$$

Here, λ is the single eigenvalue, and the Lorentz invariant sign of μ defines three subcases: $\mu > 0$, $\mu = 0$ and $\mu < 0$. The conditions (26) reduce to $\lambda \geq 0$ and $\mu \geq 0$, which excludes the third possibility $\mu < 0$ as nonphysical. Thus, every nontrivial m^{ab} is the sum of matrices corresponding to the cases $\lambda = 0$, $\mu > 0$ and $\lambda > 0$, $\mu = 0$. Let us discuss these two situations.

In the case $\lambda = 0$, $\mu > 0$, the only eigenvector of m^{ab} is lightlike, and no rest frame exists. The situation is interpreted as that of a *massless matter*. In the four-dimensional electrodynamics, for example, we can consider electric and magnetic fields of equal intensity, $E = B$, and perpendicular to each other, $\vec{E} \cdot \vec{B} = 0$. In a suitable reference frame, the stress-energy tensor has the form

$$T^{\mu\nu} = \begin{pmatrix} E^2 & E^2 & 0 & 0 \\ E^2 & E^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Such is, for example, a linearly polarized plane wave propagating in the x direction. Its stress-energy tensor belongs to the class under consideration, with $\mu = E^2$.

The boundary conditions (24) reduce to

$$\mu(n^0 - n^1)|_{\partial\mathcal{M}} = 0,$$

wherefrom $n^0 = n^1$. Thus, the normal to the boundary is lightlike, and the definition (16) implies the same for the boundary itself. We see that the boundary cannot coincide with the coordinate line $\sigma = \text{const}$, which is the reason we could not use the boundary conditions in the form (25).

The case $\lambda > 0$, $\mu = 0$. Here, the eigenvalue λ is degenerate, and the two eigenvectors can be chosen to be spacelike and timelike, respectively. The mass tensor is not only diagonal, but proportional to the metric: $m^{ab} = \lambda\eta^{ab}$. This can covariantly be written as $m^{ab} = \lambda\gamma^{ab}$, and defines the known *Nambu-Goto string*. The energy density ρ is positive, and equal to the tension:

$$\rho = -p.$$

The equations of motion are precisely the minimal-surface equations. In a local inertial frame, they reduce to the wave equation

$$\frac{\partial^2 z^\mu}{\partial\tau^2} - \frac{\partial^2 z^\mu}{\partial\sigma^2} = 0,$$

which is a special case of (27), with $\omega = 1$. Thus, the speed of sound in the string equals the speed of light. No conventional elastic material exhibits such a behavior.

In the local inertial frame ($\gamma_{ab} = \eta_{ab}$), the only appropriate form of boundary conditions is (19). Indeed, it reduces to

$$n^a u_a^\mu|_{\partial\mathcal{M}} = 0,$$

which shows that the coordinate vectors u_a^μ are not linearly independent. Thus, the inertial frame is necessarily degenerate at the boundary. The best we can do is to Lorentz rotate the normal n^a to achieve $n^0 = 0$, and bring the boundary conditions to the conventional form $u_1^\mu = 0$. If, on the other hand, we insist on using regular parametrization of the world sheet, we must leave the inertial frame. Only then, we can use the form (25) of the boundary conditions, and obtain

$$\sqrt{-\gamma}\gamma^{a1}|_{\sigma=0,\pi} = 0. \tag{29}$$

We see that the world sheet metric at the boundary is degenerate. A careful analysis of the conditions (29) yields the solution

$$\gamma_{ab} = \begin{pmatrix} 0 & 0 \\ 0 & \gamma_{11} \end{pmatrix}$$

at $\sigma = 0$ and $\sigma = \pi$. In particular, $\gamma_{00} \equiv g_{\mu\nu} u_0^\mu u_0^\nu = 0$, which means that the world sheet boundary is lightlike. This is the familiar result of the Nambu-Goto dynamics: the string ends move with the speed of light.

The boundary conditions we have derived are the familiar Neumann boundary conditions. They are the consequence of the "free falling" character of the string motion, and are obtained automatically in our approach. In contrast, the Dirichlet boundary conditions demand the string ends to be attached to an external p-brane, as explained in Sec. III C.

In both $\Delta = 0$ cases we have considered, the string ends move with the speed of light. There is a difference, however, in the behavior of the induced metric γ_{ab}. In the massless case, the metric is regular at the boundary ($\gamma \neq 0$), while in the Nambu-Goto case, it is degenerate ($\gamma = 0$). In geometric terms, the world sheet either intersects the target-space light cone, or just touches it. Both world sheets are regular 2-dimensional surfaces, though.

3. Case $\Delta < 0$

In this case, there exists a boost that brings m^{ab} to the form

$$m^{ab} = \begin{pmatrix} \lambda' & \lambda'' \\ \lambda'' & -\lambda' \end{pmatrix}.$$

Here, the two eigenvalues are complex-conjugate, $\lambda^{(0)} = \lambda' - i\lambda''$ and $\lambda^{(1)} = \lambda' + i\lambda''$. The corresponding eigenvectors are also complex.

The conditions (26) are in contradiction with the above form of m^{ab}. This means that they are never satisfied, as one can always find a reference frame where energy flux exceeds the energy density. Thus, the case is unphysical, corresponding to matter whose speed exceeds the speed of light.

To summarize, the eigenvalue problem we have considered brought about three possible cases: (a) the case $\Delta > 0$ describes massive matter, (b) the case $\Delta = 0$ combines massless matter with matter of Nambu-Goto type, and (c) the case $\Delta < 0$ represents unphysical, tachyonic matter.

B. Inhomogeneous example

Let us consider a string characterized by a highly inhomogeneous distribution of matter. Take the radical situation when all the mass is localized in one single point. In the lowest approximation, the mass tensor m^{ab} is chosen in the form

$$\sqrt{-\gamma}m^{ab} = \int ds b^{ab}(s)\delta^{(2)}(\xi - \chi(s)), \qquad (30)$$

where $b^{ab}(s)$ are some parameters, and $\xi^a = \chi^a(s)$ is a line on the world sheet, parametrized by the proper distance s. As a consequence, the corresponding tangent vector $v^a \equiv d\chi^a/ds$ has unit norm: $\gamma_{ab}v^a v^b = 1$. The world line equation is expected to be found by the analysis of the corresponding world sheet equations.

Our strategy is the same as that of Sec. II, the only difference being the dimensionality of the target-space (there 4, here 2). Therefore, we start with the conservation law $\nabla_a m^{ab} = 0$, and apply the ansatz (30). The result is the exact analogue of the 4-dimensional case: $b^{ab} \propto v^a v^b$, and the world line is a world sheet geodesic

$$\frac{dv^a}{ds} + \Gamma^a{}_{bc}v^b v^c = 0. \qquad (31)$$

The question is if this particular world sheet geodesic is also a spacetime geodesic, as one would expect. The answer is affirmative. To see this, we first note that the world line tangent vector $\vec{v}$ is also tangent to the world sheet and the spacetime. Its spacetime components v^μ can be expressed in terms of the world sheet components v^a as follows:

$$v^\mu = \frac{dz^\mu(\chi(s))}{ds} = \frac{\partial z^\mu(\chi)}{\partial \chi^a}\frac{d\chi^a(s)}{ds} = u^\mu_a v^a.$$

The spacetime proper distance s is, of course, the same as the world sheet proper distance. To check if our line is a spacetime geodesic, we use $v^\mu = u^\mu_a v^a$ and (31) to find that

$$\frac{dv^\mu}{ds} + v^\rho v^\sigma \Gamma^\mu{}_{\rho\sigma} = v^a v^b \nabla_a u^\mu_b.$$

Now, the world sheet equations $m^{ab}\nabla_a u^\mu_b = 0$, and the fact that m^{ab}, when calculated on the line $\xi^a = \chi^a(s)$, is proportional to $v^a v^b$ lead us to

$$\frac{dv^\mu}{ds} + \Gamma^\mu{}_{\rho\sigma}v^\rho v^\sigma = 0.$$

This is a spacetime geodesic equation, as we expected.

C. Nielsen-Olesen vortex line

In our second example, we shall evaluate the stress-energy tensor of the known Nielsen-Olesen vortex line field configuration [1]. We start with the Higgs type of Lagrangian in Minkowski spacetime:

$$\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{2}(\nabla_\mu\phi)(\nabla^\mu\phi)^* - \lambda(\phi\phi^* - a^2)^2,$$

where $\nabla_\mu\phi \equiv (\partial_\mu + ieA_\mu)\phi$. It was shown in [1] that the corresponding field equations allow for a static, axially symmetric solution localized around the z-axis. In polar coordinates ($x = \rho\cos\varphi$, $y = \rho\sin\varphi$), and the time gauge $A^0 = 0$, the solution has the form

$$\vec{A} = A\vec{e}_\varphi, \qquad \phi = |\phi|e^{-i\varphi},$$

where A and $|\phi|$ are functions of ρ only, and $\vec{e}_\varphi$ stands for the unit vector in the φ direction. The unknown functions $A(\rho)$ and $|\phi|(\rho)$ are determined by the field equations. Far from the vortex core (z-axis), the Nielsen-Olesen solution rapidly approaches the vacuum

$$A = -\frac{1}{e\rho}, \qquad |\phi| = a. \qquad (32)$$

It is characterized by the absence of electromagnetic field, $\vec{B} = \vec{E} = 0$, and represents the true vacuum of the theory. In the core, when $\rho \to 0$, the solution approaches the false vacuum

$$A = \frac{B}{2}\rho, \qquad |\phi| = 0, \qquad (33)$$

with constant magnetic field $\vec{B} = B\vec{e}_z$, and vanishing electric field $\vec{E} = 0$.

As we do not know the exact analytic form of the Nielsen-Olesen vortex line solution, we shall approximately view it as follows. Let us define the parameter ℓ to measure the vortex width. In the region $0 \leq \rho \leq \ell$, we assume the false vacuum solution (33), while outside that region, the solution is taken to be that of the true vacuum (32):

$$0 \leq \rho \leq \ell: A = \frac{B}{2}\rho, |\phi| = 0,$$

$$\rho > \ell: A = -\frac{1}{e\rho}, |\phi| = a.$$

The continuity of the function $A(\rho)$ requires the magnetic field to satisfy $B = -2/e\ell^2$. In the limit $\ell \to 0$ (stringlike solution), $B \to \infty$, but the magnetic flux retains the constant value $-2\pi/e$.

Now, we are ready to evaluate the stress-energy tensor

$$T_{\mu\nu} = F_{\mu\lambda}F^{\lambda}{}_{\nu} + (\nabla_{(\mu}\phi)(\nabla_{\nu)}\phi)^* - \eta_{\mu\nu}\mathcal{L},$$

where indices in round brackets are symmetrized. A simple calculation shows that it vanishes outside the vortex core, $T^{\mu\nu} = 0$ if $\rho > \ell$, while in the core, $0 \leq \rho \leq \ell$, it has diagonal form with

$$T^{00} = -T^{33} = \frac{2}{e^2\ell^4} + \lambda a^4,$$

$$T^{11} = T^{22} = \frac{2}{e^2\ell^4} - \lambda a^4.$$

In the limit $\ell \to 0$, the vortex solution looks like an infinite string whose world sheet coincides with the t-z plane. Using the parametrization $\xi^0 = t$, $\xi^1 = z$, the world sheet coordinate vectors become $u_0^\mu = \delta_0^\mu$, $u_1^\mu = \delta_3^\mu$, and the induced metric γ_{ab} reduces to η_{ab}. We see that the stress-energy tensor can be written in the form (12) with

$$b^{\mu\nu} = \pi\ell^2 T^{\mu\nu}.$$

To obtain a valid string solution, we must get rid of the undesirable tension in the transverse direction. To this end, we adjust our free parameters to obey the constraint

$$\lambda a^4 = \frac{2}{e^2\ell^4}, \tag{34}$$

whereupon $T^{00} = -T^{33} = 4/e^2\ell^4$ remain the only nonvanishing components of the stress-energy tensor. Now, our $b^{\mu\nu}$ takes the form (18) with

$$m^{ab} = m\eta^{ab}, \qquad m \equiv \frac{4\pi}{e^2\ell^2}. \tag{35}$$

This is exactly the form of m^{ab} that defines Nambu-Goto string. It is easy to check then that $t - z$ plane satisfies the world sheet equations (21).

Let us note in the end that our approximation is in good agreement with the analysis of Nielsen and Olesen [1]. They found the range of parameters that enables their vortex solution to be viewed as Nambu-Goto string. In our notation, $\lambda \sim e^2 \sim (a\ell)^{-2} \gg 1$, and $m \sim a^2$. This agrees with both our constraint (34) and our Eq. (35). In particular, we see that the coupling constant e must be very large (of the order ℓ^{-1}) to ensure finite tension in the limit $\ell \to 0$.

V. CONCLUDING REMARKS

The analysis in this paper concerns the dynamics of realistic material strings in curved backgrounds. In the simple case we have considered, the background geometry is Riemannian, defined in terms of the metric tensor alone. The dynamics of geometry and matter fields is governed by the Einstein's equations.

In the specific setting considered, we assume the existence of a stable stringlike kink configuration. The type of matter fields involved is not specified. We only assume that matter fields are sharply localized around a line, while geometry itself is not. For practical reasons, the matter fields are considered weak enough to have negligible influence on the background geometry. This way, the target-space metric is attributed the properties of an external field, insensitive to string dynamics.

The method used is, basically, the Mathisson-Papapetrou method for pointlike matter [13,14] generalized to linelike configurations. We make use of the fact that every exponentially decreasing function can be written as a series of derivatives of Dirac δ function. The Mathisson-Papapetrou multipole moments are then obtained as the coefficients in the expansion. We use this method to expand the covariantly conserved stress-energy tensor of matter fields. The world sheet equations are obtained in the lowest order—the approximation of an infinitely thin string.

The results of our analysis can be summarized as follows. The dynamics of a stringy shaped matter in torsionless spacetimes generally depends on the internal structure of the string. The coefficients m^{ab} entering the world sheet equations are the components of the covariantly conserved effective 2-dimensional stress-energy tensor of the string. As opposed to the point particle case, m^{ab} cannot generally be eliminated by world sheet reparametrizations. The diversity of possible forms of m^{ab} has been analyzed with the emphasis on two questions. The first is the question of homogeneity of matter distribution along the string. The second is the classification of possible canonical forms of m^{ab}.

The possibility of unevenly distributed matter was demonstrated in an extreme case. If all the matter is localized in one point on the string, it was shown that the world sheet equations boil down to the geodesic equation, as expected.

We have also examined the possible canonical forms of m^{ab}. The most interesting is the case of homogeneously distributed matter whose tension equals its mass density. In this case, the known Nambu-Goto string dynamics is discovered. To demonstrate that such kink configurations are indeed possible, the Nielsen-Olesen vortex line solution of a Higgs type scalar electrodynamics is given as an example.

Before we close our exposition, let us mention that our main result can easily be generalized to include arbitrary p-brane distribution of matter. The corresponding world sheet equations are of the same form

$$\nabla_a(m^{ab}u^\mu_b) = 0,$$

but this time $a, b = 0, 1, \ldots, p$, and m^{ab} is the covariantly conserved $(p+1)$-dimensional energy-momentum tensor of the brane. Obviously, the diversity of possible forms of m^{ab} is bigger than in the string case. The known minimal-surface equations are obtained for $m^{ab} \propto \gamma^{ab}$, where γ_{ab} is the induced world sheet metric.

Let us say in the end that these are just preliminary results before we address the more important problem of string dynamics in general backgrounds with torsion.

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APPENDIX: SERIES EXPANSION IN δ FUNCTION DERIVATIVES

Here, we develop a formalism to expand a given function into a series of derivatives of Dirac δ function. After that, we give an intuitive interpretation of the result, and associate it to the well-known multipole expansion in electrodynamics.

Consider a real valued function $f(x)$, and write its Fourier integral

$$f(x) = \int dk \tilde{f}(k) e^{ikx}.$$

We can expand $\tilde{f}(k)$ into the power series around $k = 0$,

$$\tilde{f}(k) = \sum_{n=0}^{\infty} a_n k^n,$$

and rewrite the function $f(x)$ as

$$f(x) = \sum_{n=0}^{\infty} a_n \int dk k^n e^{ikx}.$$

The integral on the right-hand side is evaluated by means of the identity

$$\int dk e^{ikx} = 2\pi\delta(x),$$

which gives

$$\int dk k^n e^{ikx} = 2\pi(-i)^n \frac{d^n}{dx^n}\delta(x).$$

Therefore, the function $f(x)$ can be expanded into an infinite series of derivatives of Dirac δ function as follows:

$$f(x) = \sum_{n=0}^{\infty} b_n \frac{d^n}{dx^n}\delta(x).$$

The coefficients b_n are given by

$$b_n = \frac{(-1)^n}{n!}\int dx x^n f(x),$$

and are usually called nth order moments of the function $f(x)$. The coefficients b_n are well defined if the function $f(x)$ decreases faster than any power of x. In particular, the exponentially decreasing function has well defined δ function expansion.

The above procedure can easily be extended to include a higher dimensional case. Given a function $f(x) \equiv f(x_1, \ldots, x_d)$, and a point $z \equiv (z_1, \ldots, z_d)$, one can expand $f(x)$ via the general formula

$$f(x) = \sum_{n=0}^{\infty} b^{\mu_1\cdots\mu_n}\partial_{\mu_1}\ldots\partial_{\mu_n}\delta^{(d)}(x - z). \tag{A1}$$

Here, $\delta^{(d)}(x-z)$ is a d-dimensional δ function, $\partial_\mu \equiv \partial/\partial x^\mu$, and the indices μ_i take values from 1 to d. The corresponding formula for the coefficients reads:

$$b^{\mu_1\cdots\mu_n} = \frac{(-1)^n}{n!}\int d^d x f(x)\prod_{i=1}^{n}(x^{\mu_i} - z^{\mu_i}). \tag{A2}$$

The intuitive interpretation of the expansion (A1) goes as follows. Suppose a function $f(x)$ is localized around the point z, and is rapidly approaching zero as one moves away from z. If we observe the function from a distance, we can approximate it with the δ function, which is the first term in (A1). As we get closer to z, we see more "structure" in $f(x)$. In the formalism, this is described by higher order terms in (A1). The better localized the function $f(x)$, the less significant is the contribution of higher order terms.

As an example, consider electrostatic charge density $\rho(\vec{x})$ of a localized source. Expanding it with respect to $\vec{x} = 0$, one gets the first two coefficients:

$$n = 0\colon\ b = \int d^3x\rho(\vec{x}) = Q,$$

$$n = 1\colon\ \vec{b} = -\int d^3x\vec{x}\rho(\vec{x}) = -\vec{p}.$$

We recognize Q and $\vec{p}$ as the total charge and the electrostatic dipole moment of the source. So, we can write the expansion as

$$\rho(\vec{x}) = Q\delta^{(3)}(\vec{x}) - \vec{p}\cdot\vec{\nabla}\delta^{(3)}(\vec{x}) + \cdots.$$

The electrostatic potential $\varphi(\vec{x})$ is calculated from

$$\varphi(\vec{x}) = \int d^3y \frac{\rho(\vec{y})}{|\vec{x} - \vec{y}|},$$

whereupon the well-known multipole expansion in electrodynamics is obtained [22]:

$$\varphi(\vec{x}) = \frac{Q}{r} + \frac{\vec{p}\cdot\vec{x}}{r^3} + \cdots.$$

This example illustrates the use of the δ-function expansion, and clarifies what type of approximation is done when one ignores all but the $n = 0$ term in the expansion.

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