Quantum Hamiltonian for gravitational collapse

Viqar Husain[1,*] and Oliver Winkler[2,†]

[1]*Perimeter Institute for Theoretical Physics, Waterloo, ON, Canada N2L 2W9*
[2]*Department of Mathematics and Statistics, University of New Brunswick, Fredericton, NB, Canada E3B 5A3*

Using a Hamiltonian formulation of the spherically symmetric gravity-scalar field theory adapted to flat spatial slicing, we give a construction of the reduced Hamiltonian operator. This Hamiltonian, together with the null expansion operators presented in an earlier work, form a framework for studying gravitational collapse in quantum gravity. We describe a setting for its numerical implementation, and discuss some conceptual issues associated with quantum dynamics in a partial gauge fixing.

I. INTRODUCTION

A complete understanding of the many puzzles related to black holes, such as entropy [1], Hawking radiation [2], and the final state question [3], will require a quantum theory describing gravitational collapse, where both matter and geometry are fully dynamical. The ideal would be the ability to follow the evolution of an initial matter-geometry quantum state to ''black hole formation'' and beyond. This goal has been implicit in many works on black hole physics beginning with Unruh 30 years ago [4], followed a few years later by work of Hajicek [5]. More recently it has been emphasized by Isham as one of the motivations for studying quantum gravity [6]. To date, however, no complete quantum framework is available for studying gravitational collapse of a scalar field.

This paper continues a recent line of work aimed at developing the tools necessary to realize this ideal. So far, we have developed the quantum kinematics for spherically symmetric gravitational systems, in a setting which leads to a result concerning the resolution of the classical singularity at the quantum level [7]. Furthermore, we have proposed how to define a black hole at the quantum level, without fixing any classical ''horizon boundaries'' [8]. Finally, we formulated the classical dynamics of black holes in a framework that results in a true Hamiltonian, rather than a system with a Hamiltonian constraint [9].

The current paper is concerned with the development of the quantum dynamics of the gravity-scalar field system in spherical symmetry. It consists of two main parts. The first one deals with the conceptual issues connected with the notion of true dynamics. This is necessitated by the fact that our treatment of the dynamics is based neither on Dirac's constraint quantization, nor on a complete gauge-fixing, but lies in between those two extremes by employing a partial gauge-fixing only of time. As this type of approach has-to the best of our knowledge-not been discussed before in any detail, we include a careful discussion. The second part is of a more technical nature and

describes the construction of the Hamiltonian operator starting from the reduced Hamiltonian derived in [9].

The work is an attempt to complete a proposal for the quantum theory of the gravity-scalar field system in spherical symmetry. It provides a calculational framework for studying gravitational collapse in quantum gravity, which we hope will lead to a better understanding of Hawking radiation, and of the final state of gravitational collapse.

The next section discusses some conceptual issues concerning quantum evolution in a setting where only a partial (time) gauge fixing is utilized. This is followed in Sec. III by a review of the kinematical framework for quantization of the $1 + 1$ dimensional field theory that describes the gravity-scalar field system. Section IV contains a construction of the Hamiltonian operator. Since the classical Hamiltonian contains a square root, we introduce here a new method of defining the corresponding operator. In Sec. V we give the action of the Hamiltonian on basis states, describe how to implement unitary evolution, and discuss the issue of dynamical singularity resolution. We conclude in Sec. V with a summary and outlook for numerical implementation.

II. CONCEPTUAL SETTING

There are at least two methods to introduce a notion of time into a theory with time reparameterization invariance. The most obvious way is by means of gauge-fixing, where a suitable function on the classical phase space is taken to be the time function. Another is the closely related method of using partial and complete observables [10,11], where a degree of freedom is chosen as the reference clock, and the evolution of the remaining degrees of freedom is measured with respect to that clock.

We have chosen the former since it is closest to the setting in which the well-known semiclassical results about black holes have been derived. In earlier work [9] on the classical theory, we derived a reduced Hamiltonian for the gravity-scalar field theory in spherical symmetry by performing only a time gauge fixing, and leaving the remaining spatial coordinate freedom untouched. The Hamiltonian constraint was solved as a strong condition,

---

*Electronic address: vhusain@unb.ca
†Electronic address: owinkler@perimeterinstitute.ca

and a reduced spatial diffeomorphism generator remained as the only (first class) constraint. Together with the surface term, this constraint forms the total Hamiltonian. If the surface term is written as a bulk integral and combined with the other bulk terms, one can identify the true local Hamiltonian density and a diffeomorphism generator for the remaining degrees of freedom.

Fixing only a time gauge raises the question of the extent to which unambiguous evolution can be achieved in the quantum dynamics, since evolution still contains a gauge part manifested in the freedom in the shift function. This may be seen schematically through the requirement that a gauge condition such as $f(p, q) = 0$ be preserved in time. It leads to the equation

$$0 = \dot{f} = \left\{ f, \int (NH + N^a H_a) \right\}, \tag{1}$$

which fixes the lapse function N in terms of N^a. The latter remains arbitrary, and must be specified to compute evolution classically or quantum mechanically. This situation is similar to Yang-Mills theory where the total Hamiltonian density before gauge fixing and solving the Gauss law is

$$H_{\text{YM}} = \frac{1}{2}(E^2 + B^2) + \Lambda^i G_i, \tag{2}$$

where Λ^i is the Lagrange multiplier and G^i is the Gauss law expression.

In contrast the Hamiltonian for our problem (in a fixed time gauge) contains an additional twist (see below for the details). Schematically, it is of the form

$$H_{\text{grav}} = f((N^r)'; q, \pi) + N^r C_r(q, \pi), \tag{3}$$

where C_r is the radial diffeomorphism generator, q, π denote the collection of canonical phase space variables, and N^r is a lagrange multiplier (the remaining radial component of the shift function in spherical symmetry). The similarity between the two cases (YM and gravity) is the clear separation of the gauge generators from the "true" Hamiltonians. The difference is that the gravity case contains Lagrange multiplier dependence also in the first term, but only through its radial derivative.

This poses the conceptual issue of obtaining unambiguous evolution, since the gravity reduced Hamiltonian generates a family of time evolutions for observers specified by N^r. This freedom is limited to the spatial reference system only. Once it is fixed so is the clock. This situation lies in the middle of the two common scenarios in dealing with Hamiltonian gravity, where either no gauges are fixed, or all are fully fixed.

We will address this issue in two steps. Firstly, we obtain an Hamiltonian operator that depends on N^r, determine its action on basis states, and thereby obtain a prescription for evolution with any N^r (with the prescribed asymptotic conditions). This gives a family of evolved states parameterized by N^r. A unique evolution is then obtained by specifying a specific functional form for this function. This may be viewed as the quantum analog of evolving with a fixed shift function.

Secondly, we will not impose the diffeomorphism generator as a constraint on states a'la Dirac. Rather we will give prescriptions for obtaining information pertaining to the collapse problem in a manner which is manifestly invariant under this symmetry. This means looking at the dynamics of certain phase space observables, such as null expansion, as functions of other phase space variables. Suitable semiclassical states that are peaked on classical solutions of the radial diffeomorphism constraint will be used as the physical states.

Taken together, the implementation of these ideas, provided in the following sections, appears to provide a tractable approach to the gravitational collapse problem in quantum gravity.

III. REVIEW OF THE KINEMATICAL SETTING

Before addressing the issue of quantum dynamics, we briefly review the quantum kinematical setting, as introduced in [7]. The starting point is a classical field theory in $1 + 1$ dimensions, characterized by the canonical pairs (R, P_R), describing the geometry degrees of freedom, and (ϕ, P_ϕ), describing a scalar field. The basic variables that are turned into quantum operators are the smeared fields

$$R_f = \int_0^\infty R f \, dr \tag{4}$$

and

$$\phi_g = \int_0^\infty \phi g \, dr, \tag{5}$$

where f and g are suitable test functions. These configuration variables, together with the translation generators

$$U_\lambda(P_R)(r) \equiv e^{i\lambda P_R(r)} \tag{6}$$

and

$$U_\lambda(P_\phi)(r) \equiv e^{i\lambda P_\phi(r)}, \tag{7}$$

form a closed Poisson algebra, and are taken as the basic variables to be converted into operators. All other operators are be constructed in terms of these.

The Hilbert space for the quantum theory is spanned by basis states of the form

$$\left| e^{i\sum_k a_k P_R(x_k)}; e^{iL^2\sum_l b_l P_\phi(y_l)} \right\rangle \equiv |a_1 \ldots a_{N_1}; b_1 \ldots b_{N_2}\rangle, \tag{8}$$

where a_k, b_l are real numbers, and N_1 and N_2 are positive integers. The factors of L in the exponents reflect the length dimensions of the respective field variables. The intuitive picture is that each basis state "tests" the quantum scalar field at N points in space and the basis of the "excitation space" at each point consists of (generalized) plane waves.

The inner product is

$$\langle a_1 \ldots a_{N_1}; b_1 \ldots b_{N_2} | a'_1 \ldots a'_{N_1}; b'_1 \ldots b'_{N_2} \rangle$$
$$= \delta_{a_1, a'_1} \ldots \delta_{b_{N_2}, b'_{N_2}}, \quad (9)$$

if the states contain the same number of sampled points, and is zero otherwise. The action of the basic operators is

$$\hat{R}_f | a_1 \ldots a_{N_1}; b_1 \ldots b_{N_2} \rangle = L^2 \sum_k a_k f(x_k) | a_1 \ldots a_{N_1}; b_1 \ldots b_{N_2} \rangle$$

$$\hat{\phi}_g | a_1 \ldots a_{N_1}; b_1 \ldots b_{N_2} \rangle = L^2 \sum_l b_l g(y_l) | a_1 \ldots a_{N_1}; b_1 \ldots b_{N_2} \rangle$$

$$(10)$$

and

$$e^{i\lambda_j \widehat{P_R}(x_j)} | a_1 \ldots a_{N_1}; b_1 \ldots b_{N_2} \rangle$$
$$= | a_1 \ldots, a_j - \lambda_j, \ldots a_{N_1}; b_1 \ldots b_{N_2} \rangle$$
$$e^{i\mu_k \widehat{P_\phi}(y_k)} | a_1 \ldots a_{N_1}; b_1 \ldots b_{N_2} \rangle$$
$$= | a_1 \ldots a_{N_1}; b_1 \ldots, b_k - \mu_k, \ldots b_{N_2} \rangle$$

$$(11)$$

where a_j (resp. b_k) is 0 if the point x_j (resp. y_k) is not part of the original basis state. In this case the action creates a new "excitation" at the point x_j (resp. y_k) with "mode" $-\lambda_j$ (resp. $-\mu_k$). These definitions give the commutator

$$[\hat{R}_f, e^{i\lambda \widehat{P_R}(r)}] = -\lambda f(x) L^2 e^{i\lambda \widehat{P_R}(r)}. \quad (12)$$

Comparing with the classical Poisson bracket relation, and using the Poisson bracket commutator correspondence, it turns out that $L = \sqrt{2} l_P$, where l_P is the Planck length. A similar commutator relation holds for the matter scalar field.

In this formalism we can also define other operators of interest. For example for the Hamiltonian, we need operator analogs of the radial derivatives R' and ϕ', and the square of the momentum P_ϕ^2.

Definitions for the operators corresponding to R' and other derivatives of fields are obtained by implementing the idea of finite differencing using the operator $\hat{R}_f$ (4). We use narrow Gaussians with variance proportional to the Planck scale, peaked at coordinate points $r_k + \epsilon l_P$, where $0 < \epsilon \ll 1$ is a parameter designed to sample neighboring points [12]:

$$f_\epsilon(r, r_k) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{(r - r_k - \epsilon l_P)^2}{2 l_P^2} \right] \quad (13)$$

Denoting R_{f_ϵ} by R_ϵ for this class of test functions we define

$$\hat{R}'(r_k) := \frac{1}{l_P \epsilon} (\hat{R}_\epsilon - \hat{R}_0). \quad (14)$$

A further motivation of this form is that in the gauge $R = r$ the corresponding classical expression gives unity in the limit $\epsilon \to 0$. This definition captures the simplest finite difference approximation to the derivative at the operator level. Second derivatives may be similarly defined by converting a finite differencing scheme into the corresponding operator.

The quantization defined above does not give definitions for operators corresponding to momenta, since only translation generators are directly realized as operators at the first step. This of course is similar to any quantum theory on a lattice. Operators for momenta can however be realized using the translation generators, for example, by expressions such as

$$\hat{P}_\phi^\lambda := \frac{l_P}{2i\lambda} (\hat{U}_\lambda - \hat{U}_\lambda^\dagger) \quad (15)$$

and

$$\hat{P}_\phi^2 := \frac{l_P^2}{\lambda^2} (2 - \hat{U}_\lambda - \hat{U}_\lambda^\dagger) \quad (16)$$

These λ dependent expressions will be utilized below for the Hamiltonian operator.

Finally, let us note that the background independent (ie. metric free) quantization outlined here is not the same as the one for the scalar field reported in [13,14], where the basic variables used are the integral of the scalar field momentum over space $\int_\Sigma P_\phi$, and the exponential of the scalar configuration $e^{i\lambda\phi}$. The present approach has the advantage that it is fairly straightforward to write a local Hamiltonian density operator using the translation and ϕ' operators. A further technical point of difference, apart from the choice of basic variables, arises from the fact that R and ϕ are scalars, and P_R and P_ϕ are scalar densities of weight 1. This means that the functions f in (4) and the λ in (6) are scalar densities of weights 1 and -1 respectively. These combine to give eigenvalues of $\hat{R}_f$ and $\hat{\phi}_f$ that are scalars. In contrast, for the quantization studied in [13,14], it is the space-integrated momentum density that is diagonal; this is the direct analog of the surface integrated densitized dreibein in loop quantum gravity.

This ends our review of the kinematical setup. Our goal in the next section is to define the Hamiltonian operator for the collapse problem on this Hilbert space.

IV. HAMILTONIAN

The classical Hamiltonian in a time gauge fixing corresponding to flat spatial slicing given was derived in [9]. The classical phase space before gauge fixing has an extra pair of canonical variables (Λ, P_Λ), with the spatial metric

$$ds^2 = \Lambda^2 dr^2 + R^2 d\Omega^2. \quad (17)$$

The condition $\Lambda = 1$ is second class with the Hamiltonian constraint, which is solved classically for the conjugate momentum P_Λ. This leads to the reduced Hamiltonian

$$H_R^G = \int_0^\infty [(N^r)'P_\Lambda + N^r(P_R R' + P_\phi \phi')]dr$$

$$= \int_0^\infty (N^r)'(RP_R + \sqrt{(RP_R)^2 - X})dr$$

$$+ \int_0^\infty N^r(P_R R' + P_\phi \phi')dr, \qquad (18)$$

where N^r is the remaining (and still arbitrary) nonzero component of the shift function N^a after imposing spherical symmetry,

$$X = 16R^2(2RR'' - 1 + R'^2) + 16R^2 H_\phi, \qquad (19)$$

and

$$H_\phi = \frac{P_\phi^2}{2R^2} + \frac{R^2}{2}\phi'^2. \qquad (20)$$

The Hamiltonian (18) is obtained by writing the surface term in the reduced action as a bulk integral and combining terms. It gives the time-gauge fixed evolution equations for the fields $R(r, t)$ and $\phi(r, t)$, and their canonical conjugates. Our main goal in this paper is to construct the corresponding operator.

Let us write the first part of the Hamiltonian density (excluding the radial diffeomorphism term) as

$$H_R^G = h_1 + \sqrt{h_1^2 - h_2^2 - h_3^2 + \mathcal{R}} \equiv h_1 + A \qquad (21)$$

where we have defined

$$h_1 = P_R R, \qquad \mathcal{R} = -16R^2(2RR'' - 1 + R'^2) \qquad (22)$$

$$h_2^2 = 8P_\phi^2, \qquad h_3^2 = 8R^4 \phi'^2. \qquad (23)$$

There are two ways to construct an operator corresponding to a square root. One approach is to find the eigenvalues of its argument and work with the corresponding basis of eigenvectors. The operator can then be defined as the square root of the eigenvalues if the spectrum is positive semidefinite. A potential alternative is to see if the square root operator can be defined using Dirac's idea.

In this procedure an operator for the classical function $H = \sqrt{p_i^2 + m^2}$ ($i = 1 \cdots 3$) is constructed by writing $\hat{H} = \alpha_i \hat{p}_i + \beta m$ using anticommuting matrices α_i, β, such that $\hat{H}^2 = (\hat{p}_i^2 + m^2)I$, where H^2 defined by matrix multiplication. This works because the momentum components $\hat{p}_i$ commute. This is not true for the elements h_i in the argument of our Hamiltonian, so Dirac's trick cannot be used, at least in its basic form.

There is the additional problem that the term $\mathcal{R}$, which is the Ricci scalar, is not a squared quantity, and so can be negative. (In the gauge $R = r$ this term vanishes since we are using the flat slice asymptotic conditions [9].) $\hat{\mathcal{R}}$ is however diagonal in the basis since the operator analogs of the fields R, R' and R'' are all diagonal [8]. Therefore at least for the subset of states where its eigenvalues $\lambda_{\mathcal{R}}$ are

positive we can define an operator $\hat{h}_4$ whose eigenvalues are $\sqrt{\lambda_{\mathcal{R}}}$. We will return to this issue below after describing an approach to address the square root problem. For now, and the following discussion, we define

$$h_4^2 := \mathcal{R}. \qquad (24)$$

The idea is to obtain a definition of the square root part $\hat{A}$ of the Hamiltonian operator by working in a larger Hilbert space

$$\mathcal{H} = \mathcal{H}_{Kin} \otimes V, \qquad (25)$$

for some V to be specified, and writing

$$\hat{A} = \hat{h}_k \bar{e}^k, \qquad (26)$$

where the $\hat{h}_k$ act in $\mathcal{H}_{Kin}$, and the $\bar{e}^k$ act in V. The key requirement is that the action of $\hat{A}$ in $\mathcal{H}$ must be such that

$$\hat{A}^2 = \hat{h}_k \hat{h}_l \eta^{kl} I, \qquad (27)$$

where $\eta^{kl} = \text{diag}(+ - - +)$ and I is the identity operator in the space V. All other operators $\hat{O}$ acting in $\mathcal{H}_{Kin}$ are extended to $\mathcal{H}$ by the identity action in V. Since the h_k do not commute, this means that the operators $\bar{e}^k$ must satisfy

$$\bar{e}^k \bar{e}^l = \eta^{kl} I. \qquad (28)$$

That no such operators exist for any space V may be seen by the following argument. Let $|\psi_k\rangle$ be a complete set of normalizable states, and let us assume that operators satisfying (28) exist. Then we are led to contradictions such as

$$1 = \langle \psi | \psi \rangle = \langle \psi | (\bar{e}^1)^2 | \psi \rangle = \langle \psi | \bar{e}^1 (\bar{e}^4)^2 \bar{e}^1 \psi \rangle$$

$$= \langle \psi | \bar{e}^1 \bar{e}^4 \sum_k | \psi_k \rangle \langle \psi_k | \bar{e}^4 \bar{e}^1 | \psi \rangle = 0. \qquad (29)$$

There is a way out of this situation if we demand the weaker condition that

$$\langle \Psi | \hat{A}^2 | \Psi \rangle = \langle \Psi | \hat{h}_k \hat{h}_l \eta^{kl} | \Psi \rangle, \qquad (30)$$

for every $|\Psi\rangle \in \mathcal{H}$ that is of a form specified below. The basic idea is to introduce a four-dimensional Euclidean vector space V and consider its decomposition into orthogonal subspaces

$$V = \oplus_k V_k \qquad (31)$$

$k = 1 \cdots 4$. Let us denote by P_k the projection operators onto these subspaces. By definition the P_k satisfy $P_k P_l = P_k \delta_{kl}$. Now define operators $\bar{e}_k$ by

$$\bar{e}_k = P_k \qquad (32)$$

for $k = 1, 4$, and

$$\bar{e}_k = iP_k \qquad (33)$$

for $k = 2, 3$. These give

$$\hat{A}^2 = \hat{h}_k \hat{h}_l \bar{e}_k \bar{e}_l = \hat{h}_k \hat{h}_l P_k \eta_{kl} \qquad (34)$$

Consider now the action of $\hat{A}^2$ on states $|\Psi\rangle$ of the form

$$|\Psi\rangle \equiv |\psi\rangle|\rho\rangle \qquad (35)$$

where $|\psi\rangle \in \mathcal{H}_{Kin}$,

$$|\rho\rangle = \sum_k |\rho_k\rangle, \qquad (36)$$

and the $|\rho_k\rangle$ denote the basis of V corresponding to the decomposition into the V_k. The result is

$$\hat{A}^2|\Psi\rangle = \sum_{k,l}(\hat{h}_l\hat{h}_k|\psi\rangle)P_l|\rho_k\rangle = \sum_k(\hat{h}_k\hat{h}_k|\psi\rangle)|\rho_k\rangle. \qquad (37)$$

From this it is evident that (30) holds. Finally, we still need a refinement to get a self-adjoint $\hat{A}$. To do this we define the operators $\hat{h}_2$ and $\hat{h}_3$ such that they have the property

$$\hat{h}_2^\dagger = \hat{h}_2 \qquad \hat{h}_3^\dagger = \hat{h}_3. \qquad (38)$$

These ensure that the square root operator $\hat{A}$ in the extended Hilbert space is self-adjoint.

The last issue to address is the definition of the operator corresponding to $\mathcal{R}$ (the Ricci scalar) in Eq. (22). This contains derivatives of R which are defined by the finite difference operators introduced in [8]. Since these are diagonal on basis states, so is $\hat{\mathcal{R}}$. Now, in the gauge $R = r$, which fixes the shift vector N^r to be proportional to $1/\sqrt{r}$ everywhere, the Ricci scalar vanishes. Our Hamiltonian is not in this coordinate gauge (which is still free), but as alluded to in Sec. II, the idea is to evolve quantum states with this choice of lapse function, with initial states for which the values a_k of the radial field R are distributed linearly with the graph points r_k, ie. $a_k \sim r_k$. This guarantees that the eigenvalue of $\hat{\mathcal{R}}$ vanishes, at least initially. However it will not remain so because the Hamiltonian contains terms with P_R, which is represented by the operator

$$\hat{P}_R(r_k) := \frac{1}{2i\lambda}(e^{i\lambda P_R(r_k)} - e^{-i\lambda P_R(r_k)}). \qquad (39)$$

The translation operators on the r.h.s. act to shift excitations at the point r_k which result in the eigenvalue $\lambda_{\mathcal{R}}$ of $\hat{\mathcal{R}}$ being moved from zero at the selected point. However this move is very small since $0 < \lambda \ll 1$. Because of this we define the action of $\hat{h}_4$ by

$$\hat{h}_4|\psi\rangle = \sqrt{|\lambda_{\mathcal{R}}|}|\psi\rangle, \qquad (40)$$

where $|\psi\rangle$ is a basis state. This definition makes the assumption that, starting from a basis state with $\lambda_{\mathcal{R}} = 0$, the deviation from zero after the action of $\hat{P}_R$ is positive.

This completes our prescription for the Hamiltonian operator for the scalar field collapse problem. With each of its constituents well defined, it is straightforward to compute its action on a basis state. The fact that this can be done in contrast to the case for full gauge fixing [3] represents some progress which we anticipate will lead to

concrete calculations for quantum collapse. A strategy to do this is presented in the next section.

We close with a discussion of the quantization choices, or ambiguities, inherent in the steps outlined here. Firstly, let us note that the general procedure used in obtaining the square root operator is new. The only other possible approach is to seek the spectrum of the argument of the square root, which given its form appears a formidable task. If this could be done, it would represent another, possibly physically distinct, choice of Hamiltonian for this problem. This circumstance would be analogous to the Dirac and Klein-Gordon Hamiltonians. Secondly, h_1 contains products of noncommuting operators so a choice must be made for its operator version. The natural one is to take the symmetric product $(RP_R + P_RR)/2$. Finally, the only other ambiguity in the definition of the Hamiltonian is in the choice of "lattice" parameter λ. It is natural to also use the same parameter in the Gaussian smearing functions used in operators such as $\hat{R}_f$. In implementing evolution numerically, the goal of course is to ensure that physical results do not depend on this parameter. Like any numerical computation, one would like to see that the evolution of a fixed initial state leads to a convergent answer for the final state after a fixed number of time steps. This means that when the computation is repeated for successively smaller values of λ, the answers for evolved physical variables have asymptotic "continuum" values.

V. QUANTUM EVOLUTION

In classical numerical simulations of scalar field collapse the general approach is to start with initial data representing a shell of scalar field, and to evolve it using some choice of lapse and shift functions [15]. At each step of the simulation an "apparent horizon" check is made as a criterion for black hole formation. This is a null geodesic trapping condition at each step or leaf of the evolution. In spherical symmetry the goal is to find the outermost radial location on each leaf where the condition is satisfied. The evolution of this location is taken to represent the dynamical boundary or horizon of the black hole.

There are at least two concrete versions of what is a dynamical horizon. The first was formulated by Hayward [16]. In addition to the usual criteria for null geodesic expansions, this work consists of additional conditions designed to distinguish future, past, inner and outer local horizons. The second [17] lifts some of these conditions, and points out that the resulting definition of dynamical horizon allows a nice formulation of local flux laws. The common and minimal feature of both definitions are the conditions

$$\theta_+ = 0, \qquad \theta_- < 0 \qquad (41)$$

where $\theta_\pm$ are the in(out)going null geodesic expansions. Equivalent information is captured in the observable

$\theta_+ \theta_-$, which goes from negative to positive as a dynamical black hole boundary is crossed.

With the Hamiltonian operator defined in the last section, and the operator analogs of the null expansion operators given in [8], we are in a position to give a procedure for a quantum collapse calculation. This involves specifying (i) initial states, (ii) an evolution procedure, and (iii) a quantum test for black hole formation.

The first question is what are suitable initial states. The basis states represent values of the scalar field $\phi(r, t)$ and the radial field $R(r, t)$ at a set of discrete coordinate points $r_1 \cdots r_k$, which is a sample of the half line. As such these states may be compared with the discrete data for a classical numerical simulation. This suggests the use of "profile" states where we take, for example, the scalar field excitations to have a gaussian profile, and the radial excitations to be linearly distributed. Another possibility is the use of suitable coherent [18] or other form of semiclassical states that are peaked at classical configurations. These would be infinite linear combinations of basis states, and so computation with them would be more involved, especially in the present field theory setting.

The second step is the implementation of evolution. Rather than constructing a finite evolution operator by exponentiation of the Hamiltonian, which is very cumbersome, it is more suitable to implement repeated infinitesimal evolution using a suitable scheme. The simplest possibility

$$|\psi\rangle_{t+\Delta t} = (I - i\Delta t \hat{H})|\psi\rangle_t \qquad (42)$$

is not unitary. However there are unitary schemes available for this purpose. One example is based on the well-known Crank-Nicholson method, where the Schrodinger equation in discrete time (labeled by n) is written as

$$\frac{i}{\Delta t}(|\psi\rangle_{n+1} - |\psi\rangle_n) = \frac{1}{2}(\hat{H}|\psi\rangle_{n+1} + \hat{H}|\psi\rangle_n). \qquad (43)$$

This leads to the manifestly unitary (but implicit) evolution scheme given by

$$\left(1 + \frac{i}{2}\Delta t \hat{H}\right)|\psi\rangle_{n+1} = \left(1 - \frac{i}{2}\Delta t \hat{H}\right)|\psi\rangle_n \qquad (44)$$

Since $\hat{H}$ depends also on the free (classical) function N^r, so does the evolved state $|\psi\rangle_{n+1}$. This function must be fixed (with the fall off condition $N^r \sim 1/\sqrt{r}$) to get a unique evolution [9]. The simplest choice is to take this form for all points r_k in the chosen initial state.

The third step is to implement a test for black hole formation at each time step of the evolution. As mentioned above, the minimum requirement for this is that we must have operators corresponding to the null expansions. A prescription for constructing these were given in [8]. In addition we require this test to be invariant under the remaining radial diffeomorphism constraint. This is

achieved, for example, by looking at the quantities $\langle \theta_+(r_k, t)\theta_-(r_k, t)\rangle$ as functions of $\langle R(r_k, t)\rangle$, where the expectation values are in the state arrived at by stepwise evolution from some initial state. The resulting curve is radial diffeomorphism invariant, and its intersection with the $\langle \hat{R}\rangle$ axis gives the location and size of the evolving horizon. It is the dynamics of this curve which is of interest for black hole formation and subsequent evolution at the quantum level.

There are many other useful diffeomorphism invariant quantities of interest that can be computed at each time step. Two examples are the scalar field configuration $\langle \phi(r_k, t)\rangle$, and a curvature measure such as $\langle \hat{\pi}(r_k, t)\rangle$ (the trace of the ADM momentum) [7], both viewed as functions of $\langle \hat{R}(r_k, t)\rangle$.

The second and third steps are to be repeated for multiple time steps to extract time dependent profiles of the functions of interest, such as the ones just mentioned. It is in this manner that physical information about collapse may be obtained in the kinematical Hilbert space after fixing the time gauge classically. Although evolution is with respect to a fixed N^r so that the evolved states depend on it, the physical information contained in the suggested functions is independent of the coordinate points r_k: radial diffeomorphisms act to shift the points r_k to r'_k without changing the values of the field variables.

VI. DYNAMICAL AVOIDANCE OF THE SINGULARITY

In earlier work we gave a construction of an operator corresponding to the classical variable $1/R$ that is bounded on the Hilbert space we are using for the quantum theory [7]. This result has the direct consequence that curvature singularities in spherical symmetry are avoided. This is because phase space variables that classically diverge at the singularity do so as positive powers of $1/R$. The corresponding quantum operators are constructed as products of the $1/R$ operators, and so are also bounded. This does not involve the Hamiltonian in any way so the result may be viewed as kinematical singularity avoidance. In this section we show, using the construction of the Hamiltonian operator given above, that inclusion of dynamics does not alter this result.

The fundamental question here is whether quantum evolution remains well-defined through the region of highest curvature, or whether it stalls or breaks down there. Classically dynamics is encoded either in (i) the Hamiltonian constraint, or (ii) a Hamiltonian derived from a partial (time) gauge-fixing, or (iii) a Hamiltonian derived from a complete gauge-fixing as in [4]. Although our work is concerned with the second case, we will look at the issue of singularity avoidance from all three points of view. Let us consider first the Hamiltonian constraint. Its classical expression is

$$H = \frac{1}{R^2\Lambda}\left[\frac{1}{8}(P_\Lambda\Lambda)^2 - \frac{1}{4}(P_\Lambda\Lambda)(P_R R)\right] + \frac{2}{\Lambda^2}[2RR''\Lambda$$
$$- 2RR'\Lambda' - \Lambda^3 + \Lambda R'^2] + \left[\frac{P_\phi^2}{2\Lambda R^2} + \frac{R^2}{2\Lambda}\phi'^2\right].$$
$$(45)$$

Because of the $1/R$ factors, it is divergent at the points r where $R = 0$. Upon quantization these factors turn into bounded operators as shown in [7]. From this result, and the form of the basic quantum operators it is clear that the quantum Hamiltonian constraint has finite action on all states, including those for which the eigenvalue of $\hat{R}_f$ is zero. The action of the constraint operator on a state with maximum eigenvalue of the $1/R$ operator gives a linear combination of basis states that contain shifts in excitation values of the fields generated by the action of the corresponding momentum operators (39). It is not difficult to see that due to this, each of the states in the linear combination correspond to a lower eigenvalue of the $1/R$ operator. Thus in this sense a time step "evolution" by the action of the Hamiltonian constraint on a state of maximum value of curvature gives a new state in which its expectation value is lower than the maximum.

Consider next the fully reduced Hamiltonian that was found in [4]

$$H_S = \int dr\left[\frac{P_\phi^2}{4r^2} + r^2\phi'^2\right]\exp\left(\int_\infty^r S_\phi(r')dr'\right) \quad (46)$$

where

$$S_\phi(r) = \frac{P_\phi^2}{8r^3} + \frac{r\phi'^2}{2}. \quad (47)$$

This is clearly divergent at $r = 0$. As r is a parameter rather than a configuration field variable, quantization of the totally reduced theory cannot resolve the singularity at the quantum level.

Finally, let us consider the Hamiltonian whose quantization is the subject of this paper. Firstly, an inspection of the classical reduced Hamiltonian (18) shows that, unlike the Hamiltonian constraint, it has no manifestly divergent $1/R$ factors. This surprising feature is just a consequence of the time gauge fixing, and concomitant strong solution of the Hamiltonian constraint. (If one continues the reduction process further by the gauge choice $R = r$, the divergence reappears via the shift function, which is proportional to $1/\sqrt{r}$ [9]. This becomes the source of the divergence at $r = 0$ in the fully reduced Hamiltonian.) Secondly, the Hamiltonian operator has well-defined action on basis states, and evolution does not stall on states with maximum eigenvalue of the $1/R$ operator. Rather, it gives a linear combination of states each of which has a lower eigenvalue of this operator (for the same reason as for the action of the Hamiltonian constraint discussed above). In closing this section, we compare the above scenario for dynamical

singularity resolution with the cosmological case studied in [19,20], and the Schwarzschild black hole studied in [21]. In both these cases the systems are finite dimensional, unlike the model in this paper. In the first case, the action of the Hamiltonian constraint on a basis state gives a finite difference equation with coefficients such that action on the state of zero volume gives a bounce. In the second case the interior (Kantowksi-Sachs) and exterior of the Schwarzschild spacetimes are quantized with appropriate matching at the event horizon, which is taken as the fundamental classical dividing line. Our perspective is that these cases are quite different from the matter coupled case we treat here. In particular, with nonvanishing matter fields, the extended Schwarzschild spacetime does not arise, so quantizing it to discuss singularity resolution is not relevant for the quantum treatment of the collapse problem.

VII. DISCUSSION

We have constructed the quantum Hamiltonian for the spherically symmetric system of gravity coupled to a scalar field in a fixed time gauge. Together with earlier papers [7–9], this work completes the construction of a quantum theory for studying the gravitational collapse of a scalar field in spherical symmetry. We are now in a position to address the questions surrounding black holes that have been generally acknowledged to find a resolution only within a full quantum treatment.

Any application of our formalism to a given physical situation will require a choice of initial state. As all the black holes that have been detected so far have macroscopic size, the first task is to determine how to represent what we know as a classical black hole in the quantum theory. This obviously calls for a construction of the semiclassical sector, including the search for suitable semiclassical states. This will be the subject of a future publication [22]. Once the issue of what initial state to take has been settled, one can then investigate the quantum time evolution of the system.

Among the problems of physical interest is the evolution of a quantum state that satisfies the quantum horizon conditions. One would like to know how this evolution depends on the matter part of the state, and how the horizon grows or shrinks. An indispensable tool for such questions will be the null expansion operators which were constructed in [8], which serve as horizon finders. It should be pointed out that, even if one does not subscribe to the exact definition of the horizon of a black hole proposed in [8], these operators would still play an important role in any other approach to a quantum description of black hole horizons.

Perhaps the most interesting questions to address in this framework are whether and how Hawking radiation arises, and what is the end point of collapse. A preliminary investigation [23], indicates that the end result of gravita-

tional collapse is a Planck size remnant. This appears to be intimately connected with the existence of an upper bound on curvature, which in turn has an appealing intuitive analogy with Fermi pressure; whereas the latter can be ultimately overcome by gravitational forces, the former cannot since it is a fundamental quantum gravity effect. It represents the ultimate limit to which matter can be compactified. Finally, as time evolution in our setting is unitary, one can surmise already at this point that the solution to the so-called information loss paradox is that there has never been a paradox in the first place.

Apart from applications, another potential line of investigation is to improve on the framework developed here. One of the questions here concerns time gauge fixing. There are of course many other choices, so it would be useful to see if there are others that might lead to simpler reduced Hamiltonians. One possibility is to look at only the time gauge fixing used in [4], without fixing the radial gauge $R = r$. Another approach to the same issues using the connection-triad variables is being developed by Bojowald and collaborators [24]. This work offers a parallel approach in the loop quantum gravity program.

ACKNOWLEDGMENTS

[1] For recent reviews see R. M. Wald, Living Rev. Relativity **4**, 6 (2001); T. Padmanabhan, Phys. Rep. **406**, 49 (2005).

[2] S. W. Hawking, Commun. Math. Phys. **43**, 199 (1975).

[3] See for example G. Belot, J. Earman, and L. Ruetsche, Brit. J. Phil. Sci. **50**, 189 (1999).

[4] W. Unruh, Phys. Rev. D **14**, 870 (1976); See also B. K. Berger, D. M. Chitre, Y. Nutku, and V. E. Moncrief, Phys. Rev. D **5**, 2467 (1972), where a classical Hamiltonian formulation for the scalar field gravity problem was first presented.

[5] P. Hajicek, Phys. Rev. D **30**, 1178 (1984).

[6] C. Isham, gr-qc/9510063.

[7] V. Husain and O. Winkler, Class. Quant. Grav. **22**, L127 (2005).

[8] V. Husain and O. Winkler, Class. Quant. Grav. **22**, L135 (2005).

[9] V. Husain and O. Winkler, Phys. Rev. D **71**, 104001 (2005).

[10] C. Rovelli, Phys. Rev. D **65**, 124013 (2002).

[11] B. Dittrich, gr-qc/0507106.

[12] A possible choice is to use characteristic functions of bounded subsets A of the real line. This is the easiest way to localize the configuration fields around points of interest, and to construct operators corresponding to the derivatives of field operators. It offers an alternative to the construction in [8].

[13] A. Ashtekar, J. Lewandowski, and H. Sahlmann, Class. Quant. Grav. **20**, L11 (2003).

[14] Wojciech Kaminski, Jerzy Lewandowski, and Marcin Bobienski, Class. Quant. Grav. **23**, 2761 (2006).

[15] D. Goldwirth and T. Piran, Phys. Rev. D **36**, 3575 (1987); M. Choptuik, Phys. Rev. Lett. **70**, 9 (1993); D. Garfinkle, Phys. Rev. D **51**, 5558 (1995); V. Husain and M. Olivier, Class. Quant. Grav. **18**, L1 (2001).

[16] S. Hayward, Phys. Rev. D **49**, 6467 (1994).

[17] A. Ashtekar and B. Krishnan, Phys. Rev. D **68**, 104030 (2003).

[18] H. Sahlmann, O. Winkler, and T. Thiemann, Nucl. Phys. **B606**, 401 (2001).

[19] M. Bojowald, Phys. Rev. Lett. **86**, 5227 (2001); A. Ashtekar, M. Bojowald, and J. Lewandowski, Adv. Theor. Math. Phys. **7**, 233 (2003).

[20] V. Husain and O. Winkler, Phys. Rev. D **69**, 084016 (2004).

[21] A. Ashtekar and M. Bojowald, Class. Quant. Grav. **23**, 391 (2006).

[22] V. Husain and O. Winkler, ''Semi-classical states for gravitational collapse'' (unpublished).

[23] V. Husain and O. Winkler, Int. J. Mod. Phys. D **14**, 2233 (2005).

[24] M. Bojowald and R. Swiderski, Phys. Rev. D **71**, 081501 (2005); Class. Quant. Grav. **23**, 2129 (2006).