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How robust is the entanglement entropy-area relation?

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We revisit the problem of finding the entanglement entropy of a scalar field on a lattice by tracing over its degrees of freedom inside a sphere. It is known that this entropy satisfies the area law—entropy proportional to the area of the sphere—when the field is assumed to be in its ground state. We show that the area law continues to hold when the scalar field degrees of freedom are in generic coherent states and a class of squeezed states. However, when excited states are considered, the entropy scales as a lower power of the area. This suggests that, for large horizons, the ground state entropy dominates, whereas entropy due to excited states gives power-law corrections. We discuss possible implications of this result to black hole entropy.

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Although classical black holes (BHs) have infinite entropy and zero temperature, Bekenstein—inspired by the area increase theorem of general relativity—proposed that BHs have entropy proportional to the horizon area A_H. This, together with Hawking's discovery that BHs radiate with the temperature $T_H = \hbar c^3/(8\pi GM)$, has given rise to the Bekenstein-Hawking area law for BH entropy:

$$S_{\rm BH} = \frac{A_H}{4\ell_P^2}; \qquad \ell_P \equiv \sqrt{\frac{\hbar G}{c^3}} \quad \text{(Planck length)}. \tag{1}$$

The area (as opposed to volume) proportionality of BH entropy has been an intriguing issue for decades. Attempts to understand this problem can be broadly classified into two classes: (i) those that count fundamental states such as D-branes and spin networks, which are supposed to model BHs [1], and (ii) those that study entanglement entropy [2,3] and its variants such as the brick-wall model and Shakarov's induced gravity [4].

In the case of entanglement entropy, which is of interest in this work, it is assumed that the von Neumann entropy

$$S = -\mathrm{Tr}[\rho \ln(\rho)] \tag{2}$$

of quantum fields, due to correlations between the exterior and interior of the BH horizon, accounts for black hole entropy. Such correlations imply that the state of the field, when restricted outside the horizon, is mixed, although the full state may be pure [2,3]. Although this entropy is ultraviolet divergent, a suitable short distance cutoff $[\mathcal{O}(\ell_P)]$ gives $S \propto A_H$ (it was argued in [2] that this entropy must be formally divergent). This idea gained further credence, when it was shown that, even for *Minkowski space-time* (MST), tracing over the degrees of freedom inside a hypothetical sphere (of radius R), gives rise to the entropy of the form $0.3(R/a)^2$ where a is the lattice spacing [2,3] (it was shown in [5] that quantum fluctuations inside the subvolume scale as its bounding area as well). Thus, the area law may be a direct consequence of entanglement alone.

However, a crucial assumption was made in the analyses of [2,3] that all the harmonic oscillators (HOs)—resulting from the descretization of the scalar field—are in their ground state (GS). Thus the natural question which one would ask is as follows: How sensitive is the *area law* to the choice of the quantum state of the HOs?

In a recent paper, the current authors had investigated this problem for a simpler system of two coupled oscillators, and found two interesting results [6]: (i) the entropy remains unchanged if the GS oscillator wave functions are replaced by generalized coherent states (GCS), and (ii) the entropy could increase substantially (as much as 50%) even if one of the oscillators is in its first excited state (ES). This raises the possibility that, for the more interesting case of N-coupled oscillators (modeling a free scalar field), deviations from the area law could result if excited states are taken into account. We address this issue in this work. When the oscillators are in GCS and a class of squeezed states (SS), we show analytically that entanglement entropy exactly equals that of the ground state, implying that the area law remains valid. For ESs, of the form of superpositions of a number of wave functions, each of which has exactly one HO in the first ES, we show numerically that the entanglement entropy still scales as a power of the area, but that the power is now less than unity. The more terms there are in the superposition, the less this power is [7].

The Hamiltonian for a free scalar field (φ) is

$$H = \frac{1}{2}\int d^3x[\pi^2(x) + |\vec{\nabla}\varphi(\vec{x})|^2] \tag{3}$$

where π is the momentum conjugate of φ. Decomposing φ and π in terms of real spherical harmonics (Z_{lm}), i.e.,

$$\varphi_{lm}(r)[\pi_{lm}(r)] = r\int d\Omega Z_{lm}(\theta,\phi)\varphi(\vec{r})[\pi(\vec{r})],$$

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and discretizing on a radial lattice $r \to r_j$, [with $r_{j+1} - r_j = a = M^{-1}$ and $L = (N+1)a$ is the box size], we get

$$H_{lm} = \frac{1}{2a}\sum_{j=1}^{N}\Big[\pi_{lm,j}^2 + \Big(j+\frac{1}{2}\Big)^2\Big(\frac{\varphi_{lm,j}}{j} - \frac{\varphi_{lm,j+1}}{j+1}\Big)^2 + \frac{l(l+1)}{j^2}\varphi_{lm,j}^2\Big], \qquad H = \sum_{lm} H_{lm}, \tag{4}$$

where $\varphi_{lm,j}[\pi_{lm,j}] \equiv \varphi_{lm}(r_j)[\pi_{lm,j}(r_j)]$ and $[\varphi_{lm,j}, \pi_{l'm',j'}] = i\delta_{ll'}\delta_{mm'}\delta_{jj'}$. (Note that the momenta $\pi_{lm,j}$ in Eq. (4) are a times the discretized versions of π_{lm}.) This is of the form of the Hamiltonian of N-coupled HOs (up to the overall factor of $1/a$, which does not change the entanglement entropy to be computed):

$$H = \frac{1}{2}\sum_{i=1}^{N} p_i^2 + \frac{1}{2}\sum_{i,j=1}^{N} x_i K_{ij} x_j, \tag{5}$$

with the interaction matrix elements K_{ij} given by ($i, j = 1, \ldots, N$)

$$K_{ij} = \frac{1}{i^2}\Big[l(l+1)\delta_{ij} + \frac{9}{4}\delta_{i1}\delta_{j1} + \Big(N-\frac{1}{2}\Big)^2\delta_{iN}\delta_{jN} + \Big(\Big(i+\frac{1}{2}\Big)^2 + \Big(i-\frac{1}{2}\Big)^2\Big)\delta_{i,j(i\neq 1,N)}\Big] - \Big[\frac{(j+\frac{1}{2})^2}{j(j+1)}\Big]\delta_{i,j+1} - \Big[\frac{(i+\frac{1}{2})^2}{i(i+1)}\Big]\delta_{i,j-1}. \tag{6}$$

A general eigenstate of the Hamiltonian (5) is given by

$$\psi(x_1, \ldots, x_N) = \prod_{i=1}^{N} N_i H_{\nu_i}(k_{Di}^{1/4}\underline{x}_i)\exp\Big(-\frac{1}{2}k_{Di}^{1/2}\underline{x}_i^2\Big), \quad \text{where } N_i = \frac{k_{Di}^{1/4}}{\pi^{1/4}\sqrt{2^{\nu_i}\nu_i!}}. \tag{7}$$

$K_D \equiv UKU^T$ is a diagonal matrix ($U^TU = I_N$) with elements k_{Di}, $\underline{x} = Ux$, $\Omega = U^T K_D^{1/2} U$, such that $|\Omega| = |K_D|^{1/2}$, $x^T = (x_1, \ldots, x_N)$, $\underline{x}^T = (\underline{x}_1, \ldots, \underline{x}_N)$, and $\nu_i (i = 1, \ldots, N)$ are indices of the Hermite polynomials (H_ν). Note that the frequencies are ordered such that $k_{Di} > k_{Dj}$ for $i > j$.

The density matrix, tracing over the first n of the N field points, is given by

$$\rho(t;t') = \int\prod_{i=1}^{n} dx_i \psi(x_1, \ldots, x_n; t_1, \ldots, t_{N-n})\psi^\star(x_1, \ldots, x_n; t_1', \ldots, t_{N-n}') = \int\prod_{i=1}^{n} dx_i \exp\Big[-\frac{x^T\cdot\Omega\cdot x}{2}\Big]\times\prod_{i=1}^{N} N_i H_{\nu_i}(k_{Di}^{1/4}\underline{x}_i)\times\exp\Big[-\frac{x'^T\cdot\Omega\cdot x'}{2}\Big]\times\prod_{j=1}^{N} N_j H_{\nu_j}(k_{Di}^{1/4}\underline{x}_i'), \tag{8}$$

where we have introduced the notation $t_j \equiv x_{n+j}$, $j = 1, \ldots, (N-n)$, i.e. $x^T = (x_1, \ldots, x_n; t_1, \ldots, t_{N-1}) = (x_1, \ldots, x_n; t)$, where $t \equiv t_1, \ldots, t_{N-n}$. Density matrix (8) yields the entanglement entropy via (2). For an arbitrary excited state (7), analytically, it is not possible to obtain a closed form expression for $\rho(t;t')$. When all the HOs are in the GS however, i.e. when $\nu_i = 0$, $\forall i$, a closed form expression of $\rho(t;t')$ and the corresponding entropy can be found [2,3]. Here, we will investigate two nontrivial states, namely, GCS and then ES.

Before we proceed with the evaluation of S, it is important to compare and contrast between the two nontrivial states and the ground state: (i) GCS, unlike GS, is not an energy eigenstate of HO. However, GCS and GS are both minimum uncertainty states. (ii) ES, unlike GS, is not a minimum uncertainty state. If the area law holds for both GCS and ES (or a superposition of ES), then this would indicate that it is robust and unaffected by changes of the chosen state. If it holds for GCS and not for ES (or a superposition thereof), this might signal its validity *only* for minimum uncertainty states; however, if the reverse, or some other result, holds true, a simple interpretation cannot be given and more investigation would have to be done.

First, let us assume that all the HOs are in GCS, i.e.,

$$\psi_{\rm GCS}(x_1, \ldots, x_N) = \Big|\frac{\Omega}{\pi^N}\Big|\exp\Big[-\sum_i \kappa_{Di}^{1/2}(\underline{x}_i - \alpha_i)^2\Big] \tag{9}$$

where $\alpha^T = (\alpha_1, \ldots, \alpha_N)$ represents the N complex GCS parameters α_i. Physically, the real and imaginary parts of α_i correspond to the classical position (x_0) and momentum (p_0) of the individual HOs, respectively, i.e., $\alpha_i = x_0 - ip_0/k_{Di}$. Defining

$$\tilde{x} \equiv x - U^{-1}\alpha, \qquad d\tilde{x} = dx, \tag{10}$$

it follows that

$$\psi_{\rm GCS}(x_1, \ldots, x_N) = \Big[\frac{|\Omega|}{\pi^N}\Big]^{1/4}\exp\Big[-\frac{\tilde{x}^T\cdot\Omega\cdot\tilde{x}}{2}\Big] = \psi_0(\tilde{x}_1, \ldots, \tilde{x}_N), \tag{11}$$

where $\psi_0(\tilde{x}_1, \ldots, \tilde{x}_N)$ is a GS wave function. Consequently,

$$\rho_{\rm GCS}(t;t') = \int \prod_{i=1}^{n} dx_i \psi_{\rm GCS}(x_i;t)\psi^{\star}_{\rm GCS}(x_i;t')$$
$$= \int \prod_{i=1}^{n} d\tilde{x}_i \psi_0(\tilde{x}_i;\tilde{t})\psi^{\star}_0(\tilde{x}_i;\tilde{t}')$$
$$= \left[\frac{|\Omega|}{|A|\pi^{N-n}}\right]^{1/4} \exp\left[-\frac{1}{2}(t^T\gamma t + t'^T\gamma t') + t^T\beta t'\right] = \rho_0(\tilde{t};\tilde{t}'), \quad (12)$$

where

$$\Omega = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix}, \quad (13)$$

A, B, C are $n\times n$, $n\times(N-n)$ and $(N-n)\times(N-n)$ matrices, respectively, $\beta = \frac{1}{2}B^TA^{-1}B$, $\gamma = C-\beta$, and $\tilde{t}\equiv \tilde{t}_1,\ldots,\tilde{t}_{N-n} = \tilde{x}_n,\ldots,\tilde{x}_N$. Note that B, B^T (and hence β) are zero if HOs are noninteracting. Comparing with Eq. (11) of Ref. [3], we see that $\rho_0(t,t')$ is precisely the GS density matrix. That is, the GCS density matrix has the same functional form, albeit in terms of the tilde variables. Consequently, it will have the same entropy as well, which is found in the following way. By a series of transformations, (12) can be written as

$$\rho_{\rm GCS}(t,t') = \left[\frac{|\Omega|}{|A|\pi^{N-n}}\right]^{1/2}\prod_{i=1}^{N-n}\exp\left[-\frac{v_i^2+v_i'^2}{2}+\bar{\beta}_i v_i v_i'\right] \quad (14)$$

where $V\gamma V^T \equiv \gamma_D$ = diagonal, $\bar{\beta}\equiv \gamma_D^{-(1/2)}V\beta V^T\gamma_D^{-(1/2)}$, $W\bar{\beta}W^T\equiv\bar{\beta}_D$ = diagonal with elements $\bar{\beta}_i$ and $v_i\in v\equiv W^T(V\gamma V^T)^{1/2}VT$. Since $\rho_{\rm GCS}$ in Eq. (14) is a product of the $(N-n)$, two HO ($N=2, n=1$) density matrices, the total entropy is the sum of the entropies [3], i.e.,

$$S = \sum_{i=1}^{N-n} -\ln[1-\xi_i] - \frac{\xi_i}{1-\xi_i}\ln\xi_i,$$
$$\xi_i = \frac{\bar{\beta}_i}{1+\sqrt{1-\bar{\beta}_i^2}}. \quad (15)$$

Since the eigenfunctions and eigenvalues are the same as those for the ground state values, the total entropy is also same as that of the ground state [3]:

$$S = \sum_{l=0}^{l_{\rm max}}(2l+1)S_l = 0.3\left(\frac{R}{a}\right)^2, \quad (16)$$

where $R = a(n+1/2)$, and S_l is the entropy for a given l. Strictly speaking, the upper limit of the sum should have been infinity, for which it is convergent. Thus, we choose a maximum value of $l\equiv l_{\rm max}$, such that $[S(l_{\rm max}) - S(l_{\rm max}-5)]/S(l_{\rm max}-5) < 10^{-3}$. (The numerical error in the total entropy is less than 0.1%.) The sum in Eq. (16) is convergent. In other words, the entanglement entropy follows the area law, even for *arbitrarily large* values of the coherent state parameters α_i. Thus, one can conclude that all *classical states* give rise to the area law. This is our first result. It can be easily shown that the results continue to hold for the class of SS, characterized by the same squeezing parameter r. The equivalents of Eqs. (9) and (10) are, respectively,

$$\psi_{\rm GCS}(x_1,\ldots,x_N) = \left|\frac{\Omega}{\pi^N}\right|^{r^{N/2}}\exp\left[-\sum_i r\kappa_{Di}^{1/2}\underline{x}_i^2\right], \quad (17)$$

$$\tilde{x}\equiv\sqrt{r}\underline{x}, \qquad d\tilde{x} = \sqrt{r}d\underline{x}. \quad (18)$$

Equations (11) and (12) remain unchanged (up to irrelevant multiplicative factors), and so does the entropy (16).

Next, we assume the state to be in a linear superposition of N wave functions of the form in Eq. (7), such that in each such wave function there is *exactly one* HO in the first ES and the rest $(N-1)$ HOs in their GS. Although nontrivial, we will find that this state is relatively easy to handle. Using (7), the wave function for this state can be simplified to

$$\psi_1(x_1,\ldots,x_N) = \left|\frac{\Omega}{4\pi^N}\right|^{1/4}\sum_{i=1}^{N} a_i H_1(k_{Di}^{1/4}\underline{x}_i) \times\exp\left[-\frac{1}{2}\sum_j k_{Dj}^{1/2}\underline{x}_j^2\right], \quad (19)$$
$$= \sqrt{2}(a^TK_D^{1/2}\underline{x})\psi_0(x_1,\ldots,x_N,),$$

where $a^T = (a_1,\ldots,a_N)$ are the expansion coefficients (normalization of ψ_1 requires $a^Ta=1$).

Substituting in Eq. (8), we get

$$\rho(t;t') = 2\int\prod_{i=1}^{n} dx_i[x'^T\Lambda x^T]\psi_0(x_i;t)\psi_0^{\star}(x_i;t'), \quad (20)$$

where Λ is an $N\times N$ matrix and is defined as

$$\Lambda = U^TK_D^{1/4}aa^TK_D^{1/4}U \equiv \begin{pmatrix}\Lambda_A & \Lambda_B\\ \Lambda_B^T & \Lambda_C\end{pmatrix}, \quad (21)$$

and Λ_A, Λ_B, Λ_C are $n\times n$, $n\times(N-n)$, $(N-n)\times(N-n)$ matrices, respectively. Comparing Eqs. (12) and (20), it is clear that the excited state density matrix is *not* the same as that of the ground state. More importantly, it is *not* possible to obtain Eq. (12) from (20) in any limit of Λ. Integrating Eq. (20), we get

$$\rho(t,t') = \rho_0(t,t')\mathrm{Tr}(\Lambda_AA^{-1})[1-\tfrac{1}{2}(t^T\Lambda_\gamma t + t'^T\Lambda_\gamma t') + t^T\Lambda_\beta t'], \quad (22)$$

where we have defined

$$\Lambda_\gamma = \frac{2\Lambda_B^T(A^{-1}B) - B^T(A^{-1})^T\Lambda_AA^{-1}B}{\mathrm{Tr}(\Lambda_AA^{-1})}, \qquad \Lambda_\beta = \frac{2\Lambda_C + B^T[A^{-1}]^T\Lambda_AA^{-1}B - \Lambda_B^TA^{-1}B - B^T[A^{-1}]^T\Lambda_B}{\mathrm{Tr}(\Lambda_AA^{-1})}.$$

Before proceeding further, we note the following: (i) Eq. (22) is the exact density matrix for the discretized scalar field with any one HO in the first excited state with the rest in the GS; (ii) unlike the GS (12), the excited state density matrix (22) contains nonexponential terms and hence cannot be written as a product of $(N-n)$, 2-coupled HO systems, as in Eq. (14). Consequently, the entropy cannot be written as a sum as in (15). However, for the vector t^T outside the maximum of $t^T_{\max} = (\frac{3(N-n)}{\sqrt{2\mathrm{Tr}(\gamma-\beta)}}) \times (1, 1, \ldots)$ (corresponding to 3σ limits), the Gaussian inside $\rho_0(t; t')$ in (22) is negligible. Thus, when the conditions

$$\epsilon_1 \equiv t^T_{\max}\Lambda_\beta t_{\max} \ll 1 \quad \text{and} \quad \epsilon_2 \equiv t^T_{\max}\Lambda_\gamma t_{\max} \ll 1$$

are satisfied, one can make the approximation

$$1 - \tfrac{1}{2}(t^T\Lambda_\gamma t + t'^T\Lambda_\gamma t') + t^T\Lambda_\beta t' \simeq \exp[-\tfrac{1}{2}(t^T\Lambda_\gamma t + t'^T\Lambda_\gamma t') + t^T\Lambda_\beta t']. \quad (23)$$

Correspondingly, the density matrix (22) takes the following simple form:

$$\rho(t, t') = \left[\frac{|\Omega|}{|A|\pi^{N-n}}\right]^{1/2}\mathrm{Tr}(\Lambda_A A^{-1}) \times \exp\left(-\frac{1}{2}(t^T\gamma' t + t'^T\gamma' t') + t^T\beta' t'\right), \quad (24)$$

where $\beta' \equiv \beta + \Lambda_\beta$, $\gamma' \equiv \gamma + \Lambda_\gamma$. Note that (24) is of the same form as the ground state density matrix (12), with matrices $\beta \to \beta'$, $\gamma \to \gamma'$ in the exponent, and up to irrelevant normalization factors. The corresponding entropy will then be given by Eq. (15), with the replacements $\beta \to \beta'$ and $\gamma \to \gamma'$ in the definition of ξ_i. We tested the validity of the approximation (22) numerically for large values of N ($N > 60$), using MATLAB [8]. The error in the approximation (23) was less than 0.1% for $a^T = 1/\sqrt{o}(0, \ldots, 0, 1, \ldots, 1)$ with the last o columns being nonzero. The corresponding entropy, computed from the density matrix (24), was computed numerically, for $N = 300$, $n = 100, \ldots, 200$, and $o = 10, 20, 30, 40, 50$.

Before proceeding with the results we would like to mention the following: For the above ES, the expectation value of energy is given by

$$\mathcal{E}_1 \equiv \langle H_{lm}\rangle = \mathcal{E}_0 + \frac{1}{\sqrt{o}}\sum_{i=N-o+1}^{N} k_{Di}^{1/2}, \quad (25)$$

where $k_{Di}^{1/2}$ are the normal frequencies of the Hamiltonian H_{lm} and $\mathcal{E}_0$ is the ground state energy given by $\mathcal{E}_0 = \frac{1}{2} \times \sum_{i=1}^{N} k_{Di}^{1/2}$. Note that we have not set the GS energy to zero and, as mentioned earlier, the frequencies are ordered such that $k_{D_i} > k_{D_j}$ for $i > j$. Rewriting Eq. (25), we have

$$\mathcal{E}_1 = \frac{1}{2}\sum_{i=1}^{N-o} k_{Di}^{1/2} + \left(\frac{1}{2} + \frac{1}{\sqrt{o}}\right)\sum_{i=N-o+1}^{N} k_{Di}^{1/2}. \quad (26)$$

As mentioned before, the k_{Di} are in ascending order. Moreover, the last few terms in the second sum in Eq. (26) dominate. Consequently, for $N = 300$ and $o = 10, \ldots, 50$, $(\mathcal{E}_1 - \mathcal{E}_0)/\mathcal{E}_0 \approx 0.3, \ldots, 0.6$, i.e. the energy of these ESs are about 30%–60% higher than the GS energy. Note that these energies are in units of $1/a$, with a the ultraviolet cutoff (the lattice spacing). Thus, if we choose the latter to be of the order of the Planck length, as is reasonable in any theory of quantum gravity, the GS and ES energies (as well as the energy density of the ES) are Planckian, where we have ignored the resulting gravitational self-interactions of the system.

In Fig. 1, we have plotted $\log(S)$ versus $\log(R/a)$. From the best-fit curves, we see that for $o = 10$, 20, 30, 40, 50, $S = 0.4744(R/a)^{0.9479}$, $0.6331(R/a)^{0.9223}$, $0.9669(R/a)^{0.8848}$, $1.8511(R/a)^{0.8255}$, $4.002(R/a)^{0.7571}$, respectively. Thus, although the coefficient in front increases, the power decreases with the number of excited states, and for large enough areas, the GS (or closely related GCS or SS) entropy is greater than the ES entropy. We would thus like to conjecture that, if the entanglement entropy of a superposition of the GS and ES is computed, it would (at least approximately) be a sum of the GS entropy (the area law) and the ES state entropy that we found, in which case, the latter can be interpreted as (power-law) corrections to the area law. Such corrections can be contrasted with entropy corrections obtained from other sources [9]. In Fig. 2, we have plotted the entropy for each partial wave, $(2l+1)S_l$, versus l, for $N = 300$ and various values of n. For each n, we have plotted for

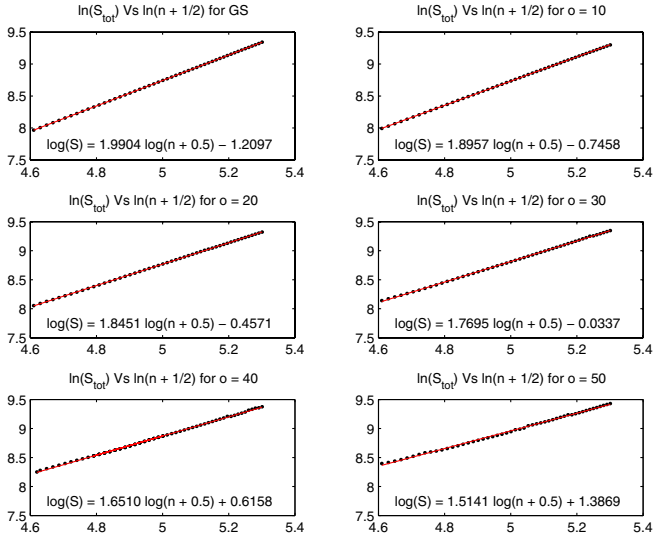

FIG. 1 (color online). Logarithm of GS and ES entropies versus the radius of the sphere (R/a), i.e. $R = a(n + 1/2)$ for $N = 300$ and $100 \le n \le 200$. We choose the maximum value of l such that $[S(l_{\max}) - S(l_{\max} - 5)]/S(l_{\max} - 5) < 10^{-3}$. The numerical error in the total entropy is less than 0.1%.

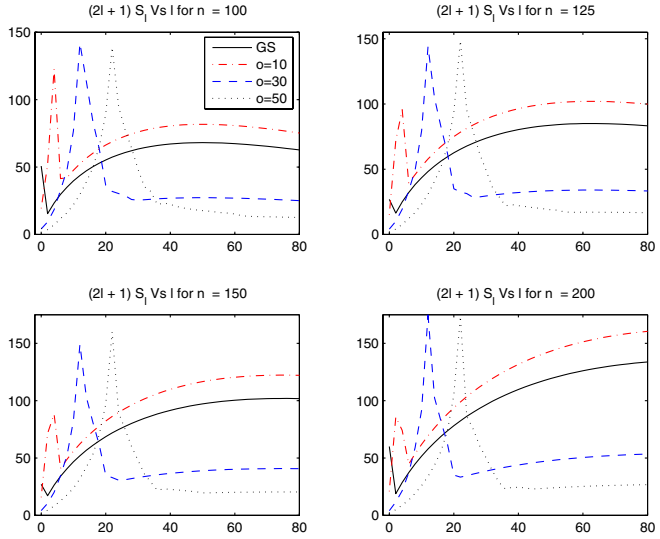

FIG. 2 (color online). Plot of the distribution of entropy per partial wave $[(21+1)S_l]$ for GS (solid curves) and ES (dotted curves). To illustrate the difference between the GS and ES (and so that all curves can be fitted in the same graph), we have multiplied the GS entropy per partial wave by a factor of 5, while the $o = 10$ and $o = 30$ curves have been multiplied by factors of 6 and 2, respectively, in each plot.

$o = 10, 30, 50$. It can be seen that, for the GS, there is a maxima at $l = 0$, after which $(2l + 1)S_l$ decreases. Once it reaches a minimum, it starts to rise again, due to the large degeneracy factor $(2l + 1)$. For the ES, however, a sharp maximum occurs between $l = 5$ and $l = 30$, depending on the parameter o. We hope to get a better understanding of this phenomenon in the future.

To summarize, in this work we have computed the entanglement entropy of scalar fields, after tracing over its degrees of freedom inside a hypothetical sphere of radius R. The oscillator modes representing these degrees of freedom were assumed to be in GCS, SS, and a superposition of the first ES. In the case of GCS and SS, the entropy turned out to be identical to that for the GS of the form $0.3(R/a)^2$, while for the ES, the entropy goes as a power of the area which is less than unity.

In light of the above results, let us discuss the implications of our results to the BH entropy: the Bekenstein-Hawking area law not only tells us that the BH entropy is proportional to the area, it also gives the precise value of the proportionality constant $[1/(4\ell_P^2)]$. Our analysis, for the MST, suggests that the constant of proportionality *as well as* the power of the area depend on the choice of the state of the scalar field. This raises an immediate question: which states determine the Bekenstein-Hawking entropy [10,11]? As stated before, it appears that the GS is most relevant, at least for large areas, while the ES gives rise to correction terms.

Open problems include extending our analysis to higher ESs. One technical problem that we anticipate in this case is that the density matrix will not be expressible as the GS matrix with shifted parameters [such was the case for (22)], since $H_n(x) \sim x^n$. Finally, analytical proofs of the area law, for GCS, SS, and ES, which do not depend on the shape of the traced out volume, along the lines of Ref. [12] would be illuminating. We hope to examine these issues elsewhere.

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[8] MATLAB uses the NAG subroutines for real (double precision—$10^{\pm 308}$) symmetric matrices. For more information, see the following URL: http://www.nag.co.uk.

[9] S. Das, P. Majumdar, and R. K. Bhaduri, Classical Quantum Gravity **19**, 2355 (2002), and references therein.

[10] R. Buniy and S. D. H. Hsu, hep-th/0510021.

[11] Saurya Das and S. Shankaranarayanan (work in progress).

[12] M. B. Plenio *et al.*, Phys. Rev. Lett. **94**, 060503 (2005); M. Cramer *et al.*, Phys. Rev. A **73**, 012309 (2006).