Two-loop QCD corrections to semileptonic *b*-quark decays near maximum recoil

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Two-gluon radiative corrections to the $b \to c\ell\bar{\nu}_\ell$ decay width have been computed analytically as an expansion in terms of $\frac{m_c}{m_b} \ll 1$ in the kinematical limit of zero lepton invariant mass. The obtained results match smoothly with a previously known expansion around $(1 - \frac{m_c}{m_b}) \ll 1$. Together they describe the process $b \to c\ell\bar{\nu}_\ell$ for all mass ratios $\frac{m_c}{m_b}$.

I. INTRODUCTION

Inclusive semileptonic decays of the *b*-quark can be described with a high degree of theoretical precision because nonperturbative effects are suppressed [1] and the *b*-quark mass is sufficiently large for the perturbative QCD expansion. However, the perturbative calculations are challenging because of the presence of massive propagators of *b*- and *c*-quarks. In particular, the $\mathcal{O}(\alpha_s^2)$ corrections to the rate of the decay $b \to c\ell\bar{\nu}_\ell$ have not yet been calculated, although a rather reliable estimate was obtained in [2]. The need for such corrections has been pointed out in connection with a precise determination of the CKM parameter V_{cb} (see, for example, [3]). Such a calculation requires a generalization of the four-loop studies of $\mu \to e\bar{\nu}_e\nu_\mu$ [4] and $b \to u\ell\bar{\nu}_\ell$ [5] to the case of a massive charged particle in the final state and is thus very difficult. One possible approach is to treat that final-state quark mass as a perturbation. In this paper we examine this approach, not with the full problem of the total decay rate, but in one special kinematical configuration: the decay $b \to c\ell\bar{\nu}_\ell$ with the lepton and neutrino escaping with a zero invariant mass, $q^2 = 0$. We treat the mass ratio $\frac{m_c}{m_b}$ as a small parameter and find several terms of expansion of $d\Gamma(b \to c\ell\bar{\nu}_\ell)/dq^2$ at $q^2 = 0$. This work builds on the recent study [6] of the decay $t \to Wb$, where an expansion in the *W* mass was constructed.

In the limit $q^2 = 0$, massless leptons are produced at rest with respect to each other and the corresponding decay rate may be represented as a product of *b* decay into a fictitious real massless *W*-boson and a consequent *W* decay. The decay amplitude is

$$i\mathcal{M}(b \to X\ell\bar{\nu}_\ell)$$
$$= \mathcal{J}^\mu \frac{-i}{q^2 - m_W^2}\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{m_W^2}\right)\bar{u}\frac{ig_w}{2\sqrt{2}}\gamma^\nu(1 + \gamma^5)v, \tag{1}$$

where $\mathcal{J}^\mu$ is the quark current. The decay rate of $b \to X\ell\bar{\nu}_\ell$ may be represented as follows:

$$d\Gamma(b \to X\ell\bar{\nu}_\ell)$$
$$= \frac{(2\pi)^4}{2m_b}|\mathcal{M}|^2 d\Phi(b \to X\ell\bar{\nu}_\ell)$$
$$\xrightarrow{q^2\to0} dq^2 \frac{G_F}{4\sqrt{2}\pi^2} \times \frac{(2\pi)^4}{2m_b}\mathcal{J}^\mu\mathcal{J}^{*\nu}\frac{q^\mu q^\nu}{m_W^2}d\Phi(b \to XW^*)$$
$$= d\Gamma(b \to XW^*)|_{m(W^*)\to0} \times \frac{G_F}{4\sqrt{2}\pi^2}dq^2. \tag{2}$$

Thus $d\Gamma(b \to c\ell\bar{\nu}_\ell)/dq^2$ at $q^2 \to 0$ differs from $d\Gamma(b \to cW^*)$ with $m(W^*) = 0$ by only a constant factor. In this paper we focus on the decay width $\Gamma(b \to cW^*)$ and integrate over final states containing up to two gluons.

To treat virtual and real gluon emission consistently, we take advantage of the optical theorem which connects the *b*-quark decay rate to the imaginary part of the *b*-quark self-energy operator: $\Gamma(b \to X) = \frac{1}{m_b}\text{Im}\mathcal{P}(b \to X \to b)$. In this approach the calculation of $\mathcal{O}(\alpha_s^2)$ corrections to $\Gamma(b \to cW^*)$ requires the evaluation of 3-loop diagrams instead of the 4-loop diagrams of the corresponding $b \to c\ell\bar{\nu}_\ell$ decay.

Several years ago [2], the $b \to cW^*$ decay rate was found from an expansion around the so-called zero recoil limit where $(1 - \frac{m_c}{m_b}) \ll 1$. Our present expansion around $\frac{m_c}{m_b} \ll 1$ is complementary and we will find that, together with [2], we now know the differential rate at all values of the *c*-quark mass.

This paper is organized as follows. In the next section we introduce the notations, discuss the gauge-invariant contributions to the decay rate, and present some technical details of the calculation. In Sec. III we present the results and combine them with those of Ref. [2] to cover the full range of final-state quark mass.

II. $\mathcal{O}(\alpha_s^2)$ CORRECTIONS TO $b \to cW^*$ DECAY WIDTH

The rate of $b \to cW^*$ decay may be written as an expansion in the strong coupling constant α_s whose coefficients are functions of $\rho = \frac{m_c}{m_b}$:

TABLE I.   Example of the asymptotic expansion of a double-scale topology. In the figures, dotted, thin solid, and thick solid lines represent massless, soft-scale massive, and hard-scale massive propagators, respectively.

	$I(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8) = \int \frac{[d^D k_1][d^D k_2][d^D k_3]}{[1]^{a_1}[2]^{a_2}[3]^{a_3}[4]^{a_4}[5]^{a_5}[6]^{a_6}[7]^{a_7}[8]^{a_8}}$, $[1] = (k_1 - p)^2$, $[2] = k_1^2 + m_c^2$, $[3] = k_2^2 + m_c^2$, $[4] = k_3^2 + m_c^2$, $[5] = (p + k_3 - k_1)^2 + m_b^2$, $[6] = (p + k_3 - k_2)^2 + m_b^2$, $[7] = (k_2 - k_1)^2$, $[8] = (k_3 - k_2)^2$, $(p^2 = -m_b^2)$
	Region 1: $|\mathbf{k_1}|, |\mathbf{k_2}|, |\mathbf{k_3}| \gg \mathbf{m_c}$ (3-loop single-scale topology) $[2] \to k_1^2$, $[3] \to k_2^2$, $[4] \to k_3^2$
	Region 2: $|\mathbf{k_1}|, |\mathbf{k_3}| \gg \mathbf{m_c}$, $|\mathbf{k_2}| \sim \mathbf{m_c}$ (vaccum bubble $\times$ 2-loop topology) $[2] \to k_1^2$, $[4] \to k_3^2$, $[6] \to (p + k_3)^2 + m_b^2$, $[7] \to k_1^2$, $[8] \to k_3^2$
	Region 3: $|\mathbf{k_1}|, |\mathbf{k_2}| \gg \mathbf{m_c}$, $|\mathbf{k_3}| \sim \mathbf{m_c}$ (vaccum bubble $\times$ 2-loop topology) $[2] \to k_1^2$, $[3] \to k_2^2$, $[8] \to k_2^2$, $[5] \to (p - k_1)^2 + m_b^2$, $[6] \to (p - k_2)^2 + m_b^2$
	Region 4: $|\mathbf{k_1}| \gg \mathbf{m_c}$, $|\mathbf{k_2}|, |\mathbf{k_3}| \sim \mathbf{m_c}$ (eikonal topology $\times$ 1-loop topology) $[2] \to k_1^2$, $[5] \to (p - k_1)^2 + m_b^2$, $[6] \to 2p(k_3 - k_2)^2 + i0$, $[7] \to k_1^2$

$$\Gamma(b \to c W^*) = \Gamma_0 \left[X_0 + C_F \frac{\alpha_s}{\pi} X_1 + C_F \left(\frac{\alpha_s}{\pi} \right)^2 X_2 + \mathcal{O}(\alpha_s^3) \right],$$
(3)

with $X_0 = (1 - \rho^2)^3$
(4)

and
$$X_1 = \frac{5}{4} - \frac{\pi^2}{3} + \left[-\frac{11}{4} + \frac{\pi^2}{3} - 9 \ln\rho \right] \rho^2$$
$$+ \left[\frac{1}{4} + \frac{\pi^2}{3} + 6 \ln\rho \right] \rho^4 + \left[\frac{5}{36} - \frac{\pi^2}{3} - \frac{5}{3} \ln\rho \right] \rho^6$$
$$+ \left[\frac{65}{72} - \frac{5}{6} \ln\rho \right] \rho^8 + \mathcal{O}(\rho^{10}).$$
(5)

Here $\Gamma_0 = \frac{G_F |V_{cb}|^2 m_b^5}{8\sqrt{2}\pi}$ is the result corresponding to $m_c = 0$. The second-order correction X_2 may be written as a sum of finite, gauge-invariant combinations:
$$X_2 = T_R N_L X_L + T_R N_H X_H + T_R N_C X_C + C_F X_A$$
$$+ C_A X_{NA}.$$
(6)

N_L represents the number of massless quarks (3 in this context) while N_H and N_C label the contributions of b- and c-quarks, respectively. The top quark contribution is sup-

pressed by $\left(\frac{m_b}{m_t}\right)^2$ and we neglect it. In $SU(3)$, the color factors are $T_R = \frac{1}{2}$, $C_F = \frac{4}{3}$, and $C_A = 3$.

Calculation

To find X_L, X_H, X_C, X_A, and X_{NA} of Eq. (6), we need to consider 39 three-loop diagrams such as those in Fig. 1, along with 19 one- and two-loop renormalization contributions. The contributing diagrams depend in general on two scales: m_b and m_c. To account for phenomena at different scales properly, we apply the well-known asymptotic methods. Table I presents contributing momentum regions in one example of a three-loop double-scale topology. In each region loop momenta are either "hard" ($|k| \gg m_c$) or "soft" ($|k| \sim m_c$), with $|p| = m_b$ being a hard momentum, allowing us to Taylor expand the propagators and thus reduce the number of scales to one. For example,

$$(|k_1| \sim m_c, |k_3| \gg m_c) \Rightarrow \frac{1}{(p + k_3 - k_1)^2 + m_b^2}$$
$$= \sum_{n=0}^{\infty} \frac{(2k_1 p + 2k_1 k_3 - k_1^2)^n}{(k_3^2 + 2k_3 p)^{n+1}}.$$
(7)

Color factor: C_F^2 $C_F^2 - C_A C_F/2$ $-C_A C_F/2$ $C_A C_F/2$

Ghost loops $(C_A C_F)$ b-quark loops $(C_F T_R N_H)$ u-,d-,s-quark loops $(C_F T_R N_L)$ c-quark loops $(C_F T_R N_C)$
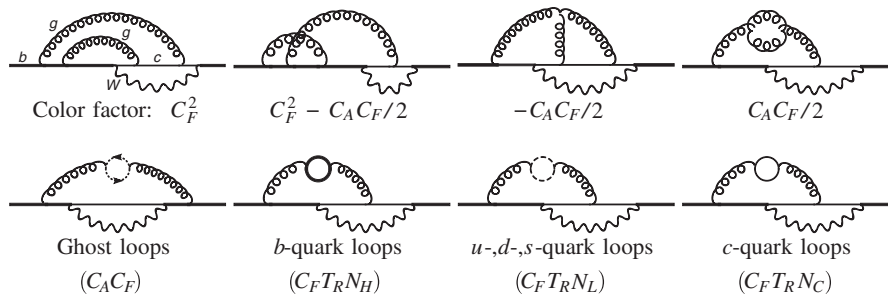
FIG. 1.   Examples of diagrams contributing to $\mathcal{O}(\alpha_s^2)$ corrections to b-decay.
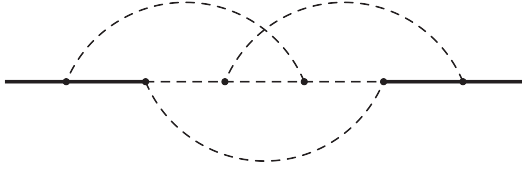
FIG. 2. The new master integral of Eq. (8). The solid and dashed lines correspond to massive and massless propagators, respectively.

Table I illustrates the asymptotic expansion process for one of the integrals encountered in this work. In this example, Region 4 involves so-called eikonal integrals,

featuring the propagator $2p(k_3 - k_2) + i0$. Although seemingly double-scale, such integrals only multiplicatively depend on the external momentum p. Care should be taken with eikonal regions, since the integral value may depend on the sign of the contour fixing term $i0$.

The large number of resulting integrals can be reduced to a small set of "master integrals" (see Ref. [7] for an example of a solution algorithm along with references to earlier work). Most of the master integrals used in this paper have been described in [6]. An additional master integral corresponds to the topology in Fig. 2; for completeness we present here the result using the same notations:

$$\mathrm{Im}F(1, 1, 1, 1, 1, 1, 1, 1, 0) = \mathrm{Im}\int \frac{[d^D k_1][d^D k_2][d^D k_3]}{k_1^2 k_2^2 k_3^2 (k_2^2 + 2k_2 p)(k_3^2 + 2k_3 p)} \frac{1}{(p + k_1 + k_2)^2 (p + k_1 + k_3)^2 (p + k_1 + k_2 + k_3)^2}$$

$$= \pi \mathcal{F}^3 \left[\frac{61\pi^4}{360} + \mathcal{O}(\epsilon) \right],$$

(8)

where $\mathcal{F} = \frac{\Gamma(1+\epsilon)}{(4\pi)^{D/2}}$ is a common loop factor and $D = 4 - 2\epsilon$ is the convention used for dimensional regularization.

III. RESULTS

Our results for the contributions to Eq. (6) are obtained as series in $\rho = \frac{m_c}{m_b}$. We have obtained terms up to ρ^9 for the expansions and present here contributions to X_2 through terms of order ρ^3:

$$X_L = -\frac{4}{9} + \frac{23\pi^2}{108} + \zeta_3 + \left[\frac{28}{9} - \frac{101\pi^2}{108} - \zeta_3 \right.$$
$$\left. + \frac{13}{2}\ln\rho - 3\ln^2\rho \right]\rho^2 + \mathcal{O}(\rho^4),$$

(9)

$$X_H = \frac{12991}{1296} - \frac{53\pi^2}{54} - \frac{\zeta_3}{3} + \left[-\frac{135467}{6480} + \frac{113\pi^2}{54} \right.$$
$$\left. + \frac{\zeta_3}{3} \right]\rho^2 + \mathcal{O}(\rho^4),$$

(10)

$$X_C = -\frac{4}{9} + \frac{23\pi^2}{108} + \zeta_3 - \frac{3\pi^2}{4}\rho + \left[\frac{65}{18} + \frac{133\pi^2}{108} - \zeta_3 \right.$$
$$\left. + \frac{13}{2}\ln\rho - 3\ln^2\rho \right]\rho^2 - \frac{25\pi^2}{18}\rho^3 + \mathcal{O}(\rho^4),$$

(11)

$$X_A = 5 - \frac{119\pi^2}{48} - \frac{53}{8}\zeta_3 - \frac{11\pi^4}{720} + \frac{19\pi^2}{4}\ln2 + \left[-\frac{315}{8} \right.$$
$$+ \frac{497\pi^2}{48} + \frac{151}{8}\zeta_3 + \frac{37\pi^4}{360} - \frac{57\pi^2}{4}\ln2$$
$$\left. + \left(\pi^2 - \frac{75}{8} \right)\ln\rho - \frac{27}{2}\ln^2\rho \right]\rho^2 - \frac{4\pi^2}{3}\rho^3 + \mathcal{O}(\rho^4),$$

(12)

$$X_{NA} = \frac{521}{576} + \frac{505\pi^2}{864} + \frac{9}{16}\zeta_3 + \frac{11\pi^4}{1440} - \frac{19\pi^2}{8}\ln2$$
$$+ \left[-\frac{2315}{576} - \frac{2119\pi^2}{864} - \frac{107}{16}\zeta_3 - \frac{\pi^4}{144} + \frac{57\pi^2}{8} \right.$$
$$\times \ln2 - \frac{185}{8}\ln\rho + \frac{33}{4}\ln^2\rho \right]\rho^2 + \frac{2\pi^2}{3}\rho^3 + \mathcal{O}(\rho^4).$$

(13)

We have used the $\overline{\mathrm{MS}}$ definition of α_s normalized at $\mu = m_b$, and the pole mass m_b.

These expansions may be directly compared to expansions around $\rho = 1$ of [2] as follows:

$$X_L(\rho) \rightarrow (1 - \rho)^3 \Delta_L(1 - \rho),$$

(14)

$$X_H(\rho) \rightarrow (1 - \rho)^3 [\Delta_H(1 - \rho) - \Delta_C(1 - \rho)],$$

(15)

$$X_C(\rho) \rightarrow (1 - \rho)^3 \Delta_C(1 - \rho),$$

(16)

$$X_A(\rho) \rightarrow (1 - \rho)^3 \Delta_F(1 - \rho),$$

(17)

$$X_{NA}(\rho) \rightarrow (1 - \rho)^3 \left[\Delta_A(1 - \rho) - \frac{1}{2}\Delta_F(1 - \rho) \right].$$

(18)

In [2] corrections from b- and c-quarks were lumped together in Δ_H. Here we divide them up by separating the c contribution in Δ_C (Eq. (A1)). As expected, X_C reaches X_L in the limit $\rho \rightarrow 0$ and X_H in the limit $\rho \rightarrow 1$.

Plots on Fig. 3 present X_L, X_H, X_C, X_A, and X_{NA} calculated to $\mathcal{O}(\rho^{10})$ around $\rho = 0$, and expansions of the corresponding functions from [2] (Eq. (A1) of that
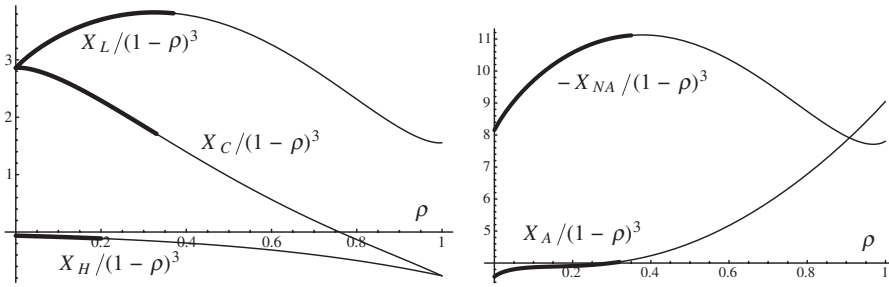
FIG. 3.   Matching of expansions around $\rho = 0$ (thick line) and $\rho = 1$ (thin line). The c-quark contribution X_C interpolates between heavy (X_H) and light (X_L) quark contributions.

work), according to Eqs. (14)–(18), calculated through $\mathcal{O}((1-\rho)^{21})$. It is sufficient to account for terms up to $\mathcal{O}(\rho^7)$ to reach the relative accuracy of 1% at the realistic value of $m_c/m_b \approx 1/3$.

For convenience we provide results for a numerical fit providing accuracy better than 0.01 for X_L, X_C, X_A, and X_{NA}, and better than 10^{-5} for X_H for $0 \leq \rho \leq 1$:

$$X_L/(1-\rho)^3 \approx 2.872 + 6.849\rho - 17.00\rho^2 + 22.56\rho^3$$
$$- 26.92\rho^4 + 13.16\rho^6, \tag{19}$$

$$X_H/(1-\rho)^3 \approx -0.06361 - 0.1902\rho - 0.2378\rho^2$$
$$- 0.1733\rho^3 - 0.09828\rho^4, \tag{20}$$

$$X_C/(1-\rho)^3 \approx 2.882 - 0.9432\rho - 14.31\rho^2 + 25.00\rho^3$$
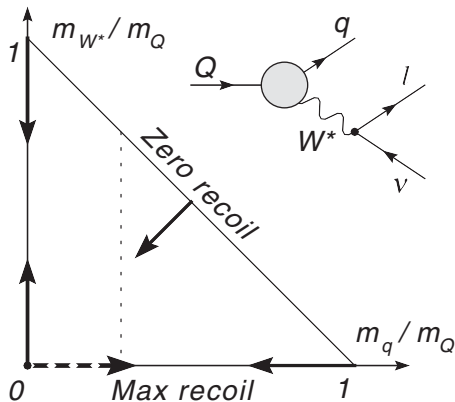$$- 18.49\rho^4 + 5.113\rho^5, \tag{21}$$

FIG. 4.   Kinematic boundaries of semileptonic heavy quark decays. The dotted arrow shows the expansion presented in this paper. Previously known expansions are indicated with solid arrows. Integration over the dotted line would correspond to integration over the leptonic phase space in the decay $b \to c\ell\bar\nu_\ell$.

$$X_A/(1-\rho)^3 \approx 3.531 + 1.305\sqrt{\rho} + 0.1496\rho - 13.76\rho^2$$
$$+ 49.64\rho^3 - 57.77\rho^4 + 33.69\rho^5$$
$$- 7.746\rho^6, \tag{22}$$

$$X_{NA}/(1-\rho)^3 \approx -8.090 - 1.696\sqrt{\rho} - 12.77\rho$$
$$+ 20.35\rho^2 + 8.257\rho^3 - 43.23\rho^4$$
$$+ 58.52\rho^5 - 29.09\rho^6. \tag{23}$$

IV. CONCLUSION

Figure 4 illustrates existing expansions [2,6,8] of semileptonic quark decays in various kinematic configurations. Analytic expressions are known along the zero recoil line and in all corners of the triangle. With the approach demonstrated in this paper and with improving computational resources, it is becoming feasible to calculate the complete $\mathcal{O}(\alpha_s^2)$ correction to the decay rate $\Gamma(b \to c\ell\bar\nu)$ as an expansion around $\rho = 0$. Like in the present study, the most challenging hard-scale regions will contribute only to even powers of ρ, with odd powers originating from factorized regions. Assuming this series will converge similarly to the expansion presented here, a 5% accuracy of the complete correction at realistic value $\rho = \frac{1}{3}$ will require calculating terms through ρ^4 of the most difficult diagrams. An extension to ρ^5 will require evaluating only factorized diagrams and will likely to improve the accuracy to the level of 3%. This is a challenging but feasible task.

In the future, if the need arises, these expansion techniques could be applied for computing precision rates of other heavy colored particle decays, e.g. squarks, accounting for mass-dependent effects.

APPENDIX A: SOFT QUARK CONTRIBUTIONS

The contribution of c-quark loops expanded in $\delta = 1 - \frac{m_c}{m_b}$ was calculated as part of the study in Ref. [2] but not explicitly shown. Here, for completeness, we present that result through terms $\mathcal{O}(\delta^8)$:

$$\Delta_C = \frac{230}{9} - \frac{8\pi^2}{3} + \left[-69 + \frac{22\pi^2}{3}\right]\delta + \left[\frac{15005}{162} - \frac{262\pi^2}{27}\right]\delta^2 + \left[-\frac{91051}{1620} + \frac{695\pi^2}{108} - \frac{32}{9}\ln 2\delta\right]\delta^3$$

$$+ \left[\frac{1517}{405} - \frac{77\pi^2}{135}\right]\delta^4 + \left[\frac{1002319}{56700} - \frac{3751\pi^2}{2160} - \frac{88}{135}\ln 2\delta\right]\delta^5 + \left[-\frac{60481}{7560} + \frac{13033\pi^2}{15120} - \frac{88}{135}\ln 2\delta\right]\delta^6$$

$$+ \left[\frac{2773441}{1587600} - \frac{493\pi^2}{3780} - \frac{586}{945}\ln 2\delta\right]\delta^7 + \left[\frac{140572}{297675} - \frac{2\pi^2}{405} - \frac{556}{945}\ln 2\delta\right]\delta^8. \tag{A1}$$

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