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Renormalization-group improved sum rule analysis for the bottom-quark mass

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We study the effect of resumming large logarithms in the determination of the bottom quark mass through a nonrelativistic sum rule analysis. Our result is complete at next-to-leading-logarithmic accuracy and includes some known contributions at next-to-next-to-leading logarithmic accuracy. Compared to finite order computations, the reliability of the theoretical evaluation is greatly improved, resulting in a substantially reduced scale dependence and a faster convergent perturbative series. This allows us to significantly improve over previous determinations of the $\overline{\rm MS}$ bottom quark mass, $\bar{m}_b$, from nonrelativistic sum rules. Our final figure reads $\bar{m}_b(\bar{m}_b) = 4.19 \pm 0.06$ GeV.

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Processes involving a $b\bar{b}$ quark pair close to threshold are very sensitive to the bottom quark mass m_b and offer a unique opportunity to accurately determine its value. One of the cleanest observables where this dependence on m_b shows up is the nonrelativistic sum rule [1]

$$M_n \equiv \frac{12\pi^2 e_b^2}{n!}\left(\frac{d}{dq^2}\right)^n \Pi(q^2)\big|_{q^2=0} = \int_0^\infty \frac{ds}{s^{n+1}} R_{b\bar{b}}(s), \quad (1)$$

where $R_{b\bar{b}}(s) \equiv \sigma(e^+e^- \to b\bar{b})/\sigma(e^+e^- \to \mu^+\mu^-)$, $\Pi(q^2)$ is the vacuum polarization and $e_b = -1/3$ the electric charge of the bottom quark. The typical scale is $p \sim 2m_b/\sqrt{n}$ and, provided n is not chosen to be too large, the left-hand side of Eq. (1) can be reliably computed using a weak coupling analysis (the right-hand side can be determined from experiment). To describe such processes theoretically, a standard fixed-order calculation in the strong coupling α_s is insufficient due to the presence of terms $(\alpha_s/v)^n \sim 1$ at each order in perturbation theory, where $v \sim 1/\sqrt{n} \ll 1$ is the velocity of the heavy quarks. Such terms appear because there are several scales involved in the problem. There is the hard scale $\mu_h \sim m_b$, the soft scale $\mu_s \sim m_b v \ll \mu_h$ of the order of the typical momentum and, finally, the ultrasoft scale $\mu_{us} \sim m_b v^2 \ll \mu_s$ of the order of the typical kinetic energy of the heavy quarks. Using effective field theories (for a review see [2]), the perturbative expansion can be systematically reorganized into an expansion in the two small parameters of the problem, α_s and v, and the $b\bar{b}$ cross section can be written as

$$R_{b\bar{b}}(s) = v\sum_n \left(\frac{\alpha_s}{v}\right)^n \times \{1(\text{LO}); \alpha_s, v(\text{NLO}); \alpha_s^2, v\alpha_s, v^2(\text{NNLO})\ldots\}. \quad (2)$$

The coefficients of this series can be computed most efficiently using the threshold expansion [3]. The singular terms $(\alpha_s/v)^n$ can be resummed and result in a well-defined function for $v \to 0$.

At present, nonrelativistic sum rules have been computed at next-to-next-to-leading order (NNLO) [4,5]. This allowed for a precise determination of the bottom quark mass using a well-understood perturbative approach. Unfortunately, in the on-shell scheme, the NNLO corrections turned out to be much larger than anticipated and, moreover, very strongly scale dependent. The use of threshold masses [6–8], which account for the cancellation of the pole mass renormalon in the observable, do not really solve these problems, specially for the strong scale dependence. Overall, this produced a very slowly convergent series, being the dominant source of error in the determination of m_b. Nonperturbative corrections are known in the limit $m_b/n \gg \Lambda_{\rm QCD}$ [9]. Even though this limit does not hold for large enough n, we can take it as an order of magnitude estimate. Numerically, these corrections are very small and can be neglected in comparison with other sources of errors.

The situation is very similar in the case of $t\bar{t}$ pair production near threshold. In this case the use of threshold masses results in a well-behaved perturbative series for the position of the peak of the $t\bar{t}$ cross section and, therefore, may enable a precise determination of the top quark mass, once experimental data is available. However, the large theoretical uncertainty in the normalization of the cross section remained (even if the series is more convergent than in the bottom case). This uncertainty is due to potentially large $\log v$ terms, which arise due to the presence of several scales and take the form $\log\mu_h/\mu_s$ and $\log\mu_s/\mu_{us}$. These logarithms can be resummed [10–13] and have been shown to be numerically important and substantially improve the scale dependence in the normalization of the cross section [10,13].

Given the importance of the $\log v$ terms for the $t\bar{t}$ cross section, it is natural to ask whether their inclusion also improves the situation in the $b\bar{b}$ case. In our case we have to replace the expansion of Eq. (2) by

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$$R_{b\bar{b}}(s) = v\sum_n\left(\frac{\alpha_s}{v}\right)^n\sum_m(\alpha_s \log v)^m \times\{1(\text{LL}); \alpha_s, v(\text{NLL}); \alpha_s^2, v\alpha_s, v^2(\text{NNLL})\ldots\}. \quad (3)$$

As we will see, these logarithms are extremely important numerically and substantially improve the reliability of the theoretical evaluation of the moments.

The nth moment, M_n, as defined in Eq. (1) is computed in the usual way [4,5]. First we match QCD to nonrelativistic QCD (NRQCD) at the hard scale which we set to $\mu_h = m$. This theory is then matched to potential NRQCD (pNRQCD) [14]. Solving the corresponding nonrelativistic Schrödinger equation perturbatively we obtain $\text{Im}G(0, 0, E)$, the imaginary part of the Green function at the origin. M_n can then be written as

$$M_n = 48\pi e_b^2 N_c \int_{-\infty}^{\infty}\frac{dE}{(E+2m_b)^{2n+3}} \times\left(c_1^2 - c_1c_2\frac{E}{3m_b}\right)\text{Im}G(0, 0, E), \quad (4)$$

where $E = \sqrt{s} - 2m_b$, $N_c = 3$ and c_1 and c_2 are the matching coefficients of the currents, normalized to 1 at leading order. In a strict nonrelativistic expansion one also expands

$$\frac{1}{(E+2m_b)^{2n+3}} = \frac{e^{-n(E/m_b)}}{(2m_b)^{2n+3}}\left(1 - \frac{3E}{2m_b} + \frac{nE^2}{4m_b^2}\ldots\right) \quad (5)$$

treating $nE \sim m_b$. We also remark that the logarithms involving μ_s always appear in the combination $\log(-4m_bE/\mu_s^2)$. This confirms that the natural scales are given by $E \sim m_b/n$ and $\mu_s \sim p \sim 2m_b/\sqrt{n}$. To ensure the applicability of perturbation theory, we cannot choose n too large and will restrict ourselves to $n \leq 14$.

The matching coefficients of pNRQCD depend on the scales $\mu_h = m_b$, μ_s and μ_{us}. In solving the renormalization group equations we have set $\mu_{us} = \mu_s^2/m$. The expressions we use are complete at NLL and NNLO. At NNLL they are also complete (in particular we include the insertions of the renormalization group improved potentials to $G(0, 0, E)$ up to the desired order in the $\overline{\text{MS}}$ scheme) except for c_1. For c_1 we are using the known NLL [11] expression as well as some partial NNLL contributions, which include the spin-dependent corrections [15], the NNLL ultrasoft corrections to the Green function, the corrections due to the two-loop beta running, and some contributions coming from the introduction of partial higher-order terms in the renormalization group improved potentials that appear in the anomalous dimension of c_1. For details we refer to Refs. [5,16]. In particular we stress that not all the ultrasoft related logarithms are included in our analysis. With this caveat in mind, we still refer to our full result as NNLL.

We also include QED corrections in our result. Counting $\alpha \sim \alpha_s^2$, these corrections enter already at NLO, due to a single exchange of a potential photon, but they have only a minor numerical impact. They increase the extracted bottom quark mass by less than 10 MeV.

The threshold masses we consider in this analysis are the potential subtracted (PS) mass $m_{b,\text{PS}}(\mu_f)$ [7] and the renormalon subtracted (RS) mass $m_{b,\text{RS}}(\mu_f)$ [8]. The subtraction scale μ_f that is needed for the definition of the PS/RS mass is set to $\mu_f = 2$ GeV, to ensure it does not exceed the characteristic scale μ_s. Once the PS/RS mass is determined, we convert it to $\bar{m}_b$, the $\overline{\text{MS}}$ mass at the renormalization scale $\bar{m}_b$. We use the three-loop conversion [17] of the pole mass to $\bar{m}_b$ and for the PS and RS mass a "large β_0" [18] and renormalon-based [8] approximation, respectively, for the four-loop term.

The moments are evaluated by performing the energy integration in the complex energy plane using a strictly expanded form as indicated in Eq. (5). The difference between this evaluation and using Eq. (4) is NNNLO and, therefore, beyond the accuracy we are aiming at. However, for small values of n, this difference is sizable. In fact, for $n = 6$ the resulting values for $m_{b,\text{PS/RS}}$ may differ by up to 45/60 MeV depending on how we expand the prefactor Eq. (5), for $n = 8$ the difference is up to 15/25 MeV, whereas for $n \geq 10$ the values for m_b agree within 10/15 MeV.

The experimental moments are determined as described in Ref. [5]. The moment is split into the contribution due to the six Υ resonances and the open $b\bar{b}$ continuum. The main uncertainty comes from the rather poor knowledge of the latter, which we parametrize as $R_{b\bar{b}}^{\text{cont}} = 0.4 \pm 0.2$ [19]. Since the continuum contribution is suppressed for larger values of n, resulting in a smaller experimental error, we refrain from using $n < 6$.

The main theoretical uncertainty in previous determinations of the bottom quark mass was due to the huge scale dependence of the NNLO result, which made it rather difficult to find a reliable procedure for estimating the theoretical error. It is the main result of this work to show that the situation improves considerably if a renormalization group improved analysis is performed. To illustrate this, in Fig. 1 we show the dependence of the theoretical value for M_{10} (evaluated at LO/LL, NLO, NLL, NNLO and NNLL, respectively) on μ_s. For the purpose of illustration we also plot the experimental value of the moment including its error. We set the strong coupling to $\alpha_s(M_Z) = 0.118$ and use three-loop evolution to determine it at lower scales. For the plot shown in Fig. 1 we set $m_{b,\text{PS}}(2\text{ GeV}) = 4.515$ GeV and $m_{b,\text{RS}}(2\text{ GeV}) = 4.370$ GeV and vary the soft scale around its characteristic value $\mu_s \sim 2m_b/\sqrt{10}$ (indicated by a dashed vertical line). Note that $\mu_s \ll \mu_h = m_b$. In this region the size of the NNLL corrections (even if large) is considerably smaller than the corresponding fixed-order NNLO ones. Moreover,

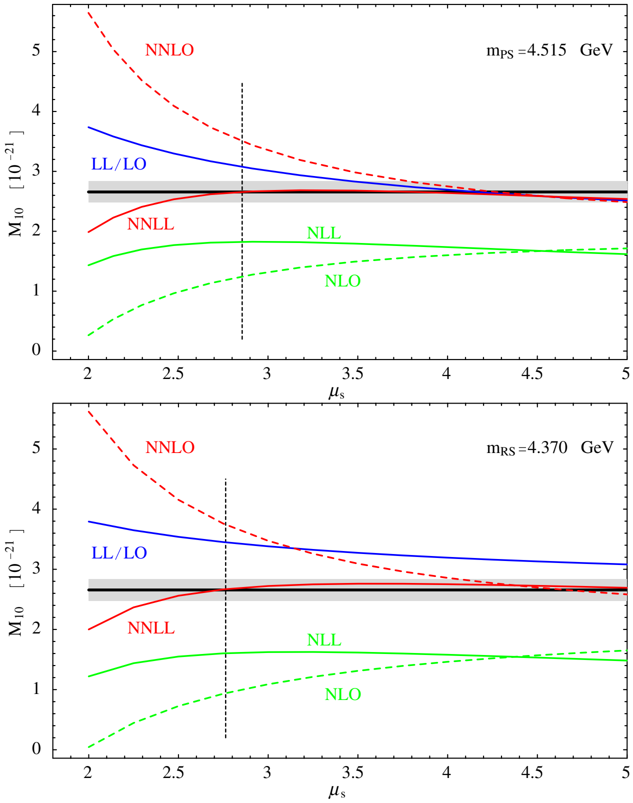

FIG. 1 (color online). The moment M_{10} as a function of μ_s at LO/LL, NLO, NLL, NNLO and NNLL for $m_{b\mathrm{PS}}(2\ \mathrm{GeV}) = 4.515\ \mathrm{GeV}$ in the PS scheme (upper figure), and for $m_{b\mathrm{RS}}(2\ \mathrm{GeV}) = 4.370\ \mathrm{GeV}$ in the RS scheme (lower figure). The experimental moment with its error is also shown (gray band).

contrary to earlier analyses, the NNLL result is now more stable with respect scale variations than the NLL one (for the range of scales for which the computation is trustworthy). This is of course what one would expect and indicates that the inclusion of the logarithms substantially improves the reliability of the theoretical prediction. Only for scales $\mu_s < 2$ GeV the situation gets out of control, but for these scales the ultrasoft scale is below 1 GeV and we can not really rely on our computation. Multiple insertions of corrections to the Coulomb potential also seem to be important in this region [20].

The situation is similar for other values of n. As a general feature, for increasing n, the scale dependence increases slightly. This is not surprising since larger n induce smaller scales and at some point the applicability of perturbation theory is questionable. On the other hand, as mentioned above, smaller values of n have the disadvantage that the nonrelativistic approximation becomes less reliable.

These findings show that it is possible to improve the accuracy of previous determinations of the bottom quark mass from nonrelativistic sum rules if the renormalization group improvement is applied. In order to determine the $\overline{\mathrm{MS}}$ mass we first determine the PS/RS mass with its error, proceeding as follows: we consider M_n for $n \in \{6, 8, 10, 12, 14\}$ and obtain our central value by equating the theoretical and experimental value of the moment at the standard scale $\mu_s = 2m_b/\sqrt{n}$. For the error in the determination of the threshold masses we consider three sources: the experimental error, the error due to the uncertainty in the strong coupling and finally the theoretical error.

The experimental error, Δ_{exp}, is simply determined by extracting the value for m_b for the two extreme values of the experimental moment. The error due to the uncertainty in the strong coupling, Δ_α, is obtained by studying the effect on the extracted bottom quark mass if we vary $0.115 < \alpha_s(M_Z) < 0.121$. Following common practise one would estimate the theoretical error, Δ_{th}, by variation of the scale by a factor of 2. As is obvious from Fig. 1, for small scales the theoretical result cannot be trusted. Therefore, in previous analyses, the scale variation was limited to scale choices above a certain cutoff, typically set to a value around 2 GeV. In the current analysis we refrain from using such an estimate. There are several reasons. First, such an error estimate depends crucially on the somewhat arbitrary lower cutoff of the scale variation. Second, it does not take into account the fact that the higher-order corrections are sizable. Given that the scale dependence is very modest (for reasonably large scales) compared to the size of the NNLL corrections, we think that such an error analysis would considerably underestimate the theoretical error in the present case. Finally, the scale variation as depicted in Fig. 1 does not take into account the independent variation of the ultrasoft scale, since in our analysis the latter is determined by the soft scale. It would be preferable to be able to vary all scales independently to obtain a better estimate of the uncertainty, in particular, since some ultrasoft logarithms are missing in our result. We have verified that a naive variation of μ_{us} results in a rather large uncertainty which, however, is consistent with the final error estimate we propose. Therefore, we prefer to determine the theoretical error by taking half the size of the highest-order correction that is included in our result. More precisely, we determine two values for m_b by equating the experimental and theoretical value (at the scale for which it reaches its maximum) of the moment at NNLL and NLL, respectively. The error is determined as half the difference between these two values. This procedure assumes a perturbative series where successive terms become less and less important. For this to hold we have to use a threshold mass, since for the pole mass the NNLL corrections are much larger than the NLL ones. In this respect, moments with low values of n and/or threshold mass definitions with values close to the $\overline{\mathrm{MS}}$ mass are better behaved. On the other hand, the actual size of the correction, and therefore the assigned error, increases for such mass definitions.

TABLE I. Extraction of $m_{b,\mathrm{PS/RS}}$ (2 GeV) with errors for various n. All values are given in MeV and rounded to 5 MeV. The total error has been obtained by adding the partial errors in quadrature. The corresponding value for the $\overline{\mathrm{MS}}$ mass with its error is given in the last column

n	$m_{b,\mathrm{PS}}$(2 GeV)	Δ_{th}	Δ_{exp}	Δ_{α}	Δ_{tot}	$\bar{m}_b$
6	4460	40	50	35	70	4135 ± 65
8	4505	45	25	30	60	4170 ± 55
10	4515	45	15	25	55	4185 ± 50
12	4520	45	10	20	50	4185 ± 45
14	4520	40	10	15	45	4185 ± 40
n	$m_{b,\mathrm{RS}}$(2 GeV)	Δ_{th}	Δ_{exp}	Δ_{α}	Δ_{tot}	$\bar{m}_b$
6	4315	55	50	25	80	4140 ± 70
8	4360	65	30	20	75	4180 ± 65
10	4370	65	20	10	70	4190 ± 60
12	4370	65	15	5	65	4190 ± 60
14	4370	65	10	5	65	4185 ± 55

We summarize our results in Table I, where we also show the combined error Δ_{tot}, which is obtained by adding the various errors in quadrature. As expected, the experimental error decreases with increasing n. The results are all consistent with each other, in particular, if we take into account the additional uncertainty mentioned above for M_n with $n \leq 8$, due to the nonrelativistic expansion in the energy integration. Related to this, we note that in computing the moments we do not use the exact fixed-order coefficient at $\mathcal{O}(\alpha_s^2)$, since we drop terms of $\mathcal{O}(\alpha_s^2/(\sqrt{n})^k)$ with $k \geq 1$. Again, this neglect is potentially more of a problem for smaller moments. Let us also reiterate that for too large values of n the applicability of weak coupling perturbation theory is questionable. We thus combine the results of Table I by simply taking the value obtained by the tenth moment

$$m_{b,\mathrm{PS}}(2\ \mathrm{GeV}) = 4.52 \pm 0.06\ \mathrm{GeV}, \tag{6}$$

$$m_{b,\mathrm{RS}}(2\ \mathrm{GeV}) = 4.37 \pm 0.07\ \mathrm{GeV}. \tag{7}$$

Note that the PS value is consistent with the result of Ref. [5], but prefers smaller values for m_b and has a reduced error.

Converting the PS and RS mass to the $\overline{\mathrm{MS}}$-mass we obtain $\bar{m}_b = 4.19$ GeV with an error of 55 MeV and 60 MeV, respectively. However, we also have to take into account the error in the conversion itself. We consider two sources, the dependence of $\bar{m}_b$ on the threshold mass used in the analysis and second, the error due to missing higher-order corrections in the conversion formula itself. To determine the first error, we start by noting that the $\bar{m}_b$ values obtained with the PS and RS scheme are very similar. We also extract the central value of $m_{b,\mathrm{PS/RS}}$(1 GeV) for the moments and convert these results to $\bar{m}_b$. These values of $\bar{m}_b$ differ at most by around 20/15 MeV from the corresponding results obtained via $m_{b,\mathrm{PS/RS}}$(2 GeV). To obtain an estimate for the error due to missing higher-order corrections in the conversion formula we drop the fourth order terms in the conversion and take as error the difference in the value of $\bar{m}_b$ thus obtained. This error is about 10/5 MeV. We thus associate a total error of 20/15 MeV to the conversion. If added in quadrature to the 55/60 MeV error, we obtain a total error for $\bar{m}_b$ of around 60 MeV in both cases.

In conclusion, we have studied the effect of resumming logarithms for nonrelativistic sum rules. The logarithms turn out to be numerically very important and improve the reliability of the theoretical computation. This manifests itself in a reduced scale dependence and an improvement of the convergence of the perturbative series. It allows us to obtain an accurate value for the $\overline{\mathrm{MS}}$ bottom quark mass using a credible error estimate

$$\bar{m}_b(\bar{m}_b) = 4.19 \pm 0.06\ \mathrm{GeV}. \tag{8}$$

At this stage, the main problem appears to be the large size of the perturbative corrections and to understand its origin. Further improvements require the full NNLL computation of the sum rule, especially the potentially large ultrasoft effects. Obviously, the inclusion of all NNNLO effects will also be important and might lead to a better control of the strong scale dependence for small values of μ_s and the large size of the perturbative corrections.

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