$1/q^2$ corrections and gauge/string duality

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We make an estimate of the quadratic correction based on gauge/string duality. Like in QCD, it proves to be negative and proportional to the string tension.

I. INTRODUCTION

As is well known, the QCD analysis of the two-current correlator results in

$$i \int d^4x e^{iq\cdot x} \langle \mathrm{T} J^\mu(x) J^\nu(0) \rangle = (q^\mu q^\nu - \eta^{\mu\nu} q^2) \Pi(q^2), \quad (1)$$

with

$$\mathcal{N} q^2 \frac{d\Pi}{dq^2} = C_0 + \frac{1}{q^2} C_2 + \sum_{n\geq 2} \frac{n}{q^{2n}} C_{2n} \langle \mathcal{O}_{2n} \rangle. \quad (2)$$

Here $\eta_{\mu\nu}$ is a four-dimensional Minkowski metric, $q \cdot x \equiv q_\mu x^\mu$, and $\mathcal{N}$ is some normalization factor. We take $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$ to connect with standard string calculations.

According to [1], C_0 is a coefficient that can be calculated perturbatively in α_s. So, it is of the form $C_0 = 1 + \sum_{n\geq 1} B_n \alpha_s^n(q^2)$. The nonperturbative effects are generated by *local* gauge invariant operators whose dimensions D are larger or equal four. For example, the $D = 4$ operators built from the quark and gluon fields are simply $\mathcal{O}_4^M = \bar{q} M q$ and $\mathcal{O}_4^G = \alpha_s G^2$. It is clear that in such an approach the existence of the quadratic correction $\frac{1}{q^2} C_2$ is puzzling. Later [2], this issue was intensively discussed in the literature. In particular, there are estimates of the quadratic correction based on the data for the e^+e^- total cross section that provide the upper bound [3]

$$|C_2| \leq 0.14 \text{ GeV}^2. \quad (3)$$

One of the implications of the anti-de Sitter/conformal field theory (AdS/CFT) correspondence is that it resumed interest in finding a string description of strong interactions. For the case of interest, let us briefly mention two approaches. In the first, usually called AdS/QCD, one starts from a five-dimensional effective field theory somehow motivated by string theory and tries to fit it to QCD as much as possible. Such an approach yields the leading asymptotic piece C_0 [4] and even condensates of the operators with $D \geq 4$ [5]. In the second, usually called gauge/string duality, one tries to keep the underlying string struc-

ture. As a result, the theory is ten dimensional and its reductions to five dimensions, in general, contain additional higher derivative terms (α' corrections).[1] In the approximation of [6], this approach also yields the leading asymptotic piece C_0 [7].

In this paper, we address the issue of the quadratic correction within the simplified model of [6,7]. We compute them by following the strategy of first quantized string theory. To this end, we build the corresponding vertex operators. Then, we define the correlator of two vector currents as an expectation value of the vertex operators. We also adopt the geometric approach to condensates introduced in [5]. Our aim is to estimate the correction and compare the result with that of QCD.

Before proceeding to the detailed analysis, let us set the basic framework. We consider the following background metric,

$$ds^2 = \frac{R^2}{r^2} h (\eta_{\mu\nu} dx^\mu dx^\nu + dr^2) + g_{ab} d\omega^a d\omega^b,$$
$$h = e^{-(1/2)cr^2}, \quad (4)$$

where ω^a are coordinates of some five-dimensional compact space X. In the region of small r the metric behaves asymptotically as $\mathrm{AdS}_5 \times X$, as expected. We take a constant dilaton and, unfortunately, discard all possible Ramond-Ramond backgrounds (if any).

As known, full control of superstring theory on curved backgrounds even like AdS_5 is beyond our grasp at present. We are forced therefore to look for a plausible approximation. The simplest possible one is that of [6]. The idea is to discard nonzero modes of sigma model fields **r** and ω^a.[2] As a result, the **X**'s only contribute to the kinetic terms of the world-sheet action which is of the form

$$S_0 = \frac{1}{4\pi\alpha_r} \int_\Sigma d^2z d^2\theta \, \eta_{\mu\nu} \bar{D} \mathbf{X}^\mu D \mathbf{X}^\nu, \qquad \alpha_r = \alpha' \frac{r^2}{R^2} h^{-1}. \quad (5)$$

Here **X** is a two-dimensional superfield and Σ is a two-dimensional Riemann surface.[3]

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[1]These corrections are of order $1/\sqrt{N_c}$. Thus, they might be relevant at $N_c = 3$.
[2]**X**, **r**, and ω are taken to be sigma model fields on a string world sheet. x, r, and ω are their zero modes, respectively.
[3]We use the superspace notations of [8].

Finally, a couple of remarks are in order. First, the action (5) formally looks like that in flat space. So, it is conformally invariant. The point is, however, that the string parameter is now a variable and, as a consequence, the theory behaves differently.

Second, even such a simplified approximation contains α' corrections and remnants of the compact space X.

II. ESTIMATE OF THE QUADRATIC CORRECTION

As a warm-up, let us fix the parameter c. A possible way to do so is to consider meson operators. As usual in first quantized string theory, we should look for the corresponding vertex operators. According to [7], these are of the form

$$\mathcal{O}(\xi, p) = \oint_{\partial\Sigma} dz d\theta \xi \cdot D\mathbf{X} \Psi_p(\mathbf{X}, \mathbf{r}, \omega), \qquad (6)$$

where ξ_μ is a polarization vector. $\partial\Sigma$ denotes a worldsheet boundary. Ψ_p is a solution to the linearized ten-dimensional Yang-Mills equations in the background (4). We choose the simplest possible solution, that is, $\Psi_p = e^{ip\cdot\mathbf{X}}\psi(\mathbf{r})$ with ψ being a solution to the following equation,[4]

$$r^2\psi'' - r(1 + \tfrac{1}{2}cr^2)\psi' + m^2r^2\psi = 0, \qquad (7)$$

where $m^2 = -p \cdot p$. A prime denotes a derivative with respect to r.

It is easy to reduce Eq. (7) to a Laguerre differential equation whose solutions are given by Laguerre polynomials L_n^a. For

$$m^2 = cn, \qquad (8)$$

we get

$$\psi_n(r) = cr^2 L_{n-1}^1(\tfrac{1}{4}cr^2), \quad \text{with} \quad n = 1, 2, \dots. \qquad (9)$$

Since the mass of the nth state is proportional to $\sqrt{n}$, it seems natural to interpret the vertex operators (6) as those of $\rho(m_n)$ mesons. If so, then the value of c is of order [9]

$$c \approx 0.9 \text{ GeV}^2. \qquad (10)$$

We close the discussion of the meson operators with a few short comments:

(i) The comparison of the mass formula (8) with the real meson masses shows a better fit for large n.[5] Evidently, the parameter c is proportional to the string tension.

(ii) By contrast, the metric used in [5] to reproduce the operators with $D \geq 4$ does not lead to a simple equation yielding the exact mass formula.

(iii) It is worth mentioning that the mass formula (8) has been derived in the AdS_5 background with a non-constant dilaton Φ [10],

$$h = 1, \qquad \Phi(r) = \tfrac{1}{2}cr^2. \qquad (11)$$

This background is indeed equivalent to ours as long as one deals with *quadratic* terms like F^2 in five-dimensional effective actions because the Weyl rescaling of the metric transforms one into another. In the generic case, the equivalence is broken by higher order terms in the field strength or by scalar fields.

Having fixed the value of the parameter c, let us make an estimate of the quadratic correction. To this end, we compute the two-current correlator.

We start with a vertex operator construction for the vector currents. In the spirit of [7], we can take the operators to be

$$J^\mu(q) = \oint_{\partial\Sigma} dz d\theta D\mathbf{X}^\mu \Psi_q(\mathbf{X}, \mathbf{r}, \omega), \qquad (12)$$

with the simplest possible form for Ψ_q which is $e^{iq\cdot\mathbf{X}}\psi(\mathbf{r})$. Again, ψ is a solution to a differential equation. This equation is given by (7) with $m^2 = -q^2$.

Next, we define the correlator as an expectation value of the vertex operators. In doing so, we choose a unit disk as the world sheet. Using the result of [7], it is easy to write down the following expression,[6]

$$\mathcal{N}q^2\frac{d\Pi}{dq^2} = -2\int_0^1 dz \int_0^\infty dt\, h^{1/2}(\psi\psi' + \mathfrak{g}t\psi^2 h^{-1}G(z))$$
$$\times \exp\{\mathfrak{g}t^2h^{-1}G(z)\}, \qquad (13)$$

where $t = qr$, $q = \sqrt{q \cdot q}$, $h = e^{-(1/2)\lambda t^2}$, $\lambda = \frac{c}{q^2}$, and $\mathfrak{g} = \frac{\alpha'}{R^2}$. $G(z)$ denotes a restriction of the scalar Green function on the boundary.[7] Note that it excludes the zero mode contribution.

Before continuing our discussion, we pause here to take a closer look at (13). A natural question to ask is to what extent this expression can be used as a basis for providing the C_{2n}'s. First, having restricted ourselves to the disk topology, we perhaps discard perturbative α_s corrections.[8] Second, in the approximation under consideration the integrand of (12) is not a dimension one half operator for generic q. As follows from (5), its dimension is given by $\Delta = \frac{1}{2} + O(\alpha_r q^2)$. It is clear that the problem is due to exclusion of the nonzero modes of $\mathbf{r}$. Thus, it seems reasonable to restrict ourselves in (13) to a few leading

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[4]Since we discard the nonzero modes of $\mathbf{r}$ and ω, we set the corresponding Yang-Mills connections to be zero. Moreover, in such an approximation there is gauge invariance, $A_\mu \to A_\mu + \partial_\mu\Lambda$. It is fixed by $\partial \cdot A = 0$.
[5]In fact, starting from $n = 3$ the agreement is better than 10%.

[6]We normalize the vertex operators (12) in such a way to fit (2).
[7]As follows from (5), it is given by $\langle X^\mu(z)X^\nu(0)\rangle = \alpha_r\eta^{\mu\nu}G(z)$.
[8]The line of thought that perturbation theory at hand is a topological expansion was pursued in [7,11].

terms in g. Finally, $G(z)$ is singular at $z = 0$. To deal with this divergence, we regulate the Green function as [12]

$$G(z) = 2 \sum_{n=1}^{\infty} \frac{e^{-\varepsilon n}}{n} \cos 2\pi n z, \tag{14}$$

where ε is some parameter.

Expanding in g and keeping only the constant and linear terms, we find

$$\mathcal{N} q^2 \frac{d\Pi}{dq^2} = -2 \int_0^{\infty} dt \, h^{1/2} \psi \psi'. \tag{15}$$

Note that the linear term vanishes as a consequence of $\int_0^1 dz G(z) = 0$.

Now the coefficients C_{2n} can be read off from (15). For the case of interest, we have

$$C_0 = 1, \qquad C_2 = -\tfrac{1}{3}c. \tag{16}$$

For completeness, we have included a calculation of (16) in the Appendix.

Finally, we substitute (10) into (16) and obtain the following estimate,

$$C_2 \approx -0.3 \text{ GeV}^2. \tag{17}$$

This is our main result. The sign of C_2 is precisely as in QCD but the absolute value is at least twice as big as the upper bound (3).

III. CONCLUDING COMMENTS

(i) Having derived the mass formula (8), it seems to be time to estimate the Regge parameters. As noted above, this formula is more or less appropriate for $n \geq 3$. Assuming that all the trajectories of interest are parallel [13], we get for the intercepts

$$\alpha_n(0) \approx -\frac{c}{2\pi\sigma} n, \qquad n = 3, 4, \ldots, \tag{18}$$

where σ is the string tension.

As noted earlier, the mass formula is in considerable disagreement with the ρ trajectory ($n = 1$). This means that the background we used is not exactly the desired string dual to QCD and we should look for a further refinement of it. From this point of view, the disagreement between our estimate and QCD is not a big surprise.

(ii) Here we used the model with the slightly deformed AdS$_5$ metric. It is clear that this is not the only option. For example, the other line of thought is to deform the dilaton [10].[9] What really fits better to QCD remains to be seen.

(iii) Interestingly enough, it was argued by Zakharov that C_2 represents the stringy effect [14]. This fact being far from obvious in QCD is manifest in our framework.

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[9]Perhaps both the models look oversimplified but they might be useful in gaining some intuition about the problem, while full control of superstring theory on curved backgrounds is missing.

APPENDIX

In this appendix we present a calculation of (16).

Having derived the integral representation (15), we can analyze its expansion in λ that is nothing but the expansion (2). For the coefficients of interest, we have

$$C_0 = -\psi_0^2|_{t=0}^{\infty},$$
$$C_2 = \frac{1}{4} c(t^2 \psi_0^2 - 8\psi_0\psi_1)|_{t=0}^{\infty} - \frac{1}{2} c \int_0^{\infty} dt t \psi_0^2. \tag{A1}$$

Here we have expanded ψ as a series in powers λ such that $\psi = \sum_{n=0} \psi_n \lambda^n$. Notice that the ψ_n's obey a set of differential equations. In particular, ψ_0 and ψ_1 are determined from

$$t\psi_0'' - \psi_0' - t\psi_0 = 0, \qquad t\psi_1'' - \psi_1' - t\psi_1 = \tfrac{1}{2}t^2 \psi_0'. \tag{A2}$$

We also impose the following boundary conditions, $\psi(0) = 1$ and $\psi(\infty) = 0$, that in one turn provide

$$\psi_0(0) = 1, \qquad \psi_n(0) = 0, \quad \text{for } n \geq 1,$$
$$\psi_n(\infty) = 0, \quad \forall \, n. \tag{A3}$$

Given the boundary conditions, the appropriate solutions to (A2) are simply given by

$$\psi_0(t) = tK_1(t), \qquad \psi_1(t) = \tfrac{1}{8}t^3 K_1(t). \tag{A4}$$

The remaining integral may be found in Ref. [15]. It is given by

$$\int_0^{\infty} dt t^3 K_1^2(t) = \frac{2}{3}. \tag{A5}$$

So finally, we get

$$C_0 = 1, \qquad C_2 = -\tfrac{1}{3}c. \tag{A6}$$

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