

Interpretation of quantum field theories with a minimal length scale

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It has been proposed that the incorporation of an observer independent minimal length scale into the quantum field theories of the standard model effectively describes phenomenological aspects of quantum gravity. The aim of this paper is to interpret this description and its implications for scattering processes.

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I. INTRODUCTION

Quantum gravity is probably the most challenging and fascinating problem of physics in the 21st century. The most impressive indicator is the number of people working on it, even though so far there is no experimental evidence that might guide us from mathematics to physical reality. During the last years, the priority in the field has undergone a shift towards the phenomenology and possible predictions [1–12]. The phenomenology of quantum gravity has been condensed into effective models which incorporate one of the most important and general features: a minimal invariant length scale that acts as a regulator in the ultraviolet. Such a minimal length scale leads to a generalized uncertainty relation and it requires a deformation of Lorentz-invariance which becomes important at high boost parameters.

The construction of a quantum field theory that self-consistently allows such a minimal length makes it necessary to carefully retrace all steps of the standard quantization scheme. So far, there are various approaches how to construct a quantum field theory that incorporates a minimal length scale and the accompanying deformed special relativity (DSR), generalized uncertainty principle (GUP) and modified dispersion relation (MDR). Most notably, there are approaches which start from the DSR [13–18], the κ -Poincaré Hopf algebra [19–24] and those which start with the GUP [25–29]. Besides this, there exists the possibility to examine specific effects like reaction thresholds or radiation spectra starting from the MDR without aiming to derive a full quantum theory in the first place [30–32]. Relations between several approaches have been investigated in [33].

In this paper we aim to closely examine the ansatz starting with the GUP by paying special attention to the interpretation of the effective theory. Since this starting point is conceptually different from the DSR-motivated one, it does not suffer from some of the problems that have been encountered within the latter, e.g. the conservation of momentum in particle interactions and the meaning of a highest energy scale for bound multiparticle states, the so-called “soccer-ball-problem”. As we will show, it is the

treatment of a single noninteracting particle which distinguishes both approaches.

This paper is organized as follows. In the next section we will investigate a picture of particle scattering with additional strong gravitational interaction and motivate an effective model to extend the quantum field theories of the standard model. In Sec. III, some properties of the model are investigated. In Sec. IV it is examined in which cases the model can be applied with special emphasis on the observer independence. In Sec. V we analyze the relation to models starting with a deformation of Lorentz transformations at high energies. We conclude in Sec. VI.

Throughout this paper we use the convention $c = \hbar = 1$ and $G = 1/m_p^2$. Small Greek indices are spacetime indices; small Latin indices label particle states.

II. MOTIVATION

A particle with energy close to the Planck mass, m_p , is expected to significantly disturb spacetime on a distance scale comparable to its own Compton wavelength and thereby make effects of quantum gravity become important. A meaningful way to quantify how nonclassical gravitational effects are is to examine the ability to describe spacetime as locally flat. The appropriate quantities are the entries of the curvature tensor in a locally orthonormal basis, or, in case it is nonvanishing, the curvature-scalar \mathcal{R} . Quantum effects should become strong, when $\mathcal{R}m_p^2 \sim 1$. However, provided that the rest mass of the particle itself is much smaller than the Planck mass, boosting the particle to high rapidity will not change the curvature, or the strength of quantum gravitational effects, it causes.¹

Instead, to make the above statement precise, one has to refine its formulation: a concentration of energy high enough to cause strong curvature will result in significant quantum effects of gravity. Such a concentration of energy might most intuitively be seen as an interaction process. In an interaction process, the relevant energy is that in the center of mass (com) system, which we will denote with \sqrt{s} . Note that this is a meaningful concept only for a theory

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¹From here on we assume that the rest mass of the particle is always much smaller than the Planck mass.

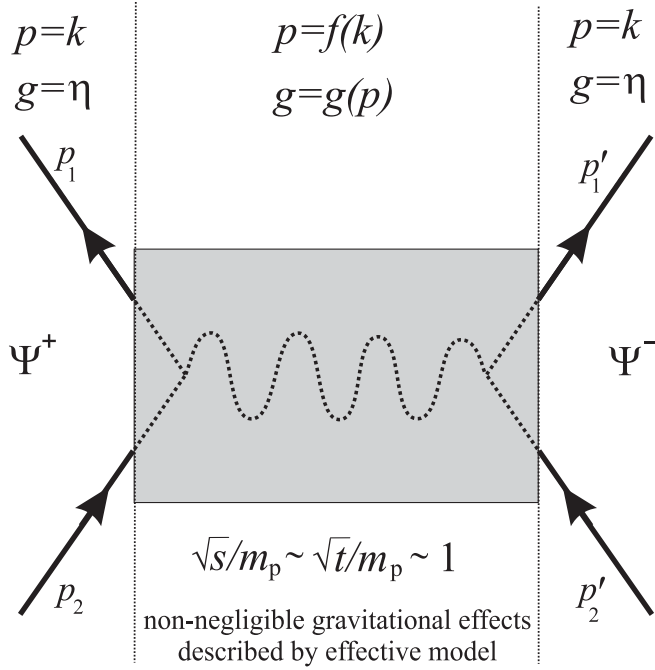


FIG. 1. In addition to the SM-interaction under investigation, strong gravitational effects accompany the processes in the collision region. These effects are condensed in the effective QFT model with a modified dispersion relation. Shown is the example of fermion scattering $f^+ f^- \rightarrow f^+ f^-$ (s -channel).

with more than one particle. However, it can also be used for a particle propagating in a background field consisting of many particles (e.g. the CMB). The scale at which effects of quantum gravity become important is when $\sqrt{s}/m_p \sim 1$, for small impact parameters $\sqrt{t} \sim 1/b \sim m_p$.

Let us consider the propagation of a particle with wave vector k_μ , when it comes into a spacetime region in which its presence will lead to a com energy close to the Planck scale. The concrete picture we want to draw is that of in- and outgoing point particles separated far enough and without noticeable gravitational interaction, that undergo a strong interaction in an intermediate region which we want to describe in an effective way.² We denote the asymptotically free in(out)-going states with Ψ^+ (Ψ^-), primes are used for the momenta of the outgoing particles. This is schematically shown in Fig. 1.

In the central collision region, the curvature of spacetime is non-negligible and the scattering process as described in the quantum field theories (QFTs) of the standard model (SM) is accompanied by gravitational interaction. We aim to find an effective description of this gravitational interaction, which we expect to modify the propagator, not of the asymptotically free states, but of the particles that transmit the interaction. The exchange

²An effective description as opposed to going beyond the theory of a pointlike particle.

particle has to propagate through a region with strong curvature, and the particle's propagation will dominantly be modified by the energy the particle carries.

Even on a classical level, the backreaction of a field propagating in spacetime is involved. For simplicity, let us consider a massless scalar field ϕ . In principle, the evolution of spacetime is described by Einstein's field equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R} = 8\pi GT_{\mu\nu}^\phi, \quad (1)$$

where $G = 1/m_p^2$, and the source term is given by

$$T_{\mu\nu}^\phi = \nabla_\nu \phi \nabla_\mu \phi - \frac{1}{2}g_{\mu\nu} g^{\kappa\epsilon} \nabla_\kappa \nabla_\epsilon \phi. \quad (2)$$

The evolution of the field itself is the wave-equation in curved space

$$g^{\kappa\epsilon} \nabla_\kappa \nabla_\epsilon \phi = 0 \quad (3)$$

which can be rewritten into partial derivatives

$$g^{\kappa\epsilon} \partial_\kappa \partial_\epsilon \phi + g^{\kappa\epsilon} \Gamma^\alpha_{\kappa\epsilon} \partial_\alpha \phi = 0. \quad (4)$$

In the limit when the backreaction is small, the dependence of the metric on the field can be neglected. In this case, one has $g_{\mu\nu} = g_{\mu\nu}(x)$. This leads to the formulation of a field theory in a possibly curved background. When spacetime is asymptotically flat, such that $g_{\mu\nu}(x \rightarrow \infty) \rightarrow \eta_{\mu\nu}$, the equation of motion reduces to the familiar wave-equation which is solved by a superposition of modes of the form

$$v_p \sim \exp(i\eta^{\mu\nu} p_\nu x_\nu), \quad (5)$$

where p fulfils the dispersion relation

$$\eta^{\mu\nu} p_\mu p_\nu = 0. \quad (6)$$

However, in general the metric $g_{\mu\nu}$ will not only be a function of the spacetime coordinates x , but also a function of the derivatives $\nabla_\alpha \phi$, as dictated by Eqs. (1). The same is true for the Christoffel symbols. The general structure of Eq. (3) is then

$$g^{\kappa\epsilon}(x, \nabla_\alpha \phi) \partial_\kappa \partial_\epsilon \phi + h^\alpha(x, \nabla_\alpha \phi) \partial_\alpha \phi = 0. \quad (7)$$

It is therefore natural to expect that in regimes where the gravitational interaction becomes important, the metric which the field propagates in will be a function of its energy (density).

It is most likely not possible to describe strong gravitational effects by using classical general relativity, and the above motivation is not suitable to derive further details of the spacetime structure. Instead, inherently new effects due to the quantum nature of spacetime will influence and eventually dominate the interaction processes. Such behavior has previously been investigated in various context using approaches from spacetime foams, loop gravity or D -brane recoil [34–39]. These investigations indicate that quantum effects result in a modified dispersion relation for the propagating particles, which can also be formulated in terms of an energy-dependent metric [15].

The aim of the here discussed approach is to examine the additional gravitational interaction by means of an effective model.³ The QFTs considered are modified in such a way that they capture one of the features that is generally expected to occur in quantum gravity [40–43]: a minimal length scale. For this purpose, it is assumed that the exchange particles which mediate the SM interactions have to propagate through a region with a non-negligible quantum gravitational effects.

As motivated above and in [15] the assumption is that the quantum gravitational effects can be captured by the following description

- 1a In a region of strong gravitational effects, the metric is dominated by the energy-dependence and one has $g_{\mu\nu} = g_{\mu\nu}(\nabla_\alpha \phi)$ in $[-d, d]$. The coordinates are those of the asymptotically flat coordinate system with the interaction box in rest.
- 2a The wave vector k^ν in the interaction region has an upper bound $1/L_{\min}$. This is the fundamental assumption of a finite possible resolution.

In a complete description, one would also expect the metric to be a function of coordinates: this behavior is simulated in the usual way by switching the interaction on and off in the central region, which allows us to remain in the momentum-space description. In this case, the metric in the interaction region is not a function of the coordinates. Since the Christoffel symbols are partial derivatives of the metric, covariant derivatives reduce to partial derivatives and the wave-equation takes the form

$$g^{\mu\nu}(\partial_\alpha \phi) \partial_\mu \partial_\nu \phi = 0, \quad (8)$$

which is solved by a superposition of modes of the form

$$u_k = \exp(i(k_\nu x^\nu)), \quad (9)$$

where k fulfils the dispersion relation

$$g^{\mu\nu}(ik_\alpha) k_\mu k_\nu = 0. \quad (10)$$

Note that k_ν does not have a bound and that indeed both k_ν and k^ν are still completely normal vectors. However, it is immediately apparent that under a transformation on k_ν , the quantity $k^\nu = g^{\nu\kappa}(k) k_\kappa$ will transform nonlinearly in k , see also [15]. To preserve parity, $g^{\mu\nu}$ should be an even function of k_α , which also assures that no i 's appear in the dispersion relation.

We will in the following refer to the dispersion relation as being a modified dispersion relation (MDR) if

$$\eta^{\mu\nu} k_\mu k_\nu \neq 0. \quad (11)$$

Note, that this need not necessarily be the case for all equations of the form (10). E.g. when the energy-dependent metric is of the form $g^{\mu\nu} = f(k)\eta^{\mu\nu}$ with

³We do not consider explicit production of real or virtual gravitons, or black hole formation.

some scaling function f , then the dispersion relation (10) implies the standard dispersion relation.

III. EFFECTIVE DESCRIPTION

Instead of dealing with an energy-dependent metric as in Eq. (10) the essence of the ansatz can also be captured by starting with the nontrivial relation between the globally conserved and the local quantities $p = f(k)$. This description has been widely used in the DSR literature, and in the following possible interpretations of this approach are examined.

The functional form of the unknown relation $f(k)$ is where knowledge from an underlying theory has to enter. So far, the precise form of the function can not be derived. However, the above mentioned general expectations allow us to constrain the form of the function. Such is that the Planck length acts as a minimal length $L_{\min} \sim 1/m_p$ in the sense that structures can not be resolved to smaller distances. Note again, that this statement is reasonable only for interaction processes since otherwise 'resolution' is not a meaningful concept.

It is nevertheless possible to construct a theory building up on single particles when one carefully keeps track of its meaning. E.g. the modified equation of motion Eq. (19) effectively describes the gravitational interaction that the particle would undergo when it comes close to a high com energy. The single particle meaning therefore is a description of what property a single particle would need to have in order to simulate the behavior of quantum gravitational effects in the interactions. To make this really clear: The right way to describe the strong gravitational effects would be to include the appropriate quantized gravitational interaction, which is desirable but so far an unsolved problem. Instead, we equip the point particle with an additional property (k no longer linear to p) that in interactions *simulates effects that we expect from quantum gravity*.

According to the above, this relation between momentum p and wave vector k to be such that no matter how high the energy of the particle gets, its wavelength can never become smaller than the minimal length. As a translation of 1a and 1b, The function $p = f(k)$ therefore has to fulfill the requirements:

- 1b For energies much smaller than $1/L_{\min}$ the usual linear relation is found.
- 2b For large energies, k asymptotically reaches $1/L_{\min}$. And for a well-defined relation we require
- 3 The function is invertible, i.e. it is monotonically increasing.

Theories of this type have been examined in various context as to their analytical structure and phenomenological consequences [44–47]. The Lorentz transformations acting on the wave vector in the collision region have to respect the above three points. This means that for the quantity k a deformed transformation is required which

has an invariant minimal length L_{\min} or an invariant maximal energy scale m_p , respectively.

We write the relation between momentum and wave vector in the form $p^\nu = f^\nu(k)$, which can be expanded in a power series

$$f^\nu(k) = \eta^{\nu\mu} \left(k_\mu + \sum_{l=1}^{\infty} \frac{A^{(2l+1)}{}^\mu{}_{\nu_1\nu_2\dots\nu_{2l+1}}}{m_p^{2l}} k_{\nu_1} k_{\nu_2} \dots k_{\nu_{2l+1}} \right)$$

where it is taken into account that p is odd in k . A is a rank- $2l+1$ -tensor with dimensionless coefficients that, in accordance with the above point 2a, are constant with respect to spacetime coordinates. Here, m_p sets the scale for the higher order terms. Theories of this type have recently been investigated in [48].

The wave vectors k coincide with the momenta of the in- or outgoing particles far away from the interaction region, where spacetime is approximately flat $g \sim \eta$. We will denote these asymptotic momenta by p_i . Putting the interaction into a box and forgetting about it, $\sum_i p_i$ is a conserved quantity.⁴ The unitary operators of the Poincaré group act as usual on the asymptotically free states. In particular, the whole box is invariant under translations a_ν and the translation operator has the form $\exp(-ia^\nu p_\nu)$ when applied to Ψ^\pm .

In contrast to the asymptotic momenta p , the wave vector k of the particle in the interaction region will behave nontrivially because strong gravitational effects disturb the propagation of the wave. In particular, it will not transform as a standard (flat space) Lorentz-vector, and obey the modified dispersion relation like Eq. (10). The action of the Lorentz-group on states inside the interaction region will be modified and has been examined e.g. in [13,14]. Though it is an important question to understand in which way the local gravitational interaction modifies operators of flat space QFT, it is for our further investigation not necessary to deal with this issue.⁵

Under quantization, the local quantity k will be translated into a partial derivate. One now wants to proceed from the single- k mode (9) to a field and to the operator $\hat{k}_\nu = -i\partial_\nu$. The corresponding momentum-operator \hat{p} should have the property

$$\hat{p}^\nu v_k = p^\nu v_k = f^\nu(k) v_k, \quad (12)$$

which is fulfilled by

$$\hat{p}^\nu = f^\nu(-i\partial), \quad (13)$$

since every derivation results in just another k . It is therefore convenient to define the higher order operator

⁴We assume that no additional explicit losses, e.g. in gravitons occur.

⁵Note that an operator of the form $\exp(-i\tilde{a}^\nu k_\nu)$, where k is not a Lorentz-vector leads to the conclusion that \tilde{a} is not a Lorentz-vector either, and therefore requires some thought [15].

$$\delta^\nu = i f^\nu(-i\partial). \quad (14)$$

Since f is even in k , this operator's expansion has only real coefficients that are up to signs those of f^ν . Note that δ^ν commutes with ∂_κ .

From this one can further define the operator $\tilde{\square}$ which generates the wave-function Eq. (8)

$$\tilde{\square} = g^{\mu\nu}(\partial_\alpha) \partial_\mu \partial_\nu = \delta^\nu \partial_\nu. \quad (15)$$

This modified D'Alembert operator plays the role of the propagator in the quantized theory. It captures the distortion of the exchange-particles in the strongly disturbed background.

It is convenient to use the higher order operator δ^ν in the setup of a field theory, instead of having to deal with an explicit infinite sum. Note, that this sum actually has to be infinite when the relation $p^\nu = f^\nu(k)$ has an asymptotic limit as one would expect for an UV-regulator. Such an asymptotic behavior could never be achieved with a finite power-series.

The higher order operator δ^ν fulfills the property (see Appendix B)

$$\phi_\mu(\delta^\mu \psi) = -(\delta^\mu \phi_\mu) \psi + \text{total divergence}. \quad (16)$$

This relation is essential for the usefulness of the operator as it allows to shift derivatives in the derivation of the equations of motion from a variational principle. In particular, the action for a scalar field⁶ takes the form

$$S = \int d^4x \sqrt{g} \mathcal{L}, \quad (17)$$

with

$$\mathcal{L} = \delta^\nu \phi \partial_\nu \phi. \quad (18)$$

Using Eq. (16), one then derives the equations from the usual variational principle to

$$\delta^\nu \partial_\nu \phi = 0. \quad (19)$$

The stress-energy tensor is calculated in a similar way

$$T_{\mu\nu} = \partial_\nu \phi \partial_\mu \phi - \frac{1}{2} g_{\mu\nu} \mathcal{L}. \quad (20)$$

This quantity is conserved with respect to δ^ν , i.e.

$$\delta^\nu T_{\mu\nu} = 0. \quad (21)$$

Note that it is not conserved with respect to $\partial^\nu = \eta^{\mu\nu} \partial_\mu$, since one uses the equation of motion Eq. (19) for the conservation law. This becomes clear when one inserts a plane wave. Since it was assumed that the dispersion relation is truly modified, it is $\eta^{\mu\nu} k_\nu k_\mu \neq 0$. Instead, the relation needed for the stress-energy conservation is $g^{\mu\nu} k_\nu k_\mu = 0$. On the other hand, the quantity $T^{\mu\nu}$ is

⁶For a discussion of the Dirac-equation, gauge fields, and applications see e.g. [25,26].

conserved with respect to ∂_ν , but it will involve the energy-dependent metric from raising the indices in Eq. (20).

The calculus with the higher order operator δ^ν effectively summarizes the explicit dealing with the infinite series, as examined in [48].

Normalized solutions to the wave-equation Eq. (19) can be found in the set of modes

$$v_p(x) = \frac{1}{\sqrt{(2\pi)^3 2E}} \exp(ik_\nu x^\nu), \quad (22)$$

where $(E, p) = f(k_0, k)$. These modes solve the equation of motion when p fulfills the usual dispersion relation, or k fulfills the MDR, respectively. Therefore, the interpretation of k in a geometrical meaning as a wave vector is justified. When imposing boundary conditions, one sees that the relevant quantity is k and not p which makes it clear that modifications will arise whenever one attempts to confine the particle in a region of size comparable to the Planck length. This, e.g. has consequences for the Casimir effect [49,50] and for the evaporation of Planck-size black holes [30,31].

According to the above discussion of the interaction region, we can now examine the properties of the S -matrix. The quantity we measure for ingoing and outgoing states of a collision is typically not the wavelength of the particle but its ability to react with other particles. For scattering processes, the quantity k therefore is a mere dummy-variable that justifies its existence as a useful interpretational device in intermediate steps, where it enters through the propagator defined in Eq. (10). It is in principle possible to calculate in k -space, however, eventually k can be completely replaced by the physical momentum p . It is important to note that from the construction of the model, k -space has finite boundaries, whereas the momentum space is infinite with a squeezed measure at high energies that regulates the usually divergent integration (see e.g. [26]).

In particular, the S -matrix is invariant under the standard Lorentz transformation and conserves the sum of in- and outgoing momenta. To see this, note that the unitary operator of the Poincaré group that belongs to a Lorentz boost Λ and a translation a acts on in- and outgoing states in momentum space in the usual way

$$U(\Lambda, a)\Psi^\pm(p) = \exp(-a^\mu (\Lambda p)_\mu) \Psi^\pm(\Lambda p). \quad (23)$$

Therefore, the scattering matrix $S = \langle \Psi^- | \Psi^+ \rangle$ transforms according to⁷

$$S_{p_1, \dots, p_n, p'_1, \dots, p'_n} \sim S_{\Lambda p_1, \dots, \Lambda p_n, \Lambda p'_1, \dots, \Lambda p'_n} \exp[-ia^\mu ((\Lambda p_1)_\mu - \dots - (\Lambda p_n)_\mu)] \exp[+ia^\mu ((\Lambda p'_1)_\mu + \dots + (\Lambda p'_n)_\mu)]. \quad (24)$$

⁷Further factors depend on the spin etc. and are not affected by a , see e.g. [51].

Since the left side is independent of a , so is the right side, which is possible only if

$$(p_1)_\mu + \dots + (p_n)_\mu - (p'_1)_\mu - \dots - (p'_n)_\mu = 0, \quad (25)$$

which remains a true statement under Lorentz boosts. The total momentum of the in- and outgoing states therefore is conserved in the standard way because these particles do not experience strong gravitational effects in the asymptotic regions.

IV. OBSERVER INDEPENDENCE OF THE MINIMAL LENGTH

The notion of a minimal length scale should be observer independent. At first sight, this seems to be in conflict with the standard Lorentz transformation since a boost would be able to contract a minimal length further. However, one has carefully to ask the right question. Consider two observers related by a standard boost, each having a ruler of minimal length in his rest frame. This is no contradiction as long as both do not compare any quantity. The observer in the one system can not actually 'see' the length of the object in the other system without probing it, which already involves an interaction process.

Instead, one would ask both observers to perform the same experiment and measure which results arise from initial conditions they have both agreed on. Such might be the ability to resolve smaller distances with larger energy, the impossibility of which indicates the closeness of Planck-scale fuzziness and should be equally impossible for all observers. Observer independence states that the outcome of such experiments has to be the same in all rest frames. This reasonable expectation makes immediately clear what consequences a model like the here discussed can have and can not have.

Consider a reaction made by one observer which results in a cross section σ (Lorentz scalar) as a function of varying com energy \sqrt{s} (Lorentz scalar). Increasing \sqrt{s} , at com energy close to the Planck scale, the reaction will stop probing smaller distances and the amplitude of the process will become (asymptotically) constant. In case the SM prediction was an increasing function, the modified amplitude will be lowered in comparison, in case the SM prediction was a decreasing function, the modified amplitude will be raised in comparison. One might say in general, the amplitude stagnates. Since the cross section $\sigma(\sqrt{s})$ takes into account the phase space of the outgoing particles, it is exponentially suppressed at energies above the Planck scale [25]. One can expect collider signatures to be dominantly visible in the s -channel at large momentum transfer t .

However, if such a cross section has a typical com energy $\sqrt{s_0}$ at which it has a sudden increase (crossing of reaction threshold), this threshold—being a Lorentz scalar—is the same for both observers. In particular, a reaction threshold that has been observed in a laboratory on

earth to occur at a certain com energy, will in every inertial frame take place at the same com energy unless observer independence is explicitly violated.

Within the here discussed model, one expects a deviation of the cross section relative to the SM result, when the com energy for the reaction gets close to the Planck scale (or a new lowered fundamental scale). This however, is certainly not the case, e.g. for the recently examined photopion production with a threshold of ~ 1 GeV that has been examined in earth's laboratories for decades. A modification of the threshold for photopion production, which has been proposed to explain the nonobservation of the GZK-cutoff [2], therefore is only possible when observer independence is violated. Indeed, one could use exactly this threshold to distinguish observers.⁸ In the here presented approach, the threshold occurs at the same com energy in all reference frames, and thus the GZK-cutoff remains unmodified. This is due to the invariance of the cross section under a boost from the com frame of the earth experiment to that of the cosmic ray interaction for the same \sqrt{s} (see also Appendix).

Nevertheless, it is possible to find modifications from phenomenological quantum gravity in processes where there is a natural candidate for a special reference frame. E.g. a long-distance propagation of particles through the CMB might reflect in a modified dispersion relation. In such a case, the particle constantly propagates through an interaction region and therefore the MDR applies. Even though the modification of the particle's propagation are tiny, they can add up over a long travel distance. An observable that has recently been investigated in this context as to its possibility to reveal such quantum gravitational effects is the time of flight or HBT [32] for γ -rays from far away sources. In case the MDR predicts a varying speed of light, the time of flight can depend on the energy of the photon which could become detectable with GLAST [52].

Also, the meaning of quantum mechanics within the above introduced framework becomes accessible using this interpretation. From a nonlinear relation $k(p)$, it follows that the uncertainty principle is generalized to

$$[\hat{x}, \hat{p}] = +i \frac{\partial p}{\partial k} \Rightarrow \Delta p \Delta x \geq \frac{1}{2} \left| \left\langle \frac{\partial p}{\partial k} \right\rangle \right|. \quad (26)$$

In quantum mechanics, the interaction is not quantized but we describe a particle in a potential or with boundary conditions, respectively. Here, the potential plays the role of a background field and sets the scale for the effects to become important. In such a scenario, the energy levels of the hydrogen atom and the spectrum of the harmonic

oscillator have been investigated [25,53–55], and also the gyromagnetic moment (precision in a strong magnetic background field) of the muon [56] receives corrections already at the quantum mechanical level.

Based on this, it is now possible to understand multi-particle bound states. Such a bound state of several particles with small rest masses can eventually have a total mass higher than the Planck mass. Assuming that the gravitational interaction of the bound particles is weak, quantum gravitational effects are negligible and the system can be described within the standard QFT. Most importantly, it will be boosted according to the standard Lorentz transformation like the free single particle (free from gravitational as well as SM interactions). There is no region of gravitationally strong interacting particles that would require modifications. One might however probe such a bound system by using a high energetic beam in a Rutherford-like experiment. One would then again find a limit to the resolution of the internal structure of the bound system.

V. DEFORMED SPECIAL RELATIVITY

From the above discussion it is now apparent that there are two conceptually different ways to include a minimal length scale into the QFTs of the SM and to obtain an effective model. The model discussed here leaves the transformation of the free single particle unmodified since for such a particle there is no natural scale which could be responsible for quantum gravitational effects to become important.

Starting from an effective description in terms of a modified dispersion relation in the interaction region, we have shown in how far this model is a reasonable candidate to describe strong gravitational effects in interactions. It has a clear interpretation for energy and momentum of the participating particles, and it does not suffer from the soccer-ball problem. As has been shown previously, the model has an ultraviolet regulator and modified Feynman rules can be derived [26]. This becomes possible by carefully asking what observables we investigate. Interestingly, it has also been previously mentioned that the arising problems in DSR might be resolved by reconsidering the measurement process [57].

It is however also possible to start with a modification of the Lorentz transformation for a free single particle and construct a QFT based on this. For this, one assumes that the free particle itself experiences the Planck mass as an upper bound on the energy scale.⁹

Such a DSR is a nonlinear representation of the Lorentz-group [13,24,58], which can be cast in a form similar to the

⁸One could achieve such a scenario by allowing a 'scalar' to depend on the boost-parameter. By this, it becomes possible to shift reaction thresholds. However, by doing so one has introduced a label to distinguish between observers through the value of that scalar.

⁹Since we have assumed that the rest mass of the particle is much smaller than the Planck mass, this can not be the particle's own energy and leads back to the question of the measurement process.

above used by introducing variables that transform under the usual representation of the Lorentz-group, $\mathcal{P} = (\epsilon, \pi)$, and a general map F to the variables that then obey the new deformed transformation law $P = F(\mathcal{P})$. However, due to the different interpretation of the free particle's quantities, the 'pseudovisible' \mathcal{P} is not the one identified with the physical four momentum. Instead, energy and momentum of the in- and outgoing particles are identified with those that have the deformed transformation behavior.

As it has turned out over the last years, such attempts result in serious conceptual problems, the most important being the question of which quantities are conserved in interactions and the soccer-ball problem. Recently, important progress has been made as to how these models can be put on a solid base [48,59,60] though the situation is not yet completely satisfactory and open questions remain [11,57,61].

From the present day status, it is not possible to decide which is the right description of nature. One might however lean on results from promising theories of quantum gravity and examine e.g. the question in how far loop gravity [38,39] or string theory have a minimal length. Though the appearance of a finite resolution in string-scattering has been examined [62–65], it remains to investigate in how far this is compatible with the DSR-description of the point-particle limit of interaction processes, or whether a DSR can be accommodated in string theory [66].

One should also keep in mind that these two mentioned approaches with a DSR might not be the only possibilities to include a minimal length into QFT. Another option might be to start with a modification of the interaction at smallest distances itself, and it is not *a priori* clear whether this can always be described in terms of the here discussed model¹⁰ or whether a more general approach is necessary.

VI. CONCLUSIONS

We have investigated an effective description of particle interactions in the presence of strong gravitational effects. As a phenomenological description of quantum gravity, we have motivated the use of a modified dispersion relation, and we have interpreted the arising picture of the interaction process. Further, we have argued that a fundamental minimal resolution is an observer independent statement even though a free particle might still transform under standard Lorentz transformations. We have shown that in this case, the model has a clear interpretation for the conserved quantities and for the behavior of multiparticle states. Based on this, we distinguished two conceptually different approaches towards a quantum field theory with a minimal length, depending on the treatment of the non-interacting single particle.

¹⁰I thank Steve Giddings for bringing this into my attention.

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APPENDIX A

The threshold for a proton with four momentum $\mathbf{p} = (E, \vec{p})$ and a photon with four momentum $\mathbf{p}_\gamma = (\epsilon_\gamma, \vec{p}_\gamma)$ to produce a pion is given by the requirement that the total energy $s = (\mathbf{p} + \mathbf{p}_\gamma)^2$ has at least to yield the rest masses of a produced pion m_π and of the outgoing proton m_{Prot} :

$$s \geq (m_\pi + m_{\text{Prot}})^2. \quad (\text{A1})$$

A map like the above introduced f or F from quantities that transform under the usual Lorentz transformation to those that obey the deformed transformation law, leaves a Lorentz scalar a Lorentz scalar (though these can differ by a factor which then necessarily is a constant, see also [59]). Applying such a map to Eq. (A1) will therefore leave the inequality valid for all inertial systems.

However, within the DSR-approach, the nonlinear transformation law for the 'physical' momenta spoils the Lorentz-invariance of this equation. Assuming that the single particle momenta transform according to the new transformation, the additive quantity s does not remain a Lorentz-scalar, which should be really carefully investigated as to whether it actually still allows observer independence.

APPENDIX B

Let us start with examining an n th-order differential operator $\mathcal{D}^{(n)}$ of the form

$$\mathcal{D}^{(n)} = b^{\nu_1} b^{\nu_2} \dots b^{\nu_n} \partial_{\nu_1} \partial_{\nu_2} \dots \partial_{\nu_n} \quad (\text{B1})$$

Then one finds for some functions g and h

$$\begin{aligned} (\mathcal{D}^{(n)} h) g &= b^{\nu_1} b^{\nu_2} \dots b^{\nu_n} (\partial_{\nu_1} \partial_{\nu_2} \dots \partial_{\nu_n} h) g \\ &= \partial_{\nu_1} (b^{\nu_1} b^{\nu_2} \dots b^{\nu_n} (\partial_{\nu_2} \dots \partial_{\nu_n} h) g) \\ &\quad - (b^{\nu_2} \dots b^{\nu_n} \partial_{\nu_2} \dots \partial_{\nu_n} h) (b^{\nu_1} \partial_{\nu_1} g) \\ &= \text{t.d.} - (b^{\nu_2} \dots b^{\nu_n} \partial_{\nu_2} \dots \partial_{\nu_n} h) (b^{\nu_1} \partial_{\nu_1} g), \end{aligned}$$

where 't.d.' means "total divergence" and is of the form $\eta^{\mu\nu} \partial_\mu A_\nu$. Repeat this step $n - 1$ times to obtain

$$(\mathcal{D}^{(n)} h) g = (-1)^n h (\mathcal{D}^{(n)} g) + \text{t.d.} \quad (\text{B2})$$

The operator δ^μ is of the form

$$\delta^\mu = \eta^{\mu\nu} \left(\partial_\nu + \sum_{l=1}^{\infty} \frac{\alpha_\nu^{(2l+1)}}{M^{2l}} \mathcal{D}^{(2l+1)} \right), \quad (\text{B3})$$

where the higher order contributions start with order 3 in agreement with the above observation that the series has only odd contributions. It is understood that the dimensionless coefficients b in Eq. (B1) of the $\mathcal{D}^{(2l+1)}$ -operators can depend on l and can be translated in the a coefficients of Eq. (14). Inserting into Eq. (16) gives

$$\begin{aligned}\phi_\mu(\delta^\mu\psi) &= \eta^{\mu\nu}\phi_\mu(\partial_\nu\psi) \\ &+ \eta^{\mu\nu}\sum_{l=1}^{\infty}\frac{\alpha_\nu^{(2l+1)}}{M^{2l}}\phi_\mu(\mathcal{D}^{(2l+1)}\psi).\end{aligned}$$

The first part is the usual part, the second part can be rewritten with Eq. (B2) to

$$\begin{aligned}\phi_\mu(\delta^\mu\psi) &= -\eta^{\mu\nu}(\partial_\nu\phi_\mu)\psi - \sum_{l=1}^{\infty}\frac{\alpha_\nu^{(2l+1)}}{M^{2l}}(\mathcal{D}^{(2l+1)}\phi_\mu)\psi \\ &+ \text{t.d.}\end{aligned}\tag{B4}$$

and rearranging finally results in Eq. (16). It is worth noting that this does only work when only even powers of the operators $\mathcal{D}^{(n)}$ appear.

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