Composite non-Abelian flux tubes in $N = 2$ SQCD

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Composite non-Abelian vortices in $\mathcal{N} = 2$ supersymmetric $U(2)$ SQCD are investigated. The internal moduli space of an elementary non-Abelian vortex is $\mathbb{CP}^1$. In this paper we find a composite state of two coincident non-Abelian vortices explicitly solving the first-order Bogomolny, Prasad and Sommerfield equations. Topology of the internal moduli space $\mathcal{T}$ is determined in terms of a discrete quotient $\mathbb{CP}^2/\mathbb{Z}_2$. The spectrum of physical strings and confined monopoles is discussed. This gives indirect information about the sigma model with target space $\mathcal{T}$.

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I. INTRODUCTION

The Abrikosov vortex, also often referred to as the Abrikosov-Nielsen-Olesen (ANO) flux tube or the ANO string, was one of the first important topological defects discovered in field theory [1]. It was also one of the first Bogomolny completion examples [2] which was later reinterpreted in a supersymmetric setting as a Bogomolny, Prasad and Sommerfield (BPS) soliton [3]. BPS saturation of the flux-tube-type solitons, such as the ANO string, is due to the $(\frac{1}{2}, \frac{1}{2})$ central charge [4] in the underlying superalgebra.

The ANO string has two translational moduli characterizing the position of the string center in the perpendicular plane. In the supersymmetric case they are accompanied by two supertranslational moduli. The effective low-energy theory on the world sheet of the ANO string is trivial; it is a free field theory of two bosonic moduli.

In recent years it was realized that $\mathcal{N} = 2$ $U(N)$ supersymmetric quantum chromodynamics (SQCD) with the Fayet-Iliopoulos (FI) term supports a rich spectrum of BPS solitons such as domain walls, Z_N and non-Abelian strings, monopoles, and their junctions, including boojums (for recent reviews see [5,6]). In particular, the issue of BPS Z_N strings was thoroughly discussed and non-Abelian strings discovered and analyzed [7–17]. (Abelian Z_N strings were studied previously in [18].)

In the theory with the $U(N)$ gauge group and $N_f = N$ flavors the solution for the elementary vortex displays, in turn, a rich structure: there are color-flavor locked zero modes for the soliton solution, and the resulting reduced moduli space is

$$\mathcal{M} = \mathbb{CP}^{N-1}.$$

As discussed in Refs. [10,11] this property allows one to directly connect the vortex solitons in the four-dimensional $U(N)$ gauge theory with the $\mathbb{CP}^{N-1}$ sigma model in two dimensions. Moreover, the kink of the $(1 + 1)$-dimensional theory is interpreted as a BPS confined monopole located at the junction of two magnetic strings [9–11].

In the $U(N)$ SQCD the ANO string is not minimal. The tension of a "minimal" string is $1/N$-th of that of the Abrikosov string. In particular, in the $U(2)$ model, on which we will focus below, the minimal string tension is $2\pi\xi$ while the ANO string tension is $4\pi\xi$ where ξ is a Fayet-Iliopoulos parameter (assumed to be positive). Then it is natural to think of the ANO string as of composite object built of two minimal strings. The question we will address in this paper is the construction of BPS composite flux tubes. We will limit ourselves to 2-strings, introduce an appropriate ansatz, and obtain, by a direct calculation, a six-parametric family of solutions.

The Abrikosov string has only trivial translational moduli. At the same time, if we consider two parallel minimal (non-Abelian) strings at a distance R from each other, they are noninteracting because of their BPS nature, and, if R is large, we are certain that the configuration is characterized by four internal moduli, in addition to two moduli which have the meaning of the relative distance between the minimal strings. Thus, the reduced moduli space is six dimensional. How can one recover the Abrikosov string?

A constructive answer to this question will be given below. The Abrikosov string will be shown to be represented by a singular point on the moduli space of the 2-string.

In the general case, the dimension of the k-string moduli space was calculated [7] through the index theorem, $\nu = 2kN$. This result has a clear-cut interpretation: if the elementary vortices are taken at large separations, the moduli space factorizes into k copies of $\mathbb{CP}^{N-1}$ plus the positions of the elementary strings in the perpendicular plane; each elementary string has two coordinates parametrizing its center. Once the number of the collective coordinates is established at large separations, it stays the same at arbitrary separations. No potential can be generated on the moduli space because of "BPS-ness." In this respect the situation is similar to the BPS nonminimal ANO strings:

the force due to the gauge boson exchange is canceled by the force due to the scalar Higgs fields, as can be checked by a direct calculation.

A general analysis of the geometry of the six-dimensional moduli space of the 2-string, from a brane perspective, was carried out in [7,19]. It will be briefly reviewed in Sec. VI B.[1]

Our task is different: explicit construction of a family of the 2-string solutions parametrized by a number of collective coordinates. Unfortunately, we could not find a generic solution with eight collective coordinates. In this paper we present a six-parametric BPS solution for the 2-string corresponding to the vanishing distance R between the elementary strings. Besides trivial translations, four other collective coordinates present in our solution have the meaning of orientation in the $SU(2)$ group space. They will be referred to as internal moduli, the corresponding moduli space being denoted by $\mathcal{T}$. Thus, we construct a four-dimensional cross section of the six-dimensional reduced moduli space (the reduced moduli space is obtained from the full moduli space by factoring out overall translations.)

We find that the moduli space $\mathcal{T}$ is given by a quotient

$$\mathcal{T} \approx \mathbb{CP}^2/\mathbb{Z}_2. \tag{1}$$

This result has a subtle distinction compared to the analysis of Ref. [19], where the moduli space of two coincident strings was found to be $\mathbb{CP}^2$. Our arguments supporting (1) are collected in a systematic manner at the end of Sec. VI B.

While the metric of the 1-string sigma model is fixed by symmetry arguments (it is the homogeneous metric in $\mathbb{CP}^1$ due to the $SU(2)_{C+F}$ group; see below), the metric on the 2-string moduli space is a much more complex object. In this issue we limit ourselves to a general remark (Sec. VI), leaving this problem essentially open.

On the other hand, the spectrum of confined monopoles can be found in the Abelian limit $\Delta m \gg \Lambda$. If we assume that the spectrum of confined monopoles does not change with ξ, as was the case for 1-strings [10], we get an indirect information on the sigma model with the target space $\mathcal{T}$.

The paper is organized as follows. In Sec. II we briefly review our basic bulk theory, with the gauge group $U(2)$, two flavors, and the Fayet-Iliopoulos term. Versions of this theory were consistently used as a laboratory for various BPS solitons in the last few years. In Sec. III we summarize aspects of the Abelian strings supported by the bulk theory under consideration. Section IV is devoted to non-Abelian elementary 1-strings. In Sec. V we thoroughly discuss the 2-string solution. Our basic ansatz is introduced in Sec. V B. We assemble BPS equations in Sec. V C. The numerical solution for the profile functions is presented in Sec. V E, while the physical interpretation of the solution obtained is discussed in Sec. V F. We turn to the discussion of geometry of $\mathcal{T}$ in Sec. VI. The issue of confined monopoles is addressed in Sec. VII. We summarize conclusions in brief in Sec. VIII. Appendixes A and B deal with the zero modes of the (1,1) and (2,0) strings, respectively.

[1]More precisely, in Ref. [19] the composite 2-string was studied through modeling the system in terms of string-theoretic D-branes in the Hanany-Witten approach [20]. The emphasis of [19] was on scattering. See also Note added.

II. THE BASIC SETUP AND THE LAGRANGIAN

The bulk theory we work with is a $U(2)$ gauge theory with $\mathcal{N} = 2$ supersymmetry and with $N_f = 2$ matter hypermultiplets and a Fayet-Iliopolous term ξ for the $U(1)$ factor. The following conventions are used:

$$\nabla_\mu = \partial_\mu - i\frac{\tau^a}{2}A^a_\mu - \frac{i}{2}A^0_\mu, \qquad A_\mu = \frac{\tau^a}{2}A^a_\mu + \frac{1}{2}A^0_\mu. \tag{2}$$

The bosonic fields of the theory are the $U(2)$ gauge field, a zero charge scalar a, a complex adjoint scalar a^a ($a = 1, 2, 3$), and the fundamental scalars Q^{kA} and $(\tilde{Q}^\dagger)^{kA}$ where $k = 1, 2$ is the color index of the $SU(2)$ gauge subgroup and $A = 1, 2$ is the flavor index. We can write these last two fields as 2×2 matrices in the color-flavor indices Q and $\tilde{Q}^\dagger$. The parameters of the theory are the gauge couplings e_0 and e_3, the mass parameters m_A for each flavor, and the Fayet-Iliopoulos term ξ. We can always consider the case in which the masses m_A are real, while ξ will be assumed to be positive. Non-Abelian flux tubes emerge in the limit $m_1 = m_2$. It is convenient to start from $m_1 \neq m_2$ (but keeping $|\Delta m| \equiv |m_1 - m_2| \ll |m_{1,2}|$), in which case we will deal with Abelian strings, and then proceed to the limit $m_1 = m_2$.

The bosonic part of the Lagrangian is

$$\begin{aligned}\mathcal{L} = \int d^4x \bigg\{ &\frac{1}{4e_3^2}|F^a_{\mu\nu}|^2 + \frac{1}{4e_0^2}|F_{\mu\nu}|^2 + \frac{1}{e_3^2}|D_\mu a^a|^2 \\ &+ \frac{1}{e_0^2}|\partial_\mu a|^2 + \mathrm{Tr}(\nabla_\mu Q)^\dagger(\nabla_\mu Q) + \mathrm{Tr}(\nabla_\mu \tilde{Q}) \\ &\times (\nabla_\mu \tilde{Q}^\dagger) + V(Q, \tilde{Q}, a^a, a)\bigg\}\end{aligned} \tag{3}$$

where the potential V is the sum of D and F terms,

$$\begin{aligned}V = {}&\frac{e_3^2}{8}\left(\frac{2}{e_3^2}\epsilon^{abc}\bar{a}^b a^c + \mathrm{Tr}(Q^\dagger \tau^a Q) - \mathrm{Tr}(\tilde{Q}\tau^a \tilde{Q}^\dagger)\right)^2 \\ &+ \frac{e_0^2}{8}(\mathrm{Tr}(Q^\dagger Q) - \mathrm{Tr}(\tilde{Q}\tilde{Q}^\dagger) - 2\xi)^2 \\ &+ \frac{e_3^2}{2}|\mathrm{Tr}(\tilde{Q}\tau^k Q)|^2 + \frac{e_0^2}{2}|\mathrm{Tr}(\tilde{Q}Q)|^2 \\ &+ \frac{1}{2}\sum_A |(a + \tau^b a^b + \sqrt{2}m_A)Q_A|^2 \\ &+ |(a + \tau^b a^b + \sqrt{2}m_A)\tilde{Q}_A^\dagger|^2.\end{aligned} \tag{4}$$

Now, let us discuss the vacuum structure of our theory. The adjoint field vacuum expectation values (VEVs) are

$$\langle a\rangle = -\sqrt{2}\frac{m_1 + m_2}{2};$$
$$\langle a_3\rangle = -\sqrt{2}\frac{m_1 - m_2}{2} = 0, \quad \text{if } \Delta m = m_1 - m_2 = 0. \tag{5}$$

If $m_1 \neq m_2$, the gauge symmetry is broken to $U(1)^2$ by the VEV of the adjoint field. Below we will consider mostly the case $\Delta m = 0$ when the gauge group is not broken by the condensation of the adjoint field a_3. The VEVs of the squark fields are

$$\langle\tilde{Q}\rangle = 0; \qquad \langle Q\rangle = \sqrt{\xi}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{6}$$

The vacuum expectation value of $\langle Q\rangle$ completely breaks the gauge symmetry, so that all gauge bosons acquire masses in the bulk.

Note that if $\Delta m = 0$, although both gauge and flavor groups are broken by the quark condensation, the *global* diagonal subgroup of the product of the gauge and flavor groups remains unbroken [21]. We call it $SU(2)_{C+F}$. Its action on the quark fields is given by

$$Q \to UQU^{-1}, \tag{7}$$

where the matrix U on the left corresponds to the global color rotation while the matrix U^{-1} on the right is associated with the flavor rotation. This mechanism is called color-flavor locking.

With two matter hypermultiplets, the $SU(2)$ part of the gauge group is asymptotically free, implying generation of a dynamical scale Λ. If descent to Λ were uninterrupted, the gauge coupling e_3^2 would explode at this scale. Moreover, strong coupling effects in the $SU(2)$ subsector at the scale Λ would break the $SU(2)$ subgroup through the Seiberg-Witten mechanism [22,23]. Since we want to stay at weak coupling we assume that

$$\sqrt{\xi} \gg \Lambda. \tag{8}$$

This guarantees that the masses of all gauge bosons in the bulk are much larger than Λ.

III. ABELIAN STRINGS

Let us start from $\Delta m \neq 0$. In this case the $SU(2)\times U(1)$ group is broken to $U(1)\times U(1)$ by the VEV of the adjoint scalar field a_3; see Eq. (5). Therefore, we have a lattice of Abelian strings labeled by two integers (p, k) associated with winding with respect to two $U(1)$ factors. BPS strings in the theory (3) were studied in [24]. Here we briefly review the main results of this paper.

The charges of the (p, k) strings can be plotted in the Cartan plane of the $SU(3)$ algebra. This is because our $SU(2)\times U(1)$ gauge theory can be considered as a theory

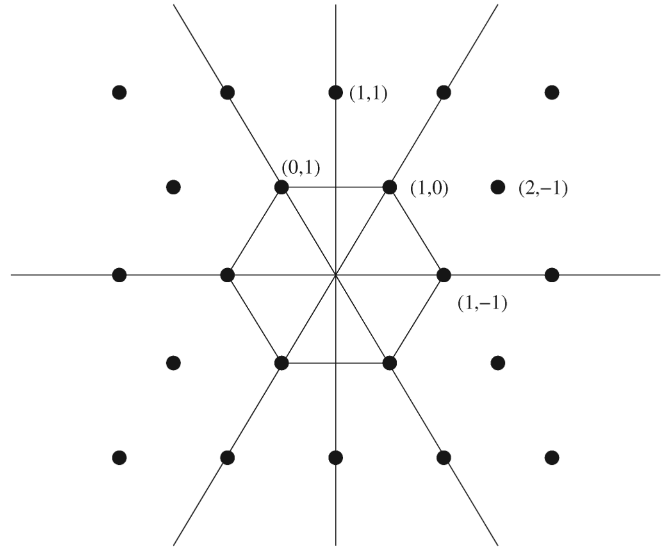

FIG. 1. Lattice of (p, k) vortices.

with the $SU(3)$ gauge group broken down to $SU(2)\times U(1)$ at some high scale. Possible (p, k) strings form a root lattice of the $SU(3)$ algebra [24]. This lattice is shown in Fig. 1. The vertical axis on this figure corresponds to charges with respect to the $U(1)$ gauge factor of the $SU(2)\times U(1)$, while the horizontal axis is associated with the τ^3 generator of the $SU(2)$ factor.

Two strings (1,0) and (0,1) are "elementary" or minimal BPS strings. They are often called Z_2 strings. All other strings can be considered as bound states of these elementary strings. If we plot two lines along the charges of these elementary strings (see Fig. 1) they divide the lattice into four sectors. It turns out [24] that the strings in the upper and lower sectors are BPS but they are marginally unstable. On the contrary, the strings lying in the right and left sectors are (meta)stable bound states of the elementary ones; they are *not* BPS saturated.

The adjoint fields play no role in the string solutions. They are equal to their VEVs (5). The same is true for the $\tilde{Q}$ quark: it vanishes on the string solution, which is consistent with the equations of motion. Hence, the relevant part of the Lagrangian takes the following form:

$$\mathcal{L} \to \int d^4x\Big\{\frac{1}{4e_3^2}|F^a_{\mu\nu}|^2 + \frac{1}{4e_0^2}|F_{\mu\nu}|^2 + \mathrm{Tr}(\nabla_\mu Q)^\dagger(\nabla_\mu Q) + \frac{e_0^2}{8}(\mathrm{Tr}(Q^\dagger Q) - 2\xi))^2 + \frac{e_3^2}{8}(\mathrm{Tr}(Q^\dagger\tau^a Q))^2\Big\}. \tag{9}$$

This gives us an expression for the tension which, in the Bogomolny-completed form [2], can be written as (the indices $i, j = 1, 2$ run over the spatial coordinates on the plane perpendicular to the string direction):

$$T = \int d^2x\biggl(\sum_{a=1}^{3}\biggl[\frac{1}{2e_3}F_{ij}^{(a)} \pm \frac{e_3}{4}\,\mathrm{Tr}(Q^\dagger\tau^a Q)\epsilon_{ij}\biggr]^2 + \biggl[\frac{1}{2e_0}F_{ij} \pm \frac{e_0}{4}(\mathrm{Tr}(Q^\dagger Q) - 2\xi)\epsilon_{ij}\biggr]^2 + \frac{1}{2}|\nabla_i Q^A \pm i\epsilon_{ij}\nabla_j Q^A|^2 \pm \xi\tilde{F}\biggr). \tag{10}$$

Equating the non-negatively-defined terms in the square brackets to zero gives us the first-order equations for the BPS strings. Then the last term in Eq. (10) gives the string tension. The ansatz used to find an explicit solution for the (p, k) string is

$$Q = \sqrt{\xi}\begin{pmatrix} e^{ip\varphi}\phi_1(r) & 0 \\ 0 & e^{ik\varphi}\phi_2(r)\end{pmatrix},\quad A_i^3 = -\frac{\epsilon_{ij}x_j}{r^2}[(p-k) - f_3(r)],\quad A_i^0 = -\frac{\epsilon_{ij}x_j}{r^2}[(p+k) - f(r)], \tag{11}$$

where φ and r are polar coordinates in the perpendicular (1,2) plane. The string axis is assumed to coincide with the z axis.

Now, using the ansatz above, the first-order equations can be written for the profile functions ϕ_1, ϕ_2, f, f_3 [8,10,24], namely

$$\begin{aligned} r\frac{\mathrm{d}}{\mathrm{d}r}\phi_1(r) - \frac{1}{2}(f(r) + f_3(r))\phi_1(r) &= 0, \\ r\frac{\mathrm{d}}{\mathrm{d}r}\phi_2(r) - \frac{1}{2}(f(r) - f_3(r))\phi_2(r) &= 0, \\ -\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}f(r) + \frac{e_0^2}{6}(\phi_1(r)^2 + \phi_2(r)^2 - 2\xi) &= 0, \\ -\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}f_3(r) + \frac{e_3^2}{2}(\phi_1(r)^2 - \phi_2(r)^2) &= 0. \end{aligned} \tag{12}$$

Furthermore, one needs to specify the boundary conditions which would determine the profile functions in these equations. It is not difficult to see that the appropriate boundary conditions are

$$f_3(0) = p - k, \qquad f(0) = p + k; \qquad f_3(\infty) = 0, \qquad f(\infty) = 0 \tag{13}$$

for the gauge fields, while the boundary conditions for the squark fields are

$$\phi_1(\infty) = \sqrt{\xi}, \qquad \phi_2(\infty) = \sqrt{\xi}, \qquad \phi_1(0) = 0, \qquad \phi_2(0) = 0. \tag{14}$$

Numerical solutions to the first-order equations (12) for the (0,1) and (1,0) elementary strings were found in Ref. [8]. Numerical solutions for (2,0), (1,1), and (0,2) 2-strings will be presented in Sec. V E.

The tension of the (p, k) string is given by the boundary term in (10). We get

$$T_{p,k} = 2\pi\xi(p + k). \tag{15}$$

IV. NON-ABELIAN 1-STRING

In this section we review the elementary non-Abelian 1-vortex solution which is associated with the elementary (1,0) and (0,1) Abelian strings and emerges in the limit $\Delta m = 0$ [8,10]. If $\Delta m = 0$ the VEV of the adjoint scalar field a_3 does not break the gauge group $SU(2)$. The relevant homotopy group in this case is the fundamental group

$$\pi_1\biggl(\frac{\mathrm{SU(2)} \times \mathrm{U(1)}}{Z_2}\biggr) = Z. \tag{16}$$

This means that the (p, k)-string lattice reduces to a tower labeled by a single integer

$$n = p + k;$$

see Fig. 2. Note that the tension of all (p, k) strings with given n are equal; see Eq. (15).

For instance, the $(1, -1)$ string becomes classically unstable (no barrier). On the $SU(2)$ group manifold it corresponds to a winding along the equator on the sphere S_3. Clearly this winding can be shrunk to zero by contracting the loop toward the north or south poles of the sphere [25]. On the other hand, the elementary (1,0) and (0,1) strings cannot be shrunk. They correspond to a half-circle winding along the equator. The (1,0) and (0,1) strings form a doublet of the residual global $SU(2)_{C+F}$.

A remarkable feature of the (1,0) and (0,1) strings is the occurrence of non-Abelian moduli which are absent for the Abelian ANO strings. Indeed, while the vacuum field (6) is invariant under the global $SU(2)_{C+F}$ [see Eq. (7)], the string configuration (11) is not. Therefore, if there is a solution of the form (11), there is in fact a two-parametric

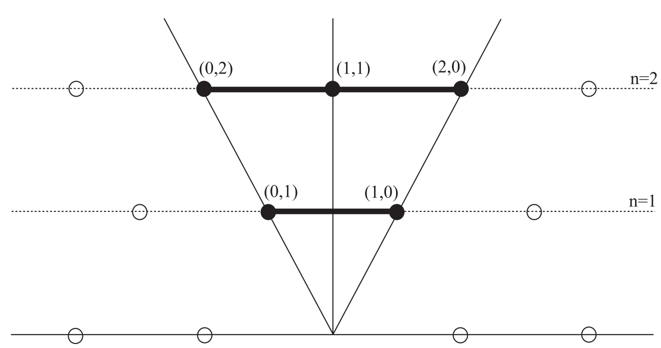

FIG. 2. Lattice of possible Abelian vortices. In the non-Abelian case $m_1 = m_2 = m$, there is a moduli space interpolating between different elements of the lattice.

family of solutions obtained from (11) by the combined global gauge-flavor rotation.

In particular, for (1,0) this gives

$$\begin{aligned} Q^{kA} &= \sqrt{\xi} U \begin{pmatrix} e^{i\varphi}\phi_1(r) & 0 \\ 0 & \phi_2(r) \end{pmatrix} U^{-1} \\ &= \sqrt{\xi} e^{(i/2)\varphi(1+n^a\tau^a)} U \begin{pmatrix} \phi_1(r) & 0 \\ 0 & \phi_2(r) \end{pmatrix} U^{-1}, \\ \mathbf{A}_i(x) &= U\left[-\frac{\tau^3}{2}\epsilon_{ij}\frac{x_j}{r^2}[1-f_3(r)]\right]U^{-1} \\ &= -\frac{1}{2}n^a\tau^a\epsilon_{ij}\frac{x_j}{r^2}[1-f_3(r)], \\ A_i^0(x) &= -\epsilon_{ij}\frac{x_j}{r^2}[1-f(r)], \end{aligned} \tag{17}$$

where the unit vector n^a is defined by

$$U\tau^3 U^\dagger = n^a\tau^a, \qquad a = 1, 2, 3. \tag{18}$$

Now it is particularly clear that this solution smoothly interpolates between the (1,0) and (0,1) strings: if $n = (0, 0, 1)$ the first-flavor squark winds at infinity while for $n = (0, 0, -1)$ it is the second-flavor squark.

Since the $SU(2)_{C+F}$ symmetry is not broken by the squark vacuum expectation values, it is physical and has nothing to do with the gauge rotations eaten by the Higgs mechanism. The orientational moduli n^a are *not* gauge artifacts. To see this it is instructive to construct *gauge-invariant* operators which have explicit n^a dependence. Such a construction is convenient in order to elucidate features of our non-Abelian string solution as well as for other purposes.

As an example, let us define the "non-Abelian" field strength,

$$\tilde{\mathcal{F}}^a = \frac{1}{\xi}\,\mathrm{Tr}\left(Q^\dagger F_3^{*b}\frac{\tau^b}{2}Q\tau^a\right), \tag{19}$$

where $F_k^* = 1/2\varepsilon_{kij}F_{i,j}$ $(i, j, k = 1, 2, 3)$ and the subscript 3 marks the z axis, the direction of the string. From the very definition it is clear that this field is *gauge invariant*.[2] Moreover, Eq. (17) implies that

$$\tilde{\mathcal{F}}^a = -n^a\frac{(\phi_1^2+\phi_2^2)}{2\xi}\frac{1}{r}\frac{df_3}{dr}. \tag{20}$$

From this formula we readily infer the physical meaning of the moduli n^a: the flux of the *color*-magnetic field[3] in the flux tube is directed along n^a. For strings in Eq. (11) the color-magnetic flux is directed along the third axis in the $SU(2)$ group space, either upward or downward. It is just this aspect that allows us to refer to the strings above as "non-Abelian."

The internal moduli space of the vortex[4] is given by the symmetry group upon performing a quotient with respect to the unbroken part [in this case, the $U(1)$ subgroup generated by $\tau^a n^a$],

$$\tilde{\mathcal{M}} = SU(2)/U(1) = \mathbb{CP}^1 = S^2. \tag{21}$$

The vector n^a is the coordinate in the moduli space $\tilde{\mathcal{M}}$.

An effective low-energy $(1 + 1)$-dimensional theory for the vortex zero modes can be readily written [7,8,10]. It turns out to be an $\mathcal{N} = 2$ $\mathbb{CP}^1$ sigma model with the standard homogeneous metric. This is because all non-translational zero modes for the system are generated by the symmetry $SU(2)_{C+F}$.

We will see that this is not the case for 2-strings which, indeed, have additional zero modes not directly associated with the symmetry of the Lagrangian. As it often happens, BPS solutions with higher topological charges have more symmetry than the underlying Lagrangian.

V. NON-ABELIAN 2-STRING

A. Preliminary remarks

If $m_1 \neq m_2$ we return to the Abelian string situation. The only solutions to Eqs. (11) at level two (i.e. with $n = p + k = 2$) are the (2,0), (1,1), and (0,2) strings. In the non-Abelian case ($\Delta m = 0$) we have the whole moduli space of solutions, with (2,0), the (1,1) and the (0,2) strings being represented by particular points on this moduli space.

Let us first consider two parallel elementary strings at a large separation, $R = R_1 - R_2 \to \infty$. As soon as two strings do not interact in this limit we conclude that the dimension of the moduli space of this configuration is eight, twice the dimension of the moduli space of each individual vortex. Two collective coordinates in this moduli space correspond to the overall translations in the (1,2) plane, two other collective coordinates correspond to relative separations R, while the other four coordinates are associated with the internal moduli space. At large R the internal moduli space is $\mathbb{CP}^1 \times \mathbb{CP}^1$ (up to a discrete quotient; see Sec. VI), described by two orientational vectors n_1^a and n_2^a of the two constituent strings. Note that as soon as strings are BPS objects their interaction potential vanishes, and the effective $(1 + 1)$-dimensional theory on the string world sheet is a (classically) massless sigma model.

In this paper we obtain the 2-string solution at zero separation, $R = 0$, when both constituent strings are lo-

[2] In the vacuum, where the matrix Q is that of VEVs, $\tilde{\mathcal{F}}^a$ and F_3^{*a} would coincide.

[3] Defined in the gauge-invariant way; see Eq. (19).

[4] In this case it coincides with the reduced moduli space obtained from the full moduli space by removing overall translations.

cated at the same point in the (1,2) plane, i.e. they are coaxial. By continuity we expect that the internal moduli space is still four dimensional.

Obtaining the four-parametric family of solutions is a serious problem. Suppose we start from the (2,0) string solution [see (11) with $p = 2$, $k = 0$] and apply rotation (7) to this solution. Then we generate only a two-dimensional $\mathbb{CP}^1$ moduli space of solutions. In particular, this transformation interpolates only between the (2,0) and (0,2) strings.

Moreover, the (1,1) string imposes even a more severe problem. The non-Abelian gauge potential is zero for this solution, and the matrix Q is diagonal; see Eq. (11) at $p = k = 1$. Therefore, the rotation (7) acts on this solution trivially generating no internal moduli space at all. This can be viewed as a naive embedding of the Abrikosov string.

Below we find the solution for the non-Abelian 2-string at $R = 0$ by explicitly solving the first-order BPS equations. We show that the internal moduli space is four dimensional, as was expected. The family of solutions is described by four parameters, one of them, α, being the angle between two orientational vectors n_1^a and n_2^a of two constituent strings. At $\alpha = 0$ and $\alpha = \pi$ the internal moduli space develops singular throats, effectively reducing its dimension. At $\alpha = 0$ it becomes $\mathbb{CP}^1$ [the (2,0)/(0,2) string] while for $\alpha = \pi$ [the (1,1) string] it shrinks to a point.

Our solution interpolates between all three Abelian strings: (2,0), (0,2), and (1,1). To describe this solution we introduce new profile functions which will depend on the polar coordinate r and, as a parameter, on the relative angle α. The general BPS equations for the 2-string are then formulated in terms of these profile functions. Finding them at arbitrary α is a rather complicated calculation. We perform an explicit analysis only near particular points corresponding to the (2,0) and (1,1) vortices (presented in Appendixes A and B).

B. The ansatz

Our 2-string solution is parametrized by two vectors, $\vec{n}_1$ and $\vec{n}_2$. The following expression is used for Q:

$$Q = \sqrt{\xi}\kappa(r) U_1 \begin{pmatrix} z_1(r)e^{i\varphi} & 0 \\ 0 & 1 \end{pmatrix} U_1^{-1} U_2 \begin{pmatrix} z_2(r)e^{i\varphi} & 0 \\ 0 & 1 \end{pmatrix} U_2^{-1}, \tag{22}$$

where

$$U_1 \tau_3 U_1^\dagger = n_1^a \tau_a, \qquad U_2 \tau_3 U_2^\dagger = n_2^a \tau_a, \tag{23}$$

and κ, z_1, z_2 are functions of the radial coordinate r and angle α between two vectors n_1 and n_2. Taking $U_1 = U_2 = U_G$ the global orientational zero modes are obtained. In order to study nontrivial α dependence we can take

$$\vec{n}_1 = (0, 0, 1), \qquad \vec{n}_2 = (\sin\alpha, 0, \cos\alpha), \tag{24}$$

with $0 \le \alpha \le \pi$ (see Fig. 3). Once the solution parametrized by the single parameter α is found we can recover the general solution, making a global rotation U_G. In particular, the functions κ, z_1, z_2 depend only on the relative angle α between $\vec{n}_1$ and $\vec{n}_2$ and not on the global orientation of the 2-string.

The particular choice (24) gives the following expression for Q:

$$Q = \sqrt{\xi}\kappa \begin{pmatrix} z_1 e^{i\varphi} & 0 \\ 0 & 1 \end{pmatrix} U \begin{pmatrix} z_2 e^{i\varphi} & 0 \\ 0 & 1 \end{pmatrix} U^{-1}, \tag{25}$$

where

$$U \begin{pmatrix} z_2 e^{i\varphi} & 0 \\ 0 & 1 \end{pmatrix} U^{-1} = \frac{(z_2 e^{i\varphi} + 1)}{2} \mathbf{1} + \frac{(z_2 e^{i\varphi} - 1)}{2} (\vec{\tau} \cdot \vec{l}) \tag{26}$$

and

$$\vec{l} = (\sin\alpha, 0, \cos\alpha). \tag{27}$$

A more explicit expression for Q has the form

$$Q = \sqrt{\xi}\kappa \begin{pmatrix} (\cos^2 \frac{\alpha}{2}) e^{2i\varphi} z_1 z_2 + (\sin^2 \frac{\alpha}{2}) e^{i\varphi} z_1 & \frac{\sin\alpha}{2} (e^{2i\varphi} z_1 z_2 - e^{i\varphi} z_1) \\ \frac{\sin\alpha}{2} (e^{i\varphi} z_2 - 1) & (\cos^2 \frac{\alpha}{2}) + (\sin^2 \frac{\alpha}{2}) e^{i\varphi} z_2 \end{pmatrix}, \tag{28}$$

where φ is the polar angle. The BPS equations are

$$(\nabla_1 + i\nabla_2) Q = 0, \tag{29}$$

which can be identically rewritten as

$$A_1 + iA_2 = -i(\partial_1 Q + i\partial_2 Q) Q^{-1}. \tag{30}$$

Substituting the ansatz (25) in this expression gives us the form of the gauge fields. The result of a rather tedious calculation is

$$-i(\partial_1 Q + i\partial_2 Q)Q^{-1} = ie^{i\varphi}\Big(\frac{2}{r} - 2\frac{\kappa'}{\kappa} - \frac{z_1'}{z_1} - \frac{z_2'}{z_2}\Big)\mathbf{1} + ie^{i\varphi}\Big(\frac{1+\cos\alpha}{r} - \frac{z_1'}{z_1} - \cos\alpha\frac{z_2'}{z_2}\Big)\tau_3 + e^{i\varphi}(\sin\alpha)\Big(\frac{1}{r} - \frac{z_2'}{z_2}\Big)\Big(i\frac{z_1^2+1}{2z_1}(\cos\varphi) - \frac{z_1^2-1}{2z_1}(\sin\varphi)\Big)\tau_1 + e^{i\varphi}(\sin\alpha)\Big(\frac{1}{r} - \frac{z_2'}{z_2}\Big)\Big(-i\frac{z_1^2+1}{2z_1}(\sin\varphi) - \frac{z_1^2-1}{2z_1}(\cos\varphi)\Big)\tau_2. \quad (31)$$

In order to satisfy Eq. (30) we choose the following gauge potentials:

$$A^0_{(i)} = -\frac{\epsilon_{ij}x_j}{r^2}(2-f),$$
$$A^3_{(i)} = -\frac{\epsilon_{ij}x_j}{r^2}((1+\cos\alpha) - f_3),$$
$$A^1_{(i)} = -\frac{\epsilon_{ij}x_j}{r^2}(\sin\alpha)(\cos\varphi)(1-g) - \frac{x_i}{r^2}(\sin\alpha)(\sin\varphi)h,$$
$$A^2_{(i)} = +\frac{\epsilon_{ij}x_j}{r^2}(\sin\alpha)(\sin\varphi)(1-g) - \frac{x_i}{r^2}(\sin\alpha)(\cos\varphi)h. \quad (32)$$

To facilitate reading, let us summarize here our set of profile functions. The set includes

$$\kappa, \qquad z_i (i = 1, 2), \qquad f, f_3, g, h. \quad (33)$$

Now we calculate the field strength tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \frac{i}{4}[A^a_\mu \tau^a, A^b_\nu \tau^b]. \quad (34)$$

Note that the commutator term does not vanish now, while in the 1-string case it was zero. Technically this is a very important distinction.

The only nonvanishing component of the field strength is $F^a_{(12)}$, namely,

$$F^0_{(12)} = -\frac{f'}{r},$$
$$F^3_{(12)} = -\frac{f_3'}{r} + \frac{(1-g)h(\sin\alpha)^2}{r^2},$$
$$F^1_{(12)} = (\cos\varphi)(\sin\alpha)\Big(-\frac{g'}{r} - \frac{\cos\alpha - f_3}{r^2}h\Big),$$
$$F^2_{(12)} = -(\sin\varphi)(\sin\alpha)\Big(-\frac{g'}{r} - \frac{\cos\alpha - f_3}{r^2}h\Big). \quad (35)$$

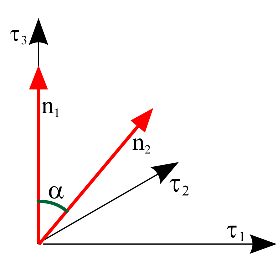

FIG. 3 (color online). It is always possible to align $\vec{n}_1$ with the τ_3 axis and put $\vec{n}_2$ on the τ_3–τ_1 plane. The angle between $\vec{n}_1$ and $\vec{n}_2$ is α. A global $SU(2)_{C+F}$ rotation introduced three extra angles.

C. The BPS equations

The full set of the BPS equations we will deal with are

$$\tilde{F}^a_{(3)} + \frac{e_3^2}{2}\mathrm{Tr}(Q^\dagger \tau^a Q) = 0,$$
$$\tilde{F}^0_{(3)} + \frac{e_0^2}{2}(\mathrm{Tr}(Q^\dagger Q) - 2\xi) = 0,$$
$$A_1 + iA_2 = -i(\partial_1 Q + i\partial_2 Q)Q^{-1}. \quad (36)$$

Substituting our ansätze we get the following system of the first-order differential equations:

$$\frac{f'}{r} = \frac{e_0^2}{4}((1+z_1^2)(1+z_2^2)\kappa^2 + \cos\alpha(1-z_1^2)(1-z_2^2)\kappa^2 - 4),$$
$$\frac{f_3'}{r} - \frac{(1-g)h(\sin\alpha)^2}{r^2} = \frac{e_3^2}{4}((z_1^2-1)(z_2^2+1)\kappa^2 + \cos\alpha(z_1^2+1)(z_2^2-1)\kappa^2),$$
$$\frac{g'}{r} + \frac{\cos\alpha - f_3}{r^2}h = \frac{e_3^2}{2}\kappa^2 z_1(z_2^2-1),$$
$$\frac{f}{r} = 2\frac{\kappa'}{\kappa} + \frac{z_1'}{z_1} + \frac{z_2'}{z_2},$$
$$\frac{f_3}{r} = \frac{z_1'}{z_1} + \cos\alpha\frac{z_2'}{z_2},$$
$$\frac{1-g}{r} = \frac{z_1^2+1}{2z_1}\Big(\frac{1}{r} - \frac{z_2'}{z_2}\Big). \quad (37)$$

The function h can be expressed in terms of other profile functions,

$$h = \frac{z_1^2-1}{z_1^2+1}(1-g). \quad (38)$$

The boundary conditions that must be imposed on the profile functions at $r \to 0$ are

$$f(r) = 2 + \mathcal{O}(r^2), \qquad f_3(r) = (1+\cos\alpha) + \mathcal{O}(r^2),$$
$$g(r) = 1 + \mathcal{O}(r^3), \qquad h(r) = \mathcal{O}(r^3), \qquad z_1(r) \to \mathcal{O}(r),$$
$$z_2(r) \to \mathcal{O}(r), \qquad \kappa(r) \to \mathcal{O}(1). \quad (39)$$

The boundary conditions at $r \to \infty$ are

$$f, f_3, g, h \to 0, \qquad \kappa, z_1, z_2 \to 1. \quad (40)$$

We see that the boundary conditions for the gauge profile functions f and f_3 at $r = 0$ are

$$f(0) = 2 \quad \text{and} \quad f_3(0) = 1 + \cos\alpha. \tag{41}$$

This is in accordance with the boundary conditions for the Abelian strings, Eq. (13). For the (2,0) string we have $p = 2$, $k = 0$, and Eq. (13) gives $f(0) = 2$, $f_3(0) = 2$. This corresponds to $\alpha = 0$ in Eq. (39); the vectors n_1^a and n_2^a of two 1-string constituents of the 2-string are parallel.

For the (1,1) string we have $p = 1$, $k = 1$, and Eq. (13) gives $f(0) = 2$, $f_3(0) = 0$. This case corresponds to $\alpha = \pi$ in Eq. (39), so that the vectors n_1^a and n_2^a are antiparallel.

D. Another gauge

With an appropriate gauge transformation (only a constant color rotation, no flavor rotation)

$$U = \exp\left(i\tau_2\left(\pi - \frac{\alpha}{2}\right)\right), \tag{42}$$

we can cast the solution in the following form:

$$Q = \sqrt{\xi}\kappa\begin{pmatrix} -\cos\frac{\alpha}{2}e^{2i\varphi}z_1 z_2 & \sin\frac{\alpha}{2}e^{i\varphi}z_1 \\ -\sin\frac{\alpha}{2}e^{i\varphi}z_2 & -\cos\frac{\alpha}{2} \end{pmatrix}. \tag{43}$$

Then the gauge field takes the form

$$A_\varphi = \begin{pmatrix} \frac{-3-\cos\alpha+f+f_3}{2r} & \frac{e^{i\varphi}(1-g)\sin\alpha}{2r} \\ \frac{e^{-i\varphi}(1-g)\sin\alpha}{2r} & \frac{-1+\cos\alpha+f-f_3}{2r} \end{pmatrix},$$
$$A_r = \begin{pmatrix} 0 & ie^{i\varphi}\frac{\sin\alpha}{2r}h \\ -ie^{-i\varphi}\frac{\sin\alpha}{2r}h & 0 \end{pmatrix}. \tag{44}$$

In this gauge the expressions are more compact; the VEV of the squark field Q at infinity takes the form

$$Q = \sqrt{\xi}\begin{pmatrix} -\cos\frac{\alpha}{2}e^{2i\varphi} & \sin\frac{\alpha}{2}e^{i\varphi} \\ -\sin\frac{\alpha}{2}e^{i\varphi} & -\cos\frac{\alpha}{2} \end{pmatrix}. \tag{45}$$

E. Numerical solution

Explicit numerical calculations can be and were performed for the vortex profile functions. The dependence on α is nontrivial. Some of the profile functions at $\alpha = 0$ (light grey, green online), $\alpha = \frac{\pi}{2}$ (medium grey, red online), $\alpha = \pi$ (black, blue online) are plotted and compared in Figs. 4 and 5. The couplings are chosen as

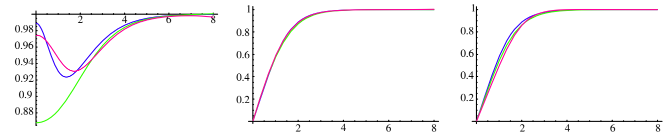

FIG. 4 (color online). $\kappa(r)$ (left panel): $z_1(r)$ (center panel), $z_2(r)$ (right panel), at $\alpha = 0$ (light grey, green online), $\alpha = \frac{\pi}{2}$ (medium grey, red online), $\alpha = \pi$ (black, blue online).

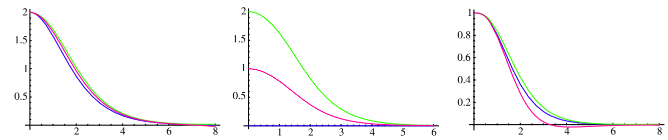

FIG. 5 (color online). $f(r)$ (left panel): $f_3(r)$ (center panel), $g(r)$ (right panel), at $\alpha = 0$ (light grey, green online), $\alpha = \frac{\pi}{2}$ (medium grey, red online), $\alpha = \pi$ (black, blue online).

$$e_0^2 = 1, \qquad e_3^2 = 2. \tag{46}$$

It seems that there is a small but nontrivial dependence on α. This is evident, in particular, for κ, but also for z_1, z_2.

F. Physical interpretation

To understand the explicit solution better, it is instructive to calculate the gauge-invariant operator $\tilde{\mathcal{F}}^a$. It is possible to make a global $SU(2)_{C+F}$ rotation of the solution, so that $\mathcal{F}^{(1,2)}$ averaged with respect to the azimuthal angle φ are zero.

The following matrix realizes this:

$$\tilde{U} = \exp\left(-i\alpha\frac{\tau_2}{2}\right), \tag{47}$$

acting on the field as

$$\mathcal{F}^a\tau^a \to \tilde{U}^\dagger \cdot \mathcal{F}^a\tau^a \cdot \tilde{U}, \qquad Q \to \tilde{U}^\dagger \cdot Q \cdot \tilde{U}. \tag{48}$$

This gives us a minimal non-Abelian 2-string solution parametrized by angle α. To obtain the full moduli space of solutions we have to apply the global $SU(2)_{C+F}$ rotation to the minimal solution.

The minimal solution has the form

$$Q = \sqrt{\xi}\kappa\begin{pmatrix} (\cos^2\frac{\alpha}{2})e^{2i\varphi}z_1 z_2 + (\sin^2\frac{\alpha}{2})e^{i\varphi}z_2 & -\frac{\sin\alpha}{2}(e^{i\varphi}z_1 - 1) \\ -\frac{\sin\alpha}{2}(e^{2i\varphi}z_1 z_2 - e^{i\varphi}z_2) & (\cos^2\frac{\alpha}{2}) + (\sin^2\frac{\alpha}{2})e^{i\varphi}z_1 \end{pmatrix}, \tag{49}$$

and

$$\tilde{\mathcal{F}}^3 = A(\alpha, r), \qquad \tilde{\mathcal{F}}^1 = (\cos\varphi)B(\alpha, r),$$
$$\tilde{\mathcal{F}}^2 = -(\sin\varphi)B(\alpha, r), \tag{50}$$

where

$$\begin{aligned} A(\alpha, z) = {}& \left(\frac{g'}{r} + \frac{h(\cos\alpha - f_3)}{r^2}\right)\kappa^2 z_1(z_2^2 + 1)(\sin\alpha)^2 \\ & + \left(\frac{f_3'}{r} - \frac{h(1-g)(\sin\alpha)^2}{r^2}\right)\frac{\cos\alpha}{2} \\ & \times (\kappa^2(z_1^2 - 1)(z_2^2 - 1) \\ & + \kappa^2(z_1^2 + 1)(z_2^2 + 1)\cos\alpha) \end{aligned} \tag{51}$$

and

$$B(\alpha, z) = (\cos\alpha \sin\alpha)\kappa^2 z_2 \biggl(2\biggl(\frac{g'}{r} + \frac{h(\cos\alpha - f_3)}{r^2}\biggr) z_1 - \biggl(\frac{f_3'}{r} - \frac{h(1-g)(\sin\alpha)^2}{r^2}\biggr)(z_1^2 + 1)\biggr). \tag{52}$$

This solution at fixed α can be rotated by applying an $SU(2)$ global color + flavor rotation. For generic $\alpha \neq 0$, π, all $SU(2)_{C+F}$ generators are broken by the vortex solution. The $\tau_{1,2}$ generators rotate the color flux direction which is independent of the cylindrical coordinate φ; the τ_3 generator shifts a phase φ in the arguments of the sine and cosine functions in Eq. (50). The resulting moduli space is parametrized by the Euler angles, in complete analogy to the phase space of a cylindrical rotator in three-dimensional space. In particular, for $\alpha = 0$ [(2,0) vortex] we have $B = 0$, and for $\alpha = \pi$ [(1,1) vortex] we have $A = B = 0$. The behavior of the solution near these points is discussed in Appendixes A and B; here we summarize our results at the qualitative level.

Let us consider the solution as a function of the angle α (see Fig. 3). At $\alpha = 0$ we have the (2,0) vortex; the action of the global $SU(2)$ is similar to the action of spatial rotation over a stick of zero thickness and the moduli space is S^2. At small nonzero α the stick acquires a thickness of order α and becomes, in color space, similar to a cigarette. The moduli space is now parametrized by three Euler angles in color space and it is three dimensional. Increasing α we can imagine that the cigarette becomes shorter and fatter, becoming a can. At $\alpha = \pi - \epsilon$ the length becomes zero at the linear order in ϵ; on the other hand, the diameter of our can is of order ϵ. The configuration in color space becomes similar to a coin with zero thickness: the moduli space is still parametrized by three Euler angles. At $\alpha = \pi$ our coin shrinks to a point and the action of global color-flavor rotation is trivial.

VI. THE 2-STRING MODULI SPACE

A. Field-theory perspective

The 2-vortex moduli space is a manifold with real dimension 8. Two coordinates correspond to a global translation and we factorize them from the other six, which correspond to the nontrivial part of the moduli space:

$$\mathcal{M} = \mathbb{C} \times \tilde{\mathcal{M}}.$$

In the limit of large relative distance between the two elementary vortices, $\tilde{\mathcal{M}}$ has the following structure [19]:

$$\tilde{\mathcal{M}} \approx \frac{\mathbb{C} \times \mathbb{CP}^1 \times \mathbb{CP}^1}{\mathbb{Z}_2}, \tag{53}$$

where $\mathbb{C}$ corresponds to the relative distance of the two elementary vortices and the two $\mathbb{CP}^1$ factors stand for the non-Abelian internal orientation of the elementary vortices. The $\mathbb{Z}_2$ quotient acts on $(z, \vec{n}_1, \vec{n}_2) \in \tilde{\mathcal{M}}$ as follows:

$$\mathbb{Z}_2\colon z \to -z, \qquad \vec{n}_1 \to \vec{n}_2, \qquad \vec{n}_2 \to \vec{n}_1. \tag{54}$$

In the following we will discuss topology of the slice of the moduli space in which the relative distance of the elementary vortices is zero. We denote this subspace by $\mathcal{T}$. In the previous section we have found an explicit solution, which can be parametrized by an $SU(2) \times SU(2)$ element (U_1, U_2),

$$Q = \sqrt{\xi}\kappa(r) U_1 \begin{pmatrix} z_1(r)e^{i\varphi} & 0 \\ 0 & 1 \end{pmatrix} U_1^{-1} U_2 \begin{pmatrix} z_2(r)e^{i\varphi} & 0 \\ 0 & 1 \end{pmatrix} U_2^{-1}, \tag{55}$$

where

$$U_1 \tau_3 U_1^\dagger = n_1^a \tau_a, \qquad U_2 \tau_3 U_2^\dagger = n_2^a \tau_a. \tag{56}$$

The functions κ, z_1, z_2 depend on the relative angle α between $\vec{n}_1$ and $\vec{n}_2$ in a nontrivial way. Taking $U_1 = U_2 = U$, the usual global orientation zero modes are obtained. Each of the $SU(2)$ subgroups is broken down locally to $U(1)$. However, the situation is different globally: for example, taking $\vec{n}_1 = -\vec{n}_2 = \vec{n}$ we find just a point in the moduli space [the (1,1) vortex] rather than a two-dimensional submanifold. So $\mathcal{T}$ is not $\mathbb{CP}^1 \times \mathbb{CP}^1$ as one could naively expect.

Let us consider topology of different slices at constant α. At $\alpha = 0$, the moduli space is given by

$$\mathcal{T}_{\alpha=0} = SU(2)/U(1) = \mathbb{CP}^1 = S^2.$$

At $0 < \alpha < \pi$, the moduli space is given by the quotient

$$\mathcal{T}_{0<\alpha<\pi} = SU(2)/\mathbb{Z}_2 = \mathbb{RP}^3 = S^3/\mathbb{Z}_2, \tag{57}$$

because the global rotations from the center of $SU(2)$ have trivial effect on the solution. At $\alpha = \pi$, the moduli space is just a point rather than a manifold. If it were a manifold, then a submanifold of constant small α would be topologically equivalent to S^3, but we know that it is $\mathbb{RP}^3$, which differs from S^3 by a Z_2 quotient. We conclude that at $\alpha = 0$ there is a conical singularity; this is similar to the singularity in the 1-instanton moduli space for the zero-size instanton. For a dedicated discussion of the occurrence of the $\mathbb{Z}_2$ factor in Eq. (57), see Sec. VI B.

Topology of $\mathcal{T}$ is equivalent to a discrete quotient of $\mathbb{CP}^2$. To make it clear, we use the following parametrization of $\mathbb{CP}^2$:

$$\vec{m} = (m_1, m_2, m_3), \tag{58}$$

where m_i $(i = 1, 2, 3)$ are complex variables subject to the constraint

$$|m_1|^2 + |m_2|^2 + |m_3|^2 = 1 \tag{59}$$

and identification

$$\vec{m} \sim e^{i\delta}\vec{m}. \tag{60}$$

Complex vector $\vec{m}$ has six real variables. Condition (59) and identification (60) reduce this number to four, which is the dimension of $\mathbb{CP}^2$.

The variable $|m_1|$ plays a role of $\sin\alpha/2$ for our solution. At $\alpha = 0$ [i.e. (2,0) string] the vector $\vec{m}$ has only two components and parametrizes the $\mathbb{CP}^1$ manifold which is a moduli space of the (2,0) string indeed. At $\alpha = \pi$

$$m_2 = m_3 = 0$$

and the space described by the vector $\vec{m}$ shrinks to a point, just like the moduli space of the (1,1) string. At intermediate α,

$$0 < \alpha < \pi,$$

the vector $\vec{m}$ produces $SU(2) = S^3$ submanifolds. We conclude that topology of the 2-string moduli space $\mathcal{T}$ is given by the following quotient:

$$\mathcal{T} = \mathbb{CP}^2/\mathbb{Z}_2,$$

where Z_2 acts as

$$(m_1, m_2, m_3) \to (m_1, -m_2, -m_3). \tag{61}$$

This Z_2 subgroup acts trivially at $\alpha = \pi$ [where $\vec{m} = (1, 0, 0)$] and at $\alpha = 0$ [where $\vec{m} = (0, m_2, m_3)$] because of the identification (60). The sections at constant α with $0 < \alpha < \pi$ have the topology of $\mathbb{RP}^3 = S^3/Z_2$. Near $\alpha = \pi$ there is a conical singularity.

When one chooses a particular *ansatz*, generally speaking, one is not guaranteed that in this given *ansatz* all moduli space of the solitonic object at hand is covered. In principle, it could happen that an *ansatz* containing an appropriate number of collective coordinates is still not general enough in order to describe in full the family of solutions. We would like to argue that this is not the case here—we do cover all the moduli space of two *coincident* vortices. Our *ansatz* has the right number of collective coordinates; it is not singular anywhere on the moduli space. Moreover, we expect that $\mathcal{T}$ is a topological space with just a single connected component. Finally, let us stress that the Z_2 quotient (a subtle point of the construction) appears as a consequence of the $SU(2)$ global rotations rather than as a specific feature of the particular form of our *ansatz*. As a nontrivial check, we will show in Sec. VI B, with satisfaction, that the result agrees with one from of the brane construction.

The effective $(1 + 1)$-dimensional theory on the string world sheet is a sigma model determined by the metric on the vortex moduli space. We know from $SU(2)_{C+F}$ symmetry arguments that the metric on $\mathcal{T}$ has the form of a cylindrical rotator with an extra parameter α (see Fig. 6),

$$w d\alpha^2 + \frac{1}{2}\left[I_{xy} d\theta^2 + \frac{(I_z + I_{xy}) + (I_z - I_{xy})\cos 2\theta}{2} d\phi^2 + I_z d\psi^2 + 2 I_z \cos\theta d\phi d\psi \right], \tag{62}$$

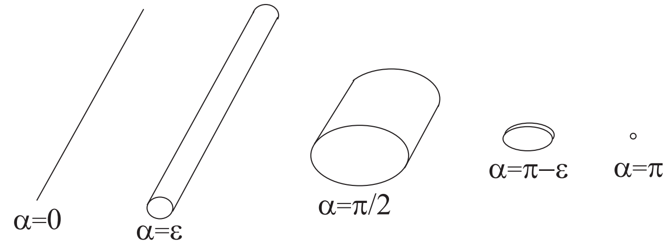

FIG. 6. The structure of the moduli space is very similar to the phase space of a cylindrical rotator whose shape depends on the parameter α. At $\alpha = 0$ one of the inertial moments is zero (as for a stick with zero thickness); at $\alpha = \pi$ all the inertial moments are zero.

where θ, ϕ, and ψ are Euler angles while α is an extra parameter, $0 < \alpha < \pi$. Explicit determination of the functions $w(\alpha)$, $J_{xy}(\alpha)$, and $J_z(\alpha)$ remains an open problem.

B. The 2-vortex in the brane construction

In Refs. [7,19] a construction for topology of the 2-vortex moduli space was proposed within the Hanany-Witten approach. In these papers it is shown that the moduli space of k vortices in the $U(N)$ theory with $N_f = N$ flavor hypermultiplets is a Kähler manifold with real dimension $2kN_c$ that we will denote as $\mathcal{H}_{k,N}$. The Kähler manifold $\mathcal{H}_{k,N}$ is built as follows.

Let us start with a $k \times k$ complex matrix Z and a $k \times N$ complex matrix Ψ, with the constraint

$$[Z, Z^\dagger] + \Psi\Psi^\dagger = 1, \tag{63}$$

where 1 is the identity matrix. The space $\mathcal{H}_{k,N}$ is defined as the quotient of the solution of this constraint divided by the $U(k)$ action,

$$Z \to UZU^\dagger, \qquad \Psi \to U\Psi. \tag{64}$$

The manifold $\mathcal{H}_{k,N}$ has the symmetry $SU(N) \times U(1)$,

$$SU(N)\colon \Psi \to \Psi V, \qquad V \in SU(2), \qquad U(1)\colon Z \to e^{i\alpha} Z. \tag{65}$$

In this formalism the action of the $SU(N)$ group is physically identified with the $SU(N)_{C+F}$ while that of the $U(1)$ is physically identified with the rotational symmetry of the plane.

In the case of 2-strings in the $N_f = N = 2$ gauge theory, both Z and Ψ are 2×2 matrices. Requiring $\mathrm{Tr}Z = 0$ we project out the trivial center-of-mass motion. The action of Eq. (64) can be used to transform Z in the upper-triangular form,

$$Z = \begin{pmatrix} z & \omega \\ 0 & -z \end{pmatrix}, \qquad \Psi = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}. \tag{66}$$

The coordinate z represents the relative positions of the strings; the other entries of the matrices have a less intuitive interpretation. This does not completely fix the $U(2)$ quotient; a remaining $U(1)_1 \times U(1)_2 \times \mathbb{Z}_2$ has to be fixed, namely,

$$U(1)_1\colon\ U = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & 1 \end{pmatrix}, \qquad U(1)_2\colon\ U = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{pmatrix},$$
$$\mathbb{Z}_2\colon\ U = \frac{-1}{\sqrt{1+|2z/\omega|^2}}\begin{pmatrix} -1 & (2z/\omega)^* \\ (2z/\omega) & 1 \end{pmatrix}. \tag{67}$$

We have the following charges with respect to $(U(1)_1, U(1)_2)$:

$$a_i \to (1,0), \qquad b_i \to (0,1),$$
$$\omega \to (1,-1), \qquad z \to (0,0).$$

The constraints in Eq. (63) read

$$|a_1|^2 + |a_2|^2 + |\omega|^2 = 1,$$
$$|b_1|^2 + |b_2|^2 - |\omega|^2 = 1, \tag{68}$$
$$\sum a_i b_i^* = 2z^*\omega.$$

If we put $z = 0$ we can recover topology of $\mathcal{T}$. Let us consider, following Ref. [19], slices at constant ω. At $\omega = 0$ a point is found which is the (1,1) vortex [note that the entries of the matrix Z all vanish and all $U(2)$ quotients have to be fixed for the matrix Ψ]. At $|\omega| = 1$ a copy of $\mathbb{CP}^1$ is found which is the (2,0) vortex and its color-flavor rotated configurations. [This is because $a_i = 0$ and b_i define a $\mathbb{CP}^1$ modulo the $U(1)_2$ action].

The slices at $0 < |\omega| < 1$ are slightly more complex. Let us consider them in detail. Let us define $U(1)_A$ as $U(1)_1 + U(1)_2$ and $U(1)_B$ as $U(1)_1 - U(1)_2$. We have the following charges with respect to $(U(1)_A, U(1)_B)$:

$$a_i \to (1,1), \qquad b_i \to (1,-1), \qquad \omega \to (0,2).$$

The most general solution to the constraints in Eq. (68) is

$$a = (1-|\omega|^2)e^{i\sigma}(\cos\theta, \sin\theta e^{i\phi}),$$
$$b = (1+|\omega|^2)e^{i\eta}(-\sin\theta, \cos\theta e^{-i\phi}), \tag{69}$$
$$\omega = e^{i\gamma}|\omega|.$$

The quotient $U(1)_B$ just gauges away the phase γ so that effectively $\gamma = 0$, with a redefinition of σ and η. Using $U(1)_A$ at this point we can bring the solution to the following form [where $\delta = (\sigma - \eta)/2$]:

$$a = (1-|\omega|^2)e^{i\delta}(\cos\theta, \sin\theta e^{i\phi}),$$
$$b = (1+|\omega|^2)e^{-i\delta}(-\sin\theta, \cos\theta e^{-i\phi}), \tag{70}$$
$$\omega = |\omega|.$$

The three angles (θ, δ, ϕ) parametrize an S^3 inside $\mathbb{C}^4$. We have to be careful, however, because we still have to perform a quotient in order to find the moduli space. Namely, in S^3 we have to identify the opposite points as

$$(a_i, b_i, |\omega|) \to (-a_i, -b_i, |\omega|) \tag{71}$$

because if we shift δ by π we have that both a_i, b_i get a -1 phase which is exactly a π rotation by $U(1)_A$. This special rotation keeps the solution in the form of Eq. (70), and, therefore, we have to take account of this special rotation "by hand." In other words, when we put the solutions of the constraints in the form (70), we fix *almost* all gauge freedom, with the exception of a $\mathbb{Z}_2$ subgroup generated by a π rotation by $U(1)_A$.

We conclude that our solitonic solution is consistent with the brane technique-based results. The $\alpha = \pi$ section in the field-theory approach corresponds to $\omega = 0$ in the brane construction [the (1,1) vortex]; $\alpha = 0$ corresponds to $\omega = 1$ [the (2,0) vortex moduli space]. Sections at intermediate α and ω are in both cases $S^3/\mathbb{Z}_2$, and at the end both approaches give $\mathcal{T} = \mathbb{CP}^2/\mathbb{Z}_2$.

VII. CONFINED MONOPOLES

If the Fayet-Iliopoulos term ξ vanishes, the squark condensate vanishes too, and the theory is in the Coulomb phase. Then there exists the 't Hooft-Polyakov monopole, and its magnetic flux is unconfined. When a nonvanishing ξ is introduced, the squarks develop a VEV, and the theory is in the Higgs phase. The monopole flux is confined. In our theory there is a stable configuration for the monopole confined by two strings oriented in opposite directions. In this configuration the monopole flux is carried by two elementary flux tubes (see [9–11,26]). This monopole can be interpreted as the junction of two different magnetic strings.

If $\Lambda \ll \Delta m \ll \xi^{1/2}$ the quasiclassical treatment is reliable. We find that the monopole is a classical soliton which is the junction of the (1,0) and (0,1) strings. The composite monopole + vortex object is 1/4 BPS; the energy is given by the BPS bound:

$$\int \mathcal{H} d^3x = \int \mathrm{Tr}\Big[\xi B_z - \frac{1}{e_3^2}\partial_\alpha(a\cdot B_\alpha)\Big]d^3x$$
$$= \int T_v dz + M_{\mathrm{mon}}, \tag{72}$$

where

$$T_v = 2\pi\xi, \qquad M_{\mathrm{mon}} = \frac{2\pi(m_1 - m_2)}{e_3^2}. \tag{73}$$

The effective world-sheet description is given by an $\mathcal{N} = 2$ $\mathbb{CP}^1$ sigma model with a large twisted mass term $\mu = \Delta m$, which has two classical vacua (see [10,11,27,28]).

In the limit $\Delta m \ll \Lambda \ll \xi^{1/2}$ the situation is more subtle; the monopole is not a classical object. The vortex world-sheet theory is an $\mathcal{N} = 2$ $\mathbb{CP}^1$ sigma model. Classically this model has an infinite number of vacua parametrized by points of $\mathbb{CP}^1$ and there are Goldstone states. In quantum theory, due to nonperturbative effects, all states become massive. The theory has two quantum vacua, as can be shown by Witten-index arguments. These two vacua correspond to two quantum non-Abelian strings. The monopole can be interpreted as a kink between these two vacua; the monopole mass is given by the mass of the kink in the $1 + 1$-dimensional sigma model [10]

$$M_{\rm mon} = \frac{2}{\pi}\Lambda_{\mathbb{CP}^1}, \tag{74}$$

where $\Lambda_{\mathbb{CP}^1} = \Lambda_{\rm QCD}$. In both the limits we have two physical string states and a confined monopole which can be interpreted as the junction between these strings.

Let us consider what happens for the case of the composite 2-vortex. If $\Delta m \gg \Lambda$ we have Abelian vortices with the same tension, the (2,0), (0,2), and (1,1) vortices. There are two possible kinds of confined monopoles: the one between the (2,0) and the (1,1) vortices and the one between the (2,0) and the (0,2). If we calculate the monopole masses using the central charge, we find that

$$M_{(2,0)\to(1,1)} = \frac{2\pi(m_1 - m_2)}{e_3^2} = \frac{M_{(2,0)\to(0,2)}}{2}. \tag{75}$$

We can think of the $(2,0) \to (0,2)$ kink as the composite state of the $(2,0) \to (1,1)$ and the $(1,1) \to (0,2)$ kinks (see Fig. 7); it is reasonable that there is no net force between the two elementary kinks because the energy of the bound state is equal to the sum of masses of two elementary kinks.

When we go to the limit $\Delta m \ll \Lambda$ the situation becomes rather complicated. Even if we neglect, for simplicity, the coordinate corresponding to the relative distance of the elementary vortices, the physics is described by a sigma model with target space $\mathcal{T} = \mathbb{CP}^2/\mathbb{Z}_2$ (a space that is not even a manifold due to a conical singularity) and with a quite complicated metric. However, in analogy to the 1-vortex case it is reasonable to think that the spectrum of BPS states in the two-dimensional world-sheet model coincides with the monopole/dyon spectrum of the four-dimensional bulk theory on the Coulomb branch because it cannot depend on the FI parameter ξ [10,11]. The latter spectrum is given by the exact Seiberg-Witten solution [22,23].

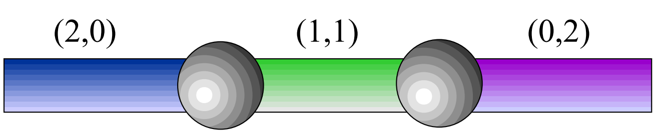

FIG. 7 (color online). Elementary BPS confined monopoles connecting the (2,0) to the (1,1) vortex and the (1,1) to the (0,2) vortex. The mass of a BPS monopole connecting the (2,0) to the (0,2) vortex is exactly the double of an elementary monopole. We can conjecture that the length of the intermediate layer of the (1,1) vortex is a modulus of the composite soliton.

VIII. CONCLUSIONS

In this paper we considered a composite non-Abelian vortex with winding number 2 in $\mathcal{N} = 2$ supersymmetric theory with gauge group $U(2)$.

The explicit BPS solution of first-order equations has been found in the case when two component elementary vortices are parallel and coincident in the space.

The internal moduli space $\mathcal{T}$ has the topology $\mathbb{CP}^2/\mathbb{Z}_2$; there is a conical singularity near the (1,1) vortex. The computation of the metric for the effective sigma model on $\mathcal{T}$ still remains an open question. However, perturbing the system with a Δm, it is possible to guess the number of vacua and the spectrum of kinks in the $1 + 1$-dimensional effective description.

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Note added.—After this work was finished, we became aware of a paper by Eto *et al.* [29]. Eto *et al.* extended the analysis of Ref. [19] and thus completed a construction allowing one to introduce the full number $2Nk$ of (real) collective coordinates in the generic k-string BPS solution. The problem of introduction and actual calculation of the profile functions was not addressed. Our result is complementary albeit not generic. One can show that our ansatz, being cast in the form suggested in [29], reduces to

$$H_0(z) = \begin{pmatrix} -\cos\frac{\alpha}{2} z^2 & \sin\frac{\alpha}{2} z \\ -\sin\frac{\alpha}{2} z & -\cos\frac{\alpha}{2} \end{pmatrix} \tag{76}$$

in the gauge discussed in Sec. V D, modulo global $SU(2)$ rotations (which introduce three other collective coordinates). The determinant of the matrix above is z^2, with a degenerate zero at the origin, which is a signal, in the language of Ref. [29], of the coincidence of the positions of two constituents of the 2-string under consideration. It seems very plausible that applying the general method of [29] one can extend our ansatz to include two missing collective coordinates responsible for the relative separation of two constituents of the 2-string in the perpendicular plane.

While this paper was in the preprint form, the issue of the $\mathbb{Z}_2$ quotient in the moduli space of two coincident vortices was studied also by K. Konishi and collaborators, whose results agree with the one of this paper.

APPENDIX A: ZERO MODES FOR THE (1,1) VORTEX

Let us consider a small perturbation around the (1,1) vortex at $\alpha = \pi$; let us write $\alpha = \pi + \tilde{\alpha}$. All profile functions are nontrivial functions of $\tilde{\alpha}$. We will calculate the corrections to the profile functions at the first nontrivial order in $\tilde{\alpha}$. Then we substitute our solution into the action [see Eq. (9)] and check that the linear and the quadratic corrections in $\tilde{\alpha}$ to the tension are zero. To this order, it is consistent to consider κ, z_1, z_2 as constants in α. We calculate profile functions h and g to $\mathcal{O}(1)$ order in $\tilde{\alpha}$; however, f should be calculated with higher accuracy, namely, to the order $\mathcal{O}(\tilde{\alpha}^2)$. Notice that at this order it is consistent to take $z_1 = z_2 = z$ and as a consequence $f_3 = \mathcal{O}(\tilde{\alpha}^2)$ [this follows from the BPS equations

$$\frac{f_3'}{r} - \tilde{\alpha}^2 \frac{h(1-g)}{r^2} = \frac{e_3^2}{2}(\kappa^2(z_1^2 - z_2^2)), \qquad \frac{f_3}{r} = \frac{z_1'}{z_1} - \frac{z_2'}{z_2}(1 + \mathcal{O}(\tilde{\alpha}^2)) \tag{A1}$$

combined with the boundary condition $f_3(r \to 0) = \mathcal{O}(\tilde{\alpha}^2)$], so we will not need to compute f_3 because it gives a contribution of order $\tilde{\alpha}^4$ to the action.

Other BPS equations are

$$\begin{aligned} \frac{f'}{r} &= \frac{e_0^2}{2}\Big(2\kappa^2 z^2 + \frac{\tilde{\alpha}^2}{4}\kappa^2(z^2-1)^2 - 2\Big), \\ \frac{g'}{r} - (1+f_3)(1-g)\frac{z^2-1}{z^2+1}\frac{1}{r^2} &= \frac{e_3^2}{2}\kappa^2 z(z^2-1), \\ \frac{f}{r} &= 2\frac{\kappa'}{\kappa} + 2\frac{z'}{z}, \\ \frac{1-g}{r} &= \frac{1}{2}\frac{z^2+1}{2z}\Big(\frac{1}{r} - \frac{z'}{z}\Big). \end{aligned} \tag{A2}$$

In what follows we put $\tilde{\alpha} = 0$ in the first equation in (A2). The following change of variables is used:

$$\kappa = \frac{w}{\phi^2}, \qquad z = \phi^2, \qquad \phi = \sqrt{z_1} = \sqrt{z_2}, \qquad w = \kappa z. \tag{A3}$$

In these variables our problem reduces to

$$f = 2\frac{w'}{w} r, \qquad f' = e_0^2 r(w^2 - 1). \tag{A4}$$

These are equations for the (1,1) Abelian vortex; see Eq. (12) for $p = k = 1$. In addition, we have new profile functions which satisfy the equations

$$\begin{aligned} \frac{1-g}{r} &= \frac{1}{2}\frac{\phi^4+1}{\phi^2}\Big(\frac{1}{r} - 2\frac{\phi'}{\phi}\Big), \\ \frac{g'}{r} - (1-g)\frac{\phi^4-1}{\phi^4+1}\frac{1}{r^2} &= \frac{e_3^2}{2}(\phi^4-1)\frac{w^2}{\phi^2}. \end{aligned} \tag{A5}$$

Let us rewrite them in a form convenient for numerical calculations,

$$\begin{aligned} \phi' &= \frac{\phi}{2r}\Big(1 - (1-g)\frac{2\phi^2}{1+\phi^4}\Big), \\ g' &= (\phi^4 - 1)\Big\{\frac{e_3^2}{2}\frac{w^2 r}{\phi^2} + \frac{1-g}{r(1+\phi^4)}\Big\}. \end{aligned} \tag{A6}$$

The squark field can be written as

$$Q = \begin{pmatrix} we^{i\varphi} & -\frac{\tilde{\alpha}}{2}(e^{2i\varphi}\phi^2 w - e^{i\varphi}w) \\ -\frac{\tilde{\alpha}}{2}(e^{i\varphi}w - \frac{w}{\phi^2}) & we^{i\varphi} \end{pmatrix}. \tag{A7}$$

The expression for the gauge field in the nonsingular gauge is completely straightforward; see Eqs. (32) and (35). The profile function h is given by

$$h = \frac{\phi^4 - 1}{\phi^4 + 1}(1 - g). \tag{A8}$$

Now, let us compute the value of the gauge-invariant operator $\tilde{\mathcal{F}}^a$ at first order in $\tilde{\alpha}$,

$$\tilde{\mathcal{F}}^3 = 0, \qquad \tilde{\mathcal{F}}^1 = 2\tilde{\alpha}w^2(\cos\varphi)\Big(\frac{g'}{r} - \frac{h}{r^2}\Big), \qquad \tilde{\mathcal{F}}^2 = -2\tilde{\alpha}w^2(\sin\varphi)\Big(\frac{g'}{r} - \frac{h}{r^2}\Big). \tag{A9}$$

In particular, Eqs. (A7) and (A9) give us the Abelian (1,1) vortex at $\tilde{\alpha} = 0$ [note that the 2-string also has the $U(1)$ gauge field $F_{12}^0 = -f'/r$].

Let us consider the action of a global color + flavor rotation, given by an $SU(2)$ matrix U,

$$\mathcal{F}^a\tau^a \to U\mathcal{F}^a\tau^a U^\dagger, \qquad Q \to UQU^\dagger.$$

The action is trivial only at $\tilde{\alpha} = 0$; otherwise the situation is similar to a rotation of a rigid body in the ordinary three-dimensional space. All $SU(2)$ global generators act nontrivially on the solution (A7) and (A9). At fixed $\tilde{\alpha}$ our solution is parametrized by some kind of Euler angles in color space. The "shape" in color space is similar to a coin with vanishing thickness (at the leading order in $\tilde{\alpha}$) and with diameter of the order of $\tilde{\alpha}$.

APPENDIX B: ZERO MODES FOR THE (2,0) VORTEX

Now we consider a small perturbation around the (2,0) vortex at $\alpha = 0$. Again, acting in the same way as in the case of the (1,1) string, we calculate our profile functions with accuracy which ensures cancellation of the first- and second-order corrections with respect to α in the action; see Eq. (9). As before, at this order it is consistent to treat κ, z_1, z_2 as constants in α. We also calculate g, h at order $\mathcal{O}(1)$ in α. On the other hand, we need to consider $\mathcal{O}(\alpha^2)$ corrections to the functions f, f_3.

The BPS equations are

$$\frac{f'}{r} = \frac{e_0^2}{2}\left(\kappa^2 + \kappa^2 z_1^2 z_2^2 - \frac{\alpha^2}{4}\kappa^2(1-z_1^2)(1-z_2^2) - 2\right), \tag{B1}$$

$$\frac{f_3'}{r} = \frac{e_3^2}{2}(z_1^2 z_2^2 - 1)\kappa^2 - \frac{\alpha^2}{8}\kappa^2(z_1^2+1)(z_2^2-1), \tag{B2}$$

$$\frac{g'}{r} + \frac{1-f_3}{r^2}\frac{z_1^2-1}{z_1^2+1}(1-g) = \frac{e_3^2}{2}\kappa^2 z_1(z_2^2-1), \tag{B3}$$

$$\frac{f}{r} = 2\frac{\kappa'}{\kappa} + \frac{z_1'}{z_1} + \frac{z_2'}{z_2}, \tag{B4}$$

$$\frac{f_3}{r} = \frac{z_1'}{z_1} + \frac{z_2'}{z_2}, \tag{B5}$$

$$\frac{1-g}{r} = \frac{z_1^2+1}{2z_1}\left(\frac{1}{r} - \frac{z_2'}{z_2}\right). \tag{B6}$$

Instead of z_1, z_2, and κ we introduce new profile functions,

$$z_1 = \frac{s}{t\phi^2}, \qquad z_2 = \phi^2, \qquad \kappa = t, \qquad s = \kappa z_1 z_2, \qquad t = \kappa, \qquad \phi = \sqrt{z_2}. \tag{B7}$$

With this change of variables we find the following equations:

$$\frac{f'}{r} = \frac{e_0^2}{2}(s^2+t^2-2), \qquad \frac{f_3'}{r} = \frac{e_3^2}{2}(s^2-t^2), \qquad \frac{f}{r} = \frac{s'}{s} + \frac{t'}{t}, \qquad \frac{f_3}{r} = \frac{s'}{s} - \frac{t'}{t}. \tag{B8}$$

These equations coincide with the first-order equations (12) for the Abelian (2,0) string ($p = 2$, $k = 0$). They can be solved separately.

Equations for the zero mode profile functions have the form

$$\frac{1-g}{r} = \frac{1}{2}\frac{s^2+t^2\phi^4}{st\phi^2}\left(\frac{1}{r} - 2\frac{\phi'}{\phi}\right), \qquad \frac{g'}{r} + \frac{1-f_3}{r^2}\frac{s^2-t^2\phi^4}{s^2+t^2\phi^4}(1-g) = \frac{e_3^2}{2}\left(\phi^2 - \frac{1}{\phi^2}\right)st. \tag{B9}$$

Let us rewrite them in a form convenient for numerical calculations,

$$\phi' = \frac{\phi}{2r}\left(1 - (1-g)\frac{2st\phi^2}{s^2+t^2\phi^4}\right), \qquad g' = \frac{e_3^2}{2}\frac{\phi^4-1}{\phi^2}rst - \frac{(1-g)(1-f_3)}{r}\frac{s^2-t^2\phi^4}{s^2+t^2\phi^4}. \tag{B10}$$

Numerical solutions can be found; see Secs. V E and V F(for numerical studies we take $e_0^2 = 1$ and $e_3^2 = 2$).

Furthermore, the squark field can be written as

$$Q = \begin{pmatrix} se^{2i\varphi} & \frac{\alpha}{2}(e^{2i\varphi}s - e^{i\varphi}\frac{s}{\phi^2}) \\ \frac{\alpha}{2}(e^{i\varphi}t\phi^2 - t) & t \end{pmatrix}. \tag{B11}$$

The profile function h is given by

$$h = \frac{s^2 - t^2\phi^4}{s^2+t^2\phi^4}(1-g). \tag{B12}$$

Calculating the value of the gauge-invariant operator $\tilde{\mathcal{F}}^a$ at first order in α, we obtain

$$\tilde{\mathcal{F}}^3 = \frac{f_3'}{r}(s^2+t^2),$$
$$\tilde{\mathcal{F}}^1 = \alpha(s^2+t^2)\frac{f_3'}{r} + \alpha(\cos\varphi)\left\{2\left(\frac{g'}{r} + \frac{h(1-f_3)}{r^2}\right)st - \frac{f_3'}{r}\left(\frac{s^2}{\phi^2} + t^2\phi^2\right)\right\},$$
$$\tilde{\mathcal{F}}^2 = -\alpha(\sin\varphi)\left\{2\left(\frac{g'}{r} + \frac{h(1-f_3)}{r^2}\right)st - \frac{f_3'}{r}\left(\frac{s^2}{\phi^2} + t^2\phi^2\right)\right\}. \tag{B13}$$

It is possible to globally rotate the solution, so that $\mathcal{F}^{(1,2)}$ have no constant parts, and their average with respect to φ vanishes (a minimal solution). The matrix which realizes this transformation has the form

$$\tilde{U} = \exp\left(i\alpha\frac{\tau_2}{2}\right), \tag{B14}$$

acting on the fields as

$$\mathcal{F}^a\tau^a \to \tilde{U}^\dagger \mathcal{F}^a\tau^a\tilde{U}, \qquad Q \to \tilde{U}^\dagger Q\tilde{U}. \tag{B15}$$

The result of the rotation is

$$\tilde{\mathcal{F}}^3 = \frac{f_3'}{r}(s^2+t^2),$$
$$\tilde{\mathcal{F}}^1 = 2\alpha(\cos\varphi)\left\{2\left(\frac{g'}{r} + \frac{h(1-f_3)}{r^2}\right)st - \frac{f_3'}{r}\left(\frac{s^2}{\phi^2} + t^2\phi^2\right)\right\},$$
$$\tilde{\mathcal{F}}^2 = -2\alpha(\sin\varphi)\left\{2\left(\frac{g'}{r} + \frac{h(1-f_3)}{r^2}\right)st - \frac{f_3'}{r}\left(\frac{s^2}{\phi^2} + t^2\phi^2\right)\right\}, \tag{B16}$$

$$Q = \begin{pmatrix} se^{2i\varphi} & \frac{\alpha}{2}(-e^{i\varphi}\frac{s}{\phi^2} + t) \\ \frac{\alpha}{2}(-e^{2i\varphi}s + e^{i\varphi}t\phi^2) & t \end{pmatrix}. \tag{B17}$$

At $\alpha = 0$ these equations give us a solution for the Abelian (2,0) string; see (11). At $\alpha = 0$ the action of the global $SU(2)$ is similar to the rotation of a stick of zero thickness in the three-dimensional space: the moduli space is $S^2 \approx \mathbb{CP}^1$. At $\alpha \neq 0$ the situation is similar to a rigid body rotation in the ordinary three-dimensional space. All $SU(2)$ global generators act nontrivially on the solution (17). At fixed α our solution is parametrized by the Euler angles in color space. The shape in color space is similar to a cigarette with thickness of order α and length 2.

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