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Noncommutative geometry induced by spin effects

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In this paper we study the nonlocal effects of noncommutative spacetime on simple physical systems. Our main point is the assumption that the noncommutative effects are consequences of a background field which generates a local spin structure. So, we reformulate some simple electrostatic models in the presence of a spin-deformation contribution to the geometry of the motion, and we obtain an interesting correlation amongst the deformed area vector, the 3D noncommutative effects, and the usual spin vector $\tilde{S}$ given in quantum mechanics framework. Remarkably we can observe that a spin-orbit coupling term comes to light on the spatial sector of a potential written in terms of noncommutative coordinates which indicates that bound states are particular cases in this procedure. Concerning confined or bounded particles in this noncommutative domain, we verify that the kinetic energy is modified by a deformation factor. Finally, we discuss perspectives.

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I. INTRODUCTION

Time and again, physics has taken the noncommutativity property of certain mathematical structures to produce suitable models [1–12]. It is well known that the mathematical structures related to quantum mechanics could be interpreted as a deformation of the classical mechanics [13,14]. And this deformation is reflected in a noncommutative algebra where the classical variables are taken as operators applied to entities that live in a special space. Formally the quantum theory involves a series of $\hbar$ whose coefficients are functions of the phase space, and this $\hbar$ can be interpreted as the deformation factor [15].

In recent years, noncommutative geometry has again come up in physics in many different contexts. Connes, Douglas, and Schwarz had introduced noncommutative tori spaces as a possible compactification manifold of the space in their pioneering papers [16–18]. In these cases, noncommutative geometry arises as a possible scenario for the short-distances behavior of physical theories. In quantum field theory, the introduction of a noncommutative structure to the space coordinates, on very small length scales, can introduce a new ultraviolet cutoff, as was formalized by Snyder [19]. It leads to new developments in quantum electrodynamics and Yang-Mills theories in noncommutative geometry and also appears in the framework of the string theory [20–30]. Recently, several tests have been suggested to detect noncommutative effects in physics [31–39]. On the other hand, it is particularly important to note Madore's work [40], which formalized the mathematics of noncommutative geometry and introduced some physical applications that we took and went beyond.

Recently, theoretical studies on physics in noncommutative space has motivated the exploration of some aspects in quantum mechanics [41–45]. In particular, the planar noncommutative space has been extensively analyzed in several contexts at the quantum mechanical level, for instance, the well-known Landau problem [46,47]. In fact, Landau levels came to light in Lagrangian models concerning interacting electrons moving in two dimensions that are subjected to an orthogonal magnetic field. It treats with a 2D noncommutative space that can be defined in terms of a projection to the lowest Landau level, which is well known in the theory of the quantum hall effect (QHE). In two-dimensional cases, one can argue [47] that the usual Landau problem might be understood as noncommutative quantum mechanics where the noncommutative parameter θ can be experimentally estimated by using a magnetic field value in the QHE [48,49], which implies

$$\theta \sim 0.22 \times 10^{-11}\ \text{cm}^2. \tag{1}$$

Hence, this suggests that in a length scale $\sqrt{\theta} \sim 1.48 \times 10^{-8}\ \text{m}^2$ we can observe planar noncommutative effects in quantum mechanical systems. Qualitatively speaking, it has also been proposed [50–52] that noncommutative effects can be observed in the Aharonov-Bohm and Aharonov-Casher experiments. Nevertheless, unfortunately, the planar description of the Landau problem cannot be simply extended for interacting electrons moving in the 3D spatial noncommutative case. In this dimension the Landau problem no longer has a direct meaning and we need an alternative physical interpretation to describe charged particles with electrostatics interaction in a 3D noncommutative space subject to a magnetic background

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field. However, one can suppose that spatial noncommutative effects might appear in quantum mechanical systems in 3D just as a motivation based on the theoretical evidence of studies of the 2D Landau problem.

In this paper we propose to analyze some aspects of the phenomenology of 3D spatial noncommutativity in a few simple physical systems. In particular, we are going to study the form of the potentials in noncommutative coordinates and its consequences. Differently from the 2D Landau problem, the central feature of this issue is to consider that the geometric deformation imposed on the coordinates, using the Bopp shift, can be related to the local spin structure and its consequences on the electromagnetism and on the quantization of the harmonic oscillator. Our starting point is the investigation of the electrostatics theory where it is extended by a noncommutative space, where the *a priori* motivation of this analysis is, because the electrostatic phenomenon is an essential ingredient in the stability of microscopic matter. To this end, we are going to assume that we are working with very small contributions to the lengths in the noncommutative regime. As a consequence, the standard electrostatics used to describe usual phenomena might be modified too. So, we are going to reassess the Coulomb potential in the noncommutative scenario where the self-energy of the electron is developed on a nonlocal space induced by the geometric deformation introduced in the model.

For a further insight into the deformation of the geometry we are going to directly associate the noncommutative resulting vector θ^i (in 3D) with the ordinary spin vector $S^i = \frac{\hbar}{2}\sigma^i$. A similar relation has also been considered in 2D noncommutative models as an exotic central extension of the planar Galilei group [53–55]. As further motivation to the present work, the relation between the area vector θ^i (whose surface encloses $\vec{S}$) and the holographic principle, where the inner hypervolume has its properties imprinted on the border hypersurface [56–59], can be mentioned. Bearing this in mind we are going to associate the noncommutative parameter θ to uncertainty information of the measurements of the spin to a deformation of the geometry. The metric measurements are redefined and we are going to show that the ordinary spin-orbit coupling emerges naturally from an effect of first order in θ of a generic potential in a noncommutative space. The contribution of the second-order θ^2 effect will also be analyzed. Furthermore, we are going to show the changes in the kinetic energy of a particle confined in a noncommutative space, and that such a phenomenon is related directly to the violation of the Lorentz symmetry.

This paper is outlined as follows: in Sec. II, we present the basic concepts of the noncommutative space. In Sec. III, we analyze the electrostatics in a deformed geometry due to noncommuting space coordinates. In Sec. IV, we present the relation between the area vector $\vec{\theta}$ (noncommutative parameter) and spin $\vec{S}$, and its consequences. In Sec. V, we study the nonlinear θ^2 effects in the kinetic energy of simple physical systems due to spin deformation of space, and the changes on the kinetic energy in noncommutative geometry. In Sec. VI, we present basic elements of quantization and suggest an interesting method to obtain the ground state of energy E_g of physical systems via the standard factor of deformation of kinetic energy. In Sec. VII, we present a general conclusion.

II. THE NONCOMMUTING SPACE

In quantum mechanics the phase space can be defined replacing the canonical variables, position x^i, and momentum p^i by their counterparts, the Hermitian operators $\hat{x}^i$ and $\hat{p}^i$ [40]. These operators obey the Heisenberg commutation relation,

$$[\hat{x}^i, \hat{p}_j] = i\hbar\delta^i_j. \tag{2}$$

In such a class of theories, one can easily infer the possible failure of the commutation property on the position operator measurements, and this fact could be reassessed by proposing that the space coordinate operators do obey the following commutation relation,

$$[\hat{x}^i, \hat{x}^j] = i\theta^{ij}, \tag{3}$$

where the parameter θ^{ij} is an antisymmetric and constant tensor with dimension equal to (length)^2. An important aspect of this proposition is that the notion of a "classical point" has no meaning any longer in the noncommutative space, and the space manifold is replaced by a Hilbert space furnished by states which obey an uncertainty relation as

$$\Delta x^i \Delta x^j = \tfrac{1}{2}|\theta^{ij}|, \tag{4}$$

which has the same form as the Heisenberg uncertainty principle. In this way, a spacetime point is replaced by a Planck cell with area dimension.

To build a noncommutative version of a model, we have to replace the ordinary product applied to functions by its noncommutative counterpart which is based on the Groenewald-Moyal [60,61] product, or star product ($\star$), or

$$f(\hat{x})g(\hat{x}) \to f(x) \star g(x) = \exp\left(\frac{i}{2}\theta^{\mu\nu}\partial_\mu\partial_\nu\right)f(x)g(x')\Big|_{x'=x}, \tag{5}$$

where the star product between the functions is written as a particular operation on functions depending on the usual commuting coordinates. In this algebra, the ordinary commutator between space coordinates is replaced by a nontrivial form given in the expression (3). The relation between the noncommutative variable functions and the usual ones is expressed by

$$f(x) \star g(x) = f(x)g(x) + \frac{i}{2}\theta^{ij}\partial_i f(x)\partial_j g(x) + \mathcal{O}(\theta^2), \tag{6}$$

where the representation of the product $f(x) * g(x)$ indicates a deformation of the algebra of functions on a general space $\mathbb{R}^3$, or a noncommutative algebra. Hence, such a deformation must be connected to a noncommutative geometry by means of a Lie algebra on the coordinates $\hat{x}^i$ in $\mathbb{R}^3$, represented by Eq. (3). Remarkably, it is possible to connect the noncommutative algebra (3) to the Heisenberg uncertainty relation (2) considering the parametric noncommutative space coordinate,

$$\hat{x}_i = x_i - \frac{\theta_{ij}p_j}{2\hbar}. \tag{7}$$

We can easily see that this relation satisfies the algebras (2) and (3) and it also exhibits a nonlocality feature of the theory in a particular and simple way. Furthermore, it gives rise to a principle which says that for a large momenta we have a large "nonlocality." This nonlocality can be depicted by observing that a plane wave no longer corresponds to a point particle, as in commutative quantum field theory, but instead to a "dipole." Indeed it refers to a rigid oriented rod whose extension is proportional to its momentum $\Delta x_i = \theta_{ij}p^j/2\hbar$. In this case, we can propose a general postulate in order that such "dipoles" interact amongst themselves by sticking their ends together, similar to the open strings [3].

III. ELECTROSTATICS IN A NONCOMMUTATIVE SPACE

To analyze the electrostatic case, we begin our investigation considering a simple model describing electric charges living on a noncommutative geometry. As a prototype application we will introduce, briefly, only some basic rules of electrostatics in a spatial noncommutative scenario. To this aim, we define the distance $\hat{r}$ in this space which undergoes the influence of deformed geometry. The simplest way to perform this issue is to assume that a modified electric force is a consequence of simply changing from the commutative coordinates to the noncommutative ones and to implement the "new" algebra of coordinates. So, we take the noncommutative electric potential $\hat{V}(\hat{r})$ as an extension of the usual electric potential, which means that it now depends on the noncommutative position $\hat{r}$, and on the usual electric charge q. The noncommutative distance $\hat{r}$ can be defined by means of the usual inner product onto noncommutative coordinates $\hat{x}^i$, in the follow form,

$$\hat{r}^2 = \langle \hat{x}^i, \hat{x}^i \rangle, \tag{8}$$

which measures the "deformed" line length, or distance. Then, using the algebraic relation (7), we find the general expression

$$\hat{r}^2 = r^2 + \rho^2 = r^2 + \frac{\vec{\theta}\cdot\vec{L}}{\hbar} + \frac{|\vec{\theta}\times\vec{P}|^2}{4\hbar^2}, \tag{9}$$

where r is the ordinary distance, and ρ is the radius of deformation of the space which is independent of r in 3D. Then, if $\rho \neq 0$, we have a nonlocal space and the classical geometry arises as a deformed one. If $\rho = 0$, we reach the local space regime. By simplicity, we assume that $\vec{L}$ is a constant in time and position. In 3D, the vector $\vec{\theta}$ can be the dual of the deformation tensor (or noncommutative tensor) θ_{ij} and, moreover, can represent an arbitrary noncommutative vector parameter, which can be written as

$$\vec{\theta} = \theta_i = (\theta_1, \theta_2, \theta_3) = -\varepsilon_{ijk}\theta_{jk}, \tag{10}$$

where the θ_i components can have any real value. The constant vector $\vec{\theta}$ represents an uncertainty parameter of the simultaneous measures of the space coordinates. In this particular choice, the noncommutative algebra takes the form of a Lie algebra; therefore, we have a kind of rotation symmetry in the coordinates, and consequently it should be correlated to the spin structure. Remarkably, we can observe the presence of the linear momentum $\vec{P}$ and angular momentum $\vec{L}$ in the expression (9); thus, it is a consequence of this deformation of the space. In this way, it is possible to perform a mapping from a general and well-defined noncommutative function $\hat{f}(\hat{r})$ to a deformed function $f(r, \rho)$ on the usual geometry,

$$\hat{f}(\hat{r}) \mapsto f(r, \rho). \tag{11}$$

From the deformed distance (9) we emphasize that the deformation radius ρ can be written as

$$\rho^2 = \frac{\vec{\theta}\cdot\vec{L}}{\hbar} + \frac{|\vec{\theta}\times\vec{P}|^2}{4\hbar^2}, \tag{12}$$

where we include all the deformation terms: one refers to the linear momentum $\vec{P}$ and the other refers to the angular momentum $\vec{L}$, and ρ denotes the radius of an elementary volume of the space at small distance measurements. Then the expression of noncommutative distance (9) is, in fact, the most general deformation which satisfies the noncommutative algebra (3). There is no deformation attributed to the θ^3 term that still satisfies the relation (3) and (7). An important point to notice is that noncommutative space models and their corresponding deformations induce a nonlocal space configuration, and this nonlocality property implies a nonconventional charge distribution where the notion of point charge turns out to be unsuitable at a very small length.

In order to build the potential energy[1] term we assume that two charges q and q' are separated by a noncommutative distance $\hat{r}$, so we can write down

[1]We are adopting Gaussian units.

$$\hat{V} = \frac{qq'}{\hat{r}} = \frac{qq'}{\sqrt{r^2 + \rho^2}}. \tag{13}$$

Then, bearing in mind that the noncommutative and nonlocal effects are consequences of the deformation ρ, the noncommutative electric potential $\hat{\Phi}(\hat{r})$ (due to a source charge q) in a noncommutative geometry can be expressed in the form

$$\hat{\Phi}(\hat{r}) = \frac{q}{\hat{r}} = \frac{q}{\sqrt{r^2 + \rho^2}}. \tag{14}$$

The norms of the electric field $\hat{E}(\hat{r})$ and the Lorentz force $\hat{F}(\hat{r})$ are written as

$$\hat{E}(\hat{r}) = |\vec{\nabla}\,\hat{\Phi}(\hat{r})| = \frac{qr}{(r^2 + \rho^2)^{3/2}}, \qquad \hat{F}(\hat{r}) = q'\frac{\partial\hat{\Phi}(\hat{r})}{\partial r} = \frac{qq'r}{(r^2 + \rho^2)^{3/2}}. \tag{15}$$

Figure 1 shows that the electrostatic force between two elementary charges e (in vacuum) decreases for very small length scales (≈ 100 fm) when we assume noncommutative geometry. This behavior obviously involves a nonconventional electric force theory. Moreover, we can note that for distances of atomic order (> 1 pm) the model reaches the well-known conventional Coulomb force, and that this modified electric force goes to zero at the Planck length ($\approx 10^{-33}$ cm). It is easy to verify that at the limit $\rho \to 0$ we also reach the commutative regime and consequently the model shows an ordinary electrostatics regime at this limit.

Assuming that ρ is small, compared to r, we can expand the scalar potential (14), and taking the terms up to the order $\vec{\theta}^2$ we have

$$\hat{\Phi}(\hat{r}) \simeq \frac{q}{r} - \frac{q}{2\hbar}\frac{\vec{\theta}\cdot\vec{L}}{r^3} - \frac{q}{8\hbar^2}\frac{|\vec{\theta}\times\vec{P}|^2}{r^3} + \frac{3q}{8\hbar^2}\frac{(\vec{\theta}\cdot\vec{L})^2}{r^5}, \tag{16}$$

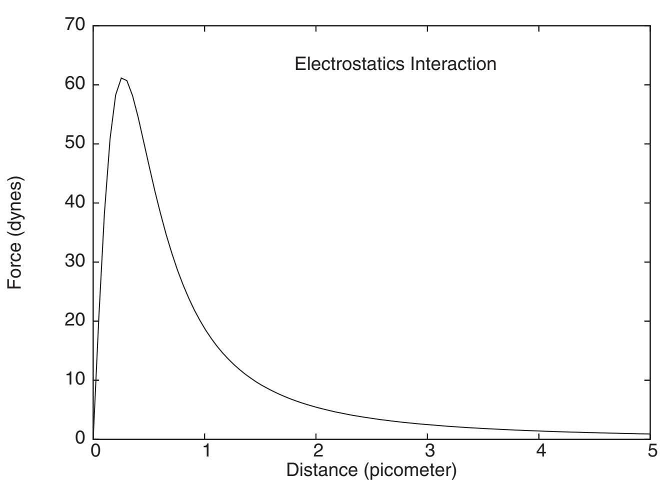

FIG. 1. Plot of electrostatics force $\hat{F}$ versus the distance r.

where the first term q/r stands for the ordinary electric potential $\Phi(r)$ in the stationary regime. The remaining terms in (16) indicate the typical structure of deformations up to the chosen order. In fact, the expression (16) is equivalent to the standard Moyal noncommutative expansion of the function $\hat{\Phi}(\hat{r})$ [60], which is given by

$$\hat{\Phi}(\hat{r}) = \Phi(r) - \frac{1}{2\hbar}\theta_{ij}P^j\partial^i\Phi(r) + \frac{1}{8\hbar^2}\theta_{ij}\theta_{kl}P^jP^l\partial^i\partial^k\Phi(r) + \ldots + \mathcal{O}(\theta^n), \tag{17}$$

where we can easily compare the expressions (16) and (17) and verify that they are equivalent, and from the expression (15) we see that $\vec{\nabla}\times\hat{E}(\hat{r}) = 0$. On the other hand, the notion of point charge is no longer sensible, due to the ''nonlocal'' effects present in this model, as we can see from $\vec{\nabla}\cdot\hat{E}(\hat{r}) \neq 0$, or from the presence of the noncommutative $\vec{\theta}$ parameter. Remarkably, at very small distances the limit $\vec{\nabla}\cdot\hat{E}$ converges Throughout toward a fixed value associated with an elementary area provided by the expression of θ, or

$$\lim_{r\to 0}\vec{\nabla}\cdot\hat{E} = 4\pi\frac{e}{\Omega}, \tag{18}$$

where e is the elementary charge distributed over the deformation volume $\Omega = 4\pi\rho^3/3$. In this limit we can infer that the ordinary space configuration loses its classical form, and may be redefined in such a way that new structural elements beyond the geometrical notion of point charge are incorporated. In our study, such elements could be regarded as kinds of ''microdeformations'' of the space, which are not connected to the usual idea of topology of space, but associated with the possible (quantum) dynamics of the deformations. If we consider the usual topology, these deformations could be correlated to anomalies that appear in discretized space. On the other hand, at large distances or $r^2 \gg \rho^2$ the conventional electrostatics effects prevail over the noncommutative ones.

The scope of the present work does not allow us to enter into mathematical formalism details; instead our aim is to present and discuss some interesting new features of the study of electrostatics in the noncommutative space extension. One of these features is the self-energy of the charged particle, which is taken as an important property induced by the deformation radius ρ. It is well known, in the usual classical electrostatics, that the self-energy of a point charge diverges, while it is easy to infer from the expression (18) that, in a noncommutative space, the electrostatics self-energy $\hat{U}$ becomes ''nonpunctual,'' which we can see from

$$\hat{U} = \frac{1}{2}\int_0^\infty |\hat{E}|^2 d^3x = \frac{3\pi^2 e^2}{8\rho}, \tag{19}$$

where we have used the expression (15); we can see that, at the infinity limit, the energy goes to zero as is usual, but at the zero limit we obtain an exact and finite value, which is a function of the radius of deformation ρ only. Therefore, we can also conjecture that the elementary electric charge e can come up as embedded in a minimal surface which is characterized by an intrinsic curvature radius dictated by the deformation ρ. Hence this nonsingular convergence of the equation $\hat{U}$, which could be an effect of the deformation on the space geometry, results in a nonlocal effect [62,63] when we consider very small lengths. Thus we can conjecture that we have obtained a clue to a possible internal structure of the charged particle depicted by the deformation radius ρ of the space.

IV. NONCOMMUTATIVE SPACE AND SPIN

An important issue to study is the possible origin of the deformation radius ρ. In the case treated here, for a very small length, it is reasonable to assume that the continuous structure of spacetime might be spoiled and, consequently, nonlocalized effects emerge. Recently, the discrete spacetime structure that resulted [64] (or atomiclike structure of spacetime) has been the object of intense study where nonlocal effects are correlated to a discrete spacetime structure via the spin foam hypothesis [65–68], where noncommutative geometry appears naturally. In our case, we are going to simply explore the noncommutative algebra (3), where we introduce further properties associated with the noncommutative antisymmetric tensor θ^{ij}, in order to obtain models that could be phenomenological. Indeed, in 2D it is usual to directly associate the tensor θ^{ij} to the Levi-Civita tensor ϵ^{ij}. In that case it is possible to build various theoretical models with simple symplectic structure. On the other hand, in 3D we can directly connect the tensor θ^{ij} to the 3-index Levi-Civita contracted to the dual vector, or $\epsilon^{ijk}\theta_k$. From this point of view, we also have to impose that the measurements of θ involve nonlocal effects, due to the possible discrete spacetime structure. Moreover, we assume that θ is the area of a connected two-dimensional spatial surface (or event horizon) that contains the spin $\vec{S}$ (actually all information of the intrinsic angular momentum). So we assume that the spin $\vec{S}$ is proportional to the area θ of this surface, which suggests that it is associated with the holographic principle [57,58]. Further we can conjecture that, due to the spin, the related event horizon can modify the classical geometry structure at very small lengths.

We must make some comments. In quantum mechanics the realization of the spin as an observable is based on the application of an external magnetic field. This implies that for the spin to be "realized" it is necessary to embed the system in an environment fulfilled with a magnetic field. From the literature, in fact, it is possible to verify that, in some cases, noncommutative theory can associate the θ parameter to the magnetic field B, or $\theta \propto 1/B$ [3,61,69]. In this sense, we propose to deform the geometry observed by the charged particle, and, consequently, the particle Lorentz symmetry is violated [70–75]. So the curved trajectory traced by the charged particle reflected by the "spin" vector in the noncommutative algebra (3) indicates that it is related to that background field environment, and to a Lorentz symmetry violation on the usual framework. Lorentz symmetry violation occurs due to incomplete information (event-horizon-like θ) of spin $\vec{S}$, and due to the "external" observer (magnetic field), both of which are limited by the uncertainty and nonlocality of spacetime. In any case, the spin deforms the spacetime structure which is proportional to the event horizon area θ [40].

The relation between θ and the spin of the particle can be shown by means of a simple and direct mathematical formalism in 3D which brings to light well-known effects. Bearing in mind the results of previous sections, we intend to fix the arbitrary character of the noncommutative area vector $\vec{\theta}$ assuming a direct connection to the spin, or

$$\vec{\theta} = \frac{\hbar}{m^2c^2}\vec{S}, \tag{20}$$

where $\vec{\theta}$ is the dual vector already given in Eq. (10) and $\vec{S}$ can be assumed to be the vector spin operator or $\vec{S} = (\hbar/2)\vec{\sigma}$, where $\vec{\sigma}$ are the Pauli matrices. We must observe that the expression (20) is a relation in the quantum domain. For very large m (classical mechanics limit) $\theta_i \approx 0$ and, consequently, the classical geometry prevails. Hence, in our proposal, we suggest that the noncommutative deformation arises from the spin structure of a particular model. Then, using the Clifford algebra $\{\sigma_i, \sigma_j\} = 2\delta_{ij}$, it is easy to conclude that $|\vec{\sigma} \times \vec{P}|^2 = 2P^2$, and thus the inner product (9), or the deformed distance, can be rewritten as

$$\hat{r}^2 = r^2 + \frac{\vec{S}\cdot\vec{L}}{m^2c^2} + \frac{\hbar^2P^2}{8m^4c^4}. \tag{21}$$

We can observe that the term $\vec{S}\cdot\vec{L}$ is the spin-orbit coupling contribution to the magnitude of the distance, while the last term can be associated with fluctuations of the kinetic energy. Actually, the deformation θ_{ij} of the space can play the same role of spin influenced by an external magnetic field generated in the electron orbital motion as in atomic physics. It is important that the expression of the deformed distance shown in the expression (21) is the most general extension which satisfies the noncommutative algebra (3). So the algebra can be reassessed in such a way that

$$[\hat{x}_i, \hat{x}_j] = -\frac{i\hbar}{m^2c^2}\epsilon_{ijk}S_k. \tag{22}$$

In this scenario the position operator has a Heisenberg's uncertainty principle, which depends on the background field applied. So, in this sense, the spin vector observation is a consequence of the presence of this field. To get a deeper insight we are going to do some applications.

A. Coulomb potential

The Hamiltonian H in ordinary spacetime with Coulomb potential and a perturbation term for the fine structure which contains the spin-orbit coupling (which is the same used to describe the hydrogen atom) can be written as

$$H = T + V = \frac{P^2}{2m} + V_{\rm em} + V_{\rm so} = \frac{P^2}{2m} - \frac{e^2}{r} + \frac{e^2}{2m^2c^2r^3}\vec{S}\cdot\vec{L}, \quad (23)$$

where m denotes the reduced mass of the system. The above Hamiltonian describes the hydrogen atom in standard quantum mechanics. Now, taking the relations (20) and (21) we are going to insert the noncommutative distance and consequently the new Hamiltonian $\hat{H}$ which brings a noncommutative Coulomb potential as suggested by expression (13). The Hamiltonian without the spin sector yields the following expression,

$$\hat{H}_{\rm em} = T + \hat{V}_{\rm em} = \frac{P^2}{2m} - \frac{e^2}{\sqrt{r^2 + \rho^2}}. \quad (24)$$

Using the expression (17) we are going to expand the noncommutative potential energy $\hat{V}_{\rm so}(\hat{r})$ up to the linear term in $\vec{\theta}$ for simplicity, so we have

$$\hat{V}_{\rm em}(\hat{r}) \approx -\frac{e^2}{r} + \frac{e^2}{2r^3\hbar}\vec{\theta}\cdot\vec{L} = -\frac{e^2}{r} + \frac{e^2}{2m^2c^2r^3}\vec{S}\cdot\vec{L}, \quad (25)$$

where we use the spin correlation ansatz (20). It is easy to see that the classical spin-orbit potential $V_{\rm so}$ coincides with the last term of the noncommutative potential (25) which is the linear term of the Taylor expansion. From a phenomenological point of view, the analysis of the hydrogen atom spectrum in noncommutative QED might also suggest the production of a spin-orbit effect, similar to that introduced by Chaichian, Sheikh-Jabbari, and Tureanu [76]. An interesting analogy between the spin dynamics and vorticelike dynamics derived from the quantization of the Eulerian dynamics of point vortices in an ideal fluid is produced. In this case, it is known that the coordinates of the geometrical center of the vortex do not commute. This similarity correspondence suggests that a relation between vortex and intrinsic spin dynamics could be dictated by the noncommutative spacetime properties.

B. The harmonic oscillator

We can extend the equivalence between noncommutative potential energy and spin-orbit effects to other models. The general spin-orbit coupling on the standard quantum mechanics can be written in the form

$$V_{\rm gso} = \frac{1}{2m^2c^2}\left(\frac{1}{r}\frac{dV(r)}{dr}\right)\vec{S}\cdot\vec{L}, \quad (26)$$

where $V(r)$ is any ordinary potential energy. We consider the potential energy for the simple harmonic oscillator. In $V(r) = kr^2/2$, as a result, that expression (26) becomes

$$V_{\rm hoso} = \frac{k}{2m^2c^2}\vec{S}\cdot\vec{L}, \quad (27)$$

where k is an elastic constant. Remarkably, we can derive the same spin-orbit expression (27) from the noncommutative potential energy, taking the deformed distance and its expansion to the linear term in θ, or

$$\hat{V}_{\rm ho}(\hat{r}) = \frac{k\hat{r}^2}{2} \approx \frac{kr^2}{2} + \frac{k}{2\hbar}\vec{\theta}\cdot\vec{L} = \frac{kr^2}{2} + \frac{k}{2m^2c^2}\vec{S}\cdot\vec{L}, \quad (28)$$

where we assume the spin correlation (20).

C. Logarithmic potential

Another example is the logarithmic potential energy $V(r) = V_0 \ln|\alpha r|$. From the general spin-orbit equation (26) we obtain

$$V_{\rm so} = \frac{V_0}{2m^2cr^2}\vec{S}\cdot\vec{L}. \quad (29)$$

And so we again can derive this term through the expansion of a noncommutative logarithm potential energy which is a function of $\hat{r}$, resulting in

$$\hat{V}(\hat{r}) = V_0 \ln|\alpha\hat{r}| \approx V_0 \ln|\alpha r| + \frac{V_0}{2m^2cr^2}\vec{S}\cdot\vec{L} \quad (30)$$

where we can see, as in the previous case, that the spin-orbit term emerges as a contribution of the noncommutative expansion of the potential energy around the commutative distance.

D. Yukawa potential

We repeat the same procedure with the Yukawa potential energy. Thus taking the potential energy $V(r) = (q^2/r)e^{-\alpha r}$ and using the general form of the spin-orbit coupling equation (26), the potential becomes

$$V_{\rm so} = -\frac{q^2e^{-\alpha r}}{2m^2c^2r^3}(1 + \alpha r)\vec{S}\cdot\vec{L}, \quad (31)$$

which we extend to the noncommutative scenario, and so we can explicitly show the expansion of the noncommutative Yukawa potential energy, which yields

$$\hat{V}(\hat{r}) \approx V(r) - \frac{q^2e^{-\alpha r}}{2m^2c^2r^3}(1 + \alpha r)\vec{S}\cdot\vec{L}. \quad (32)$$

We can conclude that for any well-defined spatial noncommutative potential energy $\hat{V}(\hat{r})$ its Taylor expansion from the commutative distance around its noncommutative background field contribution results in the ordinary potential energy $V(r)$, and an additional linear contribution of the spin-orbit coupling term. In fact, this is only valid if we assume that the spin is correlated with the noncommutative tensor θ^{ij} shown in the expression (20).

E. General case

We can infer from the potential energy, in a general case Taylor expanded up to the linear contribution on θ, that the correlation between spin-orbit effects and a noncommutative potential energy $\hat{V}(\hat{r})$ can be directly obtained taking the expression

$$\hat{V}(\hat{r}) = \hat{V}\left(\left[r^2 + \frac{\vec{\theta}\cdot\vec{L}}{\hbar}\right]^{1/2}\right) =\simeq V(r) + \frac{\vec{S}\cdot\vec{L}}{2m^2c^2}\left(\frac{1}{r}\frac{dV}{dr}\right) \tag{33}$$

where we assume the spin correlation (20). Remarkably, it shows that any noncommutative potential energy can be written (to a good approximation) as the ordinary potential energy $V(r)$ and a general spin-orbit coupling term as in Eq. (26). This suggests that noncommutative deformations could be seen as effective quantum deformations in space constrained by the dynamics of the spin structure.

V. NONCOMMUTATIVE EFFECTS ON THE KINETIC ENERGY

We have concentrated our discussion on the spin-orbit effects derived from noncommutative geometry. From now on, we are going to analyze the effective contribution of the extra kinetic term $\hbar^2P^2/8m^4c^4$, presented in expression (21), to the Coulomb potential-like using the Hamiltonian method. It is well known that, for large linear momenta P, this term is the main one, particularly in high energy physics. However, using the expression (21) we can obtain the expansion of the noncommutative Hamiltonian (24), which results in

$$\hat{H} = \frac{P^2}{2m} - \frac{e^2}{r} + \frac{e^2}{2m^2c^2r^3}\vec{S}\cdot\vec{L} + \frac{e^2\hbar^2P^2}{16m^4c^4r^3}. \tag{34}$$

As we have already seen, the third term refers to the spin-orbit coupling one; further, the last term, which also decays with r^3, represents a kinetic term contribution from the noncommutative deformation. We can verify, from the expression (34), that the extra kinetic term can be incorporated with the first term in such a way that the deformed Hamiltonian can be written as

$$\hat{H} = \frac{\alpha(r)P^2}{2m} + \hat{V}_{\mathrm{em}}(\hat{r}), \tag{35}$$

where the coefficient $\alpha(r)$ is defined as

$$\alpha(r) = 1 + \frac{e^2\hbar^2}{8m^3c^4r^3}, \tag{36}$$

which can be interpreted as a deformation factor to the kinetic energy. According to the Hamiltonian (34), the spin-orbit coupling appears as a noncommutative effect in the potential energy while the factor $\alpha(r)$ denotes the noncommutative influence on the kinetic energy part. We notice that the maximal contribution of the $\alpha(r)$ occurs when the system assumes the ground state energy. This means that the distance r is the Bohr radius $a_0 = \hbar^2/me^2$. In this scenario the factor $\alpha(a_0)$ is maximal and comes up to be a function which is dependent on the ground state of energy E_g, as well as on the particle rest energy E_r of the system. Hence the expression (36) assumes the following form,

$$\alpha(a_0) = 1 + \frac{1}{2}\left(\frac{E_g}{E_r}\right)^2. \tag{37}$$

Thus, the ground state energy of the hydrogen atom is represented by $e^2/2a_0$, while the classical rest energy is mc^2. So, for a state with a fixed energy parameter α, the kinetic energy of the Hamiltonian becomes dependent on the rate E_g/E_r of the system. We can also note that if $E_g/E_r \ll 1$, the factor (37) results in $\alpha \approx 1$, and we turn back to classical theory. If E_g is larger than E_r, we have $\alpha > 1$ which results in an interesting noncommutative effect on the kinetic energy of system. For instance, for the hydrogen atom we estimate that $E_g/E_r \approx 7.0 \times 10^{-10}$ which implies $\alpha \approx 1$. This result agrees with that of the classical Hamiltonian of the hydrogen atom where (at low energy) the kinetic term is unmodified. Therefore, in high energy systems, the rate E_g/E_r could induce an increment in the kinetic energy which can be related to a violation of Lorentz symmetry. In fact, noncommutative theories automatically manifest Lorentz symmetry violation as has been shown in many works [3,61,70–72]. It is already known that a typical spin-orbit effect violates symmetries of the classical Hamiltonian in standard quantum mechanics. However, as we have seen, both the factor α and the spin-orbit coupling could be a result of noncommutative effects of potential $\hat{V}(\hat{r})$, thus this could be the link to the manifest Lorentz symmetry violation.

Bearing this in mind, we are going to analyze the behavior of the harmonic oscillator Hamiltonian (for confined fermions), where we assume the complete definition (21). Starting from the harmonic oscillator Hamiltonian extended by the noncommutative terms, in such a way that we simply change the ordinary distance to the deformed one, so we can represent it as

$$H_0 = \frac{P^2}{2m} + \frac{m\omega^2 r}{2} \longrightarrow \hat{H} = \frac{P^2}{2m} + \frac{m\omega^2\hat{r}}{2}, \tag{38}$$

and taking the Taylor expansion in $\hat{r}$, we find that

$$\hat{H} = \frac{P^2}{2m} + \frac{m\omega^2 r}{2} + \frac{\omega^2\vec{S}\cdot\vec{L}}{2mc^2} + \frac{\omega^2\hbar^2P^2}{16m^3c^4}. \tag{39}$$

We then verify that from the Taylor expansion the classical harmonic oscillator Hamiltonian H_0 and the spin-orbit coupling V_{hoso} [see Eq. (27)] naturally emerge. The last term comes to light as an extra kinetic contribution term to the model. So, the deformed harmonic oscillator Hamiltonian can be written as

$$\hat{H} = \frac{\alpha(\omega)P^2}{2m} + \frac{m\omega^2\hat{r}}{2} + V_{\mathrm{hoso}}, \tag{40}$$

where the deformation factor appears explicitly dependent on the angular frequency ω of the system and V_{hoso} is the potential term of the harmonic oscillator that couples to the spin-orbit term. We notice that the kinetic energy is deformed, and the factor $\alpha(\omega)$ reads

$$\alpha(\omega) = 1 + \frac{1}{18}\left(\frac{E_g}{E_r}\right)^2. \tag{41}$$

We emphasize that the above expression is very similar to the expression (37). And we recall that, for the usual case of a 3D harmonic oscillator, it has a ground state energy which is dependent on the angular frequency, namely $E_g(\omega) = \frac{3\hbar\omega}{2}$, which, to low values of angular frequency, means that we have $E_g/E_r \ll 1$, and so $\alpha(\omega) \approx 1$ which implies that we have reached the usual classical harmonic oscillator. However, when ω is large, the rate E_g/E_r rapidly increases, which causes a deformation in the kinetic energy of the system. Such effect could be observed in high energy physics, although there is a possibility that these effects could be observed in nuclear or particle physics. In these cases the deformation factor (41) increases quadratically in ω, or

$$\alpha(\omega) = 1 + \beta\omega^2, \tag{42}$$

where the constant β depends on the mass of the system. In the confined electron case in the noncommutative domain, the constant β is $2.0 \times 10^{-43}\ \mathrm{s}^{-2}$. So the noncommutative effects increase quadratically with frequency of the system, as shown in Table I, where we can see that up to frequencies $\leq 10^{18}$ Hz (limit of ultraviolet frequency) the kinetic energy is unchanged. In this case, Lorentz violation cannot be observed. However, a small fluctuation in the kinetic energy of the particle can be noted at 5.0×10^{20} Hz implying that such effects can be subtly noted, both in nuclear physics and in high energy physics. For higher frequencies, the factor $\alpha(\omega)$ rapidly increases up until the system reaches an unstable state. In a further case of a proton confined in a noncommutative domain, we find that $\beta = 5.7 \times 10^{-50}\ \mathrm{s}^{-2}$ and so the above table is changed. In fact, stable effects on noncommutative space would be expected for confined objects with small mass (small β parameter) and high energy.

TABLE I. Deformation factor values for some frequencies.

$\alpha(\omega)$	ω(Hz)	
1.00	$\leq 1.0 \times 10^{18}$	$\leq$ ultraviolet frequencies
1.05	5.0×10^{20}	X-ray frequency
1.20	1.0×10^{21}	γ-ray frequency
6.00	5.0×10^{21}	γ-ray frequency
20.00	1.0×10^{22}	Highest frequencies

VI. QUANTIZATION

Now we are able to construct the Fock space starting from the Hamiltonian (40) in the absence of the spin-orbit term. So, we can represent a particular Fock space which is modified by noncommutative deformations. We reassess the creation and destruction operators as

$$\begin{aligned} a_i &= \sqrt{\frac{m\omega}{2\hbar}}r_i + i\sqrt{\frac{\alpha(\omega)}{2m\hbar\omega}}P_i, \\ a_i^\dagger &= \sqrt{\frac{m\omega}{2\hbar}}r_i - i\sqrt{\frac{\alpha(\omega)}{2m\hbar\omega}}P_i, \end{aligned} \tag{43}$$

where $r_i = (x, y, z)$ is the ordinary position vector. The particle number operator $\hat{N}$ in noncommutative geometry is then given by

$$\hat{N} = a_i^\dagger a_i = \frac{1}{\hbar\omega}\left(\frac{\alpha(\omega)P^2}{2m} + \frac{m\omega^2 r}{2}\right) - \frac{3}{2}\sqrt{\alpha(\omega)}, \tag{44}$$

and so the Hamiltonian (40) can be rewritten as

$$\hat{H} = \left(\hat{N} + \frac{3}{2}\sqrt{\alpha(\omega)}\right)\hbar\omega. \tag{45}$$

Hence the ground state $\hat{E}_g$ of the noncommutative Hamiltonian $\hat{H}$ is correlated to the ground state of energy of the ordinary harmonic oscillator E_g through the equation

$$\hat{E}_g = \frac{3}{2}\left[1 + \frac{1}{18}\left(\frac{E_g}{E_r}\right)\right]^{1/2}\hbar\omega. \tag{46}$$

This indicates that the system undergoes a deformation of the energy. Notice that within the limit $E_g/E_r \ll 1$ we will obtain $\hat{E}_g \approx E_g$. Indeed we are dealing with an expansion, so the above $\hat{E}_g$ is a truncation of the Taylor expansion around the commutative position, which implies that we could have other contributions to the noncommutative energy due to higher order terms. In this particular case, if we assume that the deformation of geometry is due to the intrinsic spin perturbation it is possible to recover the ground state of energy in noncommutative geometry iteratively through the ordinary ground state of energy E_g by using Eq. (46).

In order to further analyze the deformation factor, α, we can observe that Eqs. (37) and (41) show the same standard functional form, which is explicitly dependent on the rate E_g/E_r. So, taking the maximal deformation expansion reached in the general noncommutative Hamiltonian $\hat{H}$, we can conjecture that the maximal value of α has the form

$$\alpha(E_g) = 1 + \kappa\left(\frac{E_g}{E_r}\right)^2, \tag{47}$$

where κ is a constant coefficient to be determined. The fraction E_g/E_r is spread out over the whole model, which we assume to be a standard factor which depends, especially, on the ground state of energy of the system. An interesting point to mention is that the general factor, given in the expression (47), can determine the analytical form for the ground state of energy in any system. In this sense, we just require for the noncommutative potential energy $\hat{V}(\hat{r})$ to be well defined.

Finally, we are going to deal with the special case of the heavy quarks system, made up basically of two interacting quarks $Q\bar{Q}$, forming a quarkonium system. A nonrelativistic treatment of the bounded quark state by means of the Schrödinger equation is approximated. Hence we can derive the noncommutative Hamiltonian for the quarkonium system, taking the deformation factor in its generic form (46), as

$$\hat{H} = \frac{P^2}{2m} + \hat{V}(\hat{r}), \tag{48}$$

where $\hat{V}(\hat{r})$ is the general quark interaction potential which is given by the simple power law function, which in the noncommutative version has the form $\hat{V}(\hat{r}) = \varphi + \lambda \hat{r}^\nu$, where φ, λ, and ν are arbitrary parameters. We can apply the Taylor expansion by assuming Eqs. (9) and (20), so we have

$$\begin{aligned}\hat{H} &= \frac{P^2}{2m} + V(r) + V_{\rm so} + K_{\rm kin} \\ &= \frac{P^2}{2m} + \varphi + \lambda r^\nu + \frac{\nu\lambda r^\nu \vec{S}\cdot\vec{L}}{2m^2c^2r^2} + \frac{\nu\lambda r^\nu \hbar^2 P^2}{16 m^4 c^4 r^2},\end{aligned} \tag{49}$$

where $K_{\rm kin}$ is an extra contribution term to the kinetic sector in the Hamiltonian. The above Hamiltonian $\hat{H}$ corresponds to the general deformed model written as

$$\hat{H} = \frac{\alpha(r)P^2}{2m} + \varphi + \lambda r^\nu + V_{\rm so}, \tag{50}$$

where the deformation factor $\alpha(r)$ is given by

$$\alpha(r) = 1 + \frac{\nu\lambda\hbar^2 r^\nu}{8m^3c^4r^2}. \tag{51}$$

Again the maximum value of α occurs at a particular radius $r = a_0$, which denotes the fundamental radius for the ground state of energy of the quarkonium binding system. We can easily compute the ground state of energy of the quarkonium system by inserting the general factor of deformation (47) in the expression (51), which is given by

$$E_g = \eta\hbar\sqrt{\frac{\lambda a_0^{\nu-2}}{m}}. \tag{52}$$

Here $\eta = \sqrt{\nu/8\kappa}$ is a constant. Indeed we are able to estimate the fundamental radius a_0 for the quarkonium system as $a_0 \sim (m\lambda)^{-1/(2+\nu)}$ and by substituting in (52) we find that

$$E_g \simeq \eta\hbar\frac{(m\lambda)^{2/(2+\nu)}}{m}, \tag{53}$$

where we can observe that it is very similar to the analytical form of the ground state of the quarkonium system, which is obtained from the noncommutative assumption on the spacetime coordinates. It is interesting to see that the energy in (53) depends basically on the reduced mass m of the system, as well as on the coupling coefficient stressed by λ. This represents the energy of the ordinary ground state of the quarkonium system. Equation (53) matches the well-known classical relation of the difference of two levels of energy ΔE in quark systems [77] given by

$$\Delta E \simeq \frac{1}{m}(m\lambda)^{2/(2+\nu)}, \tag{54}$$

which also depends on the reduced mass m, and on the coupling strength λ of the system. Therefore, strictly speaking, we can see that the parameter ν above assumes some special values. For instance, to Coulomb-like potential we have $\nu = -1$, to simple harmonic oscillator potential we have $\nu = 2$, to linear potential we have $\nu = 1$, to logarithm potential we have $\upsilon = 0$, and to quark interaction potential we have $\nu = 0.1$. So, in the special case of a quark model, the equation obtained in (53) matches the quarkonium mass spectrum [77].

VII. CONCLUSION

In this work we deal with the spin structure of a charged particle as a possible deformation of the noncommutative geometry [40]. We have reassessed the electrostatics interaction theory where we have embedded it in a deformed geometry and have considered that quantum fluctuations of the spacetime involve noncommutative effects at very small lengths. In this sense, we have verified that the conventional Coulomb force is modified at the length scale of $r \leq 100$ fm, and that the effective electric force fades away with the decreasing of the distance r. Furthermore, we may infer that a screening effect of the spacetime involves the modified electrostatics force. We have reasons to believe that, on this scale, the source of the deformation of electrostatic force can be associated with quantum basics of the discretization of spacetime. On the other hand, for length scales where $r > 100$ fm, we have seen that the electrostatic force converges to the conventional Coulomb force. In this case, we understand that the noncommutative parameter θ may represent a macroscopic quantity of discrete spacetime, which is associated with the open area spanned by the vectors (or closed surface), while it becomes a nonlocal one. Hence, we can interpret physical particles as nonlocal objects immersed in this deformed geometry, and we have seen that the self-energy of the electron becomes finite when we assume the limit $r \to 0$ in this geometry. However, we note that the self-energy turns out to be finite but dependent on the intrinsic deformation of the space, ρ, which has a quantum essence. We also verify that, if $\rho \to 0$ (classical limit of spacetime), the self-energy becomes divergent again. In this sense, it is reasonable to claim that spacetime could show an internal quantum structure beyond the conventional geometry.

In fact, we would like to point out that spatial noncommutative effects have been extensively explored in the context of the quantum mechanics systems in the last years. The more commented model that connects noncommutative space and quantum mechanics is the well-known 2D Landau problem. In this dimension case, one can observe a theoretical evidence of realistic planar noncommutative effects of space coordinates for interacting electrons under a strong magnetic field. On the other hand, the theoretical evidence of 3D noncommutative effects in other quantum mechanical systems becomes subtle and presents new features that cannot be discussed as a 3D natural extension from the 2D Landau problem. Then, a striking aspect of this work was to suggest a way to derive an electrostatic model in 3D noncommutative space in the absence of an external strong magnetic field (as was presented in Sec. III), where we can also consider charged interacting particles. In the case treated in the present work, we are using the basic algebraic noncommutative structure to obtain 3D deformed expressions from the conventional electrostatics theory.

A direct and simple way to induce a deformation on the spacetime is to consider the noncommutative parameter θ to be proportional to the intrinsic spin structure [40,53–55]. Then, we can conjecture that the area (or θ) is proportional to the modulus of the spin vector. So, we may claim that the spin structure deforms the space at the quantum level resulting in several effects in conventional quantum mechanics. Taking this point as our strategy, we have obtained local spin-orbit couplings as the linear contribution to the potential energy. In the classical scenario, we can verify that a usual potential can be generally written down in terms of noncommutative coordinates, and consequently the potential-like $\hat{V}(\hat{r})$ includes a spin-orbit coupling effect, which is, in fact, an angular momenta effect due to the deformation of the trajectory.

In this work, we have obtained behaviors that agree with the conventional spin-orbit terms of the quantum mechanics, at least up to a linear approximation. So we may say that spin-orbit effects are seen as noncommutative effects of spacetime due to the spin structure that could originate from an external magnetic field. We also have to mention that similar spin effects have been studied in a noncommutative Chern-Simons matter theory with a Pauli magnetic coupling [78].

Furthermore, we have seen that the second-order contribution of the potential energy expansion on θ is an extra kinetic term which is included in the Hamiltonian of the system. We have verified that this term is relevant to the dynamical structure of confined particles which are submitted to a general potential energy $V(r)$, and to the kinetic term that is deformed by a factor α, which is dependent on the ratio between the ground state of energy of the confined system and the rest energy of the particle. Dealing with a simple model, in which we have inserted this deformation factor α, we obtain, with a very good estimate, the ground state expression of the quarkonium system. We have also noticed that when $\alpha \approx 1$ (no kinetic deformation) we easily obtain a hydrogen Hamiltonian system, thus emphasizing the quantum mechanics connection in the procedure. We conclude that when we assume the noncommutative potential energy in a spin structure we can easily obtain information about the energy ground state of many complex physical systems. Moreover we can infer that for confined systems at high energy, due to the standard deformation factor, α is different from the unity (for instance, to quarkonium systems); noncommutative effects can be observed with a possible Lorentz symmetry breaking correlation.

We may conclude that, in order to achieve the characteristic spin or, in fact, the angular momentum of a model, we have to assume that the moving charged particle is embedded in a region fulfilled by a magnetic background field and so is simulating a local Zeeman effect on the states of the charged particle. Thus the role of the noncommutativity property could be fulfilled by the application of this background magnetic field which can be detected by the observation of the spin structure content on the dynamics of the model. Thus, we can easily infer that the atomic bound state could be a special case where an electron is embedded in the nuclear electric background field. A deeper quantum analysis will be the object of study in a forthcoming work.

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