Post-Minkowski action for point particles and a helically symmetric binary solution

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(Received 6 October 2005; published 31 May 2006)

Two Fokker actions and corresponding equations of motion are obtained for two point particles in a post-Minkowski framework, in which the field of each particle is given by the half-retarded + half-advanced solution to the linearized Einstein equations. The first action is parametrization invariant, the second a generalization of the affinely parametrized quadratic action for a relativistic particle. Expressions for a conserved 4-momentum and angular momentum tensor are obtained in terms of the particles' trajectories in this post-Minkowski approximation. A formal solution to the equations of motion is found for a binary system with circular orbits. For a bound system of this kind, the post-Minkowski solution is a toy model that omits nonlinear terms of relevant post-Newtonian order; and we also obtain a Fokker action that is accurate to first post-Newtonian order, by adding to the post-Minkowski action a term cubic in the particle masses. Curiously, the conserved energy and angular momentum associated with the Fokker action are each finite and well-defined for this bound 2-particle system despite the fact that the total energy and angular momentum of the radiation field diverge. Corresponding solutions and conserved quantities are found for two scalar charges (for electromagnetic charges we exhibit the solution found by Schild). For a broad class of parametrization-invariant Fokker actions and for the affinely parametrized action, binary systems with circular orbits satisfy the relation $dE = \Omega dL$ (a form of the first law of thermodynamics), relating the energy, angular velocity and angular momentum of nearby equilibrium configurations.

DOI: 10.1103/PhysRevD.73.104039 PACS numbers: 04.25.−g, 04.25.Nx, 04.30.Db, 97.60.−s

I. INTRODUCTION

In constructing numerical initial data for compact binary systems, it may be useful to find exact solutions to the Einstein equations with helical symmetry, solutions that are stationary in a corotating frame [1–10]. In the case of two charged particles, an exact solution of this kind was found by Schild [11]: The force on each particle is associated with the half-retarded plus half-advanced field of the other. (An earlier but less explicit version is given by Schönberg [12].) We obtain here an analogous solution for point masses in a post-Minkowski framework.

A time-symmetric interaction allows one to derive the equations of motion for point-particles from a single action integral that is written solely in terms of the dynamical variables of each particle (without mediating field variables). This kind of *action at a distance theory* has been formulated by Fokker and by Wheeler and Feynman; for their treatments of the electromagnetic field, see [13,14]. We exhibit an analogous Fokker action for linearized gravity. The equations of motion are those of a post-Minkowski approximation, in which, as in the electromagnetic case, each particle moves in the half-advanced plus half-retarded field of the other.

A Fokker action is not a true action: Its variation leads to equations in which the endpoints of the action integral explicitly appear in integrals that should yield the field of each particle. Only by taking a limit of the varied action as the endpoints go to infinity does one recover the correct equations of motion. Invariance of the Fokker action under Poincaré transformations nevertheless leads to expressions for a conserved energy and angular momentum of a n-particle system. For helically symmetric binary systems (in, e.g., models with scalar, electromagnetic, or gravitational interactions), the energy and angular momentum of the field is infinite, because the particles have radiated for infinite time in past and future. The conserved energy-momentum and angular momentum associated with the Fokker action, however, are finite. For circular orbits of particles of mass m and $\bar{m}$, we find the surprising result that the form of this energy,

$$E = \frac{m}{\gamma} + \frac{\bar{m}}{\bar{\gamma}}, \tag{1}$$

is identical for scalar, electromagnetic, and linearized gravitational interactions, described by a parametrization-invariant Fokker action. The angular momentum is in each case proportional to the value of the interaction field dotted on each free index into the helical Killing vector. We show that nearby circular orbits satisfy the familiar first law, $dE = \Omega dL$.

Although one can construct formal solutions to the post-Minkowski equations in which the source is gravitationally bound, care is needed in their interpretation. A conceptual problem is related to the fact that, at zeroth order in the perturbed metric, particles move on geodesics of flat space, not in bound orbits. Because the matter density vanishes at zeroth order, the stress-energy tensor $T^{\alpha\beta} = \rho u^\alpha u^\beta$ has at first order the form $\delta T^{\alpha\beta} = (\delta\rho) u^\alpha u^\beta$, where u^α is the zeroth-order velocity field describing straight line motion. If one then finds the metric from the perturbed equation

$$0 = -2\delta[G^{\alpha\beta} - 8\pi T^{\alpha\beta}] = \Box(h^{\alpha\beta} - \tfrac{1}{2}\eta^{\alpha\beta}h) + 16\pi(\delta\rho)u^\alpha u^\beta, \quad (2)$$

the perturbed metric will not yield perturbed bound trajectories.

A resolution to this problem is to look at families of solutions, $g_{\alpha\beta}(s)$, $T^{\alpha\beta}(s)$ that describe finite fluid masses in bound orbits for all nonzero values of s, despite the fact that $s = 0$ is flat space. An example can be easily understood by first looking at Newtonian gravity, with a family of solutions corresponding to the scaling $v \sim \epsilon$, $m/r \sim \epsilon^2$ of the post-Newtonian approximation: That is, consider a family of solutions for two point masses $m = m^{(1)}s$ and $\bar{m} = \bar{m}^{(1)}s$ in circular orbits of radii a and $\bar{a}$ (independent of s), with speeds given approximately by $v^{(1)}\sqrt{s}$ and $\bar{v}^{(1)}\sqrt{s}$. At $s = 0$, the solution is flat space with no matter. For each nonzero s the solution describes masses in circular orbit, and as $s \to 0$, the period of the orbit increases without bound. An exact solution is given for each mass by equations in which the perturbed particle trajectory is found *self-consistently*. That is, with $\Phi = s\Phi^{(1)}$, the center-of-mass trajectories $\mathbf{r}$ and $\bar{\mathbf{r}}$ satisfy

$$\nabla^2\Phi^{(1)}(\mathbf{r}) = 4\pi\bar{m}^{(1)}\delta(\mathbf{r} - \bar{\mathbf{r}}(t)), \qquad \ddot{\mathbf{r}} = -\nabla\Phi,$$
$$\bar{\nabla}^2\bar{\Phi}^{(1)}(\bar{\mathbf{r}}) = 4\pi m^{(1)}\delta(\bar{\mathbf{r}} - \mathbf{r}(t)), \qquad \ddot{\bar{\mathbf{r}}} = -\bar{\nabla}\,\bar{\Phi}. \quad (3)$$

Here $\Phi(\mathbf{r})$ is the potential at $\mathbf{r}$ due to the mass $\bar{m}$. Because $v \to 0$ as $s \to 0$, the orbit remains close to a straight line for increasingly long times, but for each finite s the orbit is circular.

In general relativity, one can presumably construct a similar family of solutions with bound orbits whose energy-momentum tensor, $T^{\alpha\beta}(s) = \epsilon(s)u^\alpha(s)u^\beta(s) + p(s)[g^{\alpha\beta}(s) + u^\alpha(s)u^\beta(s)]$, and metric $g_{\alpha\beta}(s)$ are pointwise continuous in the parameter s and for which one has flat, empty space at $s = 0$,

$$g_{\alpha\beta}(0) = \eta_{\alpha\beta}, \qquad \epsilon(0) = p(0) = 0.$$

The equations for the first-order metric in a radiation gauge,

$$-2G^{(1)}_{\alpha\beta} \equiv \Box(h_{\alpha\beta} - \tfrac{1}{2}\eta_{\alpha\beta})h = -16\pi T^{(1)}_{\alpha\beta}, \quad (4)$$

have as their source $T^{(1)}_{\alpha\beta}$ a stress-energy tensor constructed from the *perturbed* velocity field and density and from the unperturbed metric. With the masses shrinking to zero as $s \to 0$, the motion of each mass is given to linear order by the linear field of the other: The self-force at linear order serves only to renormalize the mass.

The resulting "post-Minkowski" solution in which two masses move in bound orbit, each responding to the linear field of the other has the following features:

(i) It is correct to Newtonian order.

(ii) The radiation field of the linearized metric is correct to lowest nonvanishing post-Newtonian order (2 1/2 post-Newtonian order).

(iii) The orbit is not correct to first post-Newtonian order.

Using the linearized metric leads to equations of motion that are missing a term quadratic in the particle masses that enters the first post-Newtonian equations. In this sense, the first post-Minkowski approximation for bound orbits is a toy model, keeping terms of all orders in v but discarding nonlinear terms in $h_{\alpha\beta}$ that, for bound states, give corrections of order v^2 to the orbit.

Following recent codes that obtain helically symmetric solutions to nonlinear wave equations [15,16], Uryu [15,17] has obtained a neutron star code that solves the full Einstein-perfect fluid equations on an asymptotically flat initial hypersurface S, obtaining a solution with exact helical symmetry in the near zone. By solving the full Einstein equation, one expects more accurately to enforce the balance of gravity and centripetal acceleration in circular motion. The error associated with ignoring the radial motion associated with radiation reaction, however, remains. In future work, we anticipate using the point-particle model developed here to estimate the accuracy of such helically symmetric initial data sets and corresponding quasiequilibrium sequences, by comparing outgoing point-particle solution (in a post-Minkowski framework) to helically symmetric models and sequences. A further problem is related to the fact that the codes mentioned above typically diverge when helical symmetry is enforced in a region larger than the near zone (beyond one wavelength). Extending the present work to second order in the post-Minkowski approximation may be useful in developing codes with larger domains of convergence. Finally, existence of helically symmetric binaries is unproved and the second-order (nonlinear) extension may also help in understanding existence and asymptotic behavior of models of helically symmetric binaries.

In Sec. II, we review Fokker actions, obtaining the equations of motion and the form of the conserved 4-momentum and corresponding angular momentum. In Sec. III A, we introduce a parametrization-invariant Fokker action that describes point particles in the first post-Minkowski approximation. The formalism of the previous section is used to obtain equations of motion and conserved quantities; derivations of the explicit forms of the conserved quantities are relegated to Appendix D 3. Section III B then introduces the Fokker analog of the quadratic action, $\int d\tau \frac{m}{2}\dot{x}^\alpha\dot{x}_\alpha$, for affinely parametrized particles, again obtaining equations of motion, momentum, and angular momentum. In Sec. III C and III D we consider two particles of masses m and $\bar{m}$ in circular orbit, finding the equations governing the orbit, and computing the system's energy and angular momentum. The corresponding conserved quantities for scalar and electromagnetic interactions are summarized. Section IV is devoted to proving the relation $dE = \Omega dL$ for a class of parametrization-invariant Fokker actions and the affinely parametrized action. In Sec. V we introduce two forms of a post-Newtonian correction term to make the post-Minkowski action accurate to first post-Newtonian order. We obtain the corresponding corrections to the conserved momentum and angular momentum. Section VI displays the results of a numerical solution to the post-Minkowski equations of motion, with and without the added post-Newtonian term. Section VII discusses features of conserved quantities associated with Fokker actions, proposing, in particular, an explanation for their finite behavior when the field energy is infinite. A description of interacting scalar charges and many of the details of our calculations are presented in appendices.

II. ACTION AT A DISTANCE THEORY

An action-at-a-distance theory of interacting classical point charges was formulated by Fokker and by Wheeler and Feynman. [13,14]. In these treatments, one obtains the equations of motion by varying an action integral that is a function only of the trajectories (world lines) of the charges. The price one pays for eliminating the electromagnetic field is that the action integral is not a genuine action: One must vary an integral I that involves only a finite segment of each particle's trajectory, and the equations of motion emerge from the limit of δI as the trajectories are extended to infinity. In other words, the limit must be taken *after* the variation of the action integral. In addition, the action is not an integral over a single parameter time, but instead involves integrals over parameter times associated with each particle.[1] We will call an action integral having these properties a *Fokker action*.

In Dettman and Schild [18], a derivation of the equation of motion, as well as expressions for the conserved quantities, the energy-momentum and the angular momentum, is presented for a generic Fokker action that may include self-action terms and variable mass parameters.

In this paper, we obtain a similar action for two self-gravitating point particles, in a post-Minkowski approximation. We describe the particles by their constant masses m, $\bar{m}$ and by their trajectories written in terms of position vectors $x^\alpha(\tau)$, $\bar{x}^\alpha(\bar{\tau})$ on a flat background spacetime, with arbitrary parameter times τ and $\bar{\tau}$. In this approximation, the motion of each particle is determined by a gravitational interaction with the other particle, and there is no self-interaction term (any contribution from gravitational self-energy is accounted for in the mass of each particle). The two-particle system is then described by a Fokker action of the form

$$I(\tau_1, \tau_2, \bar{\tau}_1, \bar{\tau}_2) = -m \int_{\tau_1}^{\tau_2} d\tau (-\dot{x}_\alpha \dot{x}^\alpha)^{1/2} - \bar{m} \int_{\bar{\tau}_1}^{\bar{\tau}_2} d\bar{\tau} (-\dot{\bar{x}}_\alpha \dot{\bar{x}}^\alpha)^{1/2} + \int_{\tau_1}^{\tau_2} d\tau \int_{\bar{\tau}_1}^{\bar{\tau}_2} d\bar{\tau} \Lambda(x - \bar{x}, \dot{x}, \dot{\bar{x}}), \quad (5)$$

where the Fokker analog Λ of an interaction Lagrangian has the property,

$$\Lambda(x - \bar{x}, \dot{x}, \dot{\bar{x}}) = \Lambda(\bar{x} - x, \dot{\bar{x}}, \dot{x}), \quad (6)$$

and where

$$\dot{x}^\alpha := \frac{dx^\alpha}{d\tau} \quad \text{and} \quad \dot{\bar{x}}^\alpha := \frac{d\bar{x}^\alpha}{d\bar{\tau}}. \quad (7)$$

For given values of τ, $\bar{\tau}$, $\Lambda(\tau, \bar{\tau})$ is a scalar constructed only from $\eta_{\alpha\beta}$, $x^\alpha(\tau)$, $\dot{x}^\alpha(\tau)$, $\bar{x}^\alpha(\bar{\tau})$, $\dot{\bar{x}}^\alpha(\bar{\tau})$. It follows that Λ and I are Poincaré invariant—invariant under simultaneous Poincaré transformations of the particles' paths.

As noted earlier, to obtain the equations of motion, one must compute the variation of the action integral before taking the limit as $\tau_1, \bar{\tau}_1 \to -\infty$ and $\tau_2, \bar{\tau}_2 \to \infty$.

The variation of the action integral (5) is given by

$$\delta I(\tau_1, \tau_2, \bar{\tau}_1, \bar{\tau}_2) = \left[\frac{m\dot{x}_\alpha}{(-\dot{x}_\gamma \dot{x}^\gamma)^{1/2}} + \int_{\bar{\tau}_1}^{\bar{\tau}_2} d\bar{\tau} \frac{\partial \Lambda}{\partial \dot{x}^\alpha}\right] \delta x^\alpha \Bigg|_{\tau_1}^{\tau_2} + \int_{\tau_1}^{\tau_2} d\tau \delta x^\alpha \left\{-\frac{d}{d\tau} \frac{m\dot{x}_\alpha}{(-\dot{x}_\gamma \dot{x}^\gamma)^{1/2}} + \int_{\bar{\tau}_1}^{\bar{\tau}_2} d\bar{\tau} \left[\frac{\partial \Lambda}{\partial R^\alpha} - \frac{d}{d\tau} \frac{\partial \Lambda}{\partial \dot{x}^\alpha}\right]\right\}$$
$$+ \left[\frac{\bar{m}\dot{\bar{x}}_\alpha}{(-\dot{\bar{x}}_\gamma \dot{\bar{x}}^\gamma)^{1/2}} + \int_{\tau_1}^{\tau_2} d\tau \frac{\partial \Lambda}{\partial \dot{\bar{x}}^\alpha}\right] \delta \bar{x}^\alpha \Bigg|_{\bar{\tau}_1}^{\bar{\tau}_2} + \int_{\bar{\tau}_1}^{\bar{\tau}_2} d\bar{\tau} \delta \bar{x}^\alpha \left\{-\frac{d}{d\bar{\tau}} \frac{\bar{m}\dot{\bar{x}}_\alpha}{(-\dot{\bar{x}}_\gamma \dot{\bar{x}}^\gamma)^{1/2}} + \int_{\tau_1}^{\tau_2} d\tau \left[\frac{\partial \Lambda}{\partial \bar{R}^\alpha} - \frac{d}{d\bar{\tau}} \frac{\partial \Lambda}{\partial \dot{\bar{x}}^\alpha}\right]\right\}, \quad (8)$$

where $R^\alpha := x^\alpha - \bar{x}^\alpha =: -\bar{R}^\alpha$.

Requiring that the limit of the variation vanish when $\delta x^\alpha|_{\pm\infty} = 0$ and $\delta \bar{x}^\alpha|_{\pm\infty} = 0$,

$$\lim \delta I := \lim_{\substack{(\tau_1, \tau_2) \to (-\infty, +\infty) \\ (\bar{\tau}_1, \bar{\tau}_2) \to (-\infty, +\infty)}} \delta I = 0, \quad (9)$$

[1]For parametrization-invariant actions, one can choose to parametrize the trajectory of each particle by Minkowski time, but interaction terms involve double integrals over the parameter time of each particle.

yields the equation of motion for each particle,

$$\frac{d}{d\tau}\frac{m\dot{x}_\alpha}{(-\dot{x}_\gamma\dot{x}^\gamma)^{1/2}} = \int_{-\infty}^{\infty} d\bar{\tau}\left[\frac{\partial\Lambda}{\partial R^\alpha} - \frac{d}{d\tau}\frac{\partial\Lambda}{\partial\dot{x}^\alpha}\right] \tag{10}$$

$$\frac{d}{d\bar{\tau}}\frac{\bar{m}\dot{\bar{x}}_\alpha}{(-\dot{\bar{x}}_\gamma\dot{\bar{x}}^\gamma)^{1/2}} = \int_{-\infty}^{\infty} d\tau\left[\frac{\partial\Lambda}{\partial\bar{R}^\alpha} - \frac{d}{d\bar{\tau}}\frac{\partial\Lambda}{\partial\dot{\bar{x}}^\alpha}\right]. \tag{11}$$

The action integral I, with *finite* values of τ_i, $\bar{\tau}_i$, is invariant under Poincaré transformations of the paths that leave the path parameters fixed. Invariance of I under the infinitesimal spacetime translation of each path by a constant vector a^α,

$$\delta x^\alpha = \delta\bar{x}^\alpha = a^\alpha, \tag{12}$$

implies conservation of 4-momentum: That is, assuming the equations of motion, (10) and (11), and substituting Eq. (12) in Eq. (8), we obtain

$$\frac{\delta I}{\delta a^\alpha} = P_\alpha(\tau_2, \bar{\tau}_2) - P_\alpha(\tau_1, \bar{\tau}_1), \tag{13}$$

where

$$P_\alpha(\tau, \bar{\tau}) = \left[\frac{m\dot{x}_\alpha}{(-\dot{x}_\gamma\dot{x}^\gamma)^{1/2}} + \int_{-\infty}^{\infty} d\bar{\tau}\frac{\partial\Lambda}{\partial\dot{x}^\alpha}\right](\tau) + \left[\frac{\bar{m}\dot{\bar{x}}_\alpha}{(-\dot{\bar{x}}_\gamma\dot{\bar{x}}^\gamma)^{1/2}} + \int_{-\infty}^{\infty} d\tau\frac{\partial\Lambda}{\partial\dot{\bar{x}}^\alpha}\right](\bar{\tau}) + \left(\int_\tau^\infty\int_{-\infty}^{\bar{\tau}} - \int_{-\infty}^{\tau}\int_{\bar{\tau}}^{\infty}\right)\frac{\partial\Lambda}{\partial R^\alpha}d\tau d\bar{\tau}. \tag{14}$$

Translation invariance, $\delta I = 0$, implies that P_α is independent of τ and $\bar{\tau}$.

Similarly, invariance of I under an infinitesimal Lorentz transformation of each path,

$$\delta x^\alpha = \epsilon^{\alpha\beta}x_\beta \quad \text{and} \quad \delta\bar{x}^\alpha = \epsilon^{\alpha\beta}\bar{x}_\beta, \tag{15}$$

where $\epsilon^{\alpha\beta} = -\epsilon^{\beta\alpha}$ is a constant antisymmetric tensor, implies conservation of angular momentum. Again assuming the equations of motion, (10) and (11), and substituting Eq. (15) in Eq. (8), we have

$$2\frac{\delta I}{\delta\epsilon^{\beta\alpha}} = L_{\alpha\beta}(\tau_2, \bar{\tau}_2) - L_{\alpha\beta}(\tau_1, \bar{\tau}_1), \tag{16}$$

where

$$\begin{aligned} L_{\alpha\beta}(\tau, \bar{\tau}) = &\left[\frac{m(x_\alpha\dot{x}_\beta - x_\beta\dot{x}_\alpha)}{(-\dot{x}_\gamma\dot{x}^\gamma)^{1/2}} + \int_{-\infty}^{\infty} d\bar{\tau}\left(x_\alpha\frac{\partial\Lambda}{\partial\dot{x}^\beta} - x_\beta\frac{\partial\Lambda}{\partial\dot{x}^\alpha}\right)\right](\tau) + \left[\frac{\bar{m}(\bar{x}_\alpha\dot{\bar{x}}_\beta - \bar{x}_\beta\dot{\bar{x}}_\alpha)}{(-\dot{\bar{x}}_\gamma\dot{\bar{x}}^\gamma)^{1/2}} + \int_{-\infty}^{\infty} d\tau\left(\bar{x}_\alpha\frac{\partial\Lambda}{\partial\dot{\bar{x}}^\beta} - \bar{x}_\beta\frac{\partial\Lambda}{\partial\dot{\bar{x}}^\alpha}\right)\right](\bar{\tau}) \\ &+ \left(\int_\tau^\infty\int_{-\infty}^{\bar{\tau}} - \int_{-\infty}^{\tau}\int_{\bar{\tau}}^{\infty}\right)\left[\left(x_\alpha\frac{\partial\Lambda}{\partial R^\beta} - x_\beta\frac{\partial\Lambda}{\partial R^\alpha}\right) + \left(\dot{x}_\alpha\frac{\partial\Lambda}{\partial\dot{x}^\beta} - \dot{x}_\beta\frac{\partial\Lambda}{\partial\dot{x}^\alpha}\right)\right]d\tau d\bar{\tau} \end{aligned} \tag{17}$$

Here, Lorenz invariance implies

$$\epsilon^{\alpha\beta}\left(x_\beta\frac{\partial\Lambda}{\partial R^\alpha} + \bar{x}_\beta\frac{\partial\Lambda}{\partial\bar{R}^\alpha} + \dot{x}_\beta\frac{\partial\Lambda}{\partial\dot{x}^\alpha} + \dot{\bar{x}}_\beta\frac{\partial\Lambda}{\partial\dot{\bar{x}}^\alpha}\right) = 0. \tag{18}$$

Finally, $\delta I = 0$ implies that $L_{\alpha\beta}$ is independent of τ and $\bar{\tau}$.

With a definition

$$w := R^\alpha R_\alpha = (x - \bar{x})^2, \tag{19}$$

an interaction Λ that depends on the positions only through w, $\Lambda = \Lambda(w, \dot{x}, \dot{\bar{x}})$, is a restricted form of interactions that satisfies the property (6). For such an interaction term, since $\partial\Lambda/\partial R^\alpha = 2R_\alpha\partial\Lambda/\partial w$, the last line of Eq. (17) is written

$$\begin{aligned} 2\left(\int_\tau^\infty\int_{-\infty}^{\bar{\tau}} - \int_{-\infty}^{\tau}\int_{\bar{\tau}}^{\infty}\right)&\left[\frac{\partial\Lambda}{\partial w}(x_\beta\bar{x}_\alpha - x_\alpha\bar{x}_\beta)\right. \\ &\left.- \frac{1}{2}\left(\dot{x}_\beta\frac{\partial\Lambda}{\partial\dot{x}^\alpha} - \dot{x}_\alpha\frac{\partial\Lambda}{\partial\dot{x}^\beta}\right)\right]d\tau d\bar{\tau}. \end{aligned} \tag{20}$$

III. ACTION AT A DISTANCE THEORY FOR POST-MINKOWSKIAN GRAVITY

As usual in a linearized framework, all tensor indices will be raised and lowered by the flat metric $\eta_{\alpha\beta}$ of the background spacetime.

A. Fokker actions for point-particles in post-Minkowskian gravity

Havas and Goldberg [19] derived equations of motion for point particles in general relativity by expanding the metric and demanding that the covariant conservation law for the stress-energy tensor be satisfied to first order in the perturbation, with a time-symmetric, half-advanced + half-retarded field for the first-order metric. They found a Fokker action, an action integral I for which $\lim\delta I = 0$ [in the sense of Eq. (9)] gives the equation of motion to the same order. In our notation, their interaction term is

$$\Lambda(w, \dot{x}, \dot{\bar{x}}) = 2m\bar{m}\delta(w)\frac{(\dot{x}_\alpha\dot{\bar{x}}^\alpha)^2 + \frac{1}{2}\dot{x}_\alpha\dot{x}^\alpha + \frac{1}{2}\dot{\bar{x}}_\beta\dot{\bar{x}}^\beta + \frac{1}{2}}{(-\dot{x}_\gamma\dot{x}^\gamma)^{1/2}(-\dot{\bar{x}}_\delta\dot{\bar{x}}^\delta)^{1/2}}. \tag{21}$$

Ramond [20] subsequently formulated a general action-at-a-distance theory that is invariant under reparametrization and includes tensorial interactions. Following Ramond's argument, we find that the following interaction term is reparametrization invariant:

$$\Lambda(w, \dot{x}, \dot{\bar{x}}) = 2m\bar{m}\delta(w)\frac{(\dot{x}_\alpha\dot{\bar{x}}^\alpha)^2 - \frac{1}{2}\dot{x}_\alpha\dot{x}^\alpha\dot{\bar{x}}_\beta\dot{\bar{x}}^\beta}{(-\dot{x}_\gamma\dot{x}^\gamma)^{1/2}(-\dot{\bar{x}}_\delta\dot{\bar{x}}^\delta)^{1/2}}, \tag{22}$$

When parameters τ and $\bar{\tau}$ are chosen to satisfy $\dot{x}_\alpha\dot{x}^\alpha = -1$ and $\dot{\bar{x}}_\alpha\dot{\bar{x}}^\alpha = -1$, as in the Havas-Goldberg treatment, the equations of motion agree with theirs. Because $\dot{x}^\alpha$ is normalized with respect to $\eta_{\alpha\beta}$, not $\eta_{\alpha\beta} + h_{\alpha\beta}$, the Havas-Goldberg τ is not an affine parameter.

We can see as follows how, starting from the post-Minkowski equations of motion, one arrives at an interaction term with Λ given by Eq. (22). The derivation also shows why one obtains a Fokker action, not a true action.

To linear order, the metric $g_{\alpha\beta}$ is a sum of the flat metric, $\eta_{\alpha\beta}$, and a perturbation $\tilde{h}_{\alpha\beta}$,

$$g_{\alpha\beta} = \eta_{\alpha\beta} + \tilde{h}_{\alpha\beta}. \tag{23}$$

where the perturbed metric $\tilde{h}_{\alpha\beta}$ is a sum of the half-advanced + half-retarded field of each particle. As mentioned earlier, in the post-Minkowski treatment of point particles, the effect on the motion of particle m from the field of m itself is simply a mass renormalization. That is, the motion of m is described to linear order in $\tilde{h}_{\alpha\beta}$ by the linearized geodesic equation with $\tilde{h}_{\alpha\beta}$ replaced by the value at the position of m of the half-advanced + half-retarded field $h_{\alpha\beta}$ of $\bar{m}$ alone: When τ is an affine parameter (proper time with respect to the perturbed metric), the geodesic equation to linear order in $h_{\alpha\beta}$ has the form

$$\begin{gathered}(\eta_{\alpha\beta} + h_{\alpha\beta})\ddot{x}^\beta + C_{\alpha\beta\gamma}\dot{x}^\beta\dot{x}^\gamma = 0\\ \text{where } C_{\alpha\beta\gamma} := \tfrac{1}{2}(\nabla_\beta h_{\alpha\gamma} + \nabla_\gamma h_{\beta\alpha} - \nabla_\alpha h_{\beta\gamma}).\end{gathered} \tag{24}$$

Equivalently,

$$\frac{d}{d\tau}[(\eta_{\alpha\beta} + h_{\alpha\beta})\dot{x}^\beta] - \frac{1}{2}\nabla_\alpha h_{\beta\gamma}\dot{x}^\beta\dot{x}^\gamma = 0. \tag{25}$$

To linear order in $h_{..}$, the expression $(\eta_{\alpha\beta} + h_{\alpha\beta})\ddot{x}^\beta$ in Eq. (24) can be replaced by $\ddot{x}^\alpha = \eta_{\alpha\beta}\ddot{x}^\beta$, because $\ddot{x}^\alpha$ is already order $h_{..}$. The form given in Eq. (25), however, conforms to that of the action integrals below. For τ a generic time parameter, the geodesic equation has the form given in Eq. (32). This can be directly shown from (25), but we derive it below from the action for a point particle, written to linear order in $h_{\alpha\beta}$.

In the deDonder (harmonic) gauge, $\nabla_\beta h^{\alpha\beta} = 0$, $h_{\alpha\beta}$ is given by

$$\Box(h_{\alpha\beta} - \tfrac{1}{2}\eta_{\alpha\beta}h) = -16\pi\bar{T}_{\alpha\beta}, \tag{26}$$

where the stress-energy tensor $\bar{T}^{\alpha\beta}$ of $\bar{m}$ is defined by

$$\bar{T}^{\alpha\beta}(x) = \bar{m}\int_{-\infty}^{\infty} d\bar{\tau}\delta(x - \bar{x}(\bar{\tau}))\frac{\dot{\bar{x}}^\alpha\dot{\bar{x}}^\beta}{(-\dot{\bar{x}}_\gamma\dot{\bar{x}}^\gamma)^{1/2}}. \tag{27}$$

Here ∇_α is the covariant derivative operator of the flat metric $\eta_{\alpha\beta}$, and $\Box = \nabla_\alpha\nabla^\alpha$ is the corresponding flat D'Alembertian. Using the half-retarded + half-advanced Green function, $G(x, \bar{x}) = \delta(w)$ [a solution to $\Box G(x, \bar{x}) = -4\pi\delta(x - \bar{x})$], the solution to Eq. (26) is written,

$$h^{\alpha\beta}(x) = 4\bar{m}\int_{-\infty}^{\infty} d\bar{\tau}\delta(w)\frac{\dot{\bar{x}}^\alpha\dot{\bar{x}}^\beta - \frac{1}{2}\bar{\eta}^{\alpha\beta}\dot{\bar{x}}_\gamma\dot{\bar{x}}^\gamma}{(-\dot{\bar{x}}_\delta\dot{\bar{x}}^\delta)^{1/2}}. \tag{28}$$

The trajectory of the second particle, $(\bar{m}, \bar{x}^\alpha(\bar{\tau}))$, is similarly a geodesic of a background spacetime with metric $g_{\alpha\beta} = \eta_{\alpha\beta} + \bar{h}_{\alpha\beta}$, given by Eq. (25) or (32), with barred and unbarred quantities exchanged. The source for $\bar{h}_{\alpha\beta}$ is the particle $(m, x^\alpha(\tau))$:

$$\bar{h}^{\alpha\beta}(\bar{x}) = 4m\int_{-\infty}^{\infty} d\tau\delta(w)\frac{\dot{x}^\alpha\dot{x}^\beta - \frac{1}{2}\eta^{\alpha\beta}\dot{x}_\gamma\dot{x}^\gamma}{(-\dot{x}_\delta\dot{x}^\delta)^{1/2}}. \tag{29}$$

(*Note*: We use a bar to label the perturbed field acting on particle $\bar{m}$; $\bar{h}_{\alpha\beta}$ is *not* the trace-reversed form of $h_{\alpha\beta}$.)

To find an interaction term that reproduces the equations of motion, we begin with an action I_m for the first particle in the field of the second, a linearized geodesic equation on a background spacetime with metric $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$. The action I_m is then the action for geodesic motion,

$$\begin{aligned}&-m\int_{\tau_1}^{\tau_2} d\tau(-g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta)^{1/2}\\ &\quad = -m\int_{\tau_1}^{\tau_2} d\tau[-(\eta_{\alpha\beta} + h_{\alpha\beta})\dot{x}^\alpha\dot{x}^\beta]^{1/2},\end{aligned}$$

written to linear order in $h_{\alpha\beta}$:

$$\begin{aligned}I_m = &-m\int_{\tau_1}^{\tau_2} d\tau(-\dot{x}_\alpha\dot{x}^\alpha)^{1/2}\\ &+ \frac{1}{2}m\int_{\tau_1}^{\tau_2} d\tau h_{\alpha\beta}\frac{\dot{x}^\alpha\dot{x}^\beta}{(-\dot{x}_\gamma\dot{x}^\gamma)^{1/2}}.\end{aligned} \tag{30}$$

The action I_m is invariant under time reparametrization. From its variation, $\delta I_m = 0$, we can directly compute the linearized geodesic equation with arbitrary parameter τ:

$$\delta I_m = m\left[\eta_{\alpha\beta} + h_{\alpha\beta} + \frac{1}{2}\eta_{\alpha\beta}h_{\gamma\delta}\frac{\dot{x}^\gamma\dot{x}^\delta}{(-\dot{x}_\gamma\dot{x}^\gamma)^{1/2}}\right]\frac{\dot{x}^\beta}{(-\dot{x}_\gamma\dot{x}^\gamma)^{1/2}}\delta x^\alpha\Bigg|_{\tau_1}^{\tau_2}$$
$$- m\int_{\tau_1}^{\tau_2} d\tau\left\{\frac{d}{d\tau}\left[\left(\eta_{\alpha\beta} + h_{\alpha\beta} + \frac{1}{2}\eta_{\alpha\beta}h_{\gamma\delta}\frac{\dot{x}^\gamma\dot{x}^\delta}{(-\dot{x}_\gamma\dot{x}^\gamma)^{1/2}}\right)\frac{\dot{x}^\beta}{(-\dot{x}_\gamma\dot{x}^\gamma)^{1/2}}\right] - \frac{1}{2}\nabla_\alpha h_{\beta\gamma}\frac{\dot{x}^\beta\dot{x}^\gamma}{(-\dot{x}_\gamma\dot{x}^\gamma)^{1/2}}\right\}\delta x^\alpha. \quad (31)$$

Then, requiring $\delta I_m = 0$ for variations δx^α with fixed endpoints yields the equation of motion, we have

$$\frac{d}{d\tau}\left[\left(\eta_{\alpha\beta} + h_{\alpha\beta} + \frac{1}{2}\eta_{\alpha\beta}h_{\gamma\delta}\frac{\dot{x}^\gamma\dot{x}^\delta}{(-\dot{x}_\gamma\dot{x}^\gamma)^{1/2}}\right)\frac{\dot{x}^\beta}{(-\dot{x}_\gamma\dot{x}^\gamma)^{1/2}}\right] = \frac{1}{2}\nabla_\alpha h_{\beta\gamma}\frac{\dot{x}^\beta\dot{x}^\gamma}{(-\dot{x}_\gamma\dot{x}^\gamma)^{1/2}}. \quad (32)$$

If in the action I_m of Eq. (30) we substitute Eq. (28), we obtain

$$I_m = -m\int_{\tau_1}^{\tau_2} d\tau(-\dot{x}_\alpha\dot{x}^\alpha)^{1/2} + 2m\bar{m}\int_{\tau_1}^{\tau_2} d\tau\int_{-\infty}^{\infty} d\bar{\tau}\delta(w)\frac{(\dot{x}_\alpha\dot{\bar{x}}^\alpha)^2 - \frac{1}{2}\dot{x}_\alpha\dot{x}^\alpha\dot{\bar{x}}_\beta\dot{\bar{x}}^\beta}{(-\dot{x}_\gamma\dot{x}^\gamma)^{1/2}(-\dot{\bar{x}}_\delta\dot{\bar{x}}^\delta)^{1/2}}. \quad (33)$$

To obtain an action whose variations with respect to both x^α and $\bar{x}^\alpha$ yield the equations of motion for each particle, one cannot simply add a kinetic term for $\bar{m}$ and a second interaction term in which the barred and unbarred quantities are interchanged, because the new interaction term would involve x^α and alter the equations of motion of the first particle. (Roughly speaking, one would be double-counting the gravitational binding energy.) Instead, one observes that the interaction term becomes symmetric under interchange of the two particles m, $x^\alpha(\tau)$ and $\bar{m}$, $\bar{x}^\alpha(\bar{\tau})$ if, in the infinite integral, we make the replacements $-\infty \to \bar{\tau}_1$ and $\infty \to \bar{\tau}_2$. The resulting action integral, symmetric under interchange of two particles, has the form of Eq. (5), with interaction term (22)

$$I = -m\int_{\tau_1}^{\tau_2} d\tau(-\dot{x}_\alpha\dot{x}^\alpha)^{1/2} - \bar{m}\int_{\bar{\tau}_1}^{\bar{\tau}_2} d\bar{\tau}(-\dot{\bar{x}}_\alpha\dot{\bar{x}}^\alpha)^{1/2} + 2m\bar{m}\int_{\tau_1}^{\tau_2} d\tau\int_{\bar{\tau}_1}^{\bar{\tau}_2} d\bar{\tau}\delta(w)\frac{(\dot{x}_\alpha\dot{\bar{x}}^\alpha)^2 - \frac{1}{2}\dot{x}_\alpha\dot{x}^\alpha\dot{\bar{x}}_\beta\dot{\bar{x}}^\beta}{(-\dot{x}_\gamma\dot{x}^\gamma)^{1/2}(-\dot{\bar{x}}_\delta\dot{\bar{x}}^\delta)^{1/2}}. \quad (34)$$

The price for this symmetry is that a variation of I yields equations of motion in which the integrals giving $h_{\alpha\beta}$ and $\bar{h}_{\alpha\beta}$ extend only from $\bar{\tau}_1$ to $\bar{\tau}_2$ (or τ_1 to τ_2). It is for this reason that action-at-a-distance theories require a Fokker action, whose equations of motion are given by $\lim\delta I = 0$.

As mentioned, the above theory is invariant under reparametrization of τ and $\bar{\tau}$. The obvious choice of a proper-time parametrization, with $\eta_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta = -1$ and $\bar{\eta}_{\alpha\beta}\dot{\bar{x}}^\alpha\dot{\bar{x}}^\beta = -1$, simplifies computations of physical quantities.

B. A Fokker action for the affinely parametrized equations

One can, of course, specialize the parametrization-invariant action to affinely parametrized trajectories, but affine parametrization also allows a generalization to interacting particles of the quadratic action for geodesic motion, $\frac{1}{2}m\int_{\tau_1}^{\tau_2} d\tau\dot{x}_\alpha\dot{x}^\alpha$. To construct an action integral I with quadratic kinetic term, for which $\lim\delta I = 0$ reproduces the affinely parametrized equation of motion, Eq. (25), and its barred $\leftrightarrow$ unbarred form, we modify the kinetic terms in the action integral of Eq. (5), writing

$$I(\tau_1, \tau_2, \bar{\tau}_1, \bar{\tau}_2) = \frac{1}{2}m\int_{\tau_1}^{\tau_2} d\tau\dot{x}_\alpha\dot{x}^\alpha + \frac{1}{2}\bar{m}\int_{\bar{\tau}_1}^{\bar{\tau}_2} d\bar{\tau}\dot{\bar{x}}_\alpha\dot{\bar{x}}^\alpha + \int_{\tau_1}^{\tau_2} d\tau\int_{\bar{\tau}_1}^{\bar{\tau}_2} d\bar{\tau}\Lambda(w, \dot{x}, \dot{\bar{x}}); \quad (35)$$

and we take as the interaction term

$$\Lambda(w, \dot{x}, \dot{\bar{x}}) = 2m\bar{m}\delta(w)[(\dot{x}_\alpha\dot{\bar{x}}^\alpha)^2 - \tfrac{1}{2}\dot{x}_\alpha\dot{x}^\alpha\dot{\bar{x}}_\beta\dot{\bar{x}}^\beta]. \quad (36)$$

Affine parametrization, the requirement

$$(\eta_{\alpha\beta} + h_{\alpha\beta})\dot{x}^\alpha\dot{x}^\beta = \text{constant},$$
$$(\eta_{\alpha\beta} + \bar{h}_{\alpha\beta})\dot{\bar{x}}^\alpha\dot{\bar{x}}^\beta = \text{constant}, \quad (37)$$

is enforced by the equations of motion.

Formulas for the equation of motion Eqs. (10) and (11), the 4-momentum Eq. (14) and the angular momentum tensor Eq. (17) are changed as a result of this modification of the kinetic terms in Eq. (35). The variation of our kinetic terms

$$\frac{1}{2}m\delta\int_{\tau_1}^{\tau_2} d\tau\dot{x}_\alpha\dot{x}^\alpha = m\dot{x}_\alpha\delta x^\alpha|_{\tau_1}^{\tau_2} - \int_{\tau_1}^{\tau_2} d\tau m\ddot{x}_\alpha\delta x^\alpha, \quad (38)$$

$$\frac{1}{2}\bar{m}\delta\int_{\bar{\tau}_1}^{\bar{\tau}_2} d\bar{\tau}\dot{\bar{x}}_\alpha\dot{\bar{x}}^\alpha = \bar{m}\dot{\bar{x}}_\alpha\delta\bar{x}^\alpha|_{\bar{\tau}_1}^{\bar{\tau}_2} - \int_{\bar{\tau}_1}^{\bar{\tau}_2} d\bar{\tau}\,\bar{m}\,\ddot{\bar{x}}_\alpha\delta\bar{x}^\alpha, \tag{39}$$

results in the following form for the equation of motion,

$$m\ddot{x}_\alpha = \int_{-\infty}^{\infty} d\bar{\tau}\left[\frac{\partial\Lambda}{\partial R^\alpha} - \frac{d}{d\tau}\frac{\partial\Lambda}{\partial\dot{x}^\alpha}\right] = -\frac{d}{d\tau}(h_{\alpha\beta}\dot{x}^\beta) + \frac{1}{2}\nabla_\alpha h_{\beta\gamma}\dot{x}^\beta\dot{x}^\gamma, \tag{40}$$

$$\bar{m}\ddot{\bar{x}}_\alpha = \int_{-\infty}^{\infty} d\tau\left[\frac{\partial\Lambda}{\partial\bar{R}^\alpha} - \frac{d}{d\bar{\tau}}\frac{\partial\Lambda}{\partial\dot{\bar{x}}^\alpha}\right] = -\frac{d}{d\tau}(\bar{h}_{\alpha\beta}\dot{\bar{x}}^\beta) + \frac{1}{2}\nabla_\alpha\bar{h}_{\beta\gamma}\dot{\bar{x}}^\beta\dot{\bar{x}}^\gamma, \tag{41}$$

in agreement with the linearized geodesic Eq. (25).

The 4-momentum is given by

$$P_\alpha(\tau,\bar{\tau}) = \left[m\dot{x}_\alpha + \int_{-\infty}^{\infty} d\bar{\tau}\frac{\partial\Lambda}{\partial\dot{x}^\alpha}\right](\tau) + \left[\bar{m}\dot{\bar{x}}_\alpha + \int_{-\infty}^{\infty} d\tau\frac{\partial\Lambda}{\partial\dot{\bar{x}}^\alpha}\right](\bar{\tau}) + \left(\int_\tau^\infty\int_{-\infty}^{\bar{\tau}} - \int_{-\infty}^{\tau}\int_{\bar{\tau}}^{\infty}\right)\frac{\partial\Lambda}{\partial R^\alpha}d\tau d\bar{\tau}, \tag{42}$$

and the angular momentum by

$$L_{\alpha\beta}(\tau,\bar{\tau}) = \left[m(x_\alpha\dot{x}_\beta - x_\beta\dot{x}_\alpha) + \int_{-\infty}^{\infty} d\bar{\tau}\left(x_\alpha\frac{\partial\Lambda}{\partial\dot{x}^\beta} - x_\beta\frac{\partial\Lambda}{\partial\dot{x}^\alpha}\right)\right](\tau) + \left[\bar{m}(\bar{x}_\alpha\dot{\bar{x}}_\beta - \bar{x}_\beta\dot{\bar{x}}_\alpha) + \int_{-\infty}^{\infty} d\tau\left(\bar{x}_\alpha\frac{\partial\Lambda}{\partial\dot{\bar{x}}^\beta} - \bar{x}_\beta\frac{\partial\Lambda}{\partial\dot{\bar{x}}^\alpha}\right)\right](\bar{\tau})$$
$$+ 2\left(\int_\tau^\infty\int_{-\infty}^{\bar{\tau}} - \int_{-\infty}^{\tau}\int_{\bar{\tau}}^{\infty}\right)\left[\frac{\partial\Lambda}{\partial w}(x_\beta\bar{x}_\alpha - x_\alpha\bar{x}_\beta) - \frac{1}{2}\left(\dot{x}_\beta\frac{\partial\Lambda}{\partial\dot{x}^\alpha} - \dot{x}_\alpha\frac{\partial\Lambda}{\partial\dot{x}^\beta}\right)\right]d\tau d\bar{\tau}, \tag{43}$$

These expressions for momentum and angular momentum are formally identical to Eqs. (14) and (17) if one sets $(-\eta_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta)$ to 1. This is misleading: In the present section, $\dot{x}^\alpha$ is a unit vector *not* of the Minkowski metric but of the perturbed metric $\eta_{\alpha\beta} + h_{\alpha\beta}$, and the altered form of Λ compensates for the altered normalization, leading to identical Newtonian limits of the two expressions.

C. Circular orbits

We consider now a system of two point particles in circular orbit, for which the gravitational field seen by each particle is the linearized half-advanced + half-retarded field of the other. This is a gravitational analog of a solution obtained by Schild [11] for two point charges. In the gravitational context, however, linearized gravity is a toy model for bound systems: Discarded nonlinear terms are of the same post-Newtonian order as linear terms that are kept, and hence of the same magnitude as terms of the next post-Minkowskian order.

We introduce a basis $\{t^\alpha, \varpi^\alpha, \hat{\phi}^\alpha, z^\alpha\}$ of unit vectors of the flat metric $\eta_{\alpha\beta}$. A particle in circular orbit in the $z = 0$ plane with constant orbital radius a has cylindrical coordinates $\{t, \varpi = a, \phi, z = 0\}$ and position vector

$$x^\alpha = tt^\alpha + a\varpi^\alpha. \tag{44}$$

Its spacetime trajectory is tangent to the helical Killing vector

$$k^\alpha = t^\alpha + \Omega\phi^\alpha, \tag{45}$$

where $\phi^\alpha = \varpi\hat{\phi}^\alpha$ is a rotational Killing vector of the flat metric $\eta_{\alpha\beta}$ and Ω is the particle's constant angular velocity. With $\gamma := dt/d\tau$ and $v := a\Omega$, the particle's 4-velocity and acceleration are given by

$$\dot{x}^\alpha = \gamma k^\alpha = \gamma(t^\alpha + v\hat{\phi}^\alpha) \tag{46}$$

and

$$\ddot{x}^\alpha = -\gamma^2 v\Omega\varpi^\alpha. \tag{47}$$

The second particle $\bar{m}$ has a circular orbit of radius $\bar{a}$ about the same origin with coordinates $\{\bar{t}, \bar{\varpi} = \bar{a}, \bar{\phi}, \bar{z} = 0\}$ and position vector

$$\bar{x}^\alpha = \bar{t}t^\alpha + \bar{a}\bar{\varpi}^\alpha. \tag{48}$$

The particle trajectory is again tangent to the helical Killing vector

$$\bar{k}^\alpha = t^\alpha + \Omega\bar{\phi}^\alpha, \tag{49}$$

with $\bar{\phi}^\alpha$ the value of the vector field ϕ^α at the position of the second particle. With $\bar{\gamma} = d\bar{t}/d\bar{\tau}$ and $\bar{v} := \bar{a}\Omega$, the acceleration and 4-velocity of the second particle are given by

$$\dot{\bar{x}}^\alpha = \bar{\gamma}\bar{k}^\alpha = \bar{\gamma}(t^\alpha + \bar{v}\hat{\bar{\phi}}^\alpha) \tag{50}$$

and

$$\ddot{\bar{x}}^\alpha = -\bar{\gamma}^2\bar{v}\Omega\bar{\varpi}^\alpha. \tag{51}$$

For the parametrization-invariant formulation, it is convenient to parametrize the trajectories by proper time with respect to the flat metric $\eta_{\alpha\beta}$; their tangent vectors are then unit vectors of $\eta_{\alpha\beta}$,

$$\eta_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta = -1, \qquad \bar{\eta}_{\alpha\beta}\dot{\bar{x}}^\alpha\dot{\bar{x}}^\beta = -1. \tag{52}$$

The quantities γ, v, $\bar{\gamma}$ and $\bar{v}$ obey the familiar relations

$$\gamma = (1 - v^2)^{-1/2}, \qquad \bar{\gamma} = (1 - \bar{v}^2)^{-1/2}. \tag{53}$$

For the affinely parametrized formulation, a correct Newtonian limit is reproduced when a normalization with the perturbed metric (Sec. III B) is used:

$$(\eta_{\alpha\beta} + h_{\alpha\beta})\dot{x}^\alpha \dot{x}^\beta = -1, \qquad (\bar{\eta}_{\alpha\beta} + \bar{h}_{\alpha\beta})\dot{\bar{x}}^\alpha \dot{\bar{x}}^\beta = -1, \tag{54}$$

which yield relations

$$\gamma = (1 - v^2 - h_{\alpha\beta}k^\alpha k^\beta)^{-1/2}, \qquad \bar{\gamma} = (1 - \bar{v}^2 - \bar{h}_{\alpha\beta}\bar{k}^\alpha \bar{k}^\beta)^{-1/2}. \tag{55}$$

With the first particle, m, at $\phi = 0$ for $t = 0$, its trajectory has coordinates

$$\phi = \Omega t, \qquad t = \gamma\tau; \tag{56}$$

and the trajectory of $\bar{m}$ then has coordinates

$$\bar{\phi} = \pi + \Omega\bar{t}, \qquad \bar{t} = \bar{\gamma}\,\bar{\tau} - \pi/\Omega. \tag{57}$$

The positions of both particles are specified by a single parameter, and it is natural to choose a descriptor η of their relative motion, defined by

$$\eta := \bar{\phi} - \phi = \Omega(\bar{\gamma}\,\bar{\tau} - \gamma\tau), \tag{58}$$

where we pick $\tau = \bar{\tau} = 0$ when $\phi = \bar{\phi} = 0$.

A vector $R^\alpha := x^\alpha - \bar{x}^\alpha$ becomes

$$R^\alpha = \frac{1}{\Omega}[(\pi - \eta)t^\alpha + (v - \bar{v}\cos\eta)\varpi^\alpha - \bar{v}\sin\eta\,\hat{\phi}^\alpha] \tag{59}$$

$$= \frac{1}{\Omega}[(\pi - \eta)t^\alpha - (\bar{v} - v\cos\eta)\bar{\varpi}^\alpha - v\sin\eta\,\hat{\bar{\phi}}^\alpha]. \tag{60}$$

The half-retarded + half-advanced Green function $\delta(w) = \delta[(x - \bar{x})^2]$ has support on the two events that correspond to the roots of

$$w(\eta) := (x - \bar{x})^2 = \frac{1}{\Omega^2}[-(\pi - \eta)^2 + v^2 + \bar{v}^2 - 2v\bar{v}\cos\eta] = 0. \tag{61}$$

These roots are given by

$$\eta = \pi \pm \varphi, \tag{62}$$

with φ the positive root of

$$\varphi^2 = v^2 + \bar{v}^2 + 2v\bar{v}\cos\varphi. \tag{63}$$

Here $\varphi + \pi$ is the angle between m and the retarded position of $\bar{m}$, and Eq. (63) has a simple geometrical meaning, illustrated in Fig. 1

$$\varphi + \pi = \bar{\phi}_{\rm ret} - \phi,$$

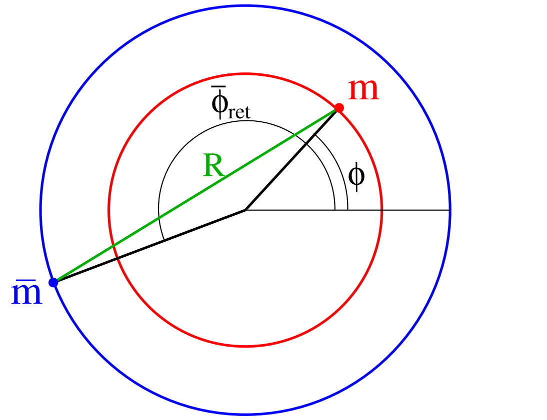

FIG. 1 (color online). The figure shows mass m at time t and mass $\bar{m}$ at the corresponding retarded time. The angle $\phi_{\rm ret}$ is the ϕ coordinate of the intersection of the past light cone of m with the trajectory of $\bar{m}$, when m is at ϕ.

in which R is the distance from m to the retarded position of $\bar{m}$. That is, using the geometric relation

$$R^2 = a^2 + \bar{a}^2 - 2a\bar{a}\cos(\bar{\phi}_{\rm ret} - \phi) = \frac{1}{\Omega^2}(v^2 + \bar{v}^2 + 2v\bar{v}\cos\varphi), \tag{64}$$

noting that $R = \bar{t}_{\rm ret} - t$, and using Eqs. (56) and (57), we have

$$\varphi = \bar{\phi}_{\rm ret} - \pi - \phi = \Omega(\bar{t}_{\rm ret} - t) = \Omega R. \tag{65}$$

Then $\varphi^2 = \Omega^2 R^2$ and Eq. (63) is immediate from (64). Analogously, the root $\eta = \pi - \varphi$ corresponds to the advanced position of $\bar{m}$ defined by $-\varphi + \pi = \bar{\phi}_{\rm adv} - \phi$.

D. Conserved quantities and angular velocity of circular orbits

We now rewrite the integrals appearing in Eqs. (10), (11), (14), and (17) for a parametrization-invariant action, in terms of the parameter η. To evaluate the integrals, we use formulas obtained in Appendix A 1. The resulting formulas for affinely parametrized formulations Eqs. (40)–(43) are quite similar to those for the parametrization-invariant formulations. However, the fact that the action appropriate to an initial choice of affine parameters is not a special case of the parametrization-invariant action leads to definitions of a conserved energy that are not identical in the two formulations.

1. Parametrization-invariant model

For circular orbits, only the radial component of the equations of motion is nontrivial, and it has the form,

$$-m\gamma^2 v\Omega = \int_{-\infty}^{\infty} d\eta \left\{ \frac{\partial\Lambda}{\partial w} \frac{2}{\bar{\gamma}\Omega^2}(v - \bar{v}\cos\eta) + \frac{\gamma}{\bar{\gamma}} \frac{\partial\Lambda}{\partial \dot{x}_\beta} \hat{\phi}_\beta \right\} \tag{66}$$

$$= 4m\bar{m}\gamma^2\bar{\gamma} \int_{-\infty}^{\infty} d\eta \left\{ \frac{1}{\Omega^2} \delta'(w)(v - \bar{v}\cos\eta)\tilde{\Phi}(\eta, v, \bar{v}) - \delta(w)\left[(1 - v\bar{v}\cos\eta)\bar{v}\cos\eta - \frac{1}{2}v(1 - \bar{v}^2) - \frac{1}{2}\frac{v}{1 - v^2}\tilde{\Phi}(\eta, v, \bar{v}) \right] \right\}, \tag{67}$$

$$-\bar{m}\bar{\gamma}^2\bar{v}\Omega = \int_{-\infty}^{\infty} d\eta \left\{ \frac{\partial\Lambda}{\partial w} \frac{2}{\gamma\Omega^2}(\bar{v} - v\cos\eta) + \frac{\bar{\gamma}}{\gamma} \frac{\partial\Lambda}{\partial \dot{\bar{x}}_\beta} \hat{\bar{\phi}}_\beta \right\} \tag{68}$$

$$= 4m\bar{m}\gamma\bar{\gamma}^2 \int_{-\infty}^{\infty} d\eta \left\{ \frac{1}{\Omega^2} \delta'(w)(\bar{v} - v\cos\eta)\tilde{\Phi}(\eta, v, \bar{v}) - \delta(w)\left[(1 - v\bar{v}\cos\eta)v\cos\eta - \frac{1}{2}\bar{v}(1 - v^2) - \frac{1}{2}\frac{\bar{v}}{1 - \bar{v}^2}\tilde{\Phi}(\eta, v, \bar{v}) \right] \right\}, \tag{69}$$

where Eqs. (A10) and (A11) in Appendix A 1 are used. We have introduced here a function $\tilde{\Phi}(\eta, v, \bar{v})$, defined by

$$\tilde{\Phi}(\eta, v, \bar{v}) := \frac{1}{\gamma^2\bar{\gamma}^2}\left[(\dot{x}_\alpha\dot{\bar{x}}^\alpha)^2 - \frac{1}{2}\dot{x}_\alpha\dot{x}^\alpha\dot{\bar{x}}_\beta\dot{\bar{x}}^\beta \right] = (1 - v\bar{v}\cos\eta)^2 - \frac{1}{2}(1 - v^2)(1 - \bar{v}^2). \tag{70}$$

Since the center-of-mass frame is chosen, the only nonzero components of the 4-momentum and the angular momentum are $E := -P_\alpha(\tau, \bar{\tau})t^\alpha = -P_0(\tau, \bar{\tau})$ and $L := L_{12}(\tau, \bar{\tau})$. Taking $(\tau, \bar{\tau}) = (0, 0)$, we have

$$\begin{aligned} E = P^0(0,0) &= \left[m\dot{x}^0 + \frac{1}{\bar{\gamma}\Omega}\int_{-\infty}^{\infty} d\eta \frac{\partial\Lambda}{\partial\dot{x}_0} \right](0) + \left[\bar{m}\dot{\bar{x}}^0 + \frac{1}{\gamma\Omega}\int_{-\infty}^{\infty} d\eta \frac{\partial\Lambda}{\partial\dot{\bar{x}}_0} \right](0) - \frac{1}{\gamma\bar{\gamma}\Omega^2}\int_{-\infty}^{\infty} d\eta\, \eta \frac{\partial\Lambda}{\partial w}2(x^0 - \bar{x}^0) \\ &= m\gamma - \frac{4m\bar{m}\gamma\bar{\gamma}}{\Omega}\int_{-\infty}^{\infty} d\eta\,\delta(w)\left[(1 - v\bar{v}\cos\eta) - \frac{1}{2}(1 - \bar{v}^2) - \frac{1}{2}\frac{1}{1 - v^2}\tilde{\Phi}(\eta, v, \bar{v}) \right] \\ &\quad + \bar{m}\,\bar{\gamma} - \frac{4m\bar{m}\gamma\bar{\gamma}}{\Omega}\int_{-\infty}^{\infty} d\eta\,\delta(w)\left[(1 - v\bar{v}\cos\eta) - \frac{1}{2}(1 - v^2) - \frac{1}{2}\frac{1}{1 - \bar{v}^2}\tilde{\Phi}(\eta, v, \bar{v}) \right] \\ &\quad - \frac{4m\bar{m}\gamma\bar{\gamma}}{\Omega^3}\int_{-\infty}^{\infty} d\eta\,\delta'(w)\eta(\pi - \eta)\tilde{\Phi}(\eta, v, \bar{v}), \end{aligned} \tag{71}$$

$$\begin{aligned} L &= L_{12}(0,0) \\ &= \left[m(x_1\dot{x}_2 - x_2\dot{x}_1) + \frac{1}{\bar{\gamma}\Omega}\int_{-\infty}^{\infty} d\eta \left(x_1\frac{\partial\Lambda}{\partial\dot{x}^2} - x_2\frac{\partial\Lambda}{\partial\dot{x}^1} \right) \right](0) + \left[\bar{m}(\bar{x}_1\dot{\bar{x}}_2 - \bar{x}_2\dot{\bar{x}}_1) + \frac{1}{\gamma\Omega}\int_{-\infty}^{\infty} d\eta \left(\bar{x}_1\frac{\partial\Lambda}{\partial\dot{\bar{x}}^2} - \bar{x}_2\frac{\partial\Lambda}{\partial\dot{\bar{x}}^1} \right) \right](0) \\ &\quad - \frac{2}{\gamma\bar{\gamma}\Omega^2}\int_{-\infty}^{\infty} d\eta\,\eta \left[\frac{\partial\Lambda}{\partial w}(x_2\bar{x}_1 - x_1\bar{x}_2) - \frac{1}{2}\left(\dot{x}_2\frac{\partial\Lambda}{\partial\dot{x}^1} - \dot{x}_1\frac{\partial\Lambda}{\partial\dot{x}^2} \right) \right] \end{aligned} \tag{72}$$

$$\begin{aligned} &= \frac{m\gamma v^2}{\Omega} - \frac{4m\bar{m}\gamma\bar{\gamma}v\bar{v}}{\Omega^2}\int_{-\infty}^{\infty} d\eta\,\delta(w)\left[(1 - v\bar{v}\cos\eta)\cos\eta - \frac{1}{2}\frac{v}{\bar{v}}(1 - \bar{v}^2) - \frac{1}{2}\frac{v}{\bar{v}}\frac{1}{1 - v^2}\tilde{\Phi}(\eta, v, \bar{v}) \right] \\ &\quad + \frac{\bar{m}\,\bar{\gamma}\,\bar{v}^2}{\Omega} - \frac{4m\bar{m}\gamma\bar{\gamma}v\bar{v}}{\Omega^2}\int_{-\infty}^{\infty} d\eta\,\delta(w)\left[(1 - v\bar{v}\cos\eta)\cos\eta - \frac{1}{2}\frac{\bar{v}}{v}(1 - v^2) - \frac{1}{2}\frac{\bar{v}}{v}\frac{1}{1 - \bar{v}^2}\tilde{\Phi}(\eta, v, \bar{v}) \right] \\ &\quad + \frac{4m\bar{m}\gamma\bar{\gamma}v\bar{v}}{\Omega^2}\int_{-\infty}^{\infty} d\eta \left[\frac{1}{\Omega^2}\delta'(w)\eta\sin\eta\tilde{\Phi}(\eta, v, \bar{v}) + \delta(w)(1 - v\bar{v}\cos\eta)\eta\sin\eta \right], \end{aligned} \tag{73}$$

where we have used a formula, Eq. (A5), derived in Appendix A 1.

Formulas for integrations involving a δ-function are summarized in Appendix A 2. First we define a function $\Phi(\varphi, v, \bar{v})$, related to $\tilde{\Phi}$ by an integration:

$$\Phi(\varphi, v, \bar{v}) = \frac{1}{\Omega^2}\int_{-\infty}^{\infty} d\eta\delta(w)\tilde{\Phi}(\eta, v, \bar{v}) = \frac{(1 + v\bar{v}\cos\varphi)^2 - \frac{1}{2}(1 - v^2)(1 - \bar{v}^2)}{\varphi + v\bar{v}\sin\varphi}. \quad (74)$$

Using this relation, we can evaluate the integrals appearing in the equations of motion (67) and (69), obtaining an algebraic relation for the radius of each orbit (or, equivalently, for the velocities $v = a\Omega$ and $\bar{v} = \bar{a}\Omega$) in terms of the angular velocity Ω:

$$-m\gamma^2 v\Omega = 4m\bar{m}\gamma^2\bar{\gamma}\Omega^2\frac{1}{(\varphi + v\bar{v}\sin\varphi)^2}\Big\{(1 + v\bar{v}\cos\varphi)\bar{v}(\varphi\cos\varphi - v^2\sin\varphi) + \frac{1}{2}v(1 - \bar{v}^2)(\varphi + v\bar{v}\sin\varphi)$$
$$- \frac{1}{2}\Big[\bar{v}\sin\varphi(\varphi + v\bar{v}\sin\varphi) + (1 + v\bar{v}\cos\varphi)(v + \bar{v}\cos\varphi) - \frac{v}{1 - v^2}(\varphi + v\bar{v}\sin\varphi)^2\Big]\Phi(\varphi, v, \bar{v})\Big\}, \quad (75)$$

$$-\bar{m}\bar{\gamma}^2\bar{v}\Omega = 4m\bar{m}\gamma\bar{\gamma}^2\Omega^2\frac{1}{(\varphi + v\bar{v}\sin\varphi)^2}\Big\{(1 + v\bar{v}\cos\varphi)v(\varphi\cos\varphi - \bar{v}^2\sin\varphi) + \frac{1}{2}\bar{v}(1 - v^2)(\varphi + v\bar{v}\sin\varphi)$$
$$- \frac{1}{2}\Big[v\sin\varphi(\varphi + v\bar{v}\sin\varphi) + (1 + v\bar{v}\cos\varphi)(\bar{v} + v\cos\varphi) - \frac{\bar{v}}{1 - \bar{v}^2}(\varphi + v\bar{v}\sin\varphi)^2\Big]\Phi(\varphi, v, \bar{v})\Big\}. \quad (76)$$

The expression for the angular momentum (73) is integrated by substituting the equations of motion Eqs. (67) and (69) into (73),

$$L = \frac{4m\bar{m}\gamma\bar{\gamma}v\bar{v}}{\Omega^2}\int_{-\infty}^{\infty} d\eta\delta(w)(1 - v\bar{v}\cos\eta)\eta\sin\eta$$
$$- \frac{4m\bar{m}\gamma\bar{\gamma}}{\Omega^4}\int_{-\infty}^{\infty} d\eta\delta'(w)\tilde{\Phi}(\eta, v, \bar{v})(v^2 + \bar{v}^2 - 2v\bar{v}\cos\eta - v\bar{v}\eta\sin\eta) \quad (77)$$

$$= -\frac{2m\bar{m}\gamma\bar{\gamma}}{\Omega^2}\int_{-\infty}^{\infty} d\eta[2w\delta'(w) + \delta(w)]\tilde{\Phi}(\eta, v, \bar{v}) \quad (78)$$

$$= \frac{2m\bar{m}\gamma\bar{\gamma}}{\Omega^2}\int_{-\infty}^{\infty} d\eta\delta(w)\tilde{\Phi}(\eta, v, \bar{v}) = 2m\bar{m}\gamma\bar{\gamma}\Phi(\varphi, v, \bar{v}). \quad (79)$$

Explicitly,

$$L = 2m\bar{m}\gamma\bar{\gamma}\frac{(1 + v\bar{v}\cos\varphi)^2 - \frac{1}{2}(1 - v^2)(1 - \bar{v}^2)}{\varphi + v\bar{v}\sin\varphi}, \quad (80)$$

with φ given by Eq. (63).

To compute the energy E, we first compute $E - \Omega L$:

$$E - \Omega L = m\gamma(1 - v^2) - \frac{4m\bar{m}\gamma\bar{\gamma}}{\Omega}\int_{-\infty}^{\infty} d\eta\delta(w)[\tilde{\Phi}(\eta, v, \bar{v}) + (1 - v\bar{v}\cos\eta)v\bar{v}\eta\sin\eta] + \bar{m}\,\bar{\gamma}(1 - \bar{v}^2)$$
$$- \frac{4m\bar{m}\gamma\bar{\gamma}}{\Omega^3}\int_{-\infty}^{\infty} d\eta\delta'(w)\eta[(\pi - \eta) + v\bar{v}\sin\eta]\tilde{\Phi}(\eta, v, \bar{v}) \quad (81)$$

$$= m\gamma(1 - v^2) + \bar{m}\,\bar{\gamma}(1 - \bar{v}^2) - \frac{2m\bar{m}\gamma\bar{\gamma}}{\Omega}\int_{-\infty}^{\infty} d\eta\delta(w)\tilde{\Phi}(\eta, v, \bar{v}) \quad (82)$$

$$= m\gamma(1 - v^2) + \bar{m}\,\bar{\gamma}(1 - \bar{v}^2) - 2m\bar{m}\gamma\bar{\gamma}\Omega\Phi(\varphi, v, \bar{v}). \quad (83)$$

Using the definitions (53) of γ and $\bar{\gamma}$, the expression for E takes the simple form,

$$E = \frac{m}{\gamma} + \frac{\bar{m}}{\bar{\gamma}} = m(1 - v^2)^{1/2} + \bar{m}(1 - \bar{v}^2)^{1/2}. \quad (84)$$

As noted in the introduction, this expression for E is identical to the expression for E in the *electromagnetic* two-body case derived by Schild [11], and to that for a scalar field, obtained in Appendix C. To clarify the structure of relevant equations used in calculations above, their formal expressions are presented in Appendix D 2.

The corresponding expressions for angular momentum are each proportional to the potential dotted on each free index with the helical symmetry vector k^α. For a scalar interaction, masses m and $\bar{m}$ have scalar charges q and $\bar{q}$. A scalar field ψ at m due to $\bar{q}$ satisfies

$$\Box\psi = -4\pi\bar{q}\int d\bar{\tau}\delta^4[x - \bar{x}(\bar{\tau})], \tag{85}$$

with the corresponding definition of $\bar{\psi}$. Similarly, for an electromagnetic interaction with charges e and $\bar{e}$, the vector potential A_α at m due to $\bar{e}$ satisfies (in the Lorentz gauge)

$$\Box A_\alpha = -4\pi\bar{e}\int d\bar{\tau}\bar{u}_\alpha(\bar{\tau})\delta^4[x - \bar{x}(\bar{\tau})]. \tag{86}$$

With ψ, A_α and $h_{\alpha\beta}$ evaluated at x, $\bar{\psi}$, $\bar{A}_\alpha$ and $\bar{h}_{\alpha\beta}$ evaluated at $\bar{x}$, the angular momentum L has the following forms for scalar, electromagnetic, and post-Minkowskian gravitational interactions:

Scalar charges q and $\bar{q}$:
$$L = -\frac{q}{\gamma}\psi = -\frac{\bar{q}}{\bar{\gamma}}\bar{\psi} = \frac{q^2}{\gamma\bar{\gamma}}\frac{1}{\varphi + v\bar{v}\cos\varphi} \tag{87}$$

Electromagnetic charges e and $-\bar{e}$:
$$L = -\frac{e}{\Omega}A_\alpha k^\alpha = -\frac{\bar{e}}{\Omega}\bar{A}_\alpha\bar{k}^\alpha = e\bar{e}\frac{1 + v\bar{v}\cos\varphi}{\varphi + v\bar{v}\cos\varphi}, \tag{88}$$

Post-Minkowski masses m and $\bar{m}$:
$$L = \frac{m\gamma}{2\Omega}h_{\alpha\beta}k^\alpha k^\beta = \frac{\bar{m}\,\bar{\gamma}}{2\Omega}\bar{h}_{\alpha\beta}\bar{k}^\alpha\bar{k}^\beta. \tag{89}$$

2. Equations of motion, energy and angular momentum for the affinely parametrized model

The analogous computations for the affinely parametrized formulation are outlined here. Performing the integrals in the radial components of the equations of motions (40) and (41), we obtain

$$\begin{aligned}-m\gamma^2 v\Omega = 4m\bar{m}\gamma^2\bar{\gamma}\Omega^2\frac{1}{(\varphi + v\bar{v}\sin\varphi)^2}\Big\{&(1 + v\bar{v}\cos\varphi)\bar{v}(\varphi\cos\varphi - v^2\sin\varphi) + \frac{1}{2}v(1 - \bar{v}^2)(\varphi + v\bar{v}\sin\varphi)\\ &-\frac{1}{2}[\bar{v}\sin\varphi(\varphi + v\bar{v}\sin\varphi) + (1 + v\bar{v}\cos\varphi)(v + \bar{v}\cos\varphi)]\Phi(\varphi, v, \bar{v})\Big\},\end{aligned} \tag{90}$$

$$\begin{aligned}-\bar{m}\bar{\gamma}^2\bar{v}\Omega = 4m\bar{m}\gamma\bar{\gamma}^2\Omega^2\frac{1}{(\varphi + v\bar{v}\sin\varphi)^2}\Big\{&(1 + v\bar{v}\cos\varphi)v(\varphi\cos\varphi - \bar{v}^2\sin\varphi) + \frac{1}{2}\bar{v}(1 - v^2)(\varphi + v\bar{v}\sin\varphi)\\ &-\frac{1}{2}[v\sin\varphi(\varphi + v\bar{v}\sin\varphi) + (1 + v\bar{v}\cos\varphi)(\bar{v} + v\cos\varphi)]\Phi(\varphi, v, \bar{v})\Big\}.\end{aligned} \tag{91}$$

The angular momentum turns out to have the same form as in the parametrization-invariant formulation,

$$\begin{aligned}L &= 2m\bar{m}\gamma\bar{\gamma}\Phi(\varphi, v, \bar{v})\\ &= 2m\bar{m}\gamma\bar{\gamma}\frac{(1 + v\bar{v}\cos\varphi)^2 - \frac{1}{2}(1 - v^2)(1 - \bar{v}^2)}{\varphi + v\bar{v}\sin\varphi}.\end{aligned} \tag{92}$$

We find, however, that the form of $E - \Omega L$ differs from the corresponding Eq. (83):

$$\begin{aligned}E - \Omega L &= m\gamma(1 - v^2) + \bar{m}\,\bar{\gamma}(1 - \bar{v}^2)\\ &\quad - 6m\bar{m}\gamma\bar{\gamma}\Omega\Phi(\varphi, v, \bar{v}).\end{aligned} \tag{93}$$

The normalizations (54) have the explicit form

$$\begin{aligned}(\eta_{\alpha\beta} + h_{\alpha\beta})\dot{x}^\alpha\dot{x}^\beta &= -\gamma^2(1 - v^2) + 4\bar{m}\gamma^2\bar{\gamma}\Omega\Phi(\varphi, v, \bar{v})\\ &= -1,\end{aligned} \tag{94}$$

$$\begin{aligned}(\bar{\eta}_{\alpha\beta} + \bar{h}_{\alpha\beta})\dot{\bar{x}}^\alpha\dot{\bar{x}}^\beta &= -\bar{\gamma}^2(1 - \bar{v}^2) + 4m\gamma\bar{\gamma}^2\Omega\Phi(\varphi, v, \bar{v})\\ &= -1.\end{aligned} \tag{95}$$

Substituting these in Eq. (93), the expression for the energy becomes

$$\begin{aligned}E &= \frac{m}{\gamma} + \frac{\bar{m}}{\bar{\gamma}} + 4m\bar{m}\gamma\bar{\gamma}\Omega\Phi(\varphi, v, \bar{v})\\ &= m\gamma(1 - v^2) + \bar{m}\,\bar{\gamma}(1 - \bar{v}^2) - 4m\bar{m}\gamma\bar{\gamma}\Omega\Phi(\varphi, v, \bar{v}).\end{aligned} \tag{96}$$

As implied by Eqs. (92) and (93), the form of E differs from the energy (84) of the parametrization-invariant action. Both, of course, agree in the Newtonian limit.

IV. FIRST LAW OF THERMODYNAMICS FOR BINARIES DESCRIBED BY FOKKER ACTIONS

The first law of thermodynamics governs nearby equilibria of conservative systems. For binary systems with circular orbits, an equilibrium solution is a solution that is stationary in a rotating frame, a solution with a helical symmetry vector k^α. When internal degrees of freedom (e.g., baryon number, entropy, vorticity) are fixed, the first law relates the change of energy to the change of angular momentum in the manner [5]

$$\delta E = \Omega \delta L. \tag{97}$$

In the present context, however, the presence of a radiation field whose energy is infinite makes the relation suspect; and the lack of a true action means that the simple Hamiltonian proof for point particles with a Newtonian potential does not hold. But the relation (97) is true, and its proof is an extension of the Hamiltonian proof that uses the parametrization invariance of the relativistic Fokker action.

Proposition—Consider a parametrization-invariant Fokker action of the form (5). Suppose there is a family of solutions $x^\alpha(s,\tau)$, $\bar{x}^\alpha(s,\bar\tau)$ for which the particles move in circular orbits with angular velocity $\Omega(s)$, with spacetime trajectories along the helical vector field $k^\alpha(s) = t^\alpha + \Omega(s)\phi^\alpha$. Then $\delta E = \Omega\delta L$, where $\delta Q := \frac{dQ}{ds}|_{s=0}$.

Parametrization invariance and constraints—As a prerequisite to the proof, we begin with a brief discussion of parametrization invariance of an Fokker action of the form (5) and the constraints that follow from it. These are analogs of the Hamiltonian constraint (vanishing of the super-Hamiltonian) associated with a true parametrization-invariant action. A parametrization $\tau \to f(\tau)$ maps the path $x^\alpha(\tau)$ to the path $X^\alpha = x^\alpha \circ f$. Similarly, $\bar\tau \to \bar f(\bar\tau)$ maps $\bar x^\alpha$ to $\bar X^\alpha = \bar x^\alpha \circ \bar f$. Invariance of the action,

$$I = -\int_{\tau_1}^{\tau_2} d\tau m(-\dot x_\alpha \dot x^\alpha)^{1/2} - \int_{\bar\tau_1}^{\bar\tau_2} d\bar\tau\, \bar m(-\dot{\bar x}_\alpha \dot{\bar x}^\alpha)^{1/2} + \int_{\tau_1}^{\tau_2} d\tau \int_{\bar\tau_1}^{\bar\tau_2} d\bar\tau \Lambda(x-\bar x, \dot x, \dot{\bar x}),$$

under reparametrization follows from the scaling

$$\Lambda(X-\bar X, \dot X^\alpha, \dot{\bar X}^\alpha)|_{\tau_0,\bar\tau_0} = \dot f(\tau_0)\dot{\bar f}(\bar\tau_0) \times \Lambda(x-\bar x, \dot x^\alpha, \dot{\bar x}^\alpha)|_{f(\tau_0),\bar f(\bar\tau_0)}. \tag{98}$$

For $f(\tau) = k\tau$, $\bar f(\bar\tau) = \bar\tau$, we have

$$\Lambda(X-\bar X, \dot X^\alpha, \dot{\bar X}^\alpha)|_{\tau_0/k,\bar\tau_0} = \Lambda(x-\bar x, k\dot x^\alpha, \dot{\bar x}^\alpha)|_{\tau_0,\bar\tau_0} = k\Lambda(x-\bar x, \dot x^\alpha, \dot{\bar x}^\alpha)|_{\tau_0,\bar\tau_0}.$$

Then the relation,

$$\frac{d}{dk}[k\Lambda(x-\bar x, \dot x, \dot{\bar x})]\Big|_{k=1} = \frac{d}{dk}[\Lambda(x-\bar x, k\dot x, \dot{\bar x})]\Big|_{k=1},$$

and its barred $\leftrightarrow$ unbarred counterpart imply

$$\Lambda = \dot x^\alpha \frac{\partial\Lambda}{\partial \dot x^\alpha}, \qquad \Lambda = \dot{\bar x}^\alpha \frac{\partial\Lambda}{\partial \dot{\bar x}^\alpha}. \tag{99}$$

Proof of proposition—The equations of motion involve the potentials

$$\mathcal{U}_m = \int_{-\infty}^{\infty} d\bar\tau\Lambda, \qquad \mathcal{U}_{\bar m} = \int_{-\infty}^{\infty} d\tau\Lambda. \tag{100}$$

That is, with 1-particle Lagrangians $\mathcal{L}_m$ and $\mathcal{L}_{\bar m}$ defined by

$$\mathcal{L}_m = -m(-\dot x^\alpha \dot x_\alpha)^{1/2} + \mathcal{U}_m, \qquad \mathcal{L}_{\bar m} = -\bar m(-\dot{\bar x}^\alpha \dot{\bar x}_\alpha)^{1/2} + \mathcal{U}_{\bar m}, \tag{101}$$

Equations (10) and (11) are the equations of motion for the actions

$$I_m = \int d\tau \mathcal{L}_m, \qquad I_{\bar m} = \int d\bar\tau \mathcal{L}_{\bar m}, \tag{102}$$

namely,

$$\frac{d}{d\tau}\left(\frac{\partial\mathcal{L}_m}{\partial \dot x^\alpha}\right) - \frac{\partial \mathcal{L}_m}{\partial x^\alpha} = 0, \qquad \frac{d}{d\bar\tau}\left(\frac{\partial\mathcal{L}_{\bar m}}{\partial \dot{\bar x}^\alpha}\right) - \frac{\partial \mathcal{L}_{\bar m}}{\partial \bar x^\alpha} = 0. \tag{103}$$

The one-particle 4-momentum p_α associated with the action I_m is

$$p_\alpha = \frac{\partial \mathcal{L}_m}{\partial \dot x^\alpha} = \frac{m\dot x_\alpha}{(-\dot x^\gamma \dot x_\gamma)^{1/2}} + \frac{\partial \mathcal{U}_m}{\partial \dot x^\alpha}. \tag{104}$$

Because $\mathcal{L}_m$ and $\dot x^\alpha$ have the same reparametrization scaling, namely

$$\mathcal{L}_m \to \dot f \mathcal{L}_m, \qquad \dot x^\alpha \to \dot f \dot x^\alpha, \tag{105}$$

the momentum p_α is independent of the choice of parameter.

Let t be a choice of Minkowski time, t^α the corresponding Killing vector (∂_t), and ϕ^α a rotational Killing vector orthogonal to t^α. The 1-particle energies associated with I_m and $I_{\bar m}$ are

$$E_m = -t^\alpha p_\alpha = -\frac{\partial \mathcal{L}_m}{\partial \dot t} = \frac{m\dot t}{(-\dot x^\gamma \dot x_\gamma)^{1/2}} - \frac{\partial \mathcal{U}_m}{\partial \dot t}, \tag{106}$$

$$E_{\bar m} = -t^\alpha \bar p_\alpha = -\frac{\partial \mathcal{L}_{\bar m}}{\partial \dot{\bar t}} = \frac{\bar m \dot{\bar t}}{(-\dot{\bar x}^\gamma \dot{\bar x}_\gamma)^{1/2}} - \frac{\partial \mathcal{U}_{\bar m}}{\partial \dot{\bar t}}; \tag{107}$$

and the 1-particle angular momenta are

$$L_m = \phi^\alpha p_\alpha = \frac{\partial \mathcal{L}_m}{\partial \dot\phi}, \qquad L_{\bar m} = \phi^\alpha \bar p_\alpha = \frac{\partial \mathcal{L}_{\bar m}}{\partial \dot{\bar\phi}}. \tag{108}$$

We first use the scaling relation and the equations of motion to show that these 1-particle momenta and angular momenta satisfy $\delta E_m = \Omega\delta L_m$, $\delta E_{\bar m} = \Omega \delta L_{\bar m}$. The scaling of $\mathcal{L}_m$ and $\mathcal{L}_{\bar m}$ (parametrization invariance of I_m) implies

$$\mathcal{L}_m = \dot x^\alpha \frac{\partial \mathcal{L}_m}{\partial \dot x^\alpha}, \qquad \mathcal{L}_{\bar m} = \dot{\bar x}^\alpha \frac{\partial \mathcal{L}_{\bar m}}{\partial \dot{\bar x}^\alpha}. \tag{109}$$

(In other words, the 1-particle super-Hamiltonians vanish:

$\mathcal{H}_m := \dot{x}^\alpha p_\alpha - \mathcal{L}_m = 0, \qquad \bar{\mathcal{H}}_{\bar{m}} := \dot{\bar{x}}^\alpha \bar{p}_\alpha - \mathcal{L}_{\bar{m}} = 0.$) Then

$$E_m = \frac{\dot{x}^a}{\dot{t}} \frac{\partial \mathcal{L}_m}{\partial \dot{x}^a} - \frac{\mathcal{L}_m}{\dot{t}} = v^a \frac{\partial \mathcal{L}_m}{\partial \dot{x}^a} - \frac{\mathcal{L}_m}{\dot{t}}. \tag{110}$$

(Note that $\mathcal{L}_m/\dot{t}$ is the form of the Lagrangian appropriate to Minkowski time: $\int dt \mathcal{L}_m/\dot{t} = \int d\tau \mathcal{L}_m$.) Consider now a family of solutions to the equation of motion for m with circular orbits, each solution stationary in a comoving frame. Neighboring orbits satisfy

$$\begin{aligned}\delta E_m &= \delta v^a \frac{\partial \mathcal{L}_m}{\partial \dot{x}^a} + v^a \delta \frac{\partial \mathcal{L}_m}{\partial \dot{x}^a} - \delta v^a \frac{\partial \mathcal{L}_m}{\partial \dot{x}^a} - \delta x^a \frac{\partial \mathcal{L}_m}{\partial x^a} \frac{1}{\dot{t}} \\ &= v^a \delta \frac{\partial \mathcal{L}_m}{\partial \dot{x}^a} = \Omega \delta L_m,\end{aligned} \tag{111}$$

where the equilibrium condition $\frac{\partial \mathcal{L}_m}{\partial \varpi} = 0$ was used to infer $\delta x^a \frac{\partial \mathcal{L}_m}{\partial x^a} = 0$.

The total energy E of Eq. (14) is not, however, the sum of the 1-particle energies: $E_m + E_{\bar{m}}$ does not include the field energy and is not, in general, conserved. Instead, the total energy has the form

$$\begin{aligned}E &= -P_\alpha t^\alpha \\ &= E_m + E_{\bar{m}} - \int_{-\infty}^{\tau} d\tau t^\alpha \nabla_\alpha \mathcal{U}_m - \int_{-\infty}^{\bar{\tau}} d\tau t^\alpha \bar{\nabla}_\alpha \mathcal{U}_{\bar{m}}.\end{aligned} \tag{112}$$

The total angular momentum is similarly

$$\begin{aligned}L &= P_\alpha \phi^\alpha \\ &= L_m + L_{\bar{m}} + \int_{-\infty}^{\tau} d\tau \phi^\alpha \nabla_\alpha \mathcal{U}_m + \int_{-\infty}^{\bar{\tau}} d\bar{\tau} \phi^\alpha \bar{\nabla}_\alpha \mathcal{U}_{\bar{m}},\end{aligned} \tag{113}$$

where we have used the notation

$$\begin{aligned}\phi^\alpha \nabla_\alpha \mathcal{U}_m &:= \frac{\partial \mathcal{U}_m}{\partial \phi} = \frac{\partial \mathcal{U}_m}{\partial x^\alpha} \frac{\partial x^\alpha}{\partial \phi} + \frac{\partial \mathcal{U}_m}{\partial \dot{x}^\alpha} \frac{\partial \dot{x}^\alpha}{\partial \phi} \\ &= \phi^\alpha \frac{\partial \mathcal{U}_m}{\partial x^\alpha} + \dot{\phi}^\alpha \frac{\partial \mathcal{U}_m}{\partial \dot{x}^\alpha}.\end{aligned} \tag{114}$$

[See also Eq. (D23) of Appendix D.]

To recover the first law, one uses the fact that E and L are independent of τ and $\bar{\tau}$, setting τ and $\bar{\tau}$ to $-\infty$ to eliminate the final term:

$$\begin{aligned}\delta E - \Omega \delta L &= \lim_{\tau, \bar{\tau} \to -\infty} (\delta E - \Omega \delta L) \\ &= \lim_{\tau, \bar{\tau} \to -\infty} \Big[\delta E_m - \Omega \delta L_m + \delta E_{\bar{m}} - \Omega \delta L_{\bar{m}} \\ &\quad + \int_{-\infty}^{\tau} d\tau k^\alpha \nabla_\alpha \delta \mathcal{U}_m + \int_{-\infty}^{\bar{\tau}} d\bar{\tau} k^\alpha \bar{\nabla}_\alpha \delta \mathcal{U}_{\bar{m}} \Big] \\ &= 0. \quad \square\end{aligned} \tag{115}$$

Note that the Killing symmetry, $k^\alpha \nabla_\alpha \mathcal{U}_m = 0$, does not in itself imply $k^\alpha \nabla_\alpha \delta \mathcal{U}_m = 0$, because $\delta k^\alpha = \phi^\alpha \delta \Omega \neq 0$.

First law for affinely parametrized action—The first law also holds for the affinely parametrized action (35). The proof is similar to that for the parametrization-invariant action, with one-particle Lagrangians now defined by

$$\mathcal{L}_m = \frac{m}{2} \dot{x}^\alpha \dot{x}_\alpha + \mathcal{U}_m, \qquad \mathcal{L}_{\bar{m}} = \frac{\bar{m}}{2} \dot{\bar{x}}^\alpha \dot{\bar{x}}_\alpha + \mathcal{U}_{\bar{m}}, \tag{116}$$

with the affinely parametrized Λ replacing the parametrization-invariant Λ in the definition of $\mathcal{U}_m$ and $\mathcal{U}_{\bar{m}}$. Note that affine parametrization is equivalent to the conditions

$$2\mathcal{L}_m = -m, \qquad 2\mathcal{L}_{\bar{m}} = -\bar{m}, \tag{117}$$

while the fact that $\mathcal{L}_m$ and $\mathcal{L}_{\bar{m}}$ are quadratic in the velocities $\dot{x}$ and $\dot{\bar{x}}$ implies

$$2\mathcal{L}_m = p_\alpha \dot{x}^\alpha, \qquad 2\mathcal{L}_{\bar{m}} = \bar{p}_\alpha \dot{\bar{x}}^\alpha. \tag{118}$$

The first of these relations, (117), together with the equilibrium condition $\partial \mathcal{L}_m / \partial \varpi = 0$, implies

$$\delta \mathcal{L}_m = \frac{\partial \mathcal{L}_m}{\partial \dot{x}^\alpha} \delta \dot{x}^\alpha = p_\alpha \delta \dot{x}^\alpha. \tag{119}$$

From the second relation, (118), we have $0 = -\delta m = \delta(p_\alpha \dot{x}^\alpha) = \delta p_\alpha \dot{x}^\alpha + p_\alpha \delta \dot{x}^\alpha$, whence, by (119),

$$\dot{x}^\alpha \delta p_\alpha = 0. \tag{120}$$

Again writing $E_m = -p_t$, $\dot{x}^a = u^t v^a$, we have

$$\delta E_m = v^a \delta p_a = \Omega \delta L_m. \tag{121}$$

Finally, the relation

$$\delta E = \Omega \delta L \tag{122}$$

follows as before from Eqs. (112)–(115), which hold as written.

V. ACTION ACCURATE TO FIRST POST-NEWTONIAN ORDER

In a first post-Newtonian approximation, the equations of motion have corrections of order v^2/c^2 to their Newtonian terms. As we noted above, the first post-Minkowski approximation, by including only terms linear in m and $\bar{m}$, fails to be accurate to first post-Newtonian order. The omitted terms in the equation of motion are quadratic in the masses m and $\bar{m}$, and they arise from a single term in the post-Newtonian Lagrangian, namely

$$L_{\rm PN} = -\frac{m\bar{m}(m + \bar{m})}{2R^2} \tag{123}$$

We can make the Fokker action accurate to first post-Newtonian order by adding any term that agrees to this order with (123), and we have tried two alternatives: A simplest parametrization-invariant choice of $\Lambda_{\rm PN}$ that reproduces (123) is

$$\Lambda_{\rm PN} = -\delta(t-\bar{t})\frac{m\bar{m}(m+\bar{m})}{2R^2}(\dot{x}_\gamma\dot{x}^\gamma\dot{\bar{x}}_\delta\dot{\bar{x}}^\delta)^{1/2}, \tag{124}$$

with value for a circular orbit given by

$$\Lambda_{\rm PN} = -\delta(\eta-\pi)\frac{\Omega^3}{2}\frac{m\bar{m}(m+\bar{m})}{v^2+\bar{v}^2-2v\bar{v}\cos\eta}. \tag{125}$$

A choice that is special-relativistically covariant (and parametrization invariant) is

$$\Lambda_{\rm SPN} = -\delta(w)\frac{m\bar{m}(m+\bar{m})}{2}\frac{(\dot{x}_\gamma\dot{x}^\gamma\dot{\bar{x}}_\delta\dot{\bar{x}}^\delta)^{3/4}}{(R_\alpha\dot{x}^\alpha R_\beta\dot{\bar{x}}^\beta)^{1/2}}. \tag{126}$$

The corresponding corrections $e_{\rm PN}$ and $e_{\rm SPN}$ to the conserved energy and the corresponding corrections $\ell_{\rm PN}$ and $\ell_{\rm SPN}$ to the angular momentum are given by

$$e_{\rm PN} = \tfrac{1}{2}\Omega\ell_{\rm PN}, \tag{127}$$

$$\ell_{\rm PN} = -\frac{m\bar{m}(m+\bar{m})\Omega}{\gamma\bar{\gamma}(v+\bar{v})^2}, \tag{128}$$

$$e_{\rm SPN} = \tfrac{1}{2}\Omega\ell_{\rm SPN}, \tag{129}$$

$$\ell_{\rm SPN} = -\frac{m\bar{m}(m+\bar{m})\Omega}{(\gamma\bar{\gamma})^{3/2}}\frac{1}{(\varphi+v\bar{v}\sin\varphi)^2}. \tag{130}$$

The derivation of Eqs. (127)–(130) and corrections to the equations of motion are given in Appendix B.

VI. NUMERICAL SOLUTION OF ORBITAL EQUATIONS

We have numerically solved the orbital equations, Eqs. (75) and (76) for the parametrization-invariant model, and Eqs. (90) and (91) for the affinely parametrized model, finding $a(\Omega)$ for equal-mass particles. Figures 2 and 3 show the relations for the post-Minkowski action.

As v increases, relativistic beaming decreases the strength of the gravitational field due to $\bar{m}$ at the position of m, and vice-versa. In the case of scalar and electromagnetically bound charges, the smaller field leads to a sharply smaller radius for a given velocity v; and the same effect implies a larger value of Ω at fixed v. Gravity, however, has the competing effect that, at small radius the field is stronger than the Newtonian field. In the exact theory, the result is that, for circular orbits about a fixed mass, the relativistic relation between Ω, r (and a velocity defined by $v \equiv \Omega r$) is identical to the Newtonian relation, when r is taken to be the circumferential radius. In the post-Minkowski models, as Figs. 2 and 3, the outcome depends on which post-Minkowski action one chooses. For the parametrization-invariant action, relativistic beaming dominates, giving a smaller radius for the same value of v and a correspondingly larger value of Ω at fixed v. The affine action, in contrast, gives a larger value of r at fixed v, an effect so pronounced for $v \gtrsim 0.3$ that r reaches a minimum value and then increases with increasing v.

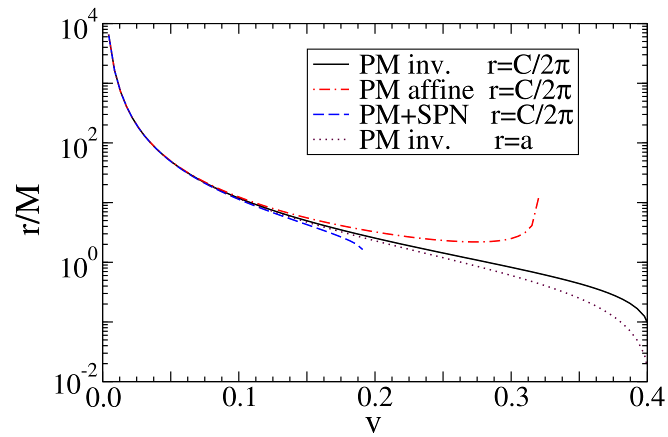

FIG. 2 (color online). The velocity v of a circular orbit versus the radius r, written in the dimensionless form r/M, for particles of equal mass $M := m + \bar{m}$, where the r is either the proper circumferential radius $C/2\pi$ or the radial parameter a. The motion is governed by the uncorrected parametrization-invariant post-Minkowski action (PM inv.), the affine parametrized action (PM affine) or the parametrization-invariant post-Minkowski action with a covariant post-Newtonian correction term (126) (PM + SPN). For the parametrization-invariant case both $C/2\pi$ and a are shown.

We were also surprised by the fact that the nonlinear term in m and $\bar{m}$ that is absent from our first post-Minkowski action has sign *opposite* to the Newtonian potential term. Figures 3 and 4 show only a small correction due to the post-Newtonian term (denoted PM + SPN in the graph), and the correction weakens the gravitational field.

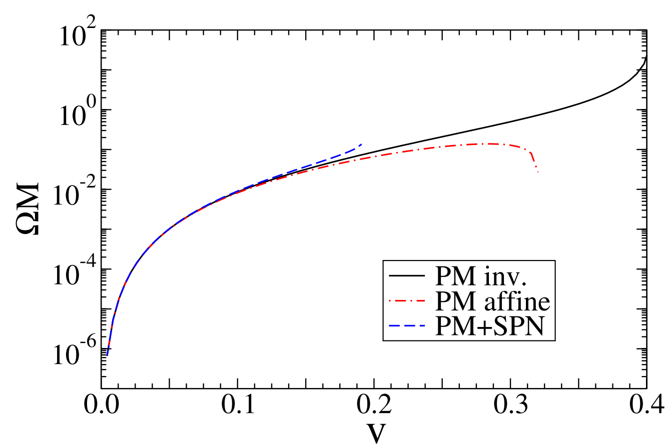

FIG. 3 (color online). The corresponding relation between Ω and v for the orbits of Fig. 2, with Ω written in the dimensionless form ΩM. The dashed curve (PM + SPN) shows the small correction to the solid curve (PM inv.) that arises from adding the post-Newtonian correction term (126) to the post-Minkowski action.

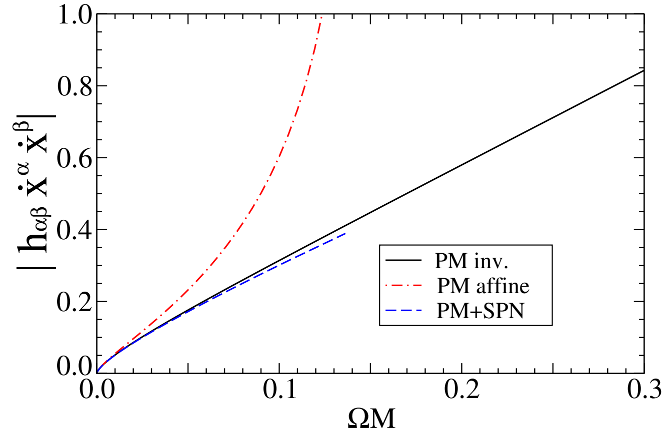

FIG. 4 (color online). The corresponding relation between Ω and $|h_{\alpha\beta}\dot{x}^{\alpha}\dot{x}^{\beta}|$ for the orbits of Fig. 2. The dashed curve (PM + SPN) shows the first-order post-Newtonian correction that is nonlinear in m (126) weakens the gravitational field.

Figures 5 and 6 display the values of the angular momentum and energy as functions of Ω, for the parametrization-invariant post-Minkowski (PM inv.) action, the affine parametrized post-Minkowski action (PM affine), the invariant post-Minkowski action with the special relativistically covariant first post-Newtonian correction (126) (PM + SPN), and the invariant post-Minkowski action with first post-Newtonian correction (124) (PM + PN). The values of Newtonian to third post-Newtonian approximations (0PN–3PN) are also plotted for references. Again relativistic beaming appears to dominate the post-Newtonian correction in the PM + PN and PM + SPN action, leading to a graph in which the energy and angular momenta have no minima. Because the relation

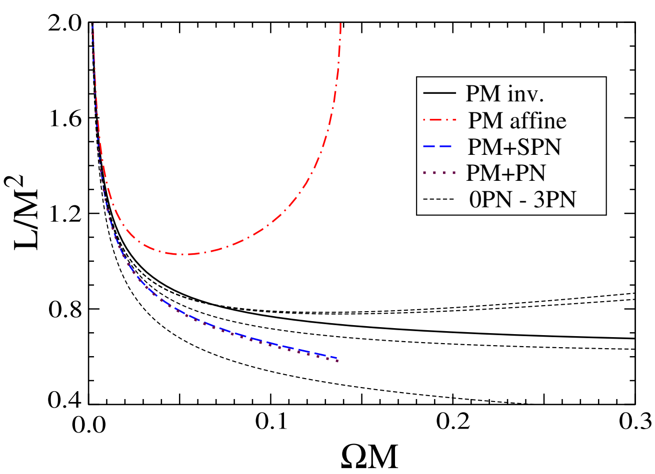

FIG. 5 (color online). Angular momentum, in dimensionless form L/M^2, is plotted against angular velocity for 8 cases. In the key labeling the curves, 0PN–3PN refer to models of equal-mass point-particles in circular orbit in the Newtonian, first post-Newtonian, second post-Newtonian, and third post-Newtonian approximations from the bottom to top.

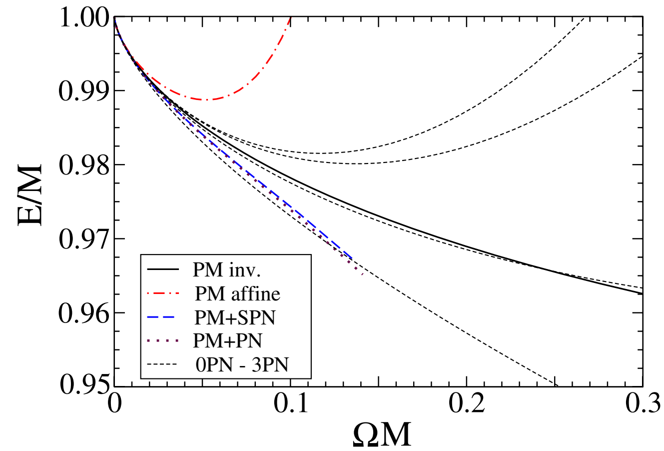

FIG. 6 (color online). Energy, in dimensionless form E/M, is plotted against angular velocity, for the models of Fig. 5.

$dE = \Omega dL$ can be used to show that the innermost stable circular orbit (ISCO) is a minimum of E and L, the ISCO present in the post-Newtonian approximation does not appear in PM inv., PM + SPN or PM + PN. Interestingly, PM affine model has a simultaneous minima in $E(\Omega)$ and $L(\Omega)$ for a sequence of circular solutions. Because the relation $dE = \Omega dL$ also holds for PM affine model as shown in Sec. IV, the circular solutions with ΩM larger than the value at the minima, $\Omega M = 0.0522$, are likely to be dynamically unstable. The values at the turning point of the other quantities are $v = 0.184$, $E/M = 0.988\,744$, and $L/M^2 = 1.027\,912\,3$. Each number above is given to computational accuracy.

Verification of our numerical solutions can be seen from Fig. 7, in which $1 - dE/\Omega dL$ is plotted against a measure Δ of the resolution of dL. As shown in Sec. IV, the relation $dE = \Omega dL$ is exact for Fokker actions of the form of PM

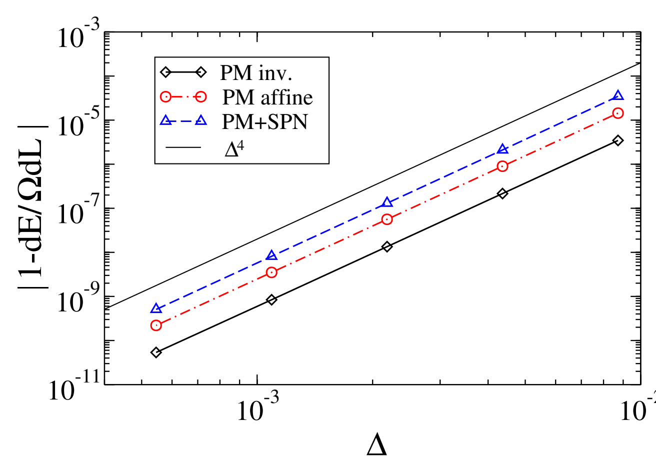

FIG. 7 (color online). Accuracy of the relation $dE = \Omega dL$. Thin solid lines have an inclination proportional to $O(\Delta^4)$, where Δ is proportional to dL. $dE/\Omega dL$ is evaluated at $v = 0.15$ using the 4th-order accurate finite difference formula (Lagrange formula).

inv., PM affine PM + SPN and PM + PN, and the numerical determination finds no discrepancy. (The result of PM + PN is omitted in Fig. 7 to avoid redundancy.)

VII. DISCUSSION

In the previous section, we discussed numerical results for the special case of gravitationally interacting particles in circular orbits. Here we consider a few surprises that arise in our more general study of Fokker actions.

Conserved quantities for particles with scalar, electromagnetic or gravitational interactions are ordinarily written as an integral over the field and its first time derivative on a spacelike hypersurface, together with sum of terms involving the position and velocity of each particle at its position on the hypersurface. In a Fokker formalism, the integral over the field is replaced by a sum of integrals over the trajectory of each particle. A striking feature of the resulting conserved 4-momentum and angular momentum is that they break up into a sum of quantities that are *separately conserved for each particle*, as in Eqs. (112) and (113) [or (D18) and (D20)]. In these equations, the separate conservation of each contribution follows from the fact that the contribution associated with each particle depends only on the proper time of that particle. How is this possible, when the field energy measures the *interaction* between the fields produced by each particle; how is it consistent with the fact that the total 4-momentum and angular momentum are, in general, an exhaustive set of integrals of the motion?

The answer is that the individual integrals that appear in the sum cannot be written as integrals over a hypersurface of a density locally constructed from the field and its first time derivative. They therefore do not represent new integrals of the motion. In fact, their existence is really a trivial consequence of the equation of motion satisfied by each particle:

$$\frac{d}{d\tau}\left(\frac{\partial \mathcal{L}_m}{\partial \dot{x}^\alpha}\right) - \frac{\partial \mathcal{L}_m}{\partial x^\alpha} = 0 \qquad (131)$$

where as before

$$\mathcal{L}_m := -m(-\dot{x}^\alpha \dot{x}_\alpha)^{1/2} + \mathcal{U}_m, \qquad \mathcal{U}_m := \int_{-\infty}^{\infty} d\bar{\tau}\Lambda. \qquad (132)$$

Integrating the equation of motion from τ_1 to τ_2, we have

$$p_\alpha|_{\tau_1}^{\tau_2} - \int_{\tau_1}^{\tau_2} d\tau \frac{\partial \mathcal{U}_m}{\partial x^\alpha} = 0, \qquad p_\alpha := \frac{\partial \mathcal{L}_m}{\partial \dot{x}^\alpha} \qquad (133)$$

implying $P_{m\alpha}$ is independent of τ, where

$$P_{m\alpha}(\tau) := p_\alpha(\tau) - \int_{-\infty}^{\tau} d\tau \frac{\partial \mathcal{U}_m}{\partial x^\alpha} \qquad (134)$$

More generally, dotting a vector ζ^α into the equation of motion (131) yields the identity

$$\frac{d}{d\tau}\left[\frac{m\dot{x}_\alpha \zeta^\alpha}{(-\dot{x}_\beta \dot{x}^\beta)^{1/2}} + \zeta^\alpha \frac{\partial \mathcal{U}_m}{\partial \dot{x}^\alpha}\right] - \left(\zeta^\alpha \frac{\partial \mathcal{U}_m}{\partial x^\alpha} + \dot{\zeta}^\alpha \frac{\partial \mathcal{U}_m}{\partial \dot{x}^\alpha}\right) - \frac{m\dot{x}_\alpha \dot{\zeta}^\alpha}{(-\dot{x}_\gamma \dot{x}^\gamma)^{1/2}} = 0. \qquad (135)$$

Then, because a Killing vector ζ^α of Minkowski space satisfies $\dot{x}^\alpha \dot{\zeta}_\alpha = \dot{x}^\alpha \dot{x}^\beta \nabla_\alpha \zeta_\beta = 0$, the quantity $\mathcal{Q}_m$ defined by

$$\mathcal{Q}_m = \left[\frac{m\dot{x}_\alpha \zeta^\alpha}{(-\dot{x}_\beta \dot{x}^\beta)^{1/2}} + \zeta^\alpha \frac{\partial \mathcal{U}_m}{\partial \dot{x}^\alpha}\right](\tau) - \int_{-\infty}^{\tau} d\tau \left(\zeta^\alpha \frac{\partial \mathcal{U}_m}{\partial x^\alpha} + \dot{\zeta}^\alpha \frac{\partial \mathcal{U}_m}{\partial \dot{x}^\alpha}\right) \qquad (136)$$

is independent of τ. In particular, for $\zeta^\alpha = -t^\alpha$ and $\zeta^\alpha = \phi^\alpha$, $\mathcal{Q}_m$ is the one-particle contribution to the energy and angular momentum, respectively. The generalization of these relations to an n-particle action is immediate.

The conserved quantities that arise from a Fokker action involve integrals along the particle paths whose integrands are not perfect time derivatives: That is, the integrals are path dependent.[2] Although the total 4-momentum and angular momentum are also expressed as path-dependent integrals in the Fokker formalism, they, presumably, differ from their 1-particle constituents by the fact that they can be expressed as integrals over a hypersurface of a local function of the field and its first time derivative (together with the kinematic momentum of each particle). The parts of these conserved quantities that directly involve the fields—the sum of terms involving $\mathcal{U}_m$ for each m—is not, however, the full field momentum or angular momentum,

$$\int T_\alpha{}^\beta \zeta^\alpha dS_\beta. \qquad (137)$$

Instead, only contributions to the field momentum (or angular momentum) that arise from the product of fields from different particles are present in the Fokker momentum (angular momentum). Terms quadratic in the field of a single particle must be absorbed in the renormalized mass.

This analysis leads to a conjecture for why the conserved Fokker momentum and angular momentum are finite for bound systems. When terms quadratic in the fields of

[2]One can similarly construct a path-dependent conserved "momentum" for a single Newtonian particle moving in any external field by writing

$$d\mathbf{p}/dt = -\nabla U(r)$$

$$\mathbf{p} + \int_{t_0}^{t} dt \nabla U = \text{constant}.$$

Because the integral depends on the path $r(t)$ from t_0 to t, one does not have a first integral of the equation of motion, in the sense of a conserved quantity that depends only on the position and velocity of the particle.

individual particles are subtracted, only a finite remainder survives.

Finally, we comment on ambiguities associated with the choice of gauge and with the choice of post-Newtonian correction terms. Solutions for two-particle circular orbits presented in this paper are obtained in the deDonder gauge. A sequence of equal mass solutions terminates near $v \sim 0.40$ for the parametrization-invariant action, and near $v \sim 0.32$ for the affine action, because in each case the equations of motion have no solution for v larger than a critical value v. When post-Newtonian corrections are added to the parametrization-invariant action, we find solutions only up to $v \sim 0.19$. When one proceeds, for example, to higher order calculation aiming to identify an ISCO, which may be expected to appear near these terminal values, a choice of gauge may be reconsidered with care.

A post-Minkowski action corrected by the terms $\Lambda_{\rm SPN}$ or $\Lambda_{\rm PN}$, does not exactly reproduce the first post-Newtonian equations of motion, because the post-Minkowski action contains terms of all post-Newtonian orders in the velocities. A more systematic approximation to the higher order post-Minkowski Fokker action may be obtained by deriving a form of interaction term that agrees with a formal expansion of the stress-energy tensor (see, for example, [21]).

ACKNOWLEDGMENTS

This work was supported by NSF Grants No. PHY 0071044 and PHY 0503366. J. F. thanks members of the PSW (periodic standing wave) consortium for relevant discussions at a meeting in Brownsville, March 12–13, 2005.

APPENDIX A: FORMULAS FOR INTEGRATION

We present here a formalism for evaluating the integrals that arise in the equations of motion, and in the expressions for momentum, energy, and angular momentum of point particles in circular orbits. This is closely patterned on formulas used by Schild in [22] in the case of two electrically charged particles.

1. Change of integration variables

We first derive a relation for rewriting integrals with respect to proper time τ and $\bar\tau$ as integrals with respect to the parameter η. From Eq. (58),

$$\frac{d\tau}{d\eta} = -\frac{1}{\gamma\Omega} \quad \text{and} \quad \frac{d\bar\tau}{d\eta} = \frac{1}{\bar\gamma\Omega}. \tag{A1}$$

The integral of a function $F(\eta)$ then becomes

$$\int_{-\infty}^{\infty} d\tau F(\eta) = \frac{1}{\gamma\Omega}\int_{-\infty}^{\infty} d\eta F(\eta) \quad \text{and}$$
$$\int_{-\infty}^{\infty} d\bar\tau F(\eta) = \frac{1}{\bar\gamma\Omega}\int_{-\infty}^{\infty} d\eta F(\eta). \tag{A2}$$

For a double integral of the kind that appears in the formulas for the linear momentum and energy, Eq. (14), and the angular momentum, Eq. (17), we may pick $\tau = \bar\tau = 0$. Then

$$\begin{aligned}\int_0^{\infty} d\tau \int_{-\infty}^{0} d\bar\tau F(\eta) &= \frac{1}{\gamma\bar\gamma\Omega^2}\int_0^{\infty} d\phi \int_{-\infty}^{0} d\bar\phi F(\eta)\\ &= \frac{1}{\gamma\bar\gamma\Omega^2}\int_0^{\infty} d\phi \int_{-\infty}^{-\phi} d\eta F(\eta)\\ &= \frac{1}{\gamma\bar\gamma\Omega^2}\int_{-\infty}^{0} d\eta \int_0^{-\eta} d\phi F(\eta)\\ &= \frac{1}{\gamma\bar\gamma\Omega^2}\int_{-\infty}^{0} d\eta(-\eta)F(\eta),\end{aligned} \tag{A3}$$

and

$$\begin{aligned}-\int_{-\infty}^{0} d\tau \int_0^{\infty} d\bar\tau F(\eta) &= -\frac{1}{\gamma\bar\gamma\Omega^2}\int_{-\infty}^{0} d\phi \int_0^{\infty} d\bar\phi F(\eta)\\ &= -\frac{1}{\gamma\bar\gamma\Omega^2}\int_{-\infty}^{0} d\phi \int_{-\phi}^{\infty} d\eta F(\eta)\\ &= -\frac{1}{\gamma\bar\gamma\Omega^2}\int_0^{\infty} d\eta \int_{-\eta}^{0} d\phi F(\eta)\\ &= -\frac{1}{\gamma\bar\gamma\Omega^2}\int_0^{\infty} d\eta\, \eta F(\eta);\end{aligned} \tag{A4}$$

adding the last two equalities, we have

$$\left(\int_0^{\infty} d\tau \int_{-\infty}^{0} d\bar\tau - \int_{-\infty}^{0} d\tau \int_0^{\infty} d\bar\tau\right)F(\eta) = -\frac{1}{\gamma\bar\gamma\Omega^2}\int_{-\infty}^{\infty} d\eta\, \eta F(\eta). \tag{A5}$$

Our next task is to evaluate integrals of the form $\int_{-\infty}^{\infty} d\tau \dot{\bar X}^\alpha$ and $\int_{-\infty}^{\infty} d\bar\tau \dot X^\alpha$, where X^α and $\bar X^\alpha$ are the vectors associated with m and $\bar m$, respectively. In terms of components along an orthonormal frame, the vectors have the form

$$X^\alpha = X^t(\eta)t^\alpha + X^\varpi(\eta)\varpi^\alpha + X^{\hat\phi}(\eta)\hat\phi^\alpha + X^z(\eta)z^\alpha, \tag{A6}$$

$$\bar X^\alpha = \bar X^t(\eta)t^\alpha + \bar X^\varpi(\eta)\bar\varpi^\alpha + \bar X^{\hat\phi}(\eta)\hat{\bar\phi}^\alpha + \bar X^z(\eta)z^\alpha. \tag{A7}$$

The corresponding expressions for the derivatives are

$$\dot X^\alpha = \frac{dX^\alpha}{d\tau} = -\gamma\Omega\left(\frac{dX^t}{d\eta}t^\alpha + \frac{dX^\varpi}{d\eta}\varpi^\alpha + \frac{dX^{\hat\phi}}{d\eta}\hat\phi^\alpha + \frac{dX^z}{d\eta}z^\alpha + X^{\hat\phi}\varpi^\alpha - X^\varpi\hat\phi^\alpha\right), \tag{A8}$$

$$\dot{\bar{X}}^{\alpha} = \frac{d\bar{X}^{\alpha}}{d\bar{\tau}} = \bar{\gamma}\Omega\left(\frac{d\bar{X}^{t}}{d\eta}t^{\alpha} + \frac{d\bar{X}^{\varpi}}{d\eta}\bar{\varpi}^{\alpha} + \frac{d\bar{X}^{\hat{\phi}}}{d\eta}\hat{\bar{\phi}}^{\alpha} + \frac{d\bar{X}^{z}}{d\eta}z^{\alpha} - \bar{X}^{\hat{\phi}}\bar{\varpi}^{\alpha} + \bar{X}^{\varpi}\hat{\bar{\phi}}^{\alpha}\right), \tag{A9}$$

where Eq. (A1) is used. Then the integral of $\dot{X}^{\alpha}$ with respect to $\bar{\tau}$, with fixed τ (and hence fixed ϕ), is then given by

$$\begin{aligned}\int_{-\infty}^{\infty} d\bar{\tau}\dot{X}^{\alpha} &= \int_{-\infty}^{\infty} d\bar{\tau}\frac{dX^{\alpha}}{d\tau} = -\frac{\gamma}{\bar{\gamma}}\int_{-\infty}^{\infty} d\eta\left[\frac{dX^{t}}{d\eta}t^{\alpha} + \frac{dX^{\varpi}}{d\eta}\varpi^{\alpha} + \frac{dX^{\hat{\phi}}}{d\eta}\hat{\phi}^{\alpha} + \frac{dX^{z}}{d\eta}z^{\alpha} + X^{\hat{\phi}}\varpi^{\alpha} - X^{\varpi}\hat{\phi}^{\alpha}\right]_{\tau=\mathrm{const}} \\ &= -\frac{\gamma}{\bar{\gamma}}\left\{[X^{\alpha}]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} d\eta(X^{\hat{\phi}}\varpi^{\alpha} - X^{\varpi}\hat{\phi}^{\alpha})\right\}_{\tau=\mathrm{const}} = \frac{\gamma}{\bar{\gamma}}\left(-\int_{-\infty}^{\infty} X^{\hat{\phi}}d\eta\varpi^{\alpha} + \int_{-\infty}^{\infty} X^{\varpi}d\eta\hat{\phi}^{\alpha}\right).\end{aligned} \tag{A10}$$

Similarly, the integral of $\dot{\bar{X}}^{\alpha}$ with respect to τ, with fixed $\bar{\tau}$ (and hence fixed $\bar{\phi}$) is

$$\begin{aligned}\int_{-\infty}^{\infty} d\tau\dot{\bar{X}}^{\alpha} &= \int_{-\infty}^{\infty} d\tau\frac{d\bar{X}^{\alpha}}{d\bar{\tau}} = \frac{\bar{\gamma}}{\gamma}\int_{-\infty}^{\infty} d\eta\left[\frac{d\bar{X}^{t}}{d\eta}t^{\alpha} + \frac{d\bar{X}^{\varpi}}{d\eta}\bar{\varpi}^{\alpha} + \frac{d\bar{X}^{\hat{\phi}}}{d\eta}\hat{\bar{\phi}}^{\alpha} + \frac{d\bar{X}^{z}}{d\eta}z^{\alpha} - \bar{X}^{\hat{\phi}}\bar{\varpi}^{\alpha} + \bar{X}^{\varpi}\hat{\bar{\phi}}^{\alpha}\right]_{\bar{\tau}=\mathrm{const}} \\ &= \frac{\bar{\gamma}}{\gamma}\left\{[\bar{X}^{\alpha}]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} d\eta(-\bar{X}^{\hat{\phi}}\bar{\varpi}^{\alpha} - \bar{X}^{\varpi}\hat{\bar{\phi}}^{\alpha})\right\}_{\bar{\tau}=\mathrm{const}} = \frac{\bar{\gamma}}{\gamma}\left(-\int_{-\infty}^{\infty} \bar{X}^{\hat{\phi}}d\eta\bar{\varpi}^{\alpha} + \int_{-\infty}^{\infty} \bar{X}^{\varpi}d\eta\hat{\bar{\phi}}^{\alpha}\right).\end{aligned} \tag{A11}$$

Note that basis vectors ϖ^{α} and $\hat{\phi}^{\alpha}$ are functions of ϕ; and $\bar{\varpi}^{\alpha}$ and $\hat{\bar{\phi}}^{\alpha}$ are functions of $\bar{\phi}$.

2. Integral formulas for half-retarded + half-advanced Green function

Integrals involving the half-retarded + half-advanced Green function $\delta(w)$ are derived in this section. As mentioned in Sec. III C, the solutions to $w(\eta) = 0$ are $\eta = \pi \pm \varphi$. It is assumed that any contribution from $\eta = \pm\infty$ vanishes. From Eq. (61), we have

$$\left(\frac{dw}{d\eta}\right)^{-1}_{\eta=\pi\pm\varphi} = \mp\frac{\Omega^2}{2}\frac{1}{\varphi + v\bar{v}\sin\varphi}, \tag{A12}$$

and hence

$$\begin{aligned}\int_{-\infty}^{\infty} d\eta\delta(w)F(\eta) &= \sum_{\eta=\pi\pm\varphi}\left|\frac{dw}{d\eta}\right|^{-1}F(\eta) \\ &= \frac{\Omega^2}{2}\frac{1}{\varphi + v\bar{v}\sin\varphi} \\ &\quad\times\{F(\pi+\varphi) + F(\pi-\varphi)\}.\end{aligned} \tag{A13}$$

For integrals involving a derivative of $\delta(w)$,

$$\begin{aligned}\int_{-\infty}^{\infty} d\eta\delta'(w)F(\eta) &= \int_{-\infty}^{\infty} d\eta\frac{d\delta[w(\eta)]}{d\eta}\left(\frac{dw}{d\eta}\right)^{-1}F(\eta) \\ &= -\int_{-\infty}^{\infty} d\eta\delta[w(\eta)]\frac{d}{d\eta} \\ &\quad\times\left[\left(\frac{dw}{d\eta}\right)^{-1}F(\eta)\right].\end{aligned} \tag{A14}$$

Using Eq. (A12) and its derivative,

$$\frac{d}{d\eta}\left(\frac{dw}{d\eta}\right)^{-1}_{\eta=\pi\pm\varphi} = \frac{\Omega^2}{2}\frac{1 + v\bar{v}\cos\varphi}{(\varphi + v\bar{v}\sin\varphi)^2}, \tag{A15}$$

the integral becomes

$$\begin{aligned}\int_{-\infty}^{\infty} d\eta\delta'(w)F(\eta) &= \frac{\Omega^4}{4}\frac{1}{(\varphi + v\bar{v}\sin\varphi)^2}\left\{\frac{dF}{d\eta}(\pi+\varphi)\right. \\ &\quad - \frac{dF}{d\eta}(\pi-\varphi) - \frac{1 + v\bar{v}\cos\varphi}{\varphi + v\bar{v}\sin\varphi} \\ &\quad\left.\times[F(\pi+\varphi) + F(\pi-\varphi)]\right\}.\end{aligned} \tag{A16}$$

Finally, the following relation is useful for computation of Eq. (78) in Sec. III D:

$$\begin{aligned}\int_{-\infty}^{\infty} d\eta\delta'(w)wF(\eta) &= -\int_{-\infty}^{\infty} d\eta\delta(w)F(\eta) \\ &\quad - \int_{-\infty}^{\infty} d\eta\delta(w)w\frac{d}{d\eta}\left[\frac{d\eta}{dw}F(\eta)\right] \\ &= -\int_{-\infty}^{\infty} d\eta\delta(w)F(\eta),\end{aligned} \tag{A17}$$

where, in the second term of the first equality, a derivative $\frac{d}{d\eta}[\frac{d\eta}{dw}F(\eta)]$ is assumed to be nonsingular [has an order larger than $O(w^{-1})$] in the neighborhood of $w = 0$.

APPENDIX B: POST-NEWTONIAN CORRECTIONS TO E AND L

In presenting relations used to compute corrections from Λ_{PN} to E, L, and the equations of motion, we avoid repetition by omitting barred $\leftrightarrow$ unbarred versions. The corrections to the parametrization-invariant post-Minkowski action are calculated, assuming the parametrization (52). We begin with the simpler correction term, (124). Using the relations from Sec. III C, with R in the form $R^2 = \frac{1}{\Omega^2}(v^2 + \bar{v}^2 - 2v\bar{v}\cos\eta)$, we obtain,

$$\Lambda_{\rm PN} = -\delta(t-\bar{t})\frac{m\bar{m}(m+\bar{m})}{2R^2}(-\dot{x}_\alpha\dot{x}^\alpha)^{1/2}(-\dot{\bar{x}}_\beta\dot{\bar{x}}^\beta)^{1/2} = -\delta(\eta-\pi)\frac{\Omega^3}{2}\frac{m\bar{m}(m+\bar{m})}{v^2+\bar{v}^2-2v\bar{v}\cos\eta}. \quad \text{(B1)}$$

Derivatives appearing in the equations of motion and the conserved quantities have the form

$$\frac{\partial\Lambda_{\rm PN}}{\partial(t-\bar{t})} = \delta'(\eta-\pi)\frac{\Omega^4}{2}\frac{m\bar{m}(m+\bar{m})}{v^2+\bar{v}^2-2v\bar{v}\cos\eta}, \quad \text{(B2)}$$

$$\frac{\partial\Lambda_{\rm PN}}{\partial(x^a-\bar{x}^a)} = \delta(\eta-\pi)\Omega^4\frac{m\bar{m}(m+\bar{m})}{(v^2+\bar{v}^2-2v\bar{v}\cos\eta)^2}\times\{(v-\bar{v}\cos\eta)\varpi_a-\bar{v}\sin\eta\hat{\phi}_a\}, \quad \text{(B3)}$$

$$\frac{\partial\Lambda_{\rm PN}}{\partial\dot{x}^\beta}\hat{\phi}^\beta = \delta(\eta-\pi)\frac{\Omega^3}{2}\frac{m\bar{m}(m+\bar{m})\gamma v}{v^2+\bar{v}^2-2v\bar{v}\cos\eta}, \quad \text{(B4)}$$

$$\frac{\partial\Lambda_{\rm PN}}{\partial\dot{x}^0} = \frac{\partial\Lambda_{\rm PN}}{\partial\dot{x}^\alpha}t^\alpha = -\delta(\eta-\pi)\frac{\Omega^3}{2}\frac{m\bar{m}(m+\bar{m})\gamma}{v^2+\bar{v}^2-2v\bar{v}\cos\eta}. \quad \text{(B5)}$$

The index a of x^a is spatial. The post-Newtonian correction term to the ϖ-component of the equation of motion is then

$$\int_{-\infty}^{\infty}d\bar{\tau}\left[\frac{\partial\Lambda_{\rm PN}}{\partial R^\alpha}-\frac{d}{d\tau}\frac{\partial\Lambda_{\rm PN}}{\partial\dot{x}^\alpha}\right]\varpi^\alpha = m\bar{m}(m+\bar{m})\frac{\Omega^3}{\bar{\gamma}(v+\bar{v})^3}\left[1+\frac{1}{2}\gamma^2v(v+\bar{v})\right]. \quad \text{(B6)}$$

The correction to the angular momentum is

$$\begin{aligned}\ell_{\rm PN} &= -\frac{1}{\gamma\bar{\gamma}\Omega^3}\int_{-\infty}^{\infty}d\eta\left(v\frac{\partial\Lambda_{\rm PN}}{\partial R^a}\varpi^a-\bar{v}\frac{\partial\Lambda_{\rm PN}}{\partial R^a}\bar{\varpi}^a\right)-\frac{v}{\gamma\bar{\gamma}\Omega^3}\int_{-\infty}^{\infty}d\eta\,\eta\frac{\partial\Lambda_{\rm PN}}{\partial R^a}\hat{\phi}^a\\ &= \frac{m\bar{m}(m+\bar{m})\Omega}{\gamma\bar{\gamma}}\int_{-\infty}^{\infty}d\eta\,\delta(\eta-\pi)\left[-\frac{1}{v^2+\bar{v}^2-2v\bar{v}\cos\eta}+\frac{v\bar{v}\sin\eta}{(v^2+\bar{v}^2-2v\bar{v}\cos\eta)^2}\right] = -\frac{m\bar{m}(m+\bar{m})\Omega}{\gamma\bar{\gamma}(v+\bar{v})^2},\end{aligned}$$

and the corresponding correction to the energy is

$$\begin{aligned}e_{\rm PN}-\Omega\ell_{\rm PN} &= -\frac{1}{\bar{\gamma}\Omega}\int_{-\infty}^{\infty}d\eta\frac{\partial\Lambda_{\rm PN}}{\partial\dot{x}^\alpha}k^\alpha-\frac{1}{\gamma\Omega}\int_{-\infty}^{\infty}d\eta\frac{\partial\Lambda_{\rm PN}}{\partial\dot{\bar{x}}^\alpha}\bar{k}^\alpha+\frac{1}{\gamma\bar{\gamma}\Omega^2}\int_{-\infty}^{\infty}d\eta\,\eta\frac{\partial\Lambda_{\rm PN}}{\partial R^\alpha}k^\alpha\\ &= \frac{m\bar{m}(m+\bar{m})\Omega^2}{\gamma\bar{\gamma}}\int_{-\infty}^{\infty}d\eta\left\{\delta(\eta-\pi)\left[\frac{1}{v^2+\bar{v}^2-2v\bar{v}\cos\eta}-\frac{v\bar{v}\eta\sin\eta}{(v^2+\bar{v}^2-2v\bar{v}\cos\eta)^2}\right]\right.\\ &\quad\left.+\frac{d\delta(\eta-\pi)}{d\eta}\frac{1}{2}\frac{\eta}{v^2+\bar{v}^2-2v\bar{v}\cos\eta}\right\}\\ &= \frac{m\bar{m}(m+\bar{m})\Omega^2}{2\gamma\bar{\gamma}(v+\bar{v})^2} = -\frac{1}{2}\Omega\ell_{\rm PN}.\end{aligned}$$

Finally,

$$e_{\rm PN} = \frac{1}{2}\Omega\ell_{\rm PN} = -\frac{m\bar{m}(m+\bar{m})\Omega^2}{2\gamma\bar{\gamma}(v+\bar{v})^2}. \quad \text{(B7)}$$

The analogous corrections from the special-relativistically covariant post-Newtonian correction are as follows. Correction to the equation of motion:

$$\begin{aligned}\int_{-\infty}^{\infty} d\bar{\tau}\Big[\frac{\partial \Lambda_{\rm SPN}}{\partial R^\alpha} - \frac{d}{d\tau}\frac{\partial}{\partial \dot{x}^\alpha}\Big]\varpi^\alpha &= \frac{1}{\bar{\gamma}\Omega}\int_{-\infty}^{\infty} d\eta\frac{\partial \Lambda_{\rm SPN}}{\partial R^\alpha}\varpi^\alpha + \frac{\gamma}{\bar{\gamma}}\int_{-\infty}^{\infty} d\eta\frac{\partial \Lambda_{\rm SPN}}{\partial \dot{x}^\alpha}\hat{\phi}^\alpha \\ &= -\frac{m\bar{m}(m+\bar{m})\Omega}{\bar{\gamma}(\gamma\bar{\gamma})^{1/2}}\int_{-\infty}^{\infty} d\eta\Big\{\frac{1}{\Omega^2}\delta'(w)\sigma\frac{v - \bar{v}\cos\eta}{\eta - \pi - v\bar{v}\sin\eta} \\ &\quad + \delta(w)\sigma\Big[\frac{\frac{1}{2}\bar{v}\sin\eta}{(\eta-\pi-v\bar{v}\sin\eta)^2} - \frac{\frac{3}{4}\gamma^2 v}{\eta - \pi - v\bar{v}\sin\eta}\Big]\Big\} \\ &= \frac{m\bar{m}(m+\bar{m})\Omega^3}{\bar{\gamma}(\gamma\bar{\gamma})^{1/2}}\frac{1}{(\varphi + v\bar{v}\sin\varphi)^2}\Big\{\frac{3}{4}\gamma^2 v + \frac{\bar{v}\sin\varphi}{\varphi + v\bar{v}\sin\varphi} \\ &\quad + \frac{(1+v\bar{v}\cos\varphi)(v+\bar{v}\cos\varphi)}{(\varphi+v\bar{v}\sin\varphi)^2}\Big\}, \end{aligned} \tag{B8}$$

where $\sigma = \pm 1$ for $\eta - \pi - v\bar{v}\sin\eta \gtrless 0$, respectively.

Correction to the angular momentum:

$$\begin{aligned}\ell_{\rm SPN} &= -\frac{1}{\gamma\bar{\gamma}\Omega^3}\int_{-\infty}^{\infty} d\eta\Big(v\frac{\partial \Lambda_{\rm SPN}}{\partial R^\alpha}\varpi^\alpha - \bar{v}\frac{\partial \Lambda_{\rm SPN}}{\partial R^\alpha}\bar{\varpi}^\alpha\Big) - \frac{1}{\gamma\bar{\gamma}\Omega^2}\int_{-\infty}^{\infty} d\eta\,\eta\Big(\frac{\partial \Lambda_{\rm SPN}}{\partial R^\alpha}\phi^\alpha - \frac{\partial \Lambda_{\rm SPN}}{\partial \dot{x}^\alpha}\dot{\phi}^\alpha\Big) \\ &= \frac{m\bar{m}(m+\bar{m})}{(\gamma\bar{\gamma})^{3/2}\Omega}\int_{-\infty}^{\infty} d\eta\Big(\frac{1}{\Omega^2}\delta'(w)\sigma\frac{v^2+\bar{v}^2 - 2v\bar{v}\cos\eta - v\bar{v}\eta\sin\eta}{\eta - \pi - v\bar{v}\sin\eta} + \delta(w)\sigma\frac{\frac{1}{2}v\bar{v}(\sin\eta - \eta\cos\eta)}{(\eta-\pi-v\bar{v}\sin\eta)^2}\Big) \\ &= -\frac{m\bar{m}(m+\bar{m})\Omega}{(\gamma\bar{\gamma})^{3/2}}\frac{1}{(\varphi+v\bar{v}\sin\varphi)^2}. \end{aligned} \tag{B9}$$

Correction to the energy:

$$\begin{aligned} e_{\rm SPN} - \Omega\ell_{\rm SPN} &= -\frac{1}{\bar{\gamma}\Omega}\int_{-\infty}^{\infty} d\eta\frac{\partial \Lambda_{\rm SPN}}{\partial \dot{x}^\alpha}k^\alpha - \frac{1}{\gamma\Omega}\int_{-\infty}^{\infty} d\eta\frac{\partial \Lambda_{\rm SPN}}{\partial \dot{\bar{x}}^\alpha}\bar{k}^\alpha + \frac{1}{\gamma\bar{\gamma}\Omega^2}\int_{-\infty}^{\infty} d\eta\,\eta\Big(\frac{\partial \Lambda_{\rm SPN}}{\partial R^\alpha}k^\alpha + \frac{\partial \Lambda_{\rm SPN}}{\partial \dot{x}^\alpha}\Omega\dot{\phi}^\alpha\Big) \\ &= -\frac{m\bar{m}(m+\bar{m})}{(\gamma\bar{\gamma})^{3/2}}\int_{-\infty}^{\infty} d\eta\Big[\frac{1}{\Omega^2}\delta'(w)\sigma\eta + \delta(w)\sigma\frac{\eta(1-v\bar{v}\cos\eta)}{2(\eta-\pi-v\bar{v}\sin\eta)^2} - \delta(w)\sigma\frac{1}{\eta-\pi-v\bar{v}\sin\eta}\Big] \\ &= \frac{1}{2}\frac{m\bar{m}(m+\bar{m})\Omega^2}{(\gamma\bar{\gamma})^{3/2}}\frac{1}{(\varphi+v\bar{v}\sin\varphi)^2} = -\frac{1}{2}\Omega\ell_{\rm SPN}. \end{aligned} \tag{B10}$$

Finally,

$$e_{\rm SPN} = \frac{1}{2}\Omega\ell_{\rm SPN} = -\frac{1}{2}\frac{m\bar{m}(m+\bar{m})\Omega^2}{(\gamma\bar{\gamma})^{3/2}}\frac{1}{(\varphi+v\bar{v}\sin\varphi)^2}. \tag{B11}$$

APPENDIX C: SCALAR INTERACTION

A scalar field is described by a Fokker action of the form (5), with

$$\Lambda = q\bar{q}\delta(w)(-\dot{x}_\alpha\dot{x}^\alpha)^{1/2}(-\dot{\bar{x}}_\alpha\dot{\bar{x}}^\alpha)^{1/2}. \tag{C1}$$

The equation of motion for two particles in circular orbit has, for m, the form,

$$\begin{aligned}\frac{d}{d\tau}\frac{m\dot{x}_\alpha}{(-\dot{x}_\alpha\dot{x}^\alpha)^{1/2}}\varpi^\alpha &= -m\gamma^2 v\Omega = \int_{-\infty}^{\infty} d\eta\left\{\frac{\partial\Lambda}{\partial w}\frac{2}{\bar{\gamma}\Omega^2}(v-\bar{v}\cos\eta)+\frac{\gamma}{\bar{\gamma}}\frac{\partial\Lambda}{\partial\dot{x}^\alpha}\hat{\phi}^\alpha\right\}\\ &= \frac{2q\bar{q}}{\bar{\gamma}}\int_{-\infty}^{\infty} d\eta\left\{\frac{1}{\Omega^2}\delta'(w)(v-\bar{v}\cos\eta)-\delta(w)\frac{1}{2}\gamma^2 v\right\}\\ &= \frac{q\bar{q}\Omega^2}{\bar{\gamma}^2}\frac{1}{\varphi+v\bar{v}\sin\varphi}\left\{\gamma^2 v+\frac{\bar{v}\sin\varphi}{\varphi+v\bar{v}\sin\varphi}+\frac{(1+v\bar{v}\cos\varphi)(v+\bar{v}\cos\varphi)}{\varphi+v\bar{v}\sin\varphi}\right\},\end{aligned}$$

with the corresponding barred $\leftrightarrow$ unbarred equation for $\bar{m}$.[3]

The formalism developed earlier quickly leads to expressions for angular momentum and energy:

$$\begin{aligned}L &= -\int_{-\infty}^{\infty} d\eta\frac{\partial\Lambda}{\partial w}\frac{2}{\gamma\bar{\gamma}\Omega^4}(v^2+\bar{v}^2-2v\bar{v}\cos\eta)-\frac{2}{\gamma\bar{\gamma}\Omega^2}\int_{-\infty}^{\infty} d\eta\eta\left[\frac{\partial\Lambda}{\partial w}(x_2\bar{x}_1-x_1\bar{x}_2)-\frac{1}{2}\left(\dot{x}_2\frac{\partial\Lambda}{\partial\dot{x}^1}-\dot{x}_1\frac{\partial\Lambda}{\partial\dot{x}^2}\right)\right]\\ &= -\frac{2q\bar{q}}{\gamma\bar{\gamma}\Omega^2}\int_{-\infty}^{\infty} d\eta\frac{1}{\Omega^2}\delta'(w)(v^2+\bar{v}^2-2v\bar{v}\cos\eta-v\bar{v}\eta\sin\eta)=\frac{q\bar{q}}{\gamma\bar{\gamma}}\frac{1}{\varphi+v\bar{v}\sin\varphi}.\end{aligned}\tag{C2}$$

$$\begin{aligned}E-\Omega L &= -\left[m\dot{x}_\alpha k^\alpha+\frac{1}{\Omega\bar{\gamma}}\int_{-\infty}^{\infty} d\eta\frac{\partial\Lambda}{\partial\dot{x}^\alpha}k^\alpha\right]-\left[\bar{m}\dot{\bar{x}}_\alpha\bar{k}^\alpha+\frac{1}{\Omega\gamma}\int_{-\infty}^{\infty} d\eta\frac{\partial\Lambda}{\partial\dot{\bar{x}}^\alpha}\bar{k}^\alpha\right]+\frac{2}{\gamma\bar{\gamma}}\Omega^2\int_{-\infty}^{\infty} d\eta\eta\frac{\partial\Lambda}{\partial w}R_\alpha k^\alpha\\ &= m\gamma(1-v^2)-\frac{q\bar{q}}{\Omega\bar{\gamma}}\int_{-\infty}^{\infty} d\eta\delta(w)\gamma(1-v^2)+\bar{m}\,\bar{\gamma}(1-\bar{v}^2)-\frac{q\bar{q}}{\Omega\gamma}\int_{-\infty}^{\infty} d\eta\delta(w)\bar{\gamma}(1-\bar{v}^2)\\ &\quad+\frac{2q\bar{q}}{\gamma\bar{\gamma}\Omega^3}\int_{-\infty}^{\infty} d\eta\delta'(w)\eta(\eta-\pi-v\bar{v}\sin\eta)\\ &= \frac{m}{\gamma}+\frac{\bar{m}}{\bar{\gamma}}-\frac{2q\bar{q}}{\gamma\bar{\gamma}\Omega}\int_{-\infty}^{\infty} d\eta\delta(w)-\frac{q\bar{q}}{\gamma\bar{\gamma}\Omega}\int_{-\infty}^{\infty} d\eta\delta'(w)\eta\frac{dw}{d\eta}=\frac{m}{\gamma}+\frac{\bar{m}}{\bar{\gamma}}-\frac{q\bar{q}}{\gamma\bar{\gamma}\Omega}\int_{-\infty}^{\infty} d\eta\delta(w)\\ &= \frac{m}{\gamma}+\frac{\bar{m}}{\bar{\gamma}}-\frac{q\bar{q}\Omega}{\gamma\bar{\gamma}}\frac{1}{\varphi+v\bar{v}\sin\varphi}=\frac{m}{\gamma}+\frac{\bar{m}}{\bar{\gamma}}-\Omega L.\end{aligned}\tag{C3}$$

Then

$$E=\frac{m}{\gamma}+\frac{\bar{m}}{\bar{\gamma}},\tag{C4}$$

$$L=\frac{q\bar{q}}{\gamma\bar{\gamma}}\frac{1}{\varphi+v\bar{v}\sin\varphi}.\tag{C5}$$

APPENDIX D: DERIVATIONS OF THE MOMENTUM AND ANGULAR MOMENTUM FORMULAS

In this appendix, we present calculations and useful expressions relating to the momentum and the angular momentum formulas, for an arbitrary interaction term $\Lambda(x-\bar{x},\dot{x},\dot{\bar{x}})$.

1. Derivations of Eqs. (14) and (17)

Dettman and Schild [18] derived the momentum and the angular momentum formulas from Poincaré invariance of the general Fokker action. We follow their calculation for the case of our action for two point particles to derive Eqs. (14) and (17), from the varied action (8).

We first rewrite Eq. (8) for δI, substituting the equation of motion for each particle, Eqs. (10) and (11):

[3]An equivalent solution has been obtained independently by Jean-Philippe Bruneton and Gilles Esposito-Farèse [23].

$$\begin{aligned}\delta I(\tau_1,\tau_2,\bar\tau_1,\bar\tau_2) &= \left[\frac{m\dot x_\alpha}{(-\dot x_\gamma \dot x^\gamma)^{1/2}} + \int_{-\infty}^{\infty} d\bar\tau \frac{\partial\Lambda}{\partial \dot x^\alpha}\right]\delta x^\alpha \Big|_{\tau_1}^{\tau_2} + \int_{\tau_1}^{\tau_2} d\tau \frac{d}{d\tau}\left[\delta x^\alpha\left(\int_{\bar\tau_1}^{\bar\tau_2} - \int_{-\infty}^{\infty}\right)d\bar\tau \frac{\partial\Lambda}{\partial\dot x^\alpha}\right] \\
&\quad + \int_{\tau_1}^{\tau_2} d\tau \delta x^\alpha\left(\int_{\bar\tau_1}^{\bar\tau_2} - \int_{-\infty}^{\infty}\right)d\bar\tau\left[\frac{\partial\Lambda}{\partial R^\alpha} - \frac{d}{d\tau}\frac{\partial\Lambda}{\partial\dot x^\alpha}\right] + \left[\frac{\bar m\dot{\bar x}_\alpha}{(-\dot{\bar x}_\gamma\dot{\bar x}^\gamma)^{1/2}} + \int_{-\infty}^{\infty} d\tau\frac{\partial\Lambda}{\partial\dot{\bar x}^\alpha}\right]\delta\bar x^\alpha\Big|_{\bar\tau_1}^{\bar\tau_2} \\
&\quad + \int_{\bar\tau_1}^{\bar\tau_2} d\bar\tau\frac{d}{d\bar\tau}\left[\delta\bar x^\alpha\left(\int_{\tau_1}^{\tau_2} - \int_{-\infty}^{\infty}\right)d\tau\frac{\partial\Lambda}{\partial\dot{\bar x}^\alpha}\right] + \int_{\bar\tau_1}^{\bar\tau_2} d\bar\tau\delta\bar x^\alpha\left(\int_{\tau_1}^{\tau_2} - \int_{-\infty}^{\infty}\right)d\tau\left[\frac{\partial\Lambda}{\partial\bar R^\alpha} - \frac{d}{d\bar\tau}\frac{\partial\Lambda}{\partial\dot{\bar x}^\alpha}\right] \\
&= \left[\frac{m\dot x_\alpha}{(-\dot x_\gamma \dot x^\gamma)^{1/2}} + \int_{-\infty}^{\infty} d\bar\tau \frac{\partial\Lambda}{\partial \dot x^\alpha}\right]\delta x^\alpha \Big|_{\tau_1}^{\tau_2} + \int_{\tau_1}^{\tau_2} d\tau\left(\int_{\bar\tau_1}^{\bar\tau_2} - \int_{-\infty}^{\infty}\right)d\bar\tau\left[\delta x^\alpha\frac{\partial\Lambda}{\partial R^\alpha} + \delta\dot x^\alpha\frac{\partial\Lambda}{\partial\dot x^\alpha}\right] \\
&\quad + \left[\frac{\bar m\dot{\bar x}_\alpha}{(-\dot{\bar x}_\gamma\dot{\bar x}^\gamma)^{1/2}} + \int_{-\infty}^{\infty} d\tau\frac{\partial\Lambda}{\partial\dot{\bar x}^\alpha}\right]\delta\bar x^\alpha\Big|_{\bar\tau_1}^{\bar\tau_2} + \int_{\bar\tau_1}^{\bar\tau_2} d\bar\tau\left(\int_{\tau_1}^{\tau_2} - \int_{-\infty}^{\infty}\right)d\tau\left[\delta\bar x^\alpha\frac{\partial\Lambda}{\partial\bar R^\alpha} + \delta\dot{\bar x}^\alpha\frac{\partial\Lambda}{\partial\dot{\bar x}^\alpha}\right], \end{aligned} \tag{D1}$$

where

$$\delta\dot x^\alpha := \frac{d\delta x^\alpha}{d\tau}, \qquad \delta\dot{\bar x}^\alpha := \frac{d\delta\bar x^\alpha}{d\bar\tau}. \tag{D2}$$

We next restrict the variations of the trajectory of each particle, δx^α and $\delta\bar x^\alpha$, to infinitesimal Poincaré transformations [which we will subsequently take to have the forms (12) and (15)]. Since the Fokker action, and hence the interaction Λ, is Poincaré invariant, the interaction term Λ satisfies the identity

$$\Lambda(x + \delta x - \bar x - \delta\bar x, \dot x + \delta\dot x, \dot{\bar x} + \delta\dot{\bar x}) = \Lambda(x - \bar x, \dot x, \dot{\bar x}). \tag{D3}$$

Expanding the identity to first order in δx and $\delta\bar x$, we have

$$\delta x^\alpha\frac{\partial\Lambda}{\partial R^\alpha} + \delta\bar x^\alpha\frac{\partial\Lambda}{\partial\bar R^\alpha} + \delta\dot x^\alpha\frac{\partial\Lambda}{\partial\dot x^\alpha} + \delta\dot{\bar x}^\alpha\frac{\partial\Lambda}{\partial\dot{\bar x}^\alpha} = 0. \tag{D4}$$

Using this relation, the variation of the action (D1) becomes

$$\begin{aligned}\delta I(\tau_1,\tau_2,\bar\tau_1,\bar\tau_2) &= \left[\frac{m\dot x_\alpha}{(-\dot x_\gamma \dot x^\gamma)^{1/2}} + \int_{-\infty}^{\infty} d\bar\tau \frac{\partial\Lambda}{\partial \dot x^\alpha}\right]\delta x^\alpha \Big|_{\tau_1}^{\tau_2} + \left[\frac{\bar m\dot{\bar x}_\alpha}{(-\dot{\bar x}_\gamma\dot{\bar x}^\gamma)^{1/2}} + \int_{-\infty}^{\infty} d\tau\frac{\partial\Lambda}{\partial\dot{\bar x}^\alpha}\right]\delta\bar x^\alpha\Big|_{\bar\tau_1}^{\bar\tau_2} \\
&\quad + \left(\int_{-\infty}^{\infty} d\tau\int_{\bar\tau_1}^{\bar\tau_2} d\bar\tau - \int_{\tau_1}^{\tau_2} d\tau\int_{-\infty}^{\infty} d\bar\tau\right)\left[\delta x^\alpha\frac{\partial\Lambda}{\partial R^\alpha} + \delta\dot x^\alpha\frac{\partial\Lambda}{\partial\dot x^\alpha}\right]. \end{aligned} \tag{D5}$$

The double integral in Eq. (D5) is rearranged to separate the contribution from $(\tau_1, \bar\tau_1)$ and $(\tau_2, \bar\tau_2)$ explicitly as follows,

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{\bar\tau_1}^{\bar\tau_2} - \int_{\tau_1}^{\tau_2}\int_{-\infty}^{\infty} &= \int_{-\infty}^{\tau_1}\left(\int_{\bar\tau_1}^{\infty} - \int_{\bar\tau_2}^{\infty}\right) + \int_{\tau_1}^{\infty}\left(\int_{-\infty}^{\bar\tau_2} - \int_{-\infty}^{\bar\tau_1}\right) - \left(\int_{\tau_1}^{\infty} - \int_{\tau_2}^{\infty}\right)\int_{-\infty}^{\bar\tau_2} - \left(\int_{-\infty}^{\tau_2} - \int_{-\infty}^{\tau_1}\right)\int_{\bar\tau_2}^{\infty} \\
&= \left(\int_{\tau_2}^{\infty}\int_{-\infty}^{\bar\tau_2} - \int_{-\infty}^{\tau_2}\int_{\bar\tau_2}^{\infty}\right) - \left(\int_{\tau_1}^{\infty}\int_{-\infty}^{\bar\tau_1} - \int_{-\infty}^{\tau_1}\int_{\bar\tau_1}^{\infty}\right). \end{aligned} \tag{D6}$$

Finally, the variation δI with respect to an infinitesimal Poincaré transformation takes the form

$$\begin{aligned}\delta I(\tau_1,\tau_2,\bar\tau_1,\bar\tau_2) &= \left[\frac{m\dot x_\alpha}{(-\dot x_\gamma \dot x^\gamma)^{1/2}} + \int_{-\infty}^{\infty} d\bar\tau \frac{\partial\Lambda}{\partial \dot x^\alpha}\right]\delta x^\alpha \Big|_{\tau_1}^{\tau_2} + \left[\frac{\bar m\dot{\bar x}_\alpha}{(-\dot{\bar x}_\gamma\dot{\bar x}^\gamma)^{1/2}} + \int_{-\infty}^{\infty} d\tau\frac{\partial\Lambda}{\partial\dot{\bar x}^\alpha}\right]\delta\bar x^\alpha\Big|_{\bar\tau_1}^{\bar\tau_2} \\
&\quad + \left(\int_{\tau}^{\infty}\int_{-\infty}^{\bar\tau} - \int_{-\infty}^{\tau}\int_{\bar\tau}^{\infty}\right)\left[\delta x^\alpha\frac{\partial\Lambda}{\partial R^\alpha} + \delta\dot x^\alpha\frac{\partial\Lambda}{\partial\dot x^\alpha}\right]d\tau d\bar\tau\Big|_{\bar\tau_1}^{\bar\tau_2}. \end{aligned} \tag{D7}$$

To derive the conserved momentum, Eq. (14), a translation by a constant vector (12) is substituted in (D7):

$$\frac{\delta I}{\delta a^\alpha} = \left[\frac{m\dot x_\alpha}{(-\dot x_\gamma \dot x^\gamma)^{1/2}} + \int_{-\infty}^{\infty} d\bar\tau \frac{\partial\Lambda}{\partial \dot x^\alpha}\right]_{\tau_1}^{\tau_2} + \left[\frac{\bar m\dot{\bar x}_\alpha}{(-\dot{\bar x}_\gamma\dot{\bar x}^\gamma)^{1/2}} + \int_{-\infty}^{\infty} d\tau\frac{\partial\Lambda}{\partial\dot{\bar x}^\alpha}\right]_{\bar\tau_1}^{\bar\tau_2} + \left(\int_{\tau}^{\infty}\int_{-\infty}^{\bar\tau} - \int_{-\infty}^{\tau}\int_{\bar\tau}^{\infty}\right)\frac{\partial\Lambda}{\partial R^\alpha}d\tau d\bar\tau\Big|_{\bar\tau_1}^{\bar\tau_2}, \tag{D8}$$

where we used $\delta\dot x^\alpha = da^\alpha/d\tau = 0$.

The angular momentum similarly corresponds to an infinitesimal rotation (15):

$$2\frac{\delta I}{\delta\epsilon^{\beta\alpha}}=\left[\frac{m(x_\alpha\dot{x}_\beta-x_\beta\dot{x}_\alpha)}{(-\dot{x}_\gamma\dot{x}^\gamma)^{1/2}}+\int_{-\infty}^{\infty}d\bar{\tau}\left(x_\alpha\frac{\partial\Lambda}{\partial\dot{x}^\beta}-x_\beta\frac{\partial\Lambda}{\partial\dot{x}^\alpha}\right)\right]_{\tau_1}^{\tau_2}+\left[\frac{\bar{m}(\bar{x}_\alpha\dot{\bar{x}}_\beta-\bar{x}_\beta\dot{\bar{x}}_\alpha)}{(-\dot{\bar{x}}_\gamma\dot{\bar{x}}^\gamma)^{1/2}}+\int_{-\infty}^{\infty}d\tau\left(\bar{x}_\alpha\frac{\partial\Lambda}{\partial\dot{\bar{x}}^\beta}-\bar{x}_\beta\frac{\partial\Lambda}{\partial\dot{\bar{x}}^\alpha}\right)\right]_{\bar{\tau}_1}^{\bar{\tau}_2}$$
$$+\left(\int_\tau^\infty\int_{-\infty}^{\bar{\tau}}-\int_{-\infty}^{\tau}\int_{\bar{\tau}}^{\infty}\right)\left[x_\alpha\frac{\partial\Lambda}{\partial R^\beta}-x_\beta\frac{\partial\Lambda}{\partial R^\alpha}+\dot{x}_\alpha\frac{\partial\Lambda}{\partial\dot{x}^\beta}-\dot{x}_\beta\frac{\partial\Lambda}{\partial\dot{x}^\alpha}\right]d\tau d\bar{\tau}\,\Bigg|_{\bar{\tau}_1}^{\bar{\tau}_2}, \tag{D9}$$

where $\delta\dot{x}^\alpha=\epsilon^{\alpha\beta}\dot{x}_\beta$ is used.

An analogous derivation of the momentum and angular momentum for the affinely parametrized action results in the replacements

$$\frac{m\dot{x}_\alpha}{(-\dot{x}_\gamma\dot{x}^\gamma)^{1/2}}\to m\dot{x}_\alpha,\qquad\frac{\bar{m}\dot{\bar{x}}_\alpha}{(-\dot{\bar{x}}_\gamma\dot{\bar{x}}^\gamma)^{1/2}}\to\bar{m}\dot{\bar{x}}_\alpha, \tag{D10}$$

in the momentum Eq. (D8), and

$$\frac{m(x_\alpha\dot{x}_\beta-x_\beta\dot{x}_\alpha)}{(-\dot{x}_\gamma\dot{x}^\gamma)^{1/2}}\to m(x_\alpha\dot{x}_\beta-x_\beta\dot{x}_\alpha),$$
$$\frac{\bar{m}(\bar{x}_\alpha\dot{\bar{x}}_\beta-\bar{x}_\beta\dot{\bar{x}}_\alpha)}{(-\dot{\bar{x}}_\gamma\dot{\bar{x}}^\gamma)^{1/2}}\to\bar{m}(\bar{x}_\alpha\dot{\bar{x}}_\beta-\bar{x}_\beta\dot{\bar{x}}_\alpha), \tag{D11}$$

in the angular momentum Eq. (D9).

The double integrals in Eqs. (D7)–(D9) have a useful alternative form:

$$\left(\int_\tau^\infty\int_{-\infty}^{\bar{\tau}}-\int_{-\infty}^{\tau}\int_{\bar{\tau}}^{\infty}\right)d\tau d\bar{\tau}=\left(\int_{-\infty}^{\infty}-\int_{-\infty}^{\tau}\right)\int_{-\infty}^{\bar{\tau}}d\tau d\bar{\tau}-\int_{-\infty}^{\tau}\left(\int_{-\infty}^{\infty}-\int_{-\infty}^{\bar{\tau}}\right)d\tau d\bar{\tau}$$
$$=-\int_{-\infty}^{\tau}d\tau\int_{-\infty}^{\infty}d\bar{\tau}+\int_{-\infty}^{\bar{\tau}}d\bar{\tau}\int_{-\infty}^{\infty}d\tau. \tag{D12}$$

The terms depending on the proper time of each path τ and $\bar{\tau}$ can be separated by writing the double integral in the above form, (D12), as seen in Eqs. (D18) and (D20). These are used in the proof of the first law in Sec. IV.

2. Formulas for E, L, and $E-\Omega L$

Next, we derive expressions for the nonzero components of the angular momentum $L:=L_{12}(\tau,\bar{\tau})$, the 4-momentum $E:=-P_\alpha(\tau,\bar{\tau})t^\alpha$, and a combination $E-\Omega L$ for parametrization-invariant action. The formulas for the affinely parametrized action can be obtained by replacements analogous to Eqs. (D10) and (D11), which are not repeated.

We begin with the angular momentum: Since the basis ϕ^α and $\bar{\phi}^\alpha$ at positions of particles $\{m,x\}$ and $\{\bar{m},\bar{x}\}$ have components in Cartesian coordinate $\phi^\alpha=(-x^2,x^1,0)=(-x_2,x_1,0)$ and $\bar{\phi}^\alpha=(-\bar{x}^2,\bar{x}^1,0)=(-\bar{x}_2,\bar{x}_1,0)$, one has relations

$$x_1A_2-x_2A_1=A_\alpha\phi^\alpha\quad\text{and}\quad\bar{x}_1\bar{A}_2-\bar{x}_2\bar{A}_1=\bar{A}_\alpha\bar{\phi}^\alpha, \tag{D13}$$

where A_α and $\bar{A}_\alpha$ are vectors at positions m and $\bar{m}$. Applying the relations (D13) to the L_{12} component of the angular momentum Eq. (17) yields

$$L=\left[\frac{m\dot{x}_\alpha\phi^\alpha}{(-\dot{x}_\beta\dot{x}^\beta)^{1/2}}+\int_{-\infty}^{\infty}d\bar{\tau}\frac{\partial\Lambda}{\partial\dot{x}^\alpha}\phi^\alpha\right](\tau)+\left[\frac{\bar{m}\dot{\bar{x}}_\alpha\bar{\phi}^\alpha}{(-\dot{\bar{x}}_\beta\dot{\bar{x}}^\beta)^{1/2}}+\int_{-\infty}^{\infty}d\tau\frac{\partial\Lambda}{\partial\dot{\bar{x}}^\alpha}\bar{\phi}^\alpha\right](\bar{\tau})$$
$$+\left(\int_\tau^\infty\int_{-\infty}^{\bar{\tau}}-\int_{-\infty}^{\tau}\int_{\bar{\tau}}^{\infty}\right)\left(\frac{\partial\Lambda}{\partial R^\alpha}\phi^\alpha+\frac{\partial\Lambda}{\partial\dot{x}^\alpha}\dot{\phi}^\alpha\right)d\tau d\bar{\tau}, \tag{D14}$$

where we define $\dot{\phi}^\alpha=\frac{d}{d\tau}\phi^\alpha$. From the definition of the linear momentum (14), the energy $E=-P_\alpha t^\alpha$ has the corresponding form

$$E=-\left[\frac{m\dot{x}_\alpha t^\alpha}{(-\dot{x}_\gamma\dot{x}^\gamma)^{1/2}}+\int_{-\infty}^{\infty}d\bar{\tau}\frac{\partial\Lambda}{\partial\dot{x}^\alpha}t^\alpha\right](\tau)-\left[\frac{\bar{m}\dot{\bar{x}}_\alpha t^\alpha}{(-\dot{\bar{x}}_\gamma\dot{\bar{x}}^\gamma)^{1/2}}+\int_{-\infty}^{\infty}d\tau\frac{\partial\Lambda}{\partial\dot{\bar{x}}^\alpha}t^\alpha\right](\bar{\tau})-\left(\int_\tau^\infty\int_{-\infty}^{\bar{\tau}}-\int_{-\infty}^{\tau}\int_{\bar{\tau}}^{\infty}\right)\frac{\partial\Lambda}{\partial R^\alpha}t^\alpha d\tau d\bar{\tau}. \tag{D15}$$

The combination $E-\Omega L$ then becomes

$$E-\Omega L=-\left[\frac{m\dot{x}_\alpha k^\alpha}{(-\dot{x}_\beta\dot{x}^\beta)^{1/2}}+\int_{-\infty}^{\infty}d\bar{\tau}\frac{\partial\Lambda}{\partial\dot{x}^\alpha}k^\alpha\right](\tau)-\left[\frac{\bar{m}\dot{\bar{x}}_\alpha\bar{k}^\alpha}{(-\dot{\bar{x}}_\beta\dot{\bar{x}}^\beta)^{1/2}}+\int_{-\infty}^{\infty}d\tau\frac{\partial\Lambda}{\partial\dot{\bar{x}}^\alpha}\bar{k}^\alpha\right](\bar{\tau})$$
$$-\left(\int_\tau^\infty\int_{-\infty}^{\bar{\tau}}-\int_{-\infty}^{\tau}\int_{\bar{\tau}}^{\infty}\right)\left(\frac{\partial\Lambda}{\partial R^\alpha}k^\alpha+\frac{\partial\Lambda}{\partial\dot{x}^\alpha}\Omega\dot{\phi}^\alpha\right)d\tau d\bar{\tau}. \tag{D16}$$

One can write the energy (D15) and the angular momentum (D14) in a form related to the one-particle energy and angular momentum defined in Sec. IV, Eqs. (104) and (106)–(108), together with the relation for the double integral (D12). Interestingly, the resulting formulas for the energy and angular momentum depend *separately* on the proper time of each path, τ and $\bar{\tau}$. Using the property

$$\frac{\partial \Lambda}{\partial R^\alpha} = \frac{\partial \Lambda}{\partial x^\alpha} = -\frac{\partial \Lambda}{\partial \bar{x}^\alpha}, \tag{D17}$$

we can rewrite the energy (D15) in terms of the one-particle potential $\mathcal{U}_m$ and $\mathcal{U}_{\bar{m}}$,

$$\begin{aligned} E = &-\bigg[\frac{m\dot{x}_\alpha t^\alpha}{(-\dot{x}_\gamma \dot{x}^\gamma)^{1/2}} + t^\alpha \frac{\partial}{\partial \dot{x}^\alpha}\int_{-\infty}^{\infty} d\bar{\tau}\Lambda\bigg](\tau) - \bigg[\frac{\bar{m}\dot{\bar{x}}_\alpha t^\alpha}{(-\dot{\bar{x}}_\gamma \dot{\bar{x}}^\gamma)^{1/2}} + t^\alpha \frac{\partial}{\partial \dot{\bar{x}}^\alpha}\int_{-\infty}^{\infty} d\tau\Lambda\bigg](\bar{\tau}) \\ &+ \bigg(\int_{-\infty}^{\tau} d\tau \int_{-\infty}^{\infty} d\bar{\tau} - \int_{-\infty}^{\bar{\tau}} d\bar{\tau}\int_{-\infty}^{\infty} d\tau\bigg)\frac{\partial \Lambda}{\partial R^\alpha} t^\alpha \\ = &-\bigg[\frac{m\dot{x}_\alpha t^\alpha}{(-\dot{x}_\gamma \dot{x}^\gamma)^{1/2}} + t^\alpha \frac{\partial \mathcal{U}_m}{\partial \dot{x}^\alpha}\bigg](\tau) + \int_{-\infty}^{\tau} d\tau t^\alpha \frac{\partial \mathcal{U}_m}{\partial x^\alpha} - \bigg[\frac{\bar{m}\dot{\bar{x}}_\alpha t^\alpha}{(-\dot{\bar{x}}_\gamma \dot{\bar{x}}^\gamma)^{1/2}} + t^\alpha \frac{\partial \mathcal{U}_{\bar{m}}}{\partial \dot{\bar{x}}^\alpha}\bigg](\bar{\tau}) + \int_{-\infty}^{\bar{\tau}} d\bar{\tau} t^\alpha \frac{\partial \mathcal{U}_{\bar{m}}}{\partial \bar{x}^\alpha}. \end{aligned} \tag{D18}$$

To relate the angular momentum (D14) to one-particle angular momentum, we notice that the relation (D4), together with properties (D13) and (D17), implies

$$\phi^\alpha \frac{\partial \Lambda}{\partial x^\alpha} + \bar{\phi}^\alpha \frac{\partial \Lambda}{\partial \bar{x}^\alpha} + \dot{\phi}^\alpha \frac{\partial \Lambda}{\partial \dot{x}^\alpha} + \dot{\bar{\phi}}^\alpha \frac{\partial \Lambda}{\partial \dot{\bar{x}}^\alpha} = 0. \tag{D19}$$

Like the energy, the angular momentum can be written in terms of the potentials $\mathcal{U}_m$ and $\mathcal{U}_{\bar{m}}$,

$$\begin{aligned} L = &\bigg[\frac{m\dot{x}_\alpha \phi^\alpha}{(-\dot{x}_\beta \dot{x}^\beta)^{1/2}} + \phi^\alpha \frac{\partial}{\partial \dot{x}^\alpha}\int_{-\infty}^{\infty} d\bar{\tau}\Lambda\bigg](\tau) + \bigg[\frac{\bar{m}\dot{\bar{x}}_\alpha \bar{\phi}^\alpha}{(-\dot{\bar{x}}_\beta \dot{\bar{x}}^\beta)^{1/2}} + \bar{\phi}^\alpha \frac{\partial}{\partial \dot{\bar{x}}^\alpha}\int_{-\infty}^{\infty} d\tau\Lambda\bigg](\bar{\tau}) \\ &- \bigg(\int_{-\infty}^{\tau} d\tau \int_{-\infty}^{\infty} d\bar{\tau} - \int_{-\infty}^{\bar{\tau}} d\bar{\tau}\int_{-\infty}^{\infty} d\tau\bigg)\bigg(\frac{\partial \Lambda}{\partial R^\alpha}\phi^\alpha + \frac{\partial \Lambda}{\partial \dot{x}^\alpha}\dot{\phi}^\alpha\bigg) \\ = &\bigg[\frac{m\dot{x}_\alpha \phi^\alpha}{(-\dot{x}_\beta \dot{x}^\beta)^{1/2}} + \phi^\alpha \frac{\partial \mathcal{U}_m}{\partial \dot{x}^\alpha}\bigg](\tau) - \int_{-\infty}^{\tau} d\tau\bigg(\phi^\alpha \frac{\partial \mathcal{U}_m}{\partial x^\alpha} + \dot{\phi}^\alpha \frac{\partial \mathcal{U}_m}{\partial \dot{x}^\alpha}\bigg) \\ &+ \bigg[\frac{\bar{m}\dot{\bar{x}}_\alpha \bar{\phi}^\alpha}{(-\dot{\bar{x}}_\beta \dot{\bar{x}}^\beta)^{1/2}} + \bar{\phi}^\alpha \frac{\partial \mathcal{U}_{\bar{m}}}{\partial \dot{\bar{x}}^\alpha}\bigg](\bar{\tau}) - \int_{-\infty}^{\bar{\tau}} d\bar{\tau}\bigg(\bar{\phi}^\alpha \frac{\partial \mathcal{U}_{\bar{m}}}{\partial \bar{x}^\alpha} + \dot{\bar{\phi}}^\alpha \frac{\partial \mathcal{U}_{\bar{m}}}{\partial \dot{\bar{x}}^\alpha}\bigg). \end{aligned} \tag{D20}$$

The terms in the brackets in the above Eqs. (D18) and (D20) are the one-particle energy and angular momentum; the contribution from each particle which is moving in the field produced by the other particle (see Sec. IV).

It is also noticeable that the formulas for E, L and $E - \Omega L$, Eqs. (D14)–(D16), can be written in a common form, because of a property $\dot{t}^\alpha := dt^\alpha/d\tau = 0$, $\dot{k}^\alpha = \Omega\dot{\phi}^\alpha$ accordingly, and those for corresponding barred quantities. Writing $\mathcal{Q}$ to represent these conserved quantities, E, L, and $E - \Omega L$, and ζ^α for the associated vectors, t^α, ϕ^α, and k^α, we have

$$\begin{aligned} \mathcal{Q} = &\bigg[\frac{m\dot{x}_\alpha \zeta^\alpha}{(-\dot{x}_\beta \dot{x}^\beta)^{1/2}} + \int_{-\infty}^{\infty} d\bar{\tau}\frac{\partial \Lambda}{\partial \dot{x}^\alpha}\zeta^\alpha\bigg](\tau) + \bigg[\frac{\bar{m}\dot{\bar{x}}_\alpha \bar{\zeta}^\alpha}{(-\dot{\bar{x}}_\beta \dot{\bar{x}}^\beta)^{1/2}} + \int_{-\infty}^{\infty} d\tau\frac{\partial \Lambda}{\partial \dot{\bar{x}}^\alpha}\bar{\zeta}^\alpha\bigg](\bar{\tau}) \\ &+ \bigg(\int_{\tau}^{\infty}\int_{-\infty}^{\bar{\tau}} - \int_{-\infty}^{\tau}\int_{\bar{\tau}}^{\infty}\bigg)\bigg(\frac{\partial \Lambda}{\partial R^\alpha}\zeta^\alpha + \frac{\partial \Lambda}{\partial \dot{x}^\alpha}\dot{\zeta}^\alpha\bigg)d\tau d\bar{\tau}. \end{aligned} \tag{D21}$$

Also for the expressions in terms of the one-particle potentials, we have

$$\begin{aligned} \mathcal{Q} = &\bigg[\frac{m\dot{x}_\alpha \zeta^\alpha}{(-\dot{x}_\beta \dot{x}^\beta)^{1/2}} + \zeta^\alpha \frac{\partial \mathcal{U}_m}{\partial \dot{x}^\alpha}\bigg](\tau) - \int_{-\infty}^{\tau} d\tau\bigg(\zeta^\alpha \frac{\partial \mathcal{U}_m}{\partial x^\alpha} + \dot{\zeta}^\alpha \frac{\partial \mathcal{U}_m}{\partial \dot{x}^\alpha}\bigg) \\ &+ \bigg[\frac{\bar{m}\dot{\bar{x}}_\alpha \bar{\zeta}^\alpha}{(-\dot{\bar{x}}_\beta \dot{\bar{x}}^\beta)^{1/2}} + \bar{\zeta}^\alpha \frac{\partial \mathcal{U}_{\bar{m}}}{\partial \dot{\bar{x}}^\alpha}\bigg](\bar{\tau}) - \int_{-\infty}^{\bar{\tau}} d\bar{\tau}\bigg(\bar{\zeta}^\alpha \frac{\partial \mathcal{U}_{\bar{m}}}{\partial \bar{x}^\alpha} + \dot{\bar{\zeta}}^\alpha \frac{\partial \mathcal{U}_{\bar{m}}}{\partial \dot{\bar{x}}^\alpha}\bigg). \end{aligned} \tag{D22}$$

In Eqs. (112)–(115) in Sec. IV, the integrand of Eq. (D22) is written in a short form defined by

$$\zeta^\alpha \nabla_\alpha \mathcal{U}_m := \zeta^\alpha \frac{\partial \mathcal{U}_m}{\partial x^\alpha} + \dot{\zeta}^\alpha \frac{\partial \mathcal{U}_m}{\partial \dot{x}^\alpha}, \qquad \bar{\zeta}^\alpha \bar{\nabla}_\alpha \mathcal{U}_{\bar{m}} := \bar{\zeta}^\alpha \frac{\partial \mathcal{U}_{\bar{m}}}{\partial \bar{x}^\alpha} + \dot{\bar{\zeta}}^\alpha \frac{\partial \mathcal{U}_{\bar{m}}}{\partial \dot{\bar{x}}^\alpha}. \tag{D23}$$

As seen in the above expression (D22), contributions to the total energy and angular momentum from time τ and $\bar{\tau}$ are separated. It means that each piece conserves independently; E L and $E - \Omega L$, are a sum of two conserved quantities associated with τ and $\bar{\tau}$.

For circular motion, the velocities are given by $\dot{x}^\alpha = \gamma k^\alpha$ and $\dot{\bar{x}}^\alpha = \gamma \bar{k}^\alpha$, and the parametrization-invariant interaction term satisfies Eq. (99) (Euler's relation for a homogeneous function of degree one in $\dot{x}^\alpha$ and $\dot{\bar{x}}^\alpha$); we then have

$$E - \Omega L = -\left[\frac{m\dot{x}_\alpha k^\alpha}{(-\dot{x}_\beta \dot{x}^\beta)^{1/2}} + \frac{1}{\gamma}\int_{-\infty}^{\infty} d\bar{\tau}\Lambda\right](\tau) - \left[\frac{\bar{m}\dot{\bar{x}}_\alpha \bar{k}^\alpha}{(-\dot{\bar{x}}_\beta \dot{\bar{x}}^\beta)^{1/2}} + \frac{1}{\bar{\gamma}}\int_{-\infty}^{\infty} d\tau\Lambda\right](\bar{\tau}) - \left(\int_\tau^\infty \int_{-\infty}^{\bar{\tau}} - \int_{-\infty}^{\tau}\int_{\bar{\tau}}^{\infty}\right)\left(\frac{\partial \Lambda}{\partial R^\alpha} k^\alpha + \frac{\partial \Lambda}{\partial \dot{x}^\alpha}\Omega \dot{\phi}^\alpha\right) d\tau d\bar{\tau}. \tag{D24}$$

Only the radial components of the equations of motion, (10) and (11), are nontrivial, and they take the form

$$\frac{d}{d\tau}\left[\frac{m\dot{x}_\alpha}{(-\dot{x}_\gamma \dot{x}^\gamma)^{1/2}}\right]\varpi^\alpha = \int_{-\infty}^{\infty} d\bar{\tau}\frac{\partial \Lambda}{\partial R^\alpha}\varpi^\alpha + \frac{\gamma \Omega^2}{v}\int_{-\infty}^{\infty} d\bar{\tau}\frac{\partial \Lambda}{\partial \dot{x}^\alpha}\phi^\alpha \tag{D25}$$

$$\frac{d}{d\bar{\tau}}\left[\frac{\bar{m}\dot{\bar{x}}_\alpha}{(-\dot{\bar{x}}_\gamma \dot{\bar{x}}^\gamma)^{1/2}}\right]\bar{\varpi}^\alpha = \int_{-\infty}^{\infty} d\tau\frac{\partial \Lambda}{\partial \bar{R}^\alpha}\bar{\varpi}^\alpha + \frac{\bar{\gamma} \Omega^2}{\bar{v}}\int_{-\infty}^{\infty} d\tau\frac{\partial \Lambda}{\partial \dot{\bar{x}}^\alpha}\bar{\phi}^\alpha. \tag{D26}$$

In the above we used Eqs. (A10) and (A11) in Appendix A 1, to eliminate the τ derivative in the $\bar{\tau}$ integral (and respectively $\bar{\tau}$ derivative in the τ integral), then transformed the integration variable η back to τ ($\bar{\tau}$) using (A2).

The radial component of the accelerations $\ddot{x}_\alpha \varpi^\alpha = -\gamma^2 v\Omega$ and $\ddot{\bar{x}}_\alpha \bar{\varpi}^\alpha = -\bar{\gamma}^2 \bar{v}\Omega$ are related to $\dot{x}_\alpha \phi^\alpha = \gamma v^2/\Omega$ and $\dot{\bar{x}}_\alpha \bar{\phi}^\alpha = \bar{\gamma}\bar{v}^2/\Omega$, respectively, by

$$\frac{m\dot{x}_\alpha \phi^\alpha}{(-\dot{x}_\gamma \dot{x}^\gamma)^{1/2}} = -\frac{v}{\gamma\Omega^2}\frac{d}{d\tau}\left[\frac{m\dot{x}_\alpha}{(-\dot{x}_\gamma \dot{x}^\gamma)^{1/2}}\right]\varpi^\alpha \quad \text{and} \quad \frac{\bar{m}\dot{\bar{x}}_\alpha \bar{\phi}^\alpha}{(-\dot{\bar{x}}_\gamma \dot{\bar{x}}^\gamma)^{1/2}} = -\frac{\bar{v}}{\bar{\gamma}\Omega^2}\frac{d}{d\bar{\tau}}\left[\frac{\bar{m}\dot{\bar{x}}_\alpha}{(-\dot{\bar{x}}_\gamma \dot{\bar{x}}^\gamma)^{1/2}}\right]\bar{\varpi}^\alpha, \tag{D27}$$

where $\dot{x}_\alpha \ddot{x}^\alpha = 0 = \dot{\bar{x}}_\alpha \ddot{\bar{x}}^\alpha$ are used. With this relation (D27), the radial equations of motion (D25) and (D26) are substituted to further simplify the angular momentum formula (D14):

$$L = -\frac{v}{\gamma\Omega^2}\left[\int_{-\infty}^{\infty} d\bar{\tau}\frac{\partial \Lambda}{\partial R^\alpha}\varpi^\alpha\right](\tau) - \frac{\bar{v}}{\bar{\gamma}\Omega^2}\left[\int_{-\infty}^{\infty} d\tau\frac{\partial \Lambda}{\partial \bar{R}^\alpha}\bar{\varpi}^\alpha\right](\bar{\tau}) + \left(\int_\tau^\infty \int_{-\infty}^{\bar{\tau}} - \int_{-\infty}^{\tau}\int_{\bar{\tau}}^{\infty}\right)\left(\frac{\partial \Lambda}{\partial R^\alpha}\phi^\alpha + \frac{\partial \Lambda}{\partial \dot{x}^\alpha}\dot{\phi}^\alpha\right) d\tau d\bar{\tau}. \tag{D28}$$

3. Conserved quantities in post-Minkowski gravity in terms of $h_{\alpha\beta}$ and $\bar{h}_{\alpha\beta}$

As shown in Sec. III A, the parametrization-invariant interaction (22) leads to equations of motion, Eqs. (28) and (29), that describe point particles in a post-Minkowski approximation with half-advanced + half-retarded fields.

The momentum and angular momentum formulas, in the form of a sum of one-particle contributions, can be written in terms of $h_{\alpha\beta}$ and $\bar{h}_{\alpha\beta}$. We present these formulas valid for arbitrary particle trajectories m and $\bar{m}$.

We begin by rewriting the variation of I with respect to an infinitesimal Poincaré transformation, (D5), in terms of one-particle potentials $\mathcal{U}_m$ and $\mathcal{U}_{\bar{m}}$,

$$\delta I(\tau_1, \tau_2, \bar{\tau}_1, \bar{\tau}_2) = \left[\frac{m\dot{x}_\alpha}{(-\dot{x}_\gamma \dot{x}^\gamma)^{1/2}} + \int_{-\infty}^{\infty} d\bar{\tau}\frac{\partial \Lambda}{\partial \dot{x}^\alpha}\right]\delta x^\alpha \Big|_{\tau_1}^{\tau_2} - \int_{\tau_1}^{\tau_2} d\tau \int_{-\infty}^{\infty} d\bar{\tau}\left[\delta x^\alpha \frac{\partial \Lambda}{\partial R^\alpha} + \delta \dot{x}^\alpha \frac{\partial \Lambda}{\partial \dot{x}^\alpha}\right] + \left[\frac{\bar{m}\dot{\bar{x}}_\alpha}{(-\dot{\bar{x}}_\gamma \dot{\bar{x}}^\gamma)^{1/2}} + \int_{-\infty}^{\infty} d\tau\frac{\partial \Lambda}{\partial \dot{\bar{x}}^\alpha}\right]\delta \bar{x}^\alpha \Big|_{\bar{\tau}_1}^{\bar{\tau}_2} - \int_{\bar{\tau}_1}^{\bar{\tau}_2} d\bar{\tau} \int_{-\infty}^{\infty} d\tau\left[\delta \bar{x}^\alpha \frac{\partial \Lambda}{\partial \bar{R}^\alpha} + \delta \dot{\bar{x}}^\alpha \frac{\partial \Lambda}{\partial \dot{\bar{x}}^\alpha}\right] \tag{D29}$$

$$= \left[\frac{m\dot{x}_\alpha}{(-\dot{x}_\gamma \dot{x}^\gamma)^{1/2}} + \frac{\partial \mathcal{U}_m}{\partial \dot{x}^\alpha}\right]\delta x^\alpha \Big|_{\tau_1}^{\tau_2} - \int_{\tau_1}^{\tau_2} d\tau\left[\delta x^\alpha \frac{\partial \mathcal{U}_m}{\partial x^\alpha} + \delta \dot{x}^\alpha \frac{\partial \mathcal{U}_m}{\partial \dot{x}^\alpha}\right] + \left[\frac{\bar{m}\dot{\bar{x}}_\alpha}{(-\dot{\bar{x}}_\gamma \dot{\bar{x}}^\gamma)^{1/2}} + \frac{\partial \mathcal{U}_{\bar{m}}}{\partial \dot{\bar{x}}^\alpha}\right]\delta \bar{x}^\alpha \Big|_{\bar{\tau}_1}^{\bar{\tau}_2} - \int_{\bar{\tau}_1}^{\bar{\tau}_2} d\bar{\tau}\left[\delta \bar{x}^\alpha \frac{\partial \mathcal{U}_{\bar{m}}}{\partial \bar{x}^\alpha} + \delta \dot{\bar{x}}^\alpha \frac{\partial \mathcal{U}_{\bar{m}}}{\partial \dot{\bar{x}}^\alpha}\right]. \tag{D30}$$

The conserved momentum and angular momentum are then derived as follows,

$$P_\alpha(\tau,\bar\tau) = \left[\frac{m\dot x_\alpha}{(-\dot x_\gamma \dot x^\gamma)^{1/2}} + \frac{\partial \mathcal{U}_m}{\partial \dot x^\alpha}\right](\tau) - \int_{-\infty}^{\tau} d\tau \frac{\partial \mathcal{U}_m}{\partial x^\alpha} + \left[\frac{\bar m\dot{\bar x}_\alpha}{(-\dot{\bar x}_\gamma \dot{\bar x}^\gamma)^{1/2}} + \frac{\partial \mathcal{U}_{\bar m}}{\partial \dot{\bar x}^\alpha}\right](\bar\tau) - \int_{-\infty}^{\bar\tau} d\bar\tau \frac{\partial \mathcal{U}_{\bar m}}{\partial \bar x^\alpha}, \tag{D31}$$

and

$$L_{\alpha\beta}(\tau,\bar\tau) = \left[\frac{m(x_\alpha \dot x_\beta - x_\beta \dot x_\alpha)}{(-\dot x_\gamma \dot x^\gamma)^{1/2}} + \left(x_\alpha \frac{\partial \mathcal{U}_m}{\partial \dot x^\beta} - x_\beta \frac{\partial \mathcal{U}_m}{\partial \dot x^\alpha}\right)\right](\tau) - \int_{-\infty}^{\tau} d\tau \left[x_\alpha \frac{\partial \mathcal{U}_m}{\partial x^\beta} - x_\beta \frac{\partial \mathcal{U}_m}{\partial x^\alpha} + \dot x_\alpha \frac{\partial \mathcal{U}_m}{\partial \dot x^\beta} - \dot x_\beta \frac{\partial \mathcal{U}_m}{\partial \dot x^\alpha}\right]$$
$$+ \left[\frac{\bar m(\bar x_\alpha \dot{\bar x}_\beta - \bar x_\beta \dot{\bar x}_\alpha)}{(-\dot{\bar x}_\gamma \dot{\bar x}^\gamma)^{1/2}} + \left(\bar x_\alpha \frac{\partial \mathcal{U}_{\bar m}}{\partial \dot{\bar x}^\beta} - \bar x_\beta \frac{\partial \mathcal{U}_{\bar m}}{\partial \dot{\bar x}^\alpha}\right)\right](\bar\tau) - \int_{-\infty}^{\bar\tau} d\bar\tau \left[\bar x_\alpha \frac{\partial \mathcal{U}_{\bar m}}{\partial \bar x^\beta} - \bar x_\beta \frac{\partial \mathcal{U}_{\bar m}}{\partial \bar x^\alpha} + \dot{\bar x}_\alpha \frac{\partial \mathcal{U}_{\bar m}}{\partial \dot{\bar x}^\beta} - \dot{\bar x}_\beta \frac{\partial \mathcal{U}_{\bar m}}{\partial \dot{\bar x}^\alpha}\right]. \tag{D32}$$

For the interaction of Eq. (22) and corresponding fields (28) and (29), the one-particle potentials are related to the fields $h_{\alpha\beta}$ and $\bar h_{\alpha\beta}$ by

$$\mathcal{U}_m = \int_{-\infty}^{\infty} d\bar\tau \Lambda = \frac{1}{2} m h_{\alpha\beta} \frac{\dot x^\alpha \dot x^\beta}{(-\dot x_\gamma \dot x^\gamma)^{1/2}},$$
$$\mathcal{U}_{\bar m} = \int_{-\infty}^{\infty} d\tau \Lambda = \frac{1}{2} \bar m \bar h_{\alpha\beta} \frac{\dot{\bar x}^\alpha \dot{\bar x}^\beta}{(-\dot{\bar x}_\gamma \dot{\bar x}^\gamma)^{1/2}}; \tag{D33}$$

and their derivatives with respect to position and velocity take the form

$$\frac{\partial \mathcal{U}_m}{\partial x^\alpha} = \frac{1}{2} m \frac{\partial h_{\gamma\delta}}{\partial x^\alpha} \frac{\dot x^\gamma \dot x^\delta}{(-\dot x_\epsilon \dot x^\epsilon)^{1/2}},$$
$$\frac{\partial \mathcal{U}_m}{\partial \dot x^\alpha} = m\left[h_{\alpha\beta} + \frac{1}{2}\eta_{\alpha\beta} \frac{h_{\gamma\delta}\dot x^\gamma \dot x^\delta}{(-\dot x_\epsilon \dot x^\epsilon)}\right] \frac{\dot x^\beta}{(-\dot x_\gamma \dot x^\gamma)^{1/2}}, \tag{D34}$$

$$\frac{\partial \mathcal{U}_{\bar m}}{\partial \bar x^\alpha} = \frac{1}{2} \bar m \frac{\partial \bar h_{\gamma\delta}}{\partial \bar x^\alpha} \frac{\dot{\bar x}^\gamma \dot{\bar x}^\delta}{(-\dot{\bar x}_\epsilon \dot{\bar x}^\epsilon)^{1/2}},$$
$$\frac{\partial \mathcal{U}_{\bar m}}{\partial \dot{\bar x}^\alpha} = \bar m\left[\bar h_{\alpha\beta} + \frac{1}{2}\bar\eta_{\alpha\beta} \frac{\bar h_{\gamma\delta}\dot{\bar x}^\gamma \dot{\bar x}^\delta}{(-\dot{\bar x}_\epsilon \dot{\bar x}^\epsilon)}\right] \frac{\dot{\bar x}^\beta}{(-\dot{\bar x}_\gamma \dot{\bar x}^\gamma)^{1/2}}. \tag{D35}$$

Substituting relations (D34) and (D35) in Eqs. (D31) and (D31), we obtain explicit expressions for the momentum and angular momentum in terms of $h_{\alpha\beta}$ and $\bar h_{\alpha\beta}$,

$$P_\alpha(\tau,\bar\tau) = m\left[\eta_{\alpha\beta} + h_{\alpha\beta} + \frac{1}{2}\eta_{\alpha\beta}\frac{h_{\gamma\delta}\dot x^\gamma \dot x^\delta}{(-\dot x_\epsilon \dot x^\epsilon)}\right]\frac{\dot x^\beta}{(-\dot x_\gamma \dot x^\gamma)^{1/2}}(\tau) - \frac{1}{2} m \int_{-\infty}^{\tau} d\tau \frac{\partial h_{\gamma\delta}}{\partial x^\alpha}\frac{\dot x^\gamma \dot x^\delta}{(-\dot x_\epsilon \dot x^\epsilon)^{1/2}}$$
$$+ \bar m\left[\eta_{\alpha\beta} + \bar h_{\alpha\beta} + \frac{1}{2}\eta_{\alpha\beta}\frac{\bar h_{\gamma\delta}\dot{\bar x}^\gamma \dot{\bar x}^\delta}{(-\dot{\bar x}_\epsilon \dot{\bar x}^\epsilon)}\right]\frac{\dot{\bar x}^\beta}{(-\dot{\bar x}_\gamma \dot{\bar x}^\gamma)^{1/2}}(\bar\tau) - \frac{1}{2} \bar m \int_{-\infty}^{\bar\tau} d\bar\tau \frac{\partial \bar h_{\gamma\delta}}{\partial \bar x^\alpha}\frac{\dot{\bar x}^\gamma \dot{\bar x}^\delta}{(-\dot{\bar x}_\epsilon \dot{\bar x}^\epsilon)^{1/2}}, \tag{D36}$$

and

$$L_{\alpha\beta}(\tau,\bar\tau) = m\left\{\frac{(x_\alpha \dot x_\beta - x_\beta \dot x_\alpha)}{(-\dot x_\gamma \dot x^\gamma)^{1/2}}\left[1 + \frac{1}{2}\frac{h_{\gamma\delta}\dot x^\gamma \dot x^\delta}{(-\dot x_\epsilon \dot x^\epsilon)}\right] + \frac{(x_\alpha h_{\beta\gamma} - x_\beta h_{\alpha\gamma})\dot x^\gamma}{(-\dot x_\delta \dot x^\delta)^{1/2}}\right\}(\tau)$$
$$- m\int_{-\infty}^{\tau} d\tau\left[\frac{1}{2}\left(x_\alpha \frac{\partial h_{\gamma\delta}}{\partial x^\beta} - x_\beta \frac{\partial h_{\gamma\delta}}{\partial x^\alpha}\right)\frac{\dot x^\gamma \dot x^\delta}{(-\dot x_\epsilon \dot x^\epsilon)^{1/2}} + \frac{(\dot x_\alpha h_{\beta\gamma} - \dot x_\beta h_{\alpha\gamma})\dot x^\gamma}{(-\dot x_\delta \dot x^\delta)^{1/2}}\right]$$
$$+ \bar m\left\{\frac{(\bar x_\alpha \dot{\bar x}_\beta - \bar x_\beta \dot{\bar x}_\alpha)}{(-\dot{\bar x}_\gamma \dot{\bar x}^\gamma)^{1/2}}\left[1 + \frac{1}{2}\frac{\bar h_{\gamma\delta}\dot{\bar x}^\gamma \dot{\bar x}^\delta}{(-\dot{\bar x}_\epsilon \dot{\bar x}^\epsilon)}\right] + \frac{(\bar x_\alpha \bar h_{\beta\gamma} - \bar x_\beta \bar h_{\alpha\gamma})\dot{\bar x}^\gamma}{(-\dot{\bar x}_\delta \dot{\bar x}^\delta)^{1/2}}\right\}(\bar\tau)$$
$$- \bar m\int_{-\infty}^{\bar\tau} d\bar\tau\left[\frac{1}{2}\left(\bar x_\alpha \frac{\partial \bar h_{\gamma\delta}}{\partial \bar x^\beta} - \bar x_\beta \frac{\partial \bar h_{\gamma\delta}}{\partial \bar x^\alpha}\right)\frac{\dot{\bar x}^\gamma \dot{\bar x}^\delta}{(-\dot{\bar x}_\epsilon \dot{\bar x}^\epsilon)^{1/2}} + \frac{(\dot{\bar x}_\alpha \bar h_{\beta\gamma} - \dot{\bar x}_\beta \bar h_{\alpha\gamma})\dot{\bar x}^\gamma}{(-\dot{\bar x}_\delta \dot{\bar x}^\delta)^{1/2}}\right]. \tag{D37}$$

For completeness, we give the corresponding form of the conserved quantity $\mathcal{Q}$ associated with a Killing vector ζ^α of Minkowski space, rewriting Eq. (D22), in terms of $h_{\alpha\beta}$ and $\bar h_{\alpha\beta}$. Applying a property of ζ^α, $\dot x^\alpha \dot\zeta_\alpha = \dot x^\alpha \dot x^\beta \nabla_\alpha \zeta_\beta = 0$, we have

$$\mathcal{Q} = m\bigg[\eta_{\alpha\beta} + h_{\alpha\beta} + \frac{1}{2}\eta_{\alpha\beta}\frac{h_{\gamma\delta}\dot{x}^\gamma\dot{x}^\delta}{(-\dot{x}_\epsilon\dot{x}^\epsilon)}\bigg]\frac{\zeta^\alpha\dot{x}^\beta}{(-\dot{x}_\gamma\dot{x}^\gamma)^{1/2}}(\tau) - m\int_{-\infty}^{\tau} d\tau\bigg[\frac{1}{2}\zeta^\alpha\frac{\partial h_{\beta\gamma}}{\partial x^\alpha}\frac{\dot{x}^\beta\dot{x}^\gamma}{(-\dot{x}_\delta\dot{x}^\delta)^{1/2}} + \frac{h_{\alpha\beta}\dot{\zeta}^\alpha\dot{x}^\beta}{(-\dot{x}_\gamma\dot{x}^\gamma)^{1/2}}\bigg]$$
$$+ \bar{m}\bigg[\eta_{\alpha\beta} + \bar{h}_{\alpha\beta} + \frac{1}{2}\eta_{\alpha\beta}\frac{\bar{h}_{\gamma\delta}\dot{\bar{x}}^\gamma\dot{\bar{x}}^\delta}{(-\dot{\bar{x}}_\epsilon\dot{\bar{x}}^\epsilon)}\bigg]\frac{\bar{\zeta}^\alpha\dot{\bar{x}}^\beta}{(-\dot{\bar{x}}_\gamma\dot{\bar{x}}^\gamma)^{1/2}}(\bar{\tau}) - \bar{m}\int_{-\infty}^{\bar{\tau}} d\bar{\tau}\bigg[\frac{1}{2}\bar{\zeta}^\alpha\frac{\partial \bar{h}_{\beta\gamma}}{\partial \bar{x}^\alpha}\frac{\dot{\bar{x}}^\beta\dot{\bar{x}}^\gamma}{(-\dot{\bar{x}}_\delta\dot{\bar{x}}^\delta)^{1/2}} + \frac{\bar{h}_{\alpha\beta}\dot{\bar{\zeta}}^\alpha\dot{\bar{x}}^\beta}{(-\dot{\bar{x}}_\gamma\dot{\bar{x}}^\gamma)^{1/2}}\bigg]. \quad \text{(D38)}$$

For the affinely parametrized interaction (36), we obtain analogous expressions for momentum and angular momentum in terms of $h_{\alpha\beta}$ and $\bar{h}_{\alpha\beta}$. Since the form (36) relates to the solutions

$$h_{\alpha\beta}(x) = 4\bar{m}\int_{-\infty}^{\infty} d\bar{\tau}\delta(w)\Big(\dot{\bar{x}}_\alpha\dot{\bar{x}}_\beta - \frac{1}{2}\eta_{\alpha\beta}\dot{\bar{x}}_\gamma\dot{\bar{x}}^\gamma\Big), \quad \text{(D39)}$$

$$\bar{h}_{\alpha\beta}(\bar{x}) = 4m\int_{-\infty}^{\infty} d\tau\delta(w)\Big(\dot{x}_\alpha\dot{x}_\beta - \frac{1}{2}\eta_{\alpha\beta}\dot{x}_\gamma\dot{x}^\gamma\Big), \quad \text{(D40)}$$

the one-particle potentials for the affinely parametrized interaction are written,

$$\mathcal{U}_m = \int_{-\infty}^{\infty} d\bar{\tau}\Lambda = \frac{1}{2}mh_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta,$$
$$\mathcal{U}_{\bar{m}} = \int_{-\infty}^{\infty} d\tau\Lambda = \frac{1}{2}\bar{m}\bar{h}_{\alpha\beta}\dot{\bar{x}}^\alpha\dot{\bar{x}}^\beta, \quad \text{(D41)}$$

and their derivatives with respect to the position and to the velocity become

$$\frac{\partial\mathcal{U}_m}{\partial x^\alpha} = \frac{1}{2}m\frac{\partial h_{\gamma\delta}}{\partial x^\alpha}\dot{x}^\gamma\dot{x}^\delta, \qquad \frac{\partial\mathcal{U}_m}{\partial\dot{x}^\alpha} = mh_{\alpha\gamma}\dot{x}^\gamma \quad \text{(D42)}$$

$$\frac{\partial\mathcal{U}_{\bar{m}}}{\partial \bar{x}^\alpha} = \frac{1}{2}\bar{m}\frac{\partial \bar{h}_{\gamma\delta}}{\partial \bar{x}^\alpha}\dot{\bar{x}}^\gamma\dot{\bar{x}}^\delta, \qquad \frac{\partial\mathcal{U}_{\bar{m}}}{\partial\dot{\bar{x}}^\alpha} = \bar{m}\bar{h}_{\alpha\gamma}\dot{\bar{x}}^\gamma. \quad \text{(D43)}$$

Replacements (D10) and (D11) and substitution of (D42) and (D43) in Eqs. (D31) and (D32) yield for the momentum and angular momentum the expressions,

$$P_\alpha(\tau,\bar{\tau}) = m(\eta_{\alpha\beta} + h_{\alpha\beta})\dot{x}^\beta(\tau) - \frac{1}{2}m\int_{-\infty}^{\tau} d\tau\frac{\partial h_{\gamma\delta}}{\partial x^\alpha}\dot{x}^\gamma\dot{x}^\delta + \bar{m}(\eta_{\alpha\beta} + \bar{h}_{\alpha\beta})\dot{\bar{x}}^\beta(\bar{\tau}) - \frac{1}{2}\bar{m}\int_{-\infty}^{\bar{\tau}} d\bar{\tau}\frac{\partial \bar{h}_{\gamma\delta}}{\partial \bar{x}^\alpha}\dot{\bar{x}}^\gamma\dot{\bar{x}}^\delta, \quad \text{(D44)}$$

and

$$\begin{aligned} L_{\alpha\beta}(\tau,\bar{\tau}) = {}& m[(x_\alpha\dot{x}_\beta - x_\beta\dot{x}_\alpha) + (x_\alpha h_{\beta\gamma} - x_\beta h_{\alpha\gamma})\dot{x}^\gamma](\tau) \\ & - m\int_{-\infty}^{\tau} d\tau\bigg[\frac{1}{2}\Big(x_\alpha\frac{\partial h_{\gamma\delta}}{\partial x^\beta} - x_\beta\frac{\partial h_{\gamma\delta}}{\partial x^\alpha}\Big)\dot{x}^\gamma\dot{x}^\delta + (\dot{x}_\alpha h_{\beta\gamma} - \dot{x}_\beta h_{\alpha\gamma})\dot{x}^\gamma\bigg] \\ & + \bar{m}[(\bar{x}_\alpha\dot{\bar{x}}_\beta - \bar{x}_\beta\dot{\bar{x}}_\alpha) + (\bar{x}_\alpha\bar{h}_{\beta\gamma} - \bar{x}_\beta\bar{h}_{\alpha\gamma})\dot{\bar{x}}^\gamma](\bar{\tau}) \\ & - \bar{m}\int_{-\infty}^{\bar{\tau}} d\bar{\tau}\bigg[\frac{1}{2}\Big(\bar{x}_\alpha\frac{\partial \bar{h}_{\gamma\delta}}{\partial \bar{x}^\beta} - \bar{x}_\beta\frac{\partial \bar{h}_{\gamma\delta}}{\partial \bar{x}^\alpha}\Big)\dot{\bar{x}}^\gamma\dot{\bar{x}}^\delta + (\dot{\bar{x}}_\alpha\bar{h}_{\beta\gamma} - \dot{\bar{x}}_\beta\bar{h}_{\alpha\gamma})\dot{\bar{x}}^\gamma\bigg]. \end{aligned} \quad \text{(D45)}$$

Finally, the charge $\mathcal{Q}$ has for the affinely parametrized interaction the form

$$\begin{aligned} \mathcal{Q} = {}& m(\eta_{\alpha\beta} + h_{\alpha\beta})\zeta^\alpha\dot{x}^\beta(\tau) - m\int_{-\infty}^{\tau} d\tau\Big(\frac{1}{2}\zeta^\alpha\frac{\partial h_{\beta\gamma}}{\partial x^\alpha}\dot{x}^\beta\dot{x}^\gamma + h_{\alpha\beta}\dot{\zeta}^\alpha\dot{x}^\beta\Big) \\ & + \bar{m}(\eta_{\alpha\beta} + \bar{h}_{\alpha\beta})\bar{\zeta}^\alpha\dot{\bar{x}}^\beta(\bar{\tau}) - \bar{m}\int_{-\infty}^{\bar{\tau}} d\bar{\tau}\Big(\frac{1}{2}\bar{\zeta}^\alpha\frac{\partial \bar{h}_{\beta\gamma}}{\partial \bar{x}^\alpha}\dot{\bar{x}}^\beta\dot{\bar{x}}^\gamma + \bar{h}_{\alpha\beta}\dot{\bar{\zeta}}^\alpha\dot{\bar{x}}^\beta\Big). \end{aligned} \quad \text{(D46)}$$

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