String gyratons in supergravity

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We study solutions of the supergravity equations with the stringlike sources moving with the speed of light. An exact solution is obtained for the gravitational field of a boosted ring string in any dimension greater than three.

I. INTRODUCTION

In this paper we study solutions of supergravity equations generated by stringlike sources moving with the speed of light. Study of solutions of the Einstein equations for objects moving with the velocity of light has long story. In 1934 Tolman [1] obtained solutions for the gravitational field of beams of light in the linear approximation. The exact solutions of the nonlinear Einstein equations for this problem were obtained later [2–4]. For the infinitely small cross-section of the beam and for the delta-type distribution of the light-pulse in time, these solutions reduce to the Aichelburg-Sexl metric [5]. These 4-dimensional solutions were generalized to the case the matter of beam is either charged or spinning by Bonnor [6,7] in 1970. Higher-dimensional generalizations of the solutions of Einstein equations for spinning relativistic beamlike sources (gyratons) were obtained and studied in [8,9]. Solutions for charged higher-dimensional gyratons and for gyratons moving in an asymptotically anti−de Sitter space were found in [10,11], respectively. The aim of this paper is to discuss generalization of electrically charged gyraton solutions to the theory of supergravity. More concretely, we study solutions of higher-dimensional gravitational equations with the rank 3 antisymmetric Kalb-Ramon field generated by stringlike sources moving with the velocity of light. The basic equations, the metric and field ansatz are presented in Secs. II and III, respectively. In Sec. IV we demonstrate, that the field equations for the problem under consideration reduce to the set of linear equations in a flat Euclidean space. Special solutions of these equations for ringlike string gyratons are obtained in Sec. V. Section VI contains brief summary and discussions.

II. BASIC EQUATIONS

We consider the massless bosonic sector of supergravity. We restrict ourselves by discussing what is called the common sector. The fields in the common sector are the metric $g_{\mu\nu}$, the Kalb-Ramond antisymmetric field $B_{\mu\nu}$ and the dilaton field ϕ. The corresponding action, which is also the low-energy superstring effective action, is

$$S = \frac{1}{16\pi G} \int d^D x \sqrt{|g|} e^{-2\phi} \Big[R - 4(\nabla\phi)^2$$
$$- \frac{1}{12} H_{\mu\nu\lambda} H^{\mu\nu\lambda} \Big] + \frac{1}{2} \int d^D x \sqrt{|g|} B_{\mu\nu} J^{\mu\nu} + \mathcal{S}_m. \tag{1}$$

Here G is the D-dimensional gravitational (Newtonian) coupling constant, and $\mathcal{S}_m$ is the action for the string matter source. The string coupling constant g_s is determined by the vacuum expectation value of the dilaton field ϕ_0, $g_s = \exp(\phi_0)$. The 3-form flux

$$H_{\mu\nu\lambda} = \partial_\mu B_{\nu\lambda} + \partial_\nu B_{\lambda\mu} + \partial_\lambda B_{\mu\nu} \tag{2}$$

is the Kalb-Ramond (KR) field strength and $B_{\mu\nu}$ is its antisymmetric 2-form potential. The field $H_{\mu\nu\lambda}$ is invariant under the gauge transformation

$$B_{\mu\nu} \rightarrow B_{\mu\nu} + \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu. \tag{3}$$

$J^{\mu\nu}$ is the antisymmetric tensor of the current which plays the role of a source for the KR field. For example, for the interaction of the KR field with a fundamental string described by the action

$$S_{\text{int}} = -\frac{q}{2} \int \delta^2 \zeta \epsilon^{ab} B_{\mu\nu} \frac{\partial X^\mu}{\partial \zeta^a} \frac{\partial X^\nu}{\partial \zeta^b} \tag{4}$$

this current is

$$J^{\mu\nu}(x) = \frac{q}{2} \int \delta^2 \zeta \frac{\delta^D(x - X(\zeta))}{\sqrt{|g|}} \epsilon^{ab} \frac{\partial X^\mu}{\partial \zeta^a} \frac{\partial X^\nu}{\partial \zeta^b}. \tag{5}$$

Here ϵ^{ab} is the antisymmetric symbol, $\zeta^a = (\tau, \sigma)$ are parameters on the string surface and the functions $X^\mu = X^\mu(\zeta)$ determine the embedding of the string worldsheet in the bulk (target) spacetime. The parameter q is the "string charge". The current $J^{\mu\nu}$ is tangent to the worldsheet of the string, $J^{[\mu\nu} X^{\lambda]}_{,c} = 0$.

We shall study a special class of gyraton solutions for which the dilaton field is constant, i.e., $e^\phi = g_s$. In this case the field equations are

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$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = T_{\mu\nu} + g_s^2\kappa\mathcal{T}_{\mu\nu}, \tag{6}$$

$$H_{\mu\nu}{}^{\lambda}{}_{;\lambda} = 8\kappa J_{\mu\nu}. \tag{7}$$

Here the stress-energy tensor for the 3-form flux is

$$T_{\mu\nu} = \frac{1}{12}\left(3H_{\mu\lambda\rho}H_\nu{}^{\lambda\rho} - \frac{1}{2}g_{\mu\nu}H_{\rho\sigma\lambda}H^{\rho\sigma\lambda}\right), \tag{8}$$

and $\kappa = 8\pi g_s^2 G$. $\mathcal{T}_{\mu\nu}$ which enters the Eq. (6) is the stress-energy of the matter (string) which we shall specify later.

Let Σ be a $(D-2)$-dimensional spacelike surface, and $\partial\Sigma$ be its boundary. We define the charge of the fundamental string intersecting Σ by Gauss's law as

$$Q := \int_{\partial\Sigma}d\sigma_{D-3}\,{*}_D\,H_3 = \int d\sigma_{\mu\nu\lambda}H_{\mu\nu\lambda}. \tag{9}$$

By using the Stoke's theorem and ((7)) one has

$$Q = \int d\sigma_{\mu\nu}J^{\mu\nu} = 8\kappa q. \tag{10}$$

Here $d\sigma_{\mu_1\cdots\mu_n} := i_{\mu_1}\cdots i_{\mu_n}(*1)$ in which $*1$ is the volume form of D-dimensional spacetime and $i_\mu\colon \Lambda^p T^* \to \Lambda^{p-1}T^*$, $i_\mu dx^{\nu_1}\wedge\cdots\wedge dx^{\nu_p} = p\delta_\mu^{[\nu_1}dx^{\nu_2}\wedge\cdots\wedge dx^{\nu_p]}$.

For a straight string along Z-axis in Minkowski spacetime with coordinates (T, Z, X^i), $(i = 3, \ldots, D)$ one has

$$H^{TZi} = \frac{Q}{\mathcal{A}_{D-3}}\frac{n^i}{[\sum_{i=3}^D X_i^2]^{(D-3)/2}}, \tag{11}$$

the other components vanish. Here n^i is a unit vector normal to the surface $\sum_{i=3}^D X_i^2 = \text{const}$, and

$$\mathcal{A}_n = 2\pi^{n/2}/\Gamma(n/2) \tag{12}$$

is the surface area of a unit n-dimensional sphere.

In what follows, we consider the gravitational and KR fields outside the sources, that is in the region where $T_{\mu\nu} = 0$ and $J_{\mu\nu} = 0$. The relation (10) will be used to relate the parameters which enter a solution to the charge of the string.

III. ANSATZ FOR SUPERGRAVITY GYRATON

A. Metric

We shall study special solutions of the supergravity Eqs. (6) and (7) which are generated by sources moving with the speed of light. We are interested in solutions which have finite energy, angular momentum, KR charge, and finite duration in time. Following [8–10] we call such ultrarelativistic objects with spin *gyratons* and use for its gravitational field in the $D = n + 2$ dimensional spacetime the following metric ansatz (Brinkmann metric [12])

$$ds^2 = d\bar{s}^2 + 2(a_u du + a_a dx^a)du. \tag{13}$$

Here $a_u = a_u(u, x^a)$, $a_a = a_a(u, x^a)$, and

$$d\bar{s}^2 = -2dudv + dx^2 \tag{14}$$

is the D-dimensional flat metric, the transverse spatial part of the metric $dx^2 = \delta_{ab}dx^a dx^b$ in the n-dimensional hypersurface is flat. We use the Greek letters for indices which take values $1, \cdots, D$, while the Roman low-case indices take value $3, \cdots, D$. The form of the metric (13) implies that

$$\det(g_{\mu\nu}) = -1. \tag{15}$$

The field

$$a_\mu = a_u\delta_\mu^u + a_a\delta_\mu^a \tag{16}$$

is the gravitational analogue of the electromagnetic potential. It is easy to see that the metric is invariant under the following gauge transformation

$$v \to v + \lambda(u, \boldsymbol{x}), \qquad a_\mu \to a_\mu - \lambda_{,\mu}, \tag{17}$$

and the quantity

$$f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu \tag{18}$$

is gauge invariant.

The metric (13) admits a null Killing vector $l = l^\mu\partial_\mu = \partial_v$, which is parallelly propagated $l_{\mu;\nu} = 0$. One also has

$$l_\mu dx^\mu = -du, \qquad l^\mu a_\mu = l^\mu f_{\mu\nu} = 0. \tag{19}$$

The flat metric $d\bar{s}^2$ and the metric (13) are related as

$$\bar{g}_{\mu\nu} = g_{\nu\nu} - l_\mu a_\nu - l_\nu a_\mu, \tag{20}$$

$$\bar{g}^{\mu\nu} = g^{\mu\nu} + l^\mu a^\nu + l^\nu a^\mu + l^\mu l^\nu a^\rho a_\rho. \tag{21}$$

B. KR field

For the KR field potential we use the ansatz similar to the one adopted for the electromagnetic gyratons [10]. Namely we postulate that $B_{\mu\nu} = B_{\mu\nu}(u, \boldsymbol{x})$ and

$$l^\mu B_{\mu\nu} = 0. \tag{22}$$

It is easy to check that

$$l^\mu H_{\mu\nu\lambda} = 0. \tag{23}$$

The imposed constraints imply that the only nonvanishing components of $B_{\mu\nu}$ are $B_{ua}(u, \boldsymbol{x})$ and $B_{ab}(u, \boldsymbol{x})$, and of $H_{\mu\nu\lambda}$ are $H_{uab}(u, \boldsymbol{x})$ and $H_{abc}(u, \boldsymbol{x})$. Moreover, to preserve the constraint (22) under gauge transformation, we should impose

$$\partial_\nu\Lambda_\mu - \partial_\mu\Lambda_\nu = 0. \tag{24}$$

Using (20) and (21) it is easy to derive the relation between the contravariant tensors raising their indices from the same covariant one by using the metric (13) and the flat metric (14) respectively. Especially, for 3-form flux

we have

$$\bar{H}_{\mu\nu}{}^{\lambda} = H_{\mu\nu}{}^{\lambda} + l^{\lambda}a_{\rho}H_{\mu\nu}{}^{\rho}, \tag{25}$$

$$\bar{H}_{\mu}{}^{\nu\lambda} = H_{\mu}{}^{\nu\lambda} + a_{\rho}(l^{\lambda}H_{\mu}{}^{\nu\rho} - l^{\nu}H_{\mu}{}^{\lambda\rho}), \tag{26}$$

$$\bar{H}^{\mu\nu\lambda} = H^{\mu\nu\lambda} + a_{\rho}(l^{\lambda}H^{\mu\nu\rho} - l^{\nu}H^{\mu\lambda\rho} + l^{\mu}H^{\nu\lambda\rho}). \tag{27}$$

Here the quantities with bar are the ones with respect to the flat metric.

One also has

$$\bar{H}_{\mu\lambda\rho}\bar{H}_{\nu}{}^{\lambda\rho} = H_{\mu\lambda\rho}H_{\nu}{}^{\lambda\rho}, \tag{28}$$

$$H^2 \equiv \bar{H}_{\mu\nu\lambda}\bar{H}^{\mu\nu\lambda} = H_{\mu\nu\lambda}H^{\mu\nu\lambda}. \tag{29}$$

These equations will be useful in solving the field equations.

Moreover, the constraint (23) implies that

$$H^2 = \boldsymbol{H}^2 \equiv H_{abc}H^{abc}. \tag{30}$$

Note that the indices a, b, c in the above relations are raised by the flat metric δ^{ab} since $g^{ua} = \bar{g}^{ua} = 0$ and $g^{ab} = \bar{g}^{ab} = \delta^{ab}$.

C. Gyrating matter

We discuss now the ansatz for $\mathcal{T}_{\mu\nu}$ which enters the Eq. (7). We require that this tensor obeys the conservation low

$$\mathcal{T}^{\mu\nu}{}_{;\nu} = 0, \tag{31}$$

and is aligned to the null Killing vector l_{μ}

$$\mathcal{T}_{\mu\nu} = l_{(\mu}p_{\nu)}, \qquad l^{\mu}p_{\mu} = 0. \tag{32}$$

The last condition guarantees that the trace of $\mathcal{T}^{\mu\nu}$ vanishes, $\mathcal{T}^{\mu}_{\mu} = 0$. For the metric (13) these conditions are satisfied when

$$p_{\mu} = p_{\mu}(u, x^a), \qquad p^a{}_{,a} = 0. \tag{33}$$

This can be checked by using the condition $l_{\mu;\nu} = 0$. Bonnor [7] called such matter in 4-dimensional spacetime spinning null fluid.

IV. REDUCED EQUATIONS

Calculations show [9] that for the metric (13) the only nonvanishing components of the Ricci tensor are

$$R_{ua} = \frac{1}{2}f_{ab}{}^{,b}, \tag{34}$$

$$R_{uu} = -(a_u){}^{;a}_{,a} + \frac{1}{4}f_{ab}f^{ab} + \partial_u(a_a{}^{,a}). \tag{35}$$

These relations imply

$$R = 0, \tag{36}$$

which together with the fact $\mathcal{T}^{\mu}_{\mu} = 0$ the field Eq. (6) yields

$$T^{\mu}_{\mu} = 0. \tag{37}$$

By taking trace of (8) the vanishing trace of the stress tensor then implies

$$\boldsymbol{H}^2 = 0, \tag{38}$$

and hence $H_{abc} = 0$ except for $D = 6$ case. However, since in additoin all the spatial components of the Ricci tensor vanish, this result follows even in $D = 6$ case.

Therefore, the only nonvanishing component of $H_{\mu\nu\lambda}$ are $H_{uab}(u, \boldsymbol{x})$. This means that $H_{\mu\nu\lambda} = l_{[\mu}P_{\nu\lambda]}$, so that the field strength $H_{\mu\nu\lambda}$ is aligned to the null Killing vector l_{μ}.

Using (15) one has

$$H_{\mu\nu}{}^{\lambda}{}_{;\lambda} = H_{\mu\nu}{}^{\lambda}{}_{,\lambda}, \tag{39}$$

and the field Eqs. (6) and (7) reduce to

$$(a_u){}^{;a}_{,a} - \partial_u(a_a{}^{,a}) = \frac{1}{4}(f_{ab}f^{ab} - H_{uab}H_u{}^{ab}) + \kappa p_u, \tag{40}$$

$$f_{ab}{}^{,b} = -\kappa p_a, \tag{41}$$

$$H_{ua}{}^{b}{}_{,b} = 8\kappa J_{ua}. \tag{42}$$

The last two relations are linear differential equation in the n-dimensional Euclidean space ($n = D - 2$). They can be solved for f_{ab} and H_{uab} once the source J_{ua} and the distribution for the source for gravito-magnetic field f_{ab} is given. After this we can solve the first equation for a_u, which for a given right-hand-side is also linear.

On the other hand, we can also solve the constraint $H_{abc} = 0$ by the following ansatz for the 2-form potential

$$B_{\mu\nu} = A_{\mu}l_{\nu} - A_{\nu}l_{\mu}. \tag{43}$$

From $l^{\mu}B_{\mu\nu} = 0$, we have

$$l^{\mu}A_{\mu} = 0. \tag{44}$$

This is equivalent to choose a gauge so that the only non-vanishing component of $B_{\mu\nu}$ is $B_{ua} = A_a(u, \boldsymbol{x})$, and of $H_{\mu\nu\lambda}$ is

$$H_{uab} = \partial_b A_a - \partial_a A_b \equiv F_{ba}. \tag{45}$$

Note that the constraints (23) and (44) are preserved by the gauge transformation

$$A_{\mu} \rightarrow A_{\mu} + \partial_{\mu}\Lambda(u, \boldsymbol{x}). \tag{46}$$

Let us denote

$$\Phi = 2a_u, \qquad \boldsymbol{a} = a_a, \qquad \boldsymbol{f} = f_{ab} \tag{47}$$

$$\boldsymbol{A} = A_a, \qquad \boldsymbol{F} = F_{ab}, \qquad \boldsymbol{J} = J_{ua}, \qquad \boldsymbol{p} = p_a. \tag{48}$$

In these notations the gyraton metric is

$$ds^2 = d\bar{s}^2 + \Phi du^2 + 2(\boldsymbol{a}, d\boldsymbol{x})du, \tag{49}$$

and the field Eqs. (40) and (42) reduce to

$$\Delta \Phi - 2\partial_u (\boldsymbol{\nabla} \cdot \boldsymbol{a}) = \frac{1}{2}(f^2 - \boldsymbol{F}^2) + 2\kappa p_u, \tag{50}$$

$$\Delta \boldsymbol{A} + \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{A}) = 8\kappa \boldsymbol{J}. \tag{51}$$

Here $\boldsymbol{\nabla} = \partial_a$, and Δ is the Laplacian operator in the n-dimensional Euclidean space.

Using the coordinate, (17), and electromagnetic, (46), gauge transformations one can put

$$\boldsymbol{\nabla} \cdot \boldsymbol{A} = 0, \tag{52}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{a} = 0. \tag{53}$$

For these gauge fixing conditions the Eqs. (41), (50), and (51) take the form

$$\Delta \Phi = \frac{1}{2}(f^2 - \boldsymbol{F}^2) + 2\kappa p_u, \tag{54}$$

$$\Delta \boldsymbol{a} = \kappa \boldsymbol{p}, \tag{55}$$

$$\Delta \boldsymbol{A} = 8\kappa \boldsymbol{J}. \tag{56}$$

It is interesting to note that the magnetic and gravito-magnetic terms enter the right-hand-side of (54) with the opposite signs. A special type of solutions is the case when these terms cancel one another, so that the equation for Φ outside the matter source becomes homogeneous. We call such solutions *saturated*. The condition of saturation is

$$f^2 = \boldsymbol{F}^2 \tag{57}$$

which can be achieved by letting $\boldsymbol{p} = 8\boldsymbol{J}$ as suggested by (55) and (56).

V. RING-STRING GYRATONS

A. Green functions

The system of Eqs. (56)–(58) is well defined for distributed sources p_u, $\mathbf{p}$, and $\mathbf{J}$. In a general case, solutions for the Eqs. (54)–(56) can be obtained by using the Green function of the Laplace operator. A solution of the equation

$$\Delta \psi = j(u, \boldsymbol{x}) \tag{58}$$

is

$$\psi(u, \boldsymbol{x}) = -\int d\boldsymbol{x}' \mathcal{G}_n(\boldsymbol{x}, \boldsymbol{x}') j(u, \boldsymbol{x}'). \tag{59}$$

Here $\mathcal{G}_n(\boldsymbol{x}, \boldsymbol{x}')$ is the Green's function for the n-dimensional Laplace operator

$$\Delta \mathcal{G}_n(\boldsymbol{x}, \boldsymbol{x}') = -\delta(\boldsymbol{x} - \boldsymbol{x}'), \tag{60}$$

which can be written in the following explicit form

$$\mathcal{G}_2(\boldsymbol{x}, \boldsymbol{x}') = -\frac{1}{2\pi} \ln|\boldsymbol{x} - \boldsymbol{x}'|, \tag{61}$$

$$\mathcal{G}_n(\boldsymbol{x}, \boldsymbol{x}') = \frac{g_n}{|\boldsymbol{x} - \boldsymbol{x}'|^{n-2}}, \qquad n > 2, \tag{62}$$

where $g_n = 1/[(n-2)\mathcal{A}_n]$ and $\mathcal{A}_n$ is given by (12). It should be emphasized that the retarded time u plays the role of the external parameter if a source term depends on it. The dependence of the field on u can be obtained by solving "static" equations. After one obtains a solution of static equations one can simply make the coefficients which enter the solution to be u-dependent.

Since the Eqs. (55) and (56) are linear, one can solve them for the limiting case when the size of the source tends to zero. However, because the presence of the terms quadratic in $\mathbf{F}$ and $\mathbf{f}$ in the right-hand-side of Eq. (54), its solution by the transition to the point like limit of sources may be not uniquely determined and depend on detailed structure of the sources. In what follows we restrict ourselves by considering the saturated solutions where this problem does not occur. We also assume that the source of the gyraton field is a string which has a form of a ring and is moving with the speed of light in the direction orthogonal to the plane of the ring. Let us consider this case in more details.

B. KR field of a ring string

We choose coordinates $\boldsymbol{x}$ in the transverse to the motion plane in such a way that the string ring is located in the $(x^3 - x^4)$-2-plane, and denote the coordinates in the orthogonal subspace by $\boldsymbol{y} = (y^A)$, $A = 5 \ldots D$. Let (ρ, θ) be polar coordinates in the string 2-plane

$$x^3 = \rho \cos\theta, \qquad x^4 = \rho \sin\theta. \tag{63}$$

The ring-string worldsheet is defined by the following equations

$$v = \tau, \qquad u = u_0, \qquad X^3 = \rho_0 \cos\sigma,$$
$$X^4 = \rho_0 \sin\sigma, \tag{64}$$

and $X^A = 0$. The nonvanishing component of the current $J^{\mu\nu}$ is $\boldsymbol{J} = J^{va}$

$$\boldsymbol{J} = \frac{1}{2}qv(u)\boldsymbol{I}, \qquad \boldsymbol{I} = I_{\hat{\theta}}\mathbf{e}_{\hat{\theta}}, \tag{65}$$

$$\mathbf{e}_{\hat{\theta}} = -\sin\theta \mathbf{e}_{\hat{3}} + \cos\theta \mathbf{e}_{\hat{4}}, \tag{66}$$

$$I_{\hat{\theta}} = \frac{\delta(\rho - \rho_0)}{\rho_0} \delta^{D-4}(\boldsymbol{y}). \tag{67}$$

Here $\mathbf{e}_{\hat{\theta}}$ is a unit vector in the θ-direction, $\mathbf{e}_{\hat{a}}$ are unit

vectors along the axes x^a, and $v(u) = \delta(u - u_0)$. If the string source is not localized at $u = u_0$ but is smeared in time, the function $v(u)$ is smooth. We assume that $v(u)$ is normalized so that $\int du v(u) = 1$.

The solution of the Eq. (56) is [13]

$$A(u, x) = \begin{cases} -\frac{1}{2} g_{D-2} Q v(u) \int \frac{dx' I(x')}{|x-x'|^{D-4}}, & \text{for } D > 4; \\ \frac{1}{4\pi} Q v(u) \int dx' I(x') \ln|x - x'|, & \text{for } D = 4. \end{cases} \tag{68}$$

Here the charge Q is given by (10) and

$$|x - x'| = [\lambda + \rho^2 + \rho'^2 - 2\rho\rho' \cos(\theta - \theta')]^{1/2}, \tag{69}$$

and $\lambda = |y - y'|^2$ (for $D > 4$) and $\lambda = 0$ (for $D = 4$).

Since the geometry is cylindrically symmetric, it is sufficient to calculate A at $\theta = 0$. Since the azimuthal integrations in (68) is symmetric about $\theta' = 0$, the component of the current in the direction e_3 does not contribute. This leaves the only component in the direction of e_4, which is $A_{\hat{\theta}}$. Since $e_{\hat{\theta}} = \rho \partial_\theta$ one has $A_\theta = \rho^{-1} A_{\hat{\theta}}$. Thus we have

$$A_\theta = -\frac{1}{2\rho} g_{D-2} Q v(u) \mathcal{B}_D, \quad \text{for } D > 4; \tag{70}$$

$$A_\theta = \frac{1}{4\pi\rho} Q v(u) \int_0^{2\pi} d\theta' \cos\theta' \ln F_0, \quad D = 4, \tag{71}$$

where

$$F_\lambda = \lambda + \rho^2 + \rho_0^2 - 2\rho\rho_0 \cos(\theta'), \quad \lambda = y^2, \tag{72}$$

$$\mathcal{B}_D = \int_0^{2\pi} \frac{d\theta' \cos\theta'}{F_\lambda^{(D-4)/2}}. \tag{73}$$

1. 4D case

In the 4-dimensional case, $D = 4$, the integral (71) can be easily taken and the answer is

$$A_\theta = -\frac{Q v(u)}{2\rho_0} [\Theta(\rho_0 - \rho) + (\rho_0/\rho)^2 \Theta(\rho - \rho_0)], \tag{74}$$

where Θ is the Heaviside step function. In this case the only nonvanishing component of the 3-form flux

$$H_{u\theta\rho} = A_{\theta,\rho} = Q v(u) \frac{\rho_0}{\rho^3} \Theta(\rho - \rho_0). \tag{75}$$

The flux vanishes for $\rho < \rho_0$ and it is discontinuous at $\rho = \rho_0$. However, the jump of the flux value at $\rho = \rho_0$ is *finite* and proportional to the charge Q.

2. Even dimensional case

For $D > 4$ we consider first the case when D is even. We put $D = 2m + 6$, then $\mathcal{B}_D = H_m$, where

$$H_m = \int_0^{2\pi} \frac{d\theta' \cos\theta'}{F_\lambda^{m+1}}. \tag{76}$$

The calculations give

$$H_0 = \frac{\pi}{\rho\rho_0} \left[\frac{\lambda + \rho^2 + \rho_0^2}{\sqrt{(\lambda + (\rho + \rho_0)^2)(\lambda + (\rho - \rho_0)^2)}} - 1 \right]. \tag{77}$$

For $m > 0$ one has

$$H_m = \frac{(-1)^m}{m!} \frac{d^m H_0}{d\lambda^m}. \tag{78}$$

Note that the above solutions are singular at the location of the ring string, i.e., at $\rho = \rho_0$ and $y = 0$.

3. Odd dimensional case

Let $D = 2m + 5$, then $\mathcal{B}_D = J_m$, where

$$J_m = \int_0^{2\pi} \frac{d\theta' \cos\theta'}{F_\lambda^{m+1/2}}. \tag{79}$$

The calculations give

$$J_0 = \frac{2[(2 - k^2)K(k) - 2E(k)]}{k\sqrt{\rho\rho_0}}, \tag{80}$$

$$k^2 = \frac{4\rho\rho_0}{\lambda + (\rho + \rho_0)^2} \leq 1. \tag{81}$$

Note that $k = 1$ only at the location of the ring string, and $k \ll 1$ far away from the ring string or near the $\rho = 0$ axis.

In the above $E(k)$ and $K(k)$ are the complete elliptic integrals defined by

$$K(k) := \int_0^{\pi/2} \frac{dz}{\sqrt{1 - k^2 \sin^2 z}},$$

$$E(k) := \int_0^{\pi/2} dz \sqrt{1 - k^2 \sin^2 z}.$$

Note that $E(k)$ is a monotonically decreasing function from $E(0) = \pi/2$ to $E(0) = 1$, and $K(k)$ is a monotonically increasing function from $K(0) = \pi/2$ to $K(1) = \infty$. Note that $K(k) \sim \ln(1 - k)$ as $k \to 1$.

For $m > 0$ one has

$$J_m = (-1)^m \frac{2^m}{(2m - 1)!!} \frac{d^m J_0}{d\lambda^m}. \tag{82}$$

The complete elliptic integrals possess the properties

$$\frac{dK(k)}{dk} = \frac{E(k)}{k(1 - k^2)} - \frac{K(k)}{k}, \tag{83}$$

$$\frac{dE(k)}{dk} = \frac{E(k) - K(k)}{k}. \tag{84}$$

For this reason the expression J_m for any m has the same structure

$$J_m = A_m K(k) + B_m E(k), \qquad (85)$$

where A_m and B_m are some algebraical functions of k and $\rho\rho_0$, which can be found by using (83) and (84).

Note that the above solutions are singular on the location of the ring string due to the divergence of $K(k)$ at $k = 1$.

C. Gravito-magnetic potential *a*

Since the equation for the gravito-magnetic potential a, (55), differs from the (56) only by a constant coefficient one can directly obtain a solution for a from A. For the source $p = j(u)I$ it is sufficient to make the following substitution $4qv(u) \rightarrow j(u)$ in the expression for A. For saturated solutions we shall further require $j(u) = 4qv(u)$.

D. Potential Φ

The Eq. (54) for Φ is linear. Let us write its solution in the form

$$\Phi = \varphi + \psi, \qquad (86)$$

where

$$\Delta\psi = 2\kappa p_u. \qquad (87)$$

Assuming that the mater source is localized on the ring string, one has

$$p_u = \varepsilon(u)I_{\hat{\theta}}. \qquad (88)$$

The corresponding solution is

$$\psi(u, \boldsymbol{x}) = \begin{cases} -2g_{D-2}\kappa\varepsilon(u)\mathcal{C}_D, & \text{for } D > 4; \\ \frac{\kappa\varepsilon(u)}{\pi}\mathcal{C}_4, & \text{for } D = 4. \end{cases} \qquad (89)$$

Here

$$\mathcal{C}_{D>4} = \int_0^{2\pi} \frac{d\theta'}{F_\lambda^{(D-4)/2}}; \qquad (90)$$

$$\mathcal{C}_4 = \int d\theta' \ln F_0. \qquad (91)$$

1. 4D case

In the 4-dimensional case, the integral (91) can be easily taken and the answer is

$$\mathcal{C}_4 = -4\pi \begin{cases} \ln\rho, & \text{for } \rho > \rho_0; \\ \ln\rho_0, & \text{for } \rho < \rho_0. \end{cases} \qquad (92)$$

2. Even dimensional case

For $D > 4$ we consider first the case when D is even. We put $D = 2m + 6$, then $\mathcal{C}_D = N_m$,

$$N_m = \int_0^{2\pi} \frac{d\theta'}{F_\lambda^{m+1}}. \qquad (93)$$

The calculations give

$$N_0 = \frac{2\pi}{\sqrt{(\lambda + (\rho + \rho_0)^2)(\lambda + (\rho - \rho_0)^2)}}. \qquad (94)$$

For $m > 0$ one has

$$N_m = \frac{(-1)^m}{m!} \frac{d^m N_0}{d\lambda^m}. \qquad (95)$$

3. Odd dimensional case

Let $D = 2m + 5$, then $\mathcal{C}_D = L_m$,

$$L_m = \int_0^{2\pi} \frac{d\theta'}{F_\lambda^{m+1/2}}. \qquad (96)$$

The calculations give

$$L_0 = \frac{2kK(k)}{\sqrt{\rho\rho_0}}, \qquad (97)$$

where k is given by (81).

For $m > 0$ one has

$$L_m = (-1)^m \frac{2^m}{(2m-1)!!} \frac{d^m L_0}{d\lambda^m}. \qquad (98)$$

Using this relation and relations (83) and (84) it is possible to write L_m as

$$L_m = C_m K(k) + D_m E(k), \qquad (99)$$

where C_m and D_m are some algebraical functions.

For the saturated ring-string solutions $\varphi = 0$, so that $\Phi = \psi$. As for the KR field, the higher dimensional ($D > 4$) solutions of Φ are singular at the location of the ring string. The obtained relations in this section allow one to write the gyraton metric (13) and the 3-form flux (45) in an explicit form. The solutions contain 2 arbitrary function of u, $v(u)$ and $\varepsilon(u)$, which are related to the angular momentum (string current) and energy density of the ring-string source. For nonsaturated case, one needs in addition to obtain a solution of the equation

$$\Delta\varphi = \frac{1}{2}(f^2 - \boldsymbol{F}^2) \qquad (100)$$

with an explicitly known right-hand-side. The relation (59) gives the integral representation for the solution.

VI. CONCLUSIONS

In general it is difficult to obtain analytically in an explicit form a solution of the Einstein equations for a moving extended graitating object because of the nonlinearity the equations. In this paper we achieve the goal by constructing the full supergravity solution due to a boosted closed string coupled to the Kalb-Ramond field. We basically generalized the method of constructing of solutions for the point charged gyraton moving with the speed of light [10] to the string case. The gyraton metric has the special property that all the curvature invariants con-

structed from the curvature and its covariant derivatives vanish [8,9]. As a pecial case, we considered a ring-string source moving with the speed of light in a D-dimensional spacetime. The full metric and the KR field due to its back reaction are analytically constructed, and the quantities in the solutions related to angular momentum and energy density are identified. The configurations are singular at the location of the ring-string source for $D > 4$ cases but not for $D = 4$. Physically, our special ring-string solution is the theoretical realization of the boosted closed string produced in the high energy experiments, and the time-dependent background variation could, in principle, be detected by the gravitational wave interferometers.

The generalization of our construction to the p-brane gyratons is straightforward, especially the 3-brane gyraton of the vacuum supergravity solution is the generalization of the pp-wave background [14,15]. However, the explicit solutions corresponding to the the boosted p-brane of special shape are model dependent. It will be interesting to examine the string theory in the vacuum 3-brane gyraton background and its dual picture in Yang-Mills theory. Finally, we would like to mention that the solutions found in [16,17] by boosting the black ring are related to the solution found in this paper. It will be also interesting to explore the connection.

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