Resolution of a paradox in classical electrodynamics

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It is an early result of electrostatics in curved space that the *gravitational* mass of a charge distribution changes by an amount equal to U_{es}/c^2, where U_{es} is the internal electrostatic potential energy and c is the speed of light, if the system is supported at rest by external forces. This fact, independently rediscovered in recent years in the case of a simple dipole, confirms a very reasonable expectation grounded in the mass-energy equivalency equation. However, it is an unsolved paradox of classical electrodynamics that the renormalized mass of an accelerated dipole calculated from the self-forces due to the distortion of the Coulomb field differs in general from that expected from the energy correction, U_{es}/c^2, unless the acceleration is transversal to the orientation of the dipole. Here we show that this apparent paradox disappears for any dipole orientation if the self-force is evaluated by means of Whittaker's exact solution for the field of the single charge in a homogeneous gravitational field described in the Rindler metric. The discussion is supported by computer algebra results, diagrams of the electric fields distorted by gravitation, and a brief analysis of the prospects for realistic experimentation. The gravitational correction to dipole-dipole interactions is also discussed.

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I. INTRODUCTION

It appears that the earliest publication on the problem of electrostatics in curved space was that by Enrico Fermi, published when he was a third year student at the Scuola Normale Superiore at Pisa [1]. In this paper, Fermi discussed the correction to the electric field of a single point charge held at rest within a gravitational field to first order in the gravitational acceleration. The problem of the single charge seems to have been completely forgotten until a few years later, when Edmund T. Whittaker solved it exactly both in the homogeneous gravitational field and in the Schwarzschild geometry cases with no mention of Fermi's previous work [2]. His analysis was further developed by E. T. Copson, who produced an expression still used today [3].

The topic was again forgotten, except for very brief mentions of it [4,5], until its rediscovery over 30 years ago, with no reference to any of the above papers [6]. The electric field lines of the single charge in the Schwarzschild geometry were calculated a short time later [7]. In more recent times, Linet has shown that the Copson potential does not in fact satisfy the correct boundary conditions at infinity and that a term corresponding to an additional charge inside the event horizon must be added to it [8]. This contribution marks the start of the modern phase of interest in this problem that continues uninterrupted to the present day [9].

Fermi's early goal was not only to obtain the electric field of a single charge held at rest in a gravitational field but to also prove that, to within the adopted approximations, the magnitude and orientation of the needed external force are but a manifestation of the gravitational equivalent of the electrostatic potential energy of the interacting charges. For instance, in the case of a simple dipole made of two charges $\pm q$ separated by a distance s, Fermi's argument would state that an effective lifting self-force is expected, equal to $+gq^2/sc^2$, corresponding to an effective decrease in the gravitational mass of the system due to its negative potential energy and produced by the interaction of each charge with the distorted electric field of the other in curved space. This force manifests itself as a decrease in the magnitude of the external force permanently holding the dipole at rest in the gravitational field.

In recent years, Fermi's original result that the gravitational mass correction one expects from energy considerations does coincide with the electrostatic self-force on a system of supported charges has been rediscovered, again to first order and in the particular case of a dipole perpendicular to the gravitational acceleration [10]. Interestingly, an attempt to generalize this important example to the case of a dipole accelerating in any direction, for instance longitudinally, has been unsuccessful and the problem is presently characterized in the literature as an "unsolved paradox" [11,12]. These latter authors have compared the "energy-derived" mass of an accelerated dipole, $m_u = -q^2/sc^2$, to the inertia offered by such a system under the action of an external force, referred to as the "self-force derived" mass, m_s. Their result that $m_u = m_s$ only if the dipole is accelerating perpendicularly to its orientation [11] certainly defies the very reasonable expectation that this should instead occur regardless of the geometrical distribution of the charges and it also contradicts Fermi's earlier results, which were not cited by these authors.

This surprising conclusion has led to the comment that [11] *"in the light of the present paper Boyer was fortunate to have chosen transverse motion"* (their Note [3]).

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Technically it is worthwhile to point out that the efforts leading to this paradox have made use of elementary transformations of the appropriate electric and magnetic fields in different reference frames, whereas Fermi had employed an action approach to obtain a field equation for the electrostatic potential. This raises legitimate doubts about the correctness of the more recent, "simplified" treatments since more sophisticated methods have in fact removed a similar paradox that existed for spherical charge distributions [11,13].

The goals of this paper are twofold. First, we obtain Whittaker's potential field equation for a single charge supported in a homogeneous gravitational field *to all orders* and we verify previous first-order results *for any dipole orientation*. We show that no paradox exists and we suggest its appearance was due to coordinate transformation errors. Secondly, we further generalize these findings to more complex charge distributions and we treat the classical dipole-dipole interaction in curved space. This provides the foundation for a brief outline of the prospects for realistic observation of these phenomena by means of trapped atom interferometry. Since the computations yield very unwieldy results, appropriate use has been made of computer algebra systems and of graphical representations of the distorted electrostatic fields to best elucidate and check the results we report herein.

II. WHITTAKER'S SOLUTION

Here we obtain Whittaker's field equation by adopting the metric (also employed by Fermi) [1,3,4,9,14]:

$$ds^2 = \left(1 + \frac{gz}{c^2}\right)^2 c^2 dt^2 - (dx^2 + dy^2 + dz^2), \qquad (1)$$

where the gravitational acceleration g is oriented towards the negative z-axis and c is the speed of light.

As originally observed by Eddington [15,16] and pointed out by Copson (see Ref. [6], footnote on p. 186), the field equation for the single charge in a homogeneous gravitational field can be written by formally assuming the charge to be immersed in a medium of dielectric permittivity and magnetic permeability ϵ_0, and μ_0, respectively, given by ([17] (Sec. 90, Problem), [18]):

$$\epsilon_0 = \mu_0 = \frac{1}{\sqrt{g_{00}}}, \qquad (2)$$

where g_{00} is given by our metric at Eq. (1).

The Laplace equation within a medium of dielectric permittivity $\epsilon_0 = 1/\sqrt{g_{00}} = 1/(1 + gz/c^2)$ is:

$$\nabla \cdot \mathbf{D} = \nabla \cdot (\epsilon_0 \mathbf{E}) = -\nabla \cdot \left(\frac{1}{1 + g\frac{z}{c^2}} \nabla\phi\right) = -\frac{1}{1 + g\frac{z}{c^2}} \nabla^2\phi + \frac{g}{c^2}\frac{\partial\phi}{\partial z}\frac{1}{(1 + g\frac{z}{c^2})^2} = 0, \qquad (3)$$

or, equivalently:

$$\nabla^2\phi - \frac{g}{c^2}\frac{1}{1 + g\frac{z}{c^2}}\frac{\partial\phi}{\partial z} = 0. \qquad (4)$$

The connection between the above coordinate system and the primed coordinate system, (t', x', y', z') used by Whittaker, which describes in Schwarzschild coordinates the homogeneous gravitational field in the neighborhood of a point far away from the mass [9], is given by the transformation

$$\left(1 + \frac{gz}{c^2}\right)^2 = 1 + \frac{2gz'}{c^2}; \qquad t = t'; \qquad x = x'; \qquad y = y'. \qquad (5)$$

By applying this transformation to the above field equation, we find:

$$\frac{\partial\phi}{\partial z} = \frac{\partial\phi}{\partial z'}\frac{\partial z'}{\partial z} = \frac{\partial\phi}{\partial z'}\left(1 + \frac{gz}{c^2}\right) = \frac{\partial\phi}{\partial z'}\sqrt{1 + \frac{2gz'}{c^2}}; \qquad (6)$$

$$\frac{\partial^2\phi}{\partial z'^2}\left(1 + \frac{2gz'}{c^2}\right) + \frac{\partial\phi}{\partial z'}\frac{g}{c^2}. \qquad (7)$$

By substituting into Eq. (4), we obtain:

$$\frac{\partial^2\phi}{\partial x'^2} + \frac{\partial^2\phi}{\partial y'^2} + \frac{\partial^2\phi}{\partial z'^2}\left(1 + \frac{2gz'}{c^2}\right) = 0 \qquad (8)$$

in the absence of charges, which is Whittaker's Eq. (13). By comparing this field equation to that valid in cylindrical coordinates in flat space (see the appendix), Whittaker obtains the solution for a charge located at the origin:

$$V_W(\mathbf{r}'; \mathbf{0}) = q\frac{1 + \frac{gz'}{c^2} + \frac{g^2}{2c^4}(x'^2 + y'^2)}{\sqrt{r'^2 + \frac{g}{c^2}z'(x'^2 + y'^2) + \frac{g^2}{4c^4}(x'^2 + y'^2)}}, \qquad (9)$$

where $r' = \sqrt{x'^2 + y'^2 + z'^2}$. Let us now rewrite Eqs. (5) as:

$$1 + \frac{gz}{c^2} + \frac{g^2z^2}{2c^4} = 1 + \frac{gz'}{c^2} \qquad (10)$$

and rewrite the numerator of Eq. (9) as:

$$1 + \frac{gz'}{c^2} + \frac{g^2}{2c^4}(x'^2 + y'^2) = 1 + \frac{gz}{c^2} + \frac{g^2}{2c^4}r^2. \qquad (11)$$

Similarly the denominator becomes:

$$\begin{aligned} r'^2 &+ \frac{g}{c^2}z'(x'^2 + y'^2) + \frac{g^2}{4c^4}(x'^2 + y'^2) \\ &= x^2 + y^2 + z^2\left(1 + \frac{g}{2c^2}z\right)^2 + \frac{gz}{c^2}\left(1 + \frac{gz}{2c^2}\right)(x^2 + y^2) \\ &\quad + \frac{g^2}{4c^4}(x^2 + y^2) \\ &= r^2\left(1 + \frac{gz}{c^2}\right) + \frac{g^2}{4c^4}r^4. \end{aligned} \qquad (12)$$

Therefore Whittaker's solution can be expressed in these coordinates as:

$$V_W(\mathbf{r};\mathbf{0}) = \frac{q}{r}\frac{1 + \frac{gz}{c^2} + \frac{g^2}{2c^4}r^2}{\sqrt{1 + \frac{gz}{c^2} + \frac{g^2}{4c^4}r^2}}. \tag{13}$$

The next step is to generalize this solution to the case in which the charge is not located at the origin, but at a generic position $\mathbf{r_0} = (x_0, y_0, z_0)$. Let us then carry out a temporary transformation of our metric at Eq. (1) to the coordinates T, X, Y, and Z:

$$t = \frac{1}{\frac{1+gz_0}{c^2}}T; \quad x = X + x_0; \quad y = Y + y_0, \quad z = Z + z_0, \tag{14}$$

or

$$ds^2 = c^2 dT^2\left[1 + \frac{2gZ}{c^2}\frac{1}{1 + \frac{gz_0}{c^2}} + \frac{g^2}{c^4}\frac{Z^2}{(1 + \frac{gz_0}{c^2})^2}\right] - d\mathbf{R}^2, \tag{15}$$

where $d\mathbf{R}^2 = dX^2 + dY^2 + dZ^2$. Let us now define the quantity:

$$G_W \equiv \frac{g}{1 + \frac{gz_0}{c^2}}, \tag{16}$$

which allows us to rewrite the metric as:

$$ds^2 = c^2\left(1 + \frac{G_W}{c^2}Z\right)dT^2 - d\mathbf{R}^2. \tag{17}$$

Since this is formally equivalent to Eq. (1), it is immediate to write the general solution:

$$V_W(\mathbf{R}) = \frac{q}{R}\frac{1 + \frac{G_W Z}{c^2} + \frac{G_W^2}{2c^4}R^2}{\sqrt{1 + \frac{G_W Z}{c^2} + \frac{G_W^2}{4c^4}R^2}}, \tag{18}$$

where $R = \sqrt{X^2 + Y^2 + Z^2}$. Substitution of Eqs. (17) and (19) yields:

$$V'_W(\mathbf{r};\mathbf{r_0}) = \frac{q}{|\mathbf{r} - \mathbf{r_0}|} \times \frac{1 + \frac{g(z+z_0)}{c^2} + \frac{g^2}{2c^4}[|\rho - \rho_\mathbf{0}|^2 + (z^2 + z_0^2)]}{(1 + \frac{gz_0}{c^2})\sqrt{1 + \frac{g}{c^2}(z + z_0) + \frac{g^2}{4c^4}|\mathbf{r} - \mathbf{r_0}|^4}}, \tag{19}$$

where $\mathbf{r_0} = x_0^i$ $(i = 1, \ldots, 3)$ is the charge position, $|\mathbf{r} - \mathbf{r_0}|^2 = (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2$, and $|\rho - \rho_0|^2 = (x - x_0)^2 + (y - y_0)^2$.

Finally, we must transform this primed expression V'_W back to the reference frame in which the metric is given by Eq. (1). Since the electrostatic potential represents the zeroth component of the four-vector potential, as seen from Eq. (14) it transforms as ([17], Sec. 88)

$$V_W = \frac{\partial T}{\partial t}V'_W = \left(1 + \frac{gz_0}{c^2}\right)V'_W. \tag{20}$$

The final result, valid to all orders, is therefore:

$$V_W(\mathbf{r};\mathbf{r_0}) = \frac{q}{|\mathbf{r} - \mathbf{r_0}|} \times \frac{1 + \frac{g(z+z_0)}{c^2} + \frac{g^2}{2c^4}[|\rho - \rho_\mathbf{0}|^2 + (z^2 + z_0^2)]}{\sqrt{1 + \frac{g}{c^2}(z + z_0) + \frac{g^2}{4c^4}|\mathbf{r} - \mathbf{r_0}|^4}}. \tag{21}$$

As expected, this expression approaches the standard, flat-space Coulomb potential as $g \to 0$.

III. RESOLUTION OF THE PARADOX

In order to fully appreciate the unfamiliar effects caused by the "drooping field lines" of the single charge field [19], let us now consider the net force acting on a classical *point* dipole in a noninertial reference frame. This can be done by introducing two equal and opposite charges, $\pm q$—one located at $\mathbf{r_A}$ and the other one at $\mathbf{r_B}$ in the accelerated frame described by the metric of Eq. (1). In principle, the computation is relatively simple as all that is needed is the evaluation of the force, $\mathbf{F_{AB}} = -q\mathbf{E_A}(\mathbf{r_B})$ due to the field produced by the first on the second charge, and, vice versa, the force $\mathbf{F_{BA}} = +q\mathbf{E_B}(\mathbf{r_A})$, where the electric fields can be calculated from the potential according to the usual prescription [1], $\mathbf{E} = -\nabla\phi$. The two forces thus found must then be added to obtain the net force acting on the point dipole in three-dimensional space. In flat space, this procedure would of course always yield a vanishing net force, but the noncentral nature of the distorted potential alters this result of basic electrostatics.

To reduce the possibility of manipulation errors, all calculations were carried out by means of a computer algebra system and its results were first checked against known results in the literature. Only special cases representative of interesting geometries in the gravitational field are given here as the general results valid to all orders are usually impractically long to reproduce [20]. Whittaker's potential at x^i due to a charge at $\mathbf{r_A}$ is found to be, to first order in gx^i/c^2:

$$V_W(\mathbf{r};\mathbf{r_A}) \simeq q\left(\frac{1}{r} + \frac{g}{2c^2}\frac{z + z_A}{r}\right). \tag{22}$$

The corresponding electric field is:

$$E_x(\mathbf{r},\mathbf{r_A}) = \frac{q}{r^3}\left[(x - x_A) + \frac{g}{2c^2}(x - x_A)(z + z_A)\right]; \tag{23}$$

$$E_y(\mathbf{r},\mathbf{r_A}) = \frac{q}{r^3}\left[(y - y_A) + \frac{g}{2c^2}(y - y_A)(z + z_A)\right]; \tag{24}$$

$$E_z(\mathbf{r}, \mathbf{r_A}) = \frac{q}{r^3}\left\{(z - z_A) - \frac{g}{2c^2}[|\rho - \rho_0|^2 - 2z_0(z - z_0)]\right\}. \tag{25}$$

If the charge is located at the origin ($\mathbf{r_0} = \mathbf{0}$), these expressions reduce to those found by Fermi [1] to first order. Our first-order potential V_W is also equivalent to that used in the computation of the effect of a gravitational field on the eigenstates of a hydrogen atom supported at the origin [21].

The resulting net force on the dipole obtained by following the procedure outlined at the beginning of this section is, to third order in the gravitational field:

$$F_{\mathrm{net},z} = +\frac{q^2}{r}\frac{g}{c^2} - \frac{q^2}{r}\frac{3|\rho - \rho_0|^2 + 2(z - z_0)^2}{8}\left(\frac{g}{c^2}\right)^3. \tag{26}$$

This result shows that, in general, Fermi's prediction that the gravitational equivalent of the electrostatic potential energy will manifest itself as a decreased effective gravitational attraction on the dipole is fully confirmed to second order by our direct calculation based on Whittaker's solution *regardless of dipole orientation and location*, thus resolving the heretofore "unsolved paradox" [11]. In this connection, it is perhaps worthwhile to notice that the final step of our calculation leading to Eq. (21) is of critical importance. Had we omitted to treat the electrostatic potential as the zeroth component of the four-vector potential, and had we proceeded to calculate the self-force on the dipole by means of Eq. (19), we would have rediscovered exactly the same "paradox," that is, the self-force would have incorrectly been given as $F'_{\mathrm{net},z} = q^2(|\rho - \rho_0|/r^2) \times (g/c^2)$, thus implying that Fermi's prediction would be correct only if the two charges are placed transversally to the direction of acceleration, or of the gravitational field, where $|\rho - \rho_0| = r$. This leads one to speculate that somehow the multiplicative factor $1 + gz_0/c^2$, corresponding to this transformation, enters the treatment leading to the formulation of the paradox by those authors incorrectly. In fact, it is suggestive to recall that the four-acceleration of a charged particle in gravitational and electromagnetic fields, which lies at the foundation of their approach, is given by the covariant derivative $Du^\mu/ds = (q/mc^2)F^{\mu\nu}u_\nu$, where $F^{\mu\nu}$ is the electromagnetic field tensor and u_ν is the four-velocity. In a gravitational field, even for an observer at rest, or for an observer comoving with an accelerating particle because of the Principle of Equivalence, $u^0 = 1/\sqrt{g_{00}}$ [17], again leading to the same important factor introduced at Eq. (20).

IV. EXAMPLES OF DISTORTED FIELDS

The field lines of a single positive charge in a homogeneous gravitational field are shown at Fig. 1, where the effect has been made evident by choosing appropriate numerical values ($q = 1$; $g/c^2 = 0.4$). The electric field lines were calculated from direct numerical integration of the differential equations $dx/dt = E_x$, $dz/dt = E_z$ to all orders which, depending on the geometrical details, can be numerically intensive [22]. Since the field is no longer radial, the interaction between two equal and opposite charges is not central and self-forces arise. The electric and potential fields of a dipole oriented downwards are shown in Fig. 2, which makes it clear that the field is not symmetric by reflection around the x-axis when the gravitational field is present. Finally, the potential fields for dipoles at different angles with respect to the gravitational

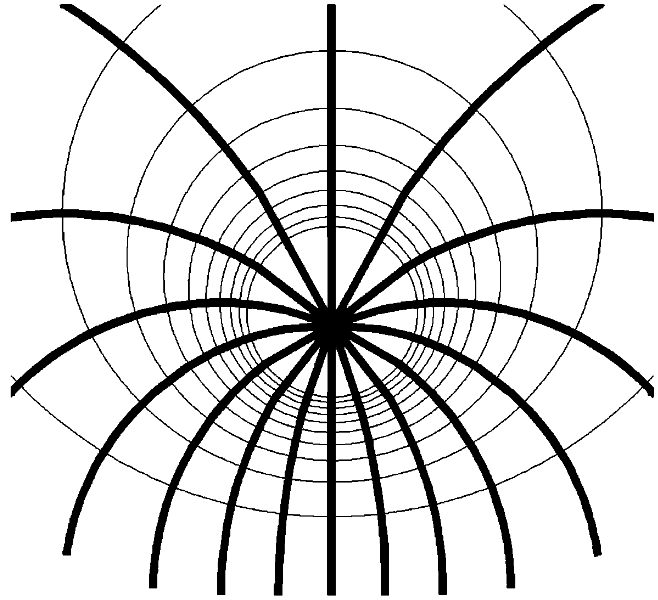

FIG. 1. Electric field of a single charge; gravitational acceleration is down.

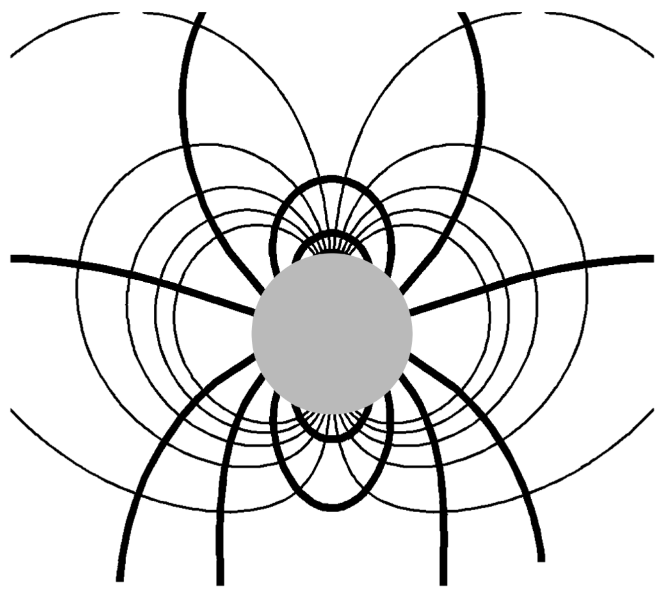

FIG. 2. Electric dipole of a point dipole; dipole moment and gravitational field are down.

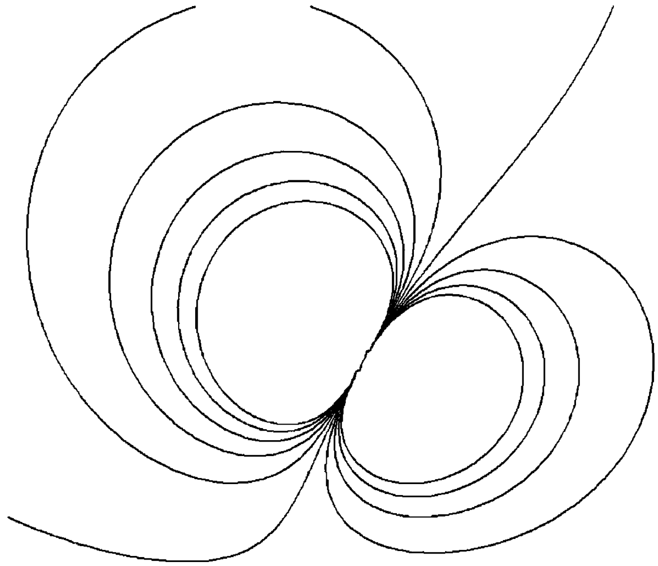

FIG. 3. Equipotentials of a dipole oriented downward, 30 degrees from the vertical.

field are shown at Figs. 3 and 4 where it becomes evident that all symmetry has disappeared. For instance, the potential field of a point dipole of vertical moment qZ_A placed at the origin is given by:

$$U_d(\mathbf{r};\mathbf{0}) = \frac{qZ_A}{r^3}\left[z + \frac{x^2+y^2+2z^2}{2}\left(\frac{g}{c^2}\right)\right], \tag{27}$$

which as usual coincides with its flat-space expression if $g \to 0$.

By calculating the distorted dipole electric field, it is then possible to express the dipole-dipole force by means of the usual expression $\mathbf{F} = (\mathbf{p}\cdot\nabla)\mathbf{E}$. This goal can be

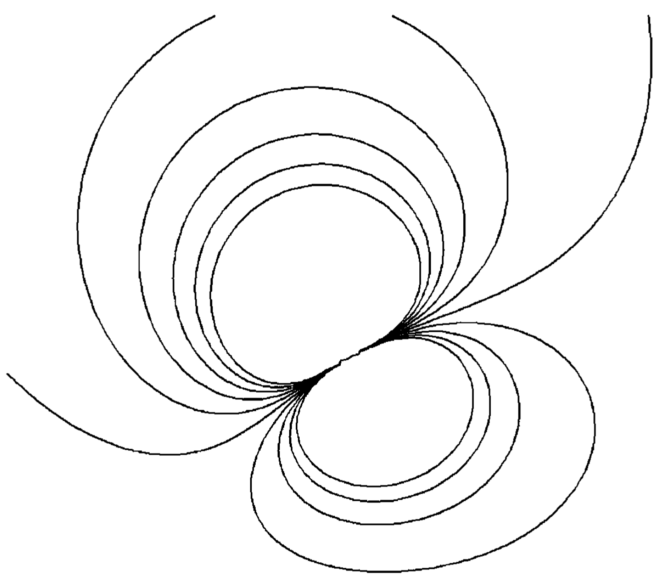

FIG. 4. Equipotentials of a dipole oriented downward, 60 degrees from the vertical.

achieved by generalizing the usual flat-space procedure for a point dipole of moment $\mathbf{p_A} = p^i_A$, with $k = 1, \ldots, 3$, located at $\mathbf{r_A}$ [23]:

$$U_{\mathrm{dip},W}(x^k, x^k_A) = p^i_A \frac{\partial V_W(\mathbf{r};\mathbf{r_0})}{\partial x^i_0}\bigg|_{\mathbf{r_A}}, \tag{28}$$

so as to calculate the gravity-induced self-force acting on a dipole pair. According to Fermi's first-order argument, this quantity is expected to be $F_{\mathrm{net},z} = -W_{dd}(g/c^2)$, where W_{dd} is the flat-space dipole-dipole interaction energy. A computer algebra evaluation for the self-force between two dipoles of moments $q(X_A, Y_A, Z_A)$ and $q(X_B, Y_B, Z_B)$ yields an extremely unwieldy expression which, to first order in the gravitational field and for dipoles lying vertically upon one another at a distance R, reduces to:

$$F_{\mathrm{net},z} = -\frac{q^2}{R^3}(X_AX_B + Y_AY_B - 2Z_AZ_B)\left(\frac{g}{c^2}\right), \tag{29}$$

while all other components vanish identically. This fully confirms our expectation since, in fact, the factor multiplying $-(g/c^2)$ is the flat-space dipole-dipole interaction energy [24]; this result remains unchanged if the two dipoles are placed, for instance, horizontally next to one another, in which case the term in parentheses becomes, correctly, $+(2X_AX_B - Y_AY_B - Z_AZ_B)$.

An example of further investigation made possible by the present approach is the calculation of the gravitational correction to the dipole-dipole interaction energy itself, which is found to be, for two dipoles arranged horizontally next to one another at a distance R, to first order:

$$W_{dd}(\mathbf{r_A};\mathbf{r_B}) = -\frac{q^2}{R^3}(2X_AX_B - Y_AY_B - Z_AZ_B) + \frac{q^2}{R^2}(X_AZ_B - X_BZ_A)\left(\frac{g}{c^2}\right). \tag{30}$$

The contour curves of the first-order term of this function, which vanishes identically in flat space, are shown in Fig. 5 for two dipoles, one placed at the origin and the other anywhere in space. Since this quantity depends on the positions of the dipoles, we can independently calculate the self-force on the pair simply as $F_{\mathrm{net},z} = -(\partial W_{dd}/\partial \mathbf{r_A} + \partial W_{dd}/\partial \mathbf{r_B})$, which is found to reduce to the same result as above.

One last application we discuss briefly is the gravitational correction to the van der Waals interaction energy between two hydrogen atoms in their ground states. As is known from elementary quantum mechanics [24], the term in first-order perturbation theory, analogous to our Eq. (30) in flat space, vanishes because of symmetry. The nonvanishing second order van der Waals energy term is:

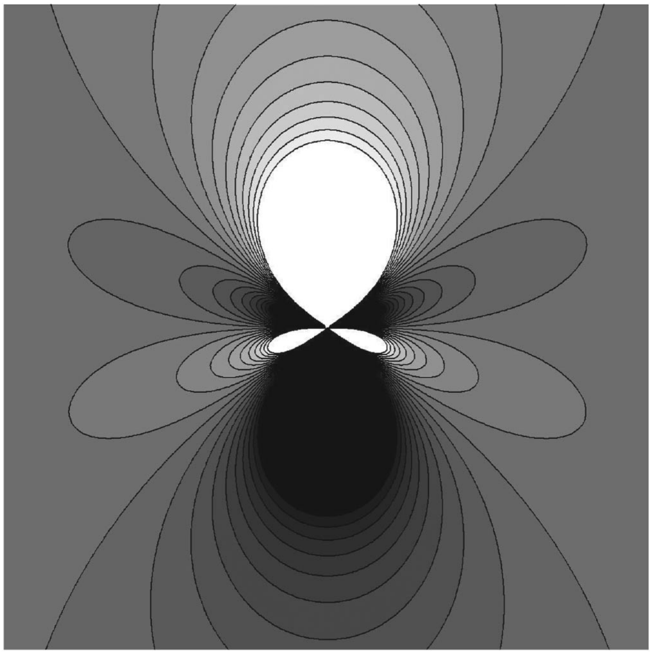

FIG. 5. Contour curves of the gravity-induced dipole-dipole interaction energy.

$$U_{\mathrm{vdW}} \approx -\frac{1}{R^6}\frac{e^4}{2E_I}\langle\psi^A_{1,0,0}\psi^B_{1,0,0}|W^2_{dd}|\psi^A_{1,0,0}\psi^B_{1,0,0}\rangle = -\frac{6e^2a_0^5}{R^6}, \tag{31}$$

which demands that we compute the classical term W^2_{dd}. By extracting the contribution to second order in the gravitational field from a general result which features hundreds of algebraic terms, we find, for two atoms placed vertically at a distance R,

$$F_{\mathrm{net},z} = -U_{\mathrm{vdW}}\left(\frac{g}{c^2}\right) \approx +\frac{6e^2a_0^5}{R^6}\left(\frac{g}{c^2}\right) \tag{32}$$

regardless of the relative position of the two atoms. This is an elegant confirmation that the original argument based on mass-energy equivalence applies not only to classical charge distributions but also to quantum mechanical systems.

V. CONCLUSIONS

In the brief introductory note placed before Fermi's first and second papers (see Ref. [4], p. 1), his colleague and friend Enrico Persico (1900–1969) comments: *"Paper No. 2 determines, by the methods of general relativity, the effect of a uniform gravitational field on a system of electric charges. It turns out that the charges have a weight equal to that of a material mass U/c^2 (where U is the electrostatic energy of the system), in perfect agreement with Einstein's principle of equivalence between mass and energy."* The fact that the calculation of the gravitational mass of a system of charges must yield the quantity U/c^2 is such an entrenched expectation that, more recently, its appearance has been hailed by the similar proclamation: "All works out perfectly" [10].

In this paper, we have explored a formulation of the problem of the single charge held at rest in a homogeneous gravitational field based on Whittaker's field equation, which offers great technical advantages over other simplistic alternatives to analyze the complexity of any charge distribution to all orders. In particular, we showed that the paradox of the apparent inequality of the energy-derived and self-force derived masses does not appear even in the case of complex dipole-dipole interactions, thus confirming previous first-order results. We also calculated the field lines for this system in some special cases.

Very recently, it has been shown that the present state of the art in trapped atom interferometry can potentially probe the self-induced forces produced by gravitation on trapped atoms [25]. Although the vertical component of the van der Waals force between two hydrogen atoms at Eq. (32) is small, corresponding to relative accelerations $\sim 10^{-15}g$, it is possible to vastly enhance these interactions by inducing atomic dipole moments with external electric fields, as in dipolar Bose-Einstein condensate (BEC) gases [26,27]. For instance, one can show that excited Rydberg atoms can undergo self-forces corresponding to accelerations $\sim 10^{-6}g$, well within the ultimate gravimetric sensitivity with trapped atoms, $\sim 10^{-13}$ [28]. These findings are important because they indicate that this phenomenon, so far believed to be completely inaccessible to experimentation [10,19], may offer a new strategy to test the equivalency of energy and *gravitational* mass by means of quantum physics in curved space. This is timely in view of the recent remarkable verification of the equivalence of energy and *inertial* mass [29].

In a masterful review of the problems of classical electron theory, which motivated much research relating to the present paper, Pearle wrote that *"[t]he state of the classical electron theory reminds one of a house under construction that was abandoned by its workmen upon receiving news of an approaching plague. The plague in this case, of course, was quantum theory."* [13]. It is therefore worthwhile to notice that Whittaker's solution for the single charge has now been shown to solve a crucial paradox of classical electrodynamics and to confirm, even for quantum systems, the validity of Fermi's early expectation that self-force methods and energy methods must be equivalent.

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at *La Sapienza*, for very stimulating follow-up conversations, and for bringing Refs. [1] to my attention. R. T. Jantzen also checked many calculations and drew my attention to the importance of the multiplicative factor at Eq. (20).

APPENDIX

Here we briefly summarize Whittaker's procedure [2] to solve Eq. (8):

$$\frac{\partial^2\phi}{\partial x'^2} + \frac{\partial^2\phi}{\partial y'^2} + \frac{\partial^2\phi}{\partial z'^2}\left(1 + \frac{2gz'}{c^2}\right) = 0. \tag{A1}$$

By carrying out a standard separation of variables [30], we write the trial solution in cylindrical coordinates, as suggested by the symmetry of the problem:

$$V'_W(\rho', \phi', z') = P(z')Q(\rho')\Phi(\phi'), \tag{A2}$$

where (ρ', ϕ', z') are the appropriate cylindrical coordinates. Direct substitution into the above field equation yields the following solutions:

$$P(z') = \left(1 + \frac{2gz'}{c^2}\right)^{1/2} J_1\left[\frac{ic^2k}{g}\left(1 + \frac{2gz'}{c^2}\right)^{1/2}\right], \tag{A3}$$

$$\Phi(\phi') = \begin{matrix}\sin m\phi' \\ \cos m\phi'\end{matrix}, \tag{A4}$$

$$Q(\rho') = J_m(k\rho'), \tag{A5}$$

where k and m are the separation constants and J_m are the Bessel functions of order m. At this point, Whittaker recalls that, in the absence of a gravitational field, the single charge solution is proportional to $1/r$, which, in cylindrical coordinates, can be expanded by means of the well-known integral [30]:

$$\frac{1}{\rho'^2 + z'^2} = \int_0^\infty e^{-kz'} J_0(k\rho')dk. \tag{A6}$$

By comparing our trial solution with the solution valid in flat space, Whittaker conjectures a similar integral superposition solution based on the integral [31]:

$$2al\int_0^\infty kI_1(lk)K_1(ak)J_0(bk)dk = \frac{a^2 + b^2 + l^2}{\sqrt{(a^2 + b^2 + l^2)^2 - 4a^2l^2}}, \tag{A7}$$

where

$$l = \frac{c^2}{g}; \qquad a = \frac{c^2}{g}\left(1 + \frac{2gz'}{c^2}\right)^{1/2}; \qquad b = \rho' = \sqrt{x'^2 + y'^2} \tag{A8}$$

and I_1 and K_1 are the modified Bessel functions. By substituting appropriately, we find the expression at our Eq. (9), to within the multiplicative constant q. Direct substitution into the field equation confirms this is indeed the correct solution for a point charge.

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